5

Truth-Functional Propositional Logic

1. INTRODUCTION

In this chapter, and the remaining chapter 6, we turn from the vista of logic as a whole and concentrate solely on the Logic of Unanalyzed Propositions. Even then, our focus is a limited one. We say nothing more about the method of inference and concern ourselves mainly with how the method of analysis can lead to knowledge of logical truth.

The present chapter takes a closer look at the truth-functional fragment of propositional logic. We try to show: (1) how the truth-functional concepts of negation, conjunction, disjunction, material conditionality, and material biconditionality may be expressed in English as well as in symbols; (2) how these concepts may be explicated in terms of the possible worlds in which they have application; and (3) how the modal attributes of propositions expressed by compound truth-functional sentences may be ascertained by considering worlds-diagrams, truth-tables, and other related methods. In effect, we try to make good our claim that modal concepts are indispensable for an understanding of logic as a whole, including those truth-functional parts within which they seemingly do not feature.

2. TRUTH-FUNCTIONAL OPERATORS

The expressions "not", "and", "or", "if . . . then . . . ", and "if and only if" may be said to be *sentential operators* just insofar as each may be used in ordinary language and logic alike to 'operate' on a sentence or sentences in such a way as to form *compound* sentences.

The sentences on which such operators operate are called the *arguments* of those operators. When such an operator operates on a *single* argument (i.e., when it operates on a single sentence, whether simple or compound), to form a more complex one, we shall say that it is a *monadic operator*. Thus the expressions "not" and "it is not the case that" are monadic operators insofar as we may take a simple sentence like

(5.1) "Jack will go up the hill"

and form from it the compound sentence

(5.2) "Jack will not go up the hill"

or (more transparently)

(5.3) "It is not the case that Jack will go up the hill."

Or we may take a compound sentence like

(5.4) "Jack will go up the hill and Jill will go up the hill"

and form from it a still more complex sentence such as

(5.5) "It is not the case that Jack will go up the hill and Jill will go up the hill."[1]

When an expression takes as its arguments *two* sentences and operates on them to form a more complex sentence we shall say that it is a *dyadic operator.* Thus, the expression "and" is a dyadic operator insofar as we may take two simple sentences like

(5.1) "Jack will go up the hill"

and

(5.6) "Jill will go up the hill"

and form from them a compound sentence such as

(5.7) "Jack and Jill will go up the hill"

or (more transparently)

(5.8) "Jack will go up the hill and Jill will go up the hill."

Or we may take two compound sentences like

(5.2) "Jack will not go up the hill"

and

(5.9) "Jill will not go up the hill"

and form from them a still more complex sentence such as

(5.10) "Jack will not go up the hill and Jill will not go up the hill."

The expressions "or", "if . . . then . . . ", and "if and only if" are also dyadic operators. Dyadic operators are sometimes called sentential *connectives* since they connect simpler sentences to form more complex ones.[2]

1. Note that this sentence is ambiguous between "It is not the case that Jack will go up the hill and it is the case that Jill will go up the hill" and "It is not the case both that Jack will go up the hill and Jill will go up the hill." This ambiguity, along with many others, is easily removed in the conceptual notation of symbolic logic, as we shall shortly see.

2. Some authors like to regard "it is not the case that" as a sort of degenerate or limiting case of a connective — a case where it 'connects' just one sentence. We, however, will reserve the term "connective" for dyadic operators only.

Now each of the sentential operators cited above is commonly said to be *truth-functional* in the sense that each generates compound sentences out of simpler ones in such a way that the truth-values of the propositions expressed by the compound sentences are determined by, or are a function of, the truth-values of the propositions expressed by the simpler sentential components. Thus it is commonly said that "it is not the case that" is truth-functional since the compound sentence "It is not the case that Jack will go up the hill" expresses a proposition which is true in just those possible worlds in which the proposition expressed by its simple sentential component "Jack will go up the hill" is false, and expresses a proposition which is false in just those possible worlds in which the proposition expressed by the latter sentence is true; that "and" is truth-functional since the compound sentence "Jack will go up the hill and Jill will go up the hill" expresses a proposition which is true in just those possible worlds in which the propositions expressed by the sentential components "Jack will go up the hill" and "Jill will go up the hill", are both true, and expresses a proposition which is false in all other possible worlds; that "or" is truth-functional since the compound sentence "Jack will go up the hill or Jill will go up the hill" expresses a proposition which is true in all those possible worlds in which at least one of the propositions expressed by the sentential components is true, and expresses a proposition which is false in all other possible worlds; and so on.

This common way of putting it gives us a fairly good grip on the notion of truth-functionality. But it is seriously misleading nonetheless. For it is just plain false to say of each of these sentential operators that it *is* truth-functional in the sense explained. We should say rather that each *may be used* truth-functionally while allowing that some at least may also be used non-truth-functionally. Let us explain case by case.

The uses of "not" and "it is not the case that"

It is easy enough to find cases in which the word "not" operates truth-functionally. When, for instance, we start with a simple sentence like

(5.11) "God does exist"

and insert the word "not" so as to form the compound sentence

(5.12) "God does not exist"

we are using "not" truth-functionally. The proposition expressed by the compound sentence *(5.12)* will be true in all those possible worlds in which the proposition expressed by the simple sentential component of that sentence is false, and will be false in all those possible worlds in which the latter is true. But suppose now that we start with a simple sentence,

(5.13) "All the children are going up the hill"

and insert the word "not" so as to form the compound sentence

(5.14) "All the children are not going up the hill."

This latter sentence is ambiguous. And the answer to the question whether the operator "not" is being used truth-functionally on *(5.13)* depends on which of two propositions *(5.14)* is being used to express. On the one hand, *(5.14)* could be used by someone to express what could better, that is, unambiguously, be expressed by the sentence

(5.15) "It is not the case that all the children are going up the hill."

In such a circumstance we would say that the "not" in *(5.14)* is being used (even though infelicitously) truth-functionally. But if, on the other hand, *(5.14)* were to be used to express the proposition which would be expressed by the sentence

(5.16) "None of the children is going up the hill"

then we would want to say that the "not" in *(5.14)* would be used non-truth-functionally. In this latter case, the truth-value of the proposition expressed by *(5.16)* viz., the proposition,

(5.17) None of the children is going up the hill

is not determined by, is not a truth-function of, the proposition expressed by the simple sentence *(5.13)*, viz., the proposition

(5.18) All the children are going up the hill.

The two disambiguations of the sentence *(5.14)*, viz., the sentences *(5.15)* and *(5.16)*, express propositions which are logically non-equivalent. Only the former of these propositions is a truth-function of the proposition expressed by the simple sentential component of *(5.14)*, viz., the simple sentence *(5.13)*, "All the children are going up the hill"; the other is not. Why is the proposition expressed by the sentence *(5.16)* — i.e., the proposition *(5.17)*, that none of the children is going up the hill — *not* a truth-function of the proposition *(5.18)*, viz., that all the children are going up the hill? The answer is simply that the truth-value of *(5.17)* is not determined by, i.e., is not a function of, the truth-value of *(5.18)*. It would suffice for *(5.17)*'s not being a truth-function of *(5.18)* if either the truth of *(5.18)* did not determine the truth-value of *(5.17)* or the falsity of *(5.18)* did not determine the truth-value of *(5.17)*. As it turns out, however, both these conditions obtain: *neither* the truth *nor* the falsity of *(5.18)* determines the truth-value of *(5.17)*. For there are possible worlds in which *(5.18)* is true and in which *(5.17)* is false, e.g., worlds in which there are children and they all are going up the hill. But in addition, there are possible worlds in which *(5.18)* is true, but so is *(5.17)*, e.g., worlds in which there are no children (see chapter 1, p. 19, footnote 12). Then, too, there are possible worlds in which *(5.18)* is false, and in which *(5.17)* is true, e.g., worlds in which there are children, but none of them is going up the hill. And finally there are possible worlds in which *(5.18)* is false and *(5.17)* is likewise, e.g., worlds in which some, but not all, of the children are going up the hill. In short, the truth-value of *(5.17)* is undetermined by the truth-value of *(5.18)*. Not so, however, with the proposition expressed by the sentence *(5.15)*. This proposition is a truth-function of the proposition *(5.18)*. In any possible world in which *(5.18)* is true, the proposition expressed by *(5.15)* is false; and in any possible world in which *(5.18)* is false, the proposition expressed by *(5.15)* is true.

By way of contrast with the word "not", the expression "it is not the case that" (which we used in *(5.15)*) seems always to operate truth-functionally. Prefix it to any proposition-expressing sentence, whether simple or compound, and the resultant compound sentence will express a proposition which is true in all those possible worlds in which the proposition expressed by its sentential component is false; and vice versa. Thus it is that an effective test for determining whether "not" is being used truth-functionally in a compound sentence is to see whether the proposition being expressed by that sentence can equally well be expressed by a compound sentence using "it is not the case that" instead. If it can be so expressed then "not" is being used truth-functionally; if it cannot then "not" is being used non-truth-functionally.

But why this preoccupation with the truth-functional sense of "not", the sense that is best brought out by the more pedantic "it is not the case that"? We earlier said (pp. 14–15) that any proposition which is true in all those possible worlds in which a given proposition is false and which is false in all those possible worlds in which a given proposition is true, is a *contradictory* of that proposition. When therefore we now say that in its truth-functional uses "not" generates a compound sentence out of a simpler one in such a way that the proposition expressed by the compound sentence will be true in all those possible worlds in which the proposition expressed by the simpler one is false, and will be false in all those possible worlds in which the latter is true, we are simply saying that in its truth-functional uses "not" expresses the *concept* of negation and that the proposition expressed by either one of these sentences is a contradictory of the proposition expressed by the other. Hence the significance, for logic, of the truth-functional uses of "not". For between them, it will be remembered, a proposition and any of its contradictories are *exclusive* in the sense that there is no possible world in which both are true, and *exhaustive* in the sense that in each of all possible worlds it must be that one or the other of them is true.

We have earlier introduced a simple piece of *conceptual notation* for the truth-functional uses of the monadic sentence-forming operators "not" and "it is not the case that", i.e., for those uses of these expressions in which they express the concept of negation. Recall that we write the symbol " ∼ " (called *tilde*) in front of the symbol for any proposition-expressing sentence "P", just when we want to express the negation of that proposition. Then " ∼ P" expresses the negation of P. We read " ∼ P" as "it is not the case that P" or, more briefly, as "not-P". Alternatively, " ∼ P", can be read as, "It is false that P", or as "P is false".

It is important to note that tilde is not to be regarded simply as a piece of shorthand for an expression in some natural language such as English. For the reasons already given it should not be regarded, for instance, simply as a shorthand way of writing whatever we would write in English by the word "not". Rather it is to be regarded as a piece of notation for that which certain expressions in natural languages such as English may, on occasion, be used to express, viz., the *concept* of negation.

The truth-functional properties of the concept of negation can be displayed in the simple sort of chart which logicians call a *truth-table*. The truth-table for negation may be set out thus:

	P	∼P
(row 1)	T	F
(row 2)	F	T

TABLE *(5.a)*

In effect, a truth-table is an abbreviated *worlds-diagram*.[3] In the (vertical) column, to the left of the double line, under the letter "P", we write a "T" and an "F" to indicate, respectively, all those possible worlds in which the proposition P is *true*, and all those possible worlds in which the proposition P is *false*. "T" represents the set of all possible worlds (if any) in which P is true; "F" represents the set of all possible worlds (if any) in which P is false. Together these two subsets of possible worlds exhaust the set of all possible worlds. Each possible world is to be thought of as being included either in the (horizontal) row marked by the "T" in the left-hand column of table *(5.a)* or in the (horizontal) row marked by the "F" in that column. In short, the rows of the left-hand column together represent an exhaustive classification of all possible worlds.

3. More exactly, it is a schematic collapsed set of worlds-diagrams. Note how table *(5.a)* captures some, but not all, of the information in figure *(5.b)*.

Obviously, however, for some instantiations of "P", one or other of these rows will represent an *empty* set of possible worlds. In the case where P is contingent, both rows of the truth-table will represent *non*-empty sets of possible worlds. But if P is *non*contingent, then one or the other row of the truth-table will represent an empty set of possible worlds. Thus, for example, if P is necessarily true, then the first row of the truth-table will represent the set of *all* possible worlds, and the second row will represent an empty set of possible worlds. On the other hand, if P is necessarily false, the latter pattern will be reversed. If P is necessarily false, then the first row of the truth-table represents an empty set of possible worlds and the second row represents the set of all possible worlds. This fact will be seen to have important consequences when we try to use truth-tables to ascertain the modal attributes of propositions.

In the right-hand column of the truth-table, under the symbol "$\sim$ P", we write down the truth-value $\sim$ P will have in each of the two sets of possible worlds defined by the rows of the left-hand column. Thus, reading across the first row of the table, we can see that in those possible worlds (if any) in which P is true, $\sim$ P is false; and reading across the second row, we can see that in those possible worlds (if any) in which P is false, $\sim$ P is true.

It is easy to see that truth-functional negation is an operation which 'reverses' the truth-value of any proposition on which it 'operates', i.e., which is its argument. That is to say, $\sim$ P has the opposite truth-value to P, whatever the truth-value of P happens to be. It follows, too, that $\sim\sim$ P has the *same* truth-value as P in all possible cases. This latter fact is usually referred to as the *Law of Double Negation.* It is in this sense, and this sense only, that one may correctly say "two negatives make a positive".

Table *(5.a)* enables us to introduce a rule for the depiction of the negation of a proposition on a worlds-diagram. The rule is this:

> Represent the negation of a proposition by a bracket spanning all the possible worlds, if any, which are not spanned by a bracket representing the proposition itself.

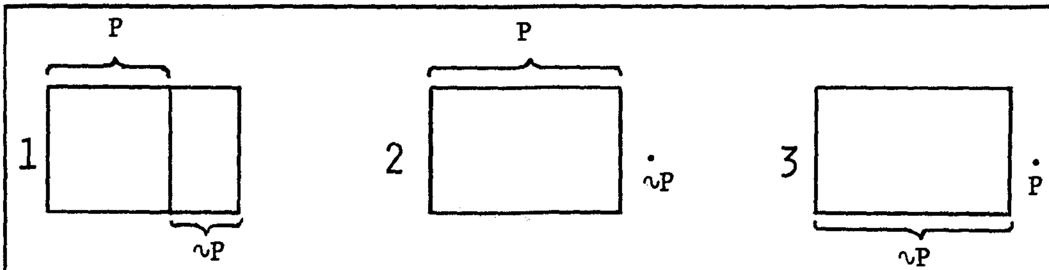


FIGURE *(5.b)*

Later in this chapter (section 9), we shall use this rule, together with rules for the depiction of other truth-functional operators, in order to devise a procedure for ascertaining the modal attributes of certain propositions.

The uses of "and"

In its truth-functional uses, "and" is a dyadic sentence-forming operator on sentences, i.e., a sentence-forming connective, which expresses the concept of *conjunction.* We symbolize conjunction in our conceptual notation by writing the symbol "·" (to be called *dot*) between the symbols for the sentences conjoined. Thus where "P" is the symbol for a proposition-expressing sentence (whether simple or compound) and "Q" is the symbol for a proposition-expressing sentence (again whether simple or compound), then "P · Q" expresses the conjunction of "P" and "Q". The truth-table for conjunction is

P	Q	P · Q
T	T	T
T	F	F
F	T	F
F	F	F

TABLE *(5.c)*

As with all truth-tables it is helpful to regard this one also as an abbreviated worlds-diagram. The four horizontal rows constitute a mutually exclusive and jointly exhaustive classification of all possible worlds. The first two rows (i.e., the rows bearing "T"s under the single "P") together represent all those worlds in which P is true. This subset of worlds in which P is true is in turn subdivided into that set in which Q is also true (represented on the truth-table by row 1 and marked by the "T" in column 2 under the "Q"), and into that set in which Q is false (row 2). And the set of possible worlds in which P is false is, in turn, subdivided into two smaller sets, that in which Q is true (row 3) and that in which Q is false (row 4). Together these four rows represent every possible distribution of truth-values for P and for Q among all possible worlds. Every possible world must be a world in which either (1) P and Q are both true, (2) P is true and Q is false, (3) P is false and Q is true, or (4) P is false and Q is false. There can be no other combination. Thus every possible world is represented by one or another row of our truth-table.

Again, as on table *(5.a)* (the truth-table for negation), we point out that for some instantiations of the symbols on the left-hand side, some of the various rows of the truth-table will represent an empty set of possible worlds. Thus, for example, if P is necessarily true and Q is contingent, both the third and the fourth rows of table *(5.c)* will represent empty sets of possible worlds. For in both these sets, P has the value "F", and there are, of course, no possible worlds in which a necessarily true proposition is false. Other combinations of modal status for P and for Q will, of course, affect the table in other, easily ascertainable, ways. We investigate the consequences of this in section 5.

To the right of the double line in the truth-table for conjunction we are able to read the truth-value of the proposition expressed by "P · Q" for each of the four specified sets of possible worlds. Only in those worlds in which both P and Q are true, is P · Q true. In all other cases (worlds), P · Q is false.

Table *(5.c)* enables us to introduce a rule for the depiction of conjunction on our worlds-diagrams. It is this:

> Represent the conjunction of two (or more) propositions by a bracket spanning the set of possible worlds, if any, in which both propositions are true.

The fifteen diagrams are:

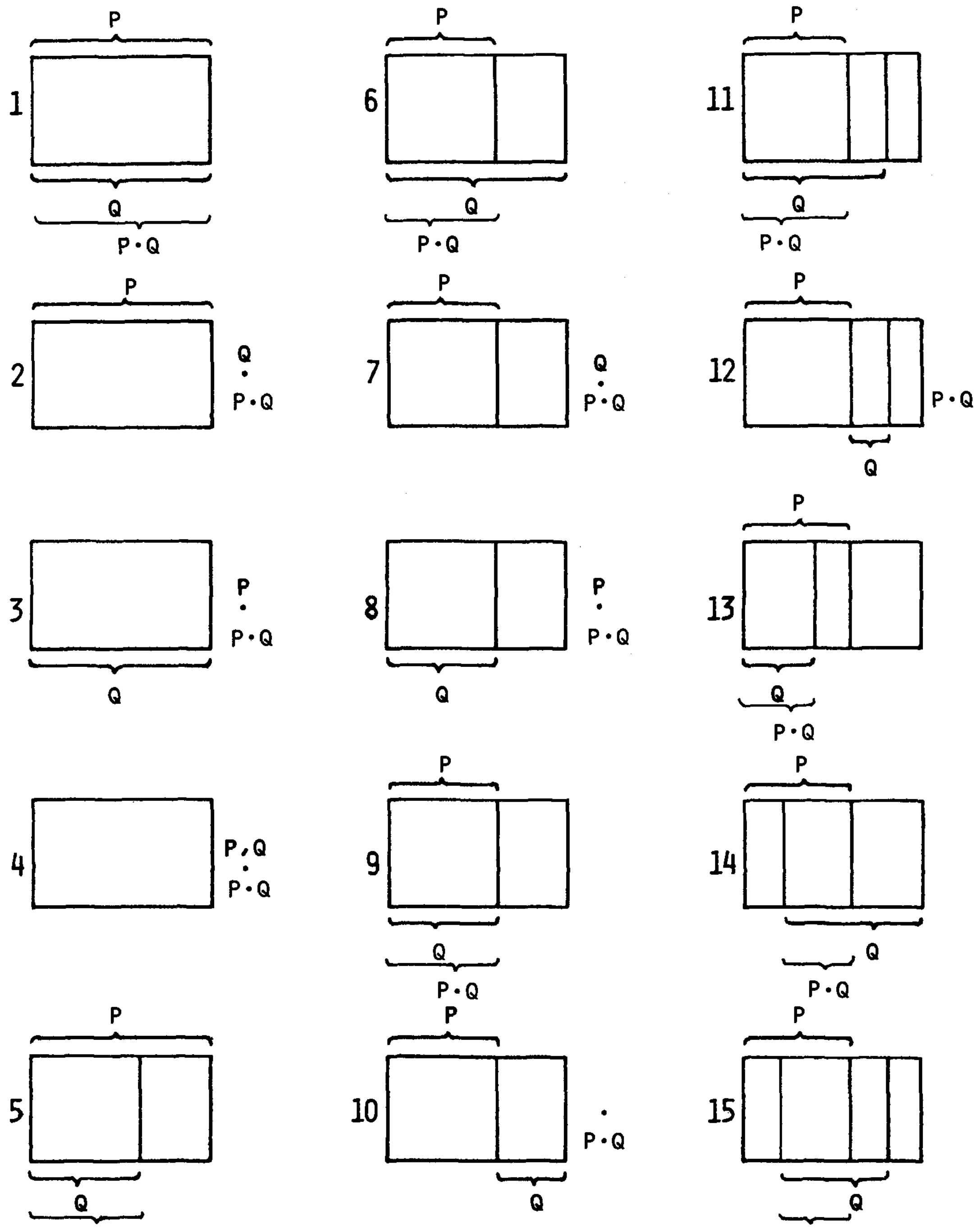


FIGURE *(5.d)*

It can be seen that if P and Q are inconsistent with one another (as in 2, 3, 4, 7, 8, 10, and 12), then there will not be any set of *possible* worlds in which both are true, and hence there will be no area on the rectangle which is common to the segments representing those propositions. In such a case the bracket representing the conjunction (P · Q) is relegated to a point, external to the rectangle, which represents the set of impossible worlds. But if the propositions involved are consistent with one another (as in 1, 5, 6, 9, 11, 13, 14, and 15), then there *will* be a set of possible worlds in which those propositions are both true, and hence there will be an area on the rectangle which is common to the segments representing those propositions. After all, to say that two propositions are consistent is *just to say* that it is possible that they should both be true together, i.e., that there is a possible world in which both are true. Not surprisingly then, the segment on our rectangle which represents the conjunction of two propositions is just that segment whose presence is indicative of the fact that those propositions are consistent with one another. In other words, two propositions are consistent with one another if and only if their conjunction is possibly true.

Note that the symbol " · " (dot) should no more be regarded merely as a shorthand abbreviation for "and" than "∼" (tilde) should be regarded as a shorthand abbreviation for "not". Three main considerations lead us to say that it is an item of *conceptual* notation.

In the first place, there are other ways, in English, of expressing the concept of conjunction. Suppose we want to assert the conjunction of the proposition that there are five oranges in the basket and the proposition that there are six apples in the bowl. One way of expressing their conjunction would be to use the sentence "There are five oranges in the basket *and* six apples in the bowl." But the conjunction of these two propositions might be expressed in other ways as well. We might use the sentence "There are five oranges in the basket *but* six apples in the bowl." Or we might say, "There are five oranges in the basket; *however*, there are six apples in the bowl." Or, again, we might say, "*Although* there are five oranges in the basket, there are six apples in the bowl." The words "but", "however", and "although", just as much as the word "and", may be used in truth-functional ways to express the concept of conjunction. When these words are so used, the truth-conditions for the propositions expressed by the sentences they yield are precisely the same as those for the propositions expressed by the sentences which, in its truth-functional uses, "and" may be used to construct. The truth-conditions for the propositions they then express are those specified in the truth-table for " · ".

In the second place, the concept of conjunction can be conveyed without using any sentence-connective whatever. One way — indeed one of the commonest of all ways — of expressing the conjunction of two propositions is simply to use first the sentence expressing one and then the sentence expressing the other. If we want to assert both that there are five oranges in the basket and that there are six apples in the bowl, then we need only utter, one after the other, the two separate sentences: "There are five oranges in the basket" and "There are six apples in the bowl." We will then be taken, correctly, to have asserted both that there are five oranges in the basket and that there are six apples in the bowl. The fact that someone who asserts first one proposition and then another has thereby asserted both of them, licenses the *Rule of Conjunction* (see chapter 4, section 4). Here again we find no one-to-one correspondence between uses of "and" and the concept of conjunction. The concept of conjunction can be expressed by connectives other than "and" and can even be expressed in the absence of any sentence-connective at all.

In the third place, the sentence-connective "and" admits of uses in which it is *not* truth-functional — uses in which the compound sentences which it helps to form express propositions whose truth-values are not determined solely by the truth-values of the propositions expressed by the simple sentences which "and" connects. This can easily be seen if we reflect on the fact that conjunction, which "and" expresses in its truth-functional uses, is *commutative*, in the sense that the order of the conjuncts makes no difference to the truth-value of their conjunction. We have only to inspect the truth-table for conjunction to see that the truth-conditions for P · Q are precisely the same as the truth-conditions for Q · P. But consider the case where "and" is used to conjoin the two sentences

(5.19) "John mowed the lawn"

and

(5.20) "John sharpened the lawn mower."

Sentences *(5.19)* and *(5.20)* can be conjoined in either of two ways to yield, respectively,

(5.21) "John mowed the lawn and John sharpened the lawn mower"[4]

and

(5.22) "John sharpened the lawn mower and mowed the lawn."

Are the truth-conditions for the proposition expressed by *(5.21)* the same as the truth-conditions for the proposition expressed by *(5.22)*? Hardly. The proposition which would ordinarily be expressed by *(5.21)* could well be true while that ordinarily expressed by *(5.22)* might be false. In cases such as these, the *order* in which the sentential components occur when they are connected by "and" makes a great deal of difference to the truth-values of the propositions expressed by the resulting compound sentences. For the order in which the simple sentences, "John mowed the lawn" and "John sharpened the lawn mower", occur is taken to convey a certain temporal ordering of the events which these sentences assert to have occurred. The most natural reading of *(5.21)* would be to read it as asserting that John sharpened the mower *after* he mowed the lawn; while the most natural reading of *(5.22)* would have this latter sentence asserting that John sharpened the mower *before* he mowed the lawn. In short, as used in *(5.21)* and *(5.22)*, "and" can be taken to mean "and *then*". In these cases the compound sentences formed through the use of "and" are not commutative as are those sentences resulting from using "and" in a purely truth-functional way. The meaning of "and" in such sentences is not exhausted. as it is in its truth-functional uses, by the truth-conditions for conjunction.

How, if at all, can our conceptual notation for conjunction capture the 'extra' meaning which "and" has in its non-truth-functional, noncommutative uses? How, for instance, can we convey in our conceptual notation the idea of temporal ordering which is intrinsic to our understandng of sentences such as *(5.21)* and *(5.22)*?[5] The answer lies, not in tampering with the meaning of "·", but in modifying the sentences conjoined. The simplest way of doing this is to use temporal indices such as "... at time 1" (abbreviated "at t_1") or "... at time 2" (abbreviated "at t_2"). We can express what we mean in sentences *(5.21)* and *(5.22)* in other sentences which use "and" truth-functionally, if we treat the components of *(5.21)* and *(5.22)* as context-dependent sentences (chapter 2, p. 75ff) — sentences which have to be made context-free by the use of some temporal index if we are to know what proposition each expresses. Thus we can make explicit the meaning of *(5.21)*, and at the same time use "and" truth-functionally, by saying

(5.23) "John mowed the lawn at t_1 and sharpened the mower at t_2."

Here the temporal indices do the job of conveying the fact that the first-mentioned event occurred before the latter-mentioned one. And similarly we could convey the sense of *(5.22)* by saying

(5.24) "John sharpened the mower at t_1 and mowed the lawn at t_2."

4. To comply with ordinary English style, we delete the reiteration of the grammatical subject, i.e., "John" in the second conjunct of the conjunctions below, specifically in *(5.22) – (5.25)*.

5. The particular non-truth-functional use of "and" here being examined should not be thought to be the only non-truth-functional use of "and." There are others. For example, "and" is also sometimes used to convey *causal* relations, as when we might say, "He fell on the ski slopes and broke his ankle."

These latter two sentences, in which the temporal indices occur explicitly, *are* commutative. Thus the truth-conditions are *identical* for the following two assertions:

(5.23) "John mowed the lawn at t_1 and sharpened the mower at t_2."

(5.25) "John sharpened the mower at t_2 and mowed the lawn at t_1."

Not only do these latter reformulations of *(5.21)* make explicit what that original sentence implicitly asserts, but they substitute truth-functional uses of "and" for a non-truth-functional one and thus render the original sentence susceptible to treatment within our conceptual notation.

The uses of "or"

First, some reminders. A compound sentence consisting of two proposition-expressing sentences joined by "or" is said to be a *disjunction*. The two component sentences in the disjunction are said to be its *disjuncts*. And the operation of putting together two proposition-expressing sentences by means of the dyadic operator "or" is called the *disjoining* of those two sentences.

The dyadic sentence connective "or", like "and", is often used truth-functionally. But, unlike "and", "or" has two distinct truth-functional uses. Sometimes it is used to mean that, of the two propositions expressed by the sentences it connects, *at least one* is true; sometimes it is used to mean that, of the two propositions expressed by the sentences it connects, *one and only one* is true. Let us distinguish between these two uses by speaking of *weak* or *inclusive disjunction* in the first case, and of *strong* or *exclusive disjunction* in the second case.

The concept of weak disjunction is captured in our conceptual notation by the symbol "$\vee$" (to be called *vel* or *wedge* or *vee*). Its truth-conditions are given in the following truth-table:

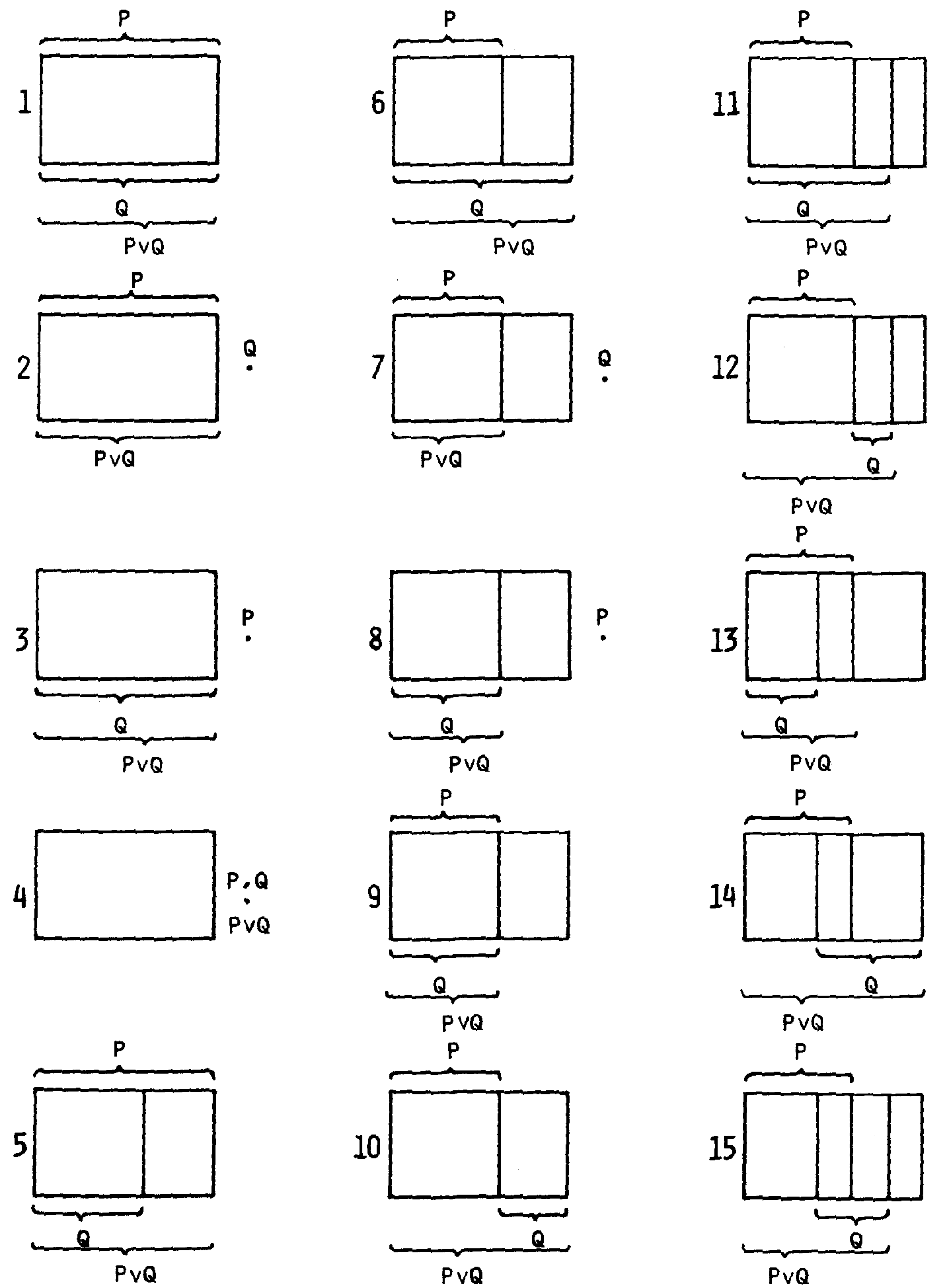
P	Q	P ∨ Q
T	T	T
T	F	T
F	T	T
F	F	F

TABLE *(5.e)*

Table *(5.e)* enables us to introduce a rule for the depiction of (weak) disjunction on a worlds-diagram. It is this:

> Represent the (weak) disjunction of two propositions by a bracket spanning the set of possible worlds, if any, in which at least one of the two propositions is true.

It can now be seen that unless both the propositions disjoined are necessarily false (as in 4) there will


1
P
Q
PvQ
2
P
Q
PvQ
3
P
Q
PvQ
4
P, Q
PvQ
5
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PvQ
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P
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PvQ
15
P
Q
PvQ

FIGURE *(5.f)*

always be at least some possible worlds in which their disjunction is true. That is to say, the proposition expressed by a disjunctive sentence is *possibly true* unless both the propositions expressed by its disjuncts are necessarily false.

Many, if not most, of our ordinary uses of "or" are weakly disjunctive and hence are captured by the truth-table for "$\vee$". If, for example, we were to explain John's absence from an examination which we knew he was intent on writing by saying

(5.26) "John is ill or he missed the bus"

we would be saying something whose truth is compatible with the possible state of affairs of John's being ill *and* his missing the bus. That is, if it should turn out both that John was ill and that he missed the bus, we should hardly want to say that what *(5.26)* expresses is false. Quite the contrary: if John both was ill and missed the bus, then what *(5.26)* expresses would be true.

But other uses of "or" are strongly disjunctive and are not captured by the truth-table for "$\vee$". Consider the following example:

(5.27) "The origin of the *Trumpet Voluntary,* traditionally attributed to Henry Purcell, has been the subject of much recent dispute. This piece of music was composed by Purcell or it was composed by Jeremiah Clarke."

In this latter instance, the connective "or" is almost certainly intended by the speaker to represent the 'stronger' species of truth-functional disjunction. The most natural reading of this example would be that in which the speaker is asserting that either Purcell or Clarke, *but not both of them,* composed the *Trumpet Voluntary.* The symbol we use for the stronger, exclusive, sense of "or" is "$\underline{\vee}$" (to be called *vee-bar*). Its truth-table is this:

P	Q	$P \underline{\vee} Q$
T	T	F
T	F	T
F	T	T
F	F	F

TABLE *(5.g)*

If "or" is interpreted in its stronger sense in *(5.27)*, then the second sentence of *(5.27)* will express a falsehood if Purcell and Clarke both composed the *Trumpet Voluntary* — see row 1 of table *(5.g)*. (Both would have composed it if each had independently composed the identical piece of music.)

There are, then, two senses of "or", each of which is truth-functional. However, in our conceptual notation we shall make use of only one of them: the inclusive sense represented by the symbol "$\vee$". The exclusive sense occurs less frequently, and when it does it can easily be defined in terms of concepts already at our disposal: specifically, the concept of negation, represented by "$\sim$"; the concept of conjunction, represented by "$\cdot$"; and the concept of inclusive disjunction, represented by "$\vee$". (We shall see precisely how to state this definition later in this chapter [p. 309].)

We have been speaking so far of weak and strong (i.e., inclusive and exclusive) disjunction.

Hereinafter when we use the term "disjunction" without qualification we shall mean "weak" or "inclusive disjunction" which may be symbolized by the use of "$\vee$".

We have given examples in which the English operators, "not" and "and", are used non-truth-functionally. Are there similar examples in which "or" also is used non-truth-functionally? There are, indeed, some such examples, but they are relatively more rare than the corresponding non-truth-functional uses of "not" and "and". That is to say, although "or" is sometimes used to connect proposition-expressing sentences in a non-truth-functional fashion, its uses in this role are very much less frequent than the uses of "not" and "and" in non-truth-functional roles. Let us examine an instance. Consider:

(5.28) "Any solution is acidic which will turn litmus paper red, or, nothing but an acidic solution will turn litmus paper red."

The occurrence of "or" in *(5.28)* is non-truth-functional. It is insufficient for the truth of what this sentence expresses, that the two disjuncts express truths. What more is required is that both sentences express equivalent propositions. In effect, the "or" in this instance is being used with the same sense as "i.e." to mean "that is". In effect, only if the two disjuncts express equivalent, as well as true propositions, is what the disjunction expresses true. And clearly, two disjuncts can both express true propositions without those propositions being equivalent. Just consider: the proposition that litmus paper is purple is true; but had the sentence, "litmus paper is purple", replaced the like-valued second conjunct of *(5.28)*, we should hardly want to say that the resulting sentence still asserted something true. For *(5.28)* would then become

(5.29) "Any solution is acidic which will turn litmus paper red, or, litmus paper is purple."

Although both disjuncts of *(5.29)* express truths, that sentence itself expresses a false proposition. Since substituting another sentence expressing a different true proposition for the second disjunct in *(5.28)* yielded (in *(5.29)*) a disjunctive sentence expressing a false proposition, the use of "or" in *(5.28)* is *not* truth-functional.[6]

Since the operators in the conceptual notation we are introducing represent only truth-functional operators, it is clear that we cannot capture the whole sense of *(5.28)* through the use of these operators alone. Nonetheless these truth-functional operators can, and need to, be called upon to express *part* of the sense of that sentence. When we were explaining, just above, the truth-conditions of the proposition expressed by *(5.28)* we said that it would be true if (1) the disjuncts of *(5.28)* express equivalent propositions, and (2) those propositions are true. But notice: in saying this, we have just invoked the truth-functional use of "and". And what this means is that the non-truth-functional use of "or" in *(5.28)* is to be explicated in terms of, among other things, a truth-functional operator. This particular result is not exceptional. Virtually all non-truth-functional operators have, we might say, a truth-functional 'component' or 'core'.

EXERCISE

For each of the following cases, construct a worlds-diagram and bracket that portion of the diagram representing $P \veebar Q$*:*

6. The authors wish to express their thanks to their colleague, Raymond Jennings, for calling their attention to this non-truth-functional use of "or".

a. *P is necessarily true; Q is contingent*

b. *P is necessarily true; Q is necessarily true*

c. *P is contingent; Q is a contradictory of P*

d. *P is contingent; Q is necessarily false*

* * * * *

Interlude: compound sentences containing two or more sentential operators

As long as a sentence — for example, "A · B" — in our conceptual notation contains only a single operator, there is no opportunity for that sentence to be ambiguous. But when a sentence contains two or more operators, generally that sentence will be ambiguous unless measures are taken to correct it. Before we give an example of such an ambiguity in our conceptual notation, let us examine a parallel case in arithmetic. Consider the sentence

(5.30) "$X = 3 + 5 \times 2$"

What is the value of "X"? There is no clear answer to this question, for the expression "$3 + 5 \times 2$" is obviously ambiguous. This expression could mean either (1) that X is equal to the sum of three and five (which is of course eight), which in turn is multiplied by two, yielding a value of sixteen for X; or (2) that three is to be added to the product of five and two, which would then yield a value of thirteen for X. Such an ambiguity is, of course, intolerable and must be corrected. The easiest way to correct it (but not the only way) is to introduce bracketing, using parentheses, to group the parts into unambiguous components. Thus the two ways of reading *(5.30)* can be distinguished clearly from one another in the following fashion:

(5.31) "$X = (3 + 5) \times 2$"

(5.32) "$X = 3 + (5 \times 2)$"

Now let us examine a parallel ambiguity in an English sentence which uses two operators. Let us return to one of the examples which introduced our discussion of "and".

(5.5) "It is not the case that Jack will go up the hill and Jill will go up the hill."

At the time we introduced this sentence we mentioned that it is ambiguous. In our conceptual notation it is a simple matter to resolve the ambiguity. Letting "B" stand for "Jack will go up the hill", and "G" stand for "Jill go up the hill", we may express the two different propositions which may be expressed by *(5.5)* in the following two different, unambiguous, sentences:

(5.33) "(∼B) · G"[7]

(5.34) "∼(B · G)"

7. As we shall see in a moment, the parentheses around the first conjunct of *(5.33)* are not essential. However, at this point, since we lack the explicit rules which would allow us to read *(5.33)* unambiguously if the parentheses were to be deleted, we require them.

In English, each of these two sentences may be expressed this way:

(5.35) "It is not the case that Jack will go up the hill, and [or "but" if you prefer] it is the case that Jill will."

(5.36) "It is not the case both that Jack will go up the hill and that Jill will go up the hill."

Notice that if *(5.33)* and *(5.34)* had been written without the parentheses, they would be indistinguishable from one another and would be ambiguous in exactly the same sort of way that *(5.5)* is.

Clearly our conceptual notation stands in need of some device to enable us to disambiguate otherwise ambiguous sentences. Bracketing (i.e., the use of parentheses) is one such device. We shall adopt it here.[8]

The formation rules in a logic are designed to yield only those unambiguous strings of symbols which we earlier (chapter 4, section 5) called *wffs* (well-formed formulae). Truth-functional Propositional Logic allows for the construction of two kinds of well-formed formulae, those called "sentences" and those called "sentence-forms". The difference between the two (which we will see in due course is an important difference) is determined by the fact that the former contain no sentence-variables and the latter contain at least one sentence-variable.[9]

The formation rules for securing well-formedness in formulae (i.e., in sentences and in sentence-forms) in Truth-functional Propositional Logic are:

R1: Any capital letter of the English alphabet standing alone is a wff.

R2: Any wff prefixed by a tilde is a wff.

R3: Any two wffs written with a dyadic truth-functional connective between them and the whole surrounded by parentheses is a wff.

Examples: The following are well-formed formulae (wffs) according to the rules R1 – R3:

$$A$$

$$\sim P$$

$$(P \vee B)$$

$$(\sim(P \cdot Q) \vee (R \cdot \sim S))$$

(Of these, the first is a sentence; the other three are sentence-forms.)

The following are *not* wffs:

$$A\sim$$

$$P\vee$$

8. For an exposition of a parentheses-free notation, see I.M. Copi, *Symbolic Logic,* fourth edition, New York, Macmillan, 1973, pp. 231–2.

9. The English letters "A" through "O" are designated as being sentence-constants; the letters "P" through "Z", sentence-variables. For the significance of this distinction, see section 6, pp. 301ff.

$P \cdot B \vee C$

$(A \cdot B \vee C)$

We also adopt the following *conventions:*

A: *We may, if we like, drop the outermost pair of parentheses on a well-formed formula.*

Example: "(P∨B)" may be rewritten as "P∨B".

[Note, however, that if a wff has had its outermost parentheses deleted, those parentheses must be restored if that formula is to be used as a component in another formula.]

B1: *We may, if we like, drop the parentheses around any conjunct which is itself a conjunction.*

Example 1: "(A · ((B∨C) · D))" may be rewritten as "(A · (B∨C) · D)".

Example 2: By two successive applications of this convention, we may rewrite "(A · ((B∨C) · (D · E)))" as "(A · (B∨C) · D · E)".

B2: *We may, if we like, drop the parentheses around any disjunct which is itself a disjunction.*

Example: "(A∨((B · C)∨D))" may be rewritten as "(A∨(B · C)∨D)".

The uses of "if . . . then . . . "

Sometimes we want to assert that a proposition P isn't true unless a proposition Q is also true; i.e., that it is not the case both that P is true and that Q is false. A natural way of saying this in English is to utter a sentence of the form "If P then Q" (or sometimes, more simply, "If P, Q"). We shall call any sentence of this form a *conditional* sentence. A conditional sentence, then, is a compound sentence formed out of two simpler ones by means of the dyadic sentence-connective "if . . . then . . . " (or, sometimes, "if" where the "then" is unexpressed but understood). The simpler sentence which occurs in the if-clause we shall call the *antecedent;* the one which occurs in the then-clause we shall call the *consequent.*

It is obvious enough that in those instances when a conditional of the form "If P then Q" is used *simply* to assert that it is not the case both that P is true and that Q is false, the connective "if . . . then . . . " is functioning in a purely truth-functional way. For in a sentence of the form "It is not the case both that P is true and that Q is false" both the operators "it is not the case that"(a monadic operator, it will be remembered) and "and" (a dyadic operator) are functioning purely truth-functionally. Hence the compound sentence "It is not the case both that P is true and that Q is false" is a truth-functional sentence. Indeed, it can be recorded in the conceptual notation already at our disposal by writing "$\sim(P \cdot \sim Q)$". It follows that in those instances when "If P then Q" is used to assert no more than "It is not the case both that P is true and that Q is false", the conditional "If P then Q" is itself truth-functional and can be recorded as "$\sim(P \cdot \sim Q)$". The proposition, If P then Q, will then have the *same* truth-conditions as $\sim(P \cdot \sim Q)$: it will be true in all those possible worlds in which it is not the case both that P is true and that Q is false, i.e., true in all those possible worlds in which $\sim(P \cdot \sim Q)$ is true; and it will be false in all and only those possible worlds in which the negation of $\sim(P \cdot \sim Q)$ is true, i.e., false in all and only those possible worlds in which $(P \cdot \sim Q)$ is true, i.e., false in all and only those possible worlds in which P is true and Q is false.

We call any sentence expressing a proposition which has these truth-conditions, a *material conditional;* and we call the relation which holds between the proposition expressed by the antecedent and the proposition expressed by the consequent of such a conditional the relation of *material conditionality.*

The relation of material conditionality is rendered in our conceptual notation by writing the symbol "$\supset$" (to be called *hook* or *horseshoe*) between the symbols for the sentences it connects. Thus where "P" is the symbol for a proposition-expressing sentence and "Q" is the symbol for a proposition-expressing sentence, "$P \supset Q$" is the symbol for the material conditional within which "P" occurs as antecedent and "Q" occurs as consequent.

As we have just shown, the relation of material conditionality will hold between any two propositions P and Q (in that order) in every possible world except in those possible worlds in which P is true and Q is false. Hence the truth-table for material conditionality is:

P	Q	$P \supset Q$
T	T	T
T	F	F
F	T	T
F	F	T

TABLE *(5.h)*

Table *(5.h)* allows us to introduce a rule for the depiction of material conditionality on a worlds-diagram:

> Represent the relation of material conditionality obtaining between two propositions by a bracket spanning all those possible worlds, if any, in which it is not the case that the first is true and the second is false.

This rule may be easier to grasp if we break it down into two stages:

1. Find all the possible worlds in which P is true and Q is false.
2. Draw the bracket for $P \supset Q$ so as to span all the possible worlds, if any, which remain.

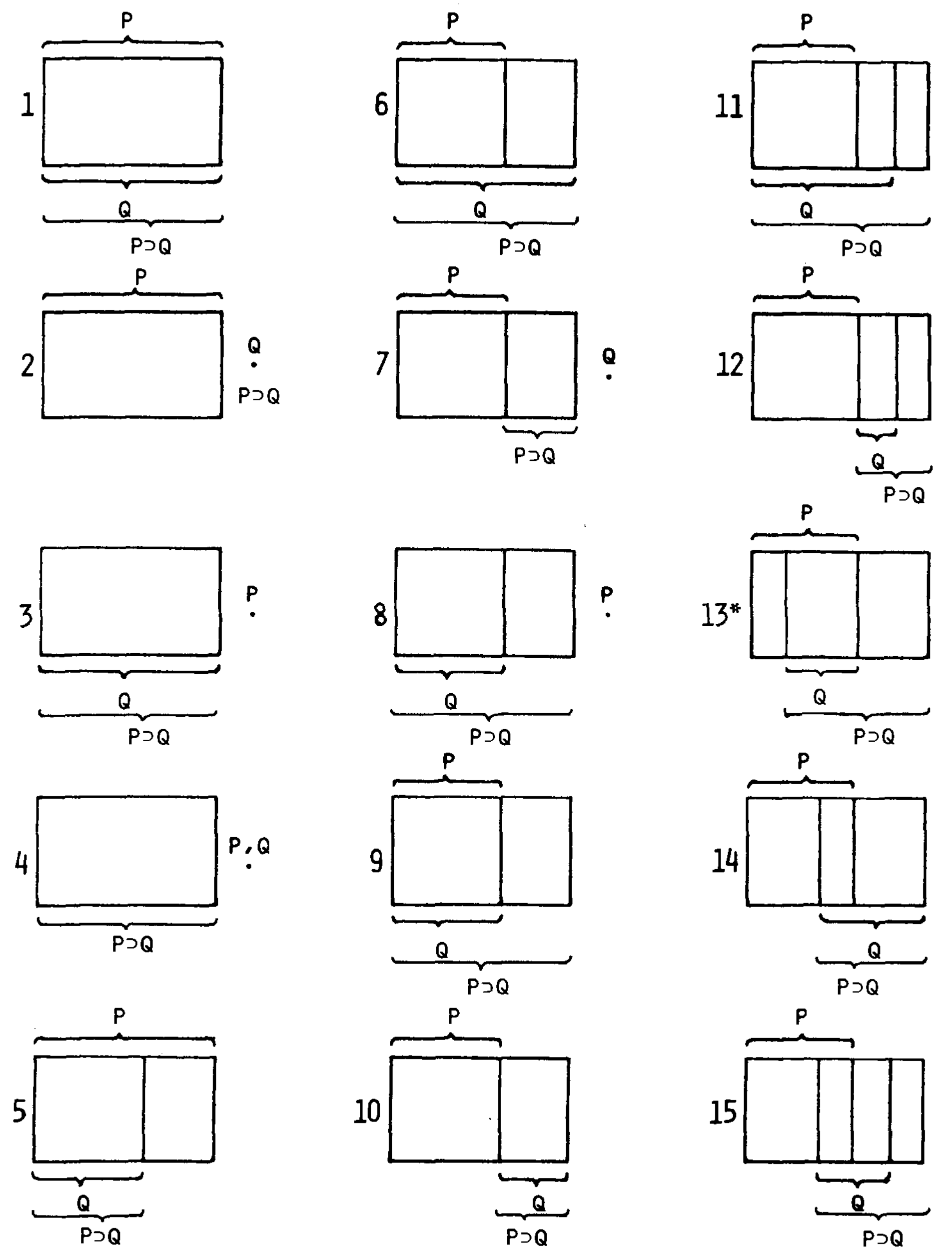


FIGURE *(5.i)*

* Note that for ease in placing the bracket for "P⊃Q" on diagram 13, we have moved the segment for Q to the right-hand side of the segment for P. No logical relations are disturbed by our doing this.

Figure *(5.i)* shows that in all those worlds-diagrams which depict an instance in which P implies Q (viz., 1, 3, 4, 6, 8, 9, and 11), we find that the bracket for $P \supset Q$ spans all possible worlds. That is to say, in all and only those cases in which P implies Q, $P \supset Q$ is true in all possible worlds. (Note carefully: this latter fact is reflected in our definition (c) of "implication" which appears in chapter 1, p. 31.)

Further, we can see that unless P is necessarily true and Q is necessarily false (as in 2), there will always be some possible worlds in which the relation of material conditionality holds between P and Q. That is to say, any proposition asserting that the relation of material conditionality holds between two propositions, P and Q, is *possibly true* unless P is necessarily true and Q is necessarily false. Of course such a proposition will not be *true in fact* unless the possible worlds in which it is true include the *actual world.* A proposition which asserts that the relation of material conditionality holds between two propositions, P and Q, is true in fact only when in the actual world it is not the case both that P is true and that Q is false.

In discussing the sentential operators, "it is not the case that", "and", and "or" we had little difficulty in citing examples of their uses which were purely truth-functional — uses, that is, in which they simply expressed the truth-functional concepts of negation, conjunction and (weak or inclusive) disjunction, respectively. In this respect the sentence connective "if . . . then . . . " is somewhat different. Only rarely do we ever assert in ordinary discourse a conditional sentence which is purely truth-functional. An example would be:

(5.37) "If he wrote that without any help, then I am a monkey's uncle."

Here the truth-functional property of the connective "if . . . then . . . " is relied upon, together with our knowledge that the proposition expressed by the consequent is blatantly false, in order to assert the falsity of the proposition expressed by the antecedent. For the only condition under which $P \supset Q$ may be true while Q is false, is for P also to be false. On nearly every occasion when we use a conditional sentence in a strictly truth-functional way, we are using it in the facetious manner of *(5.37)*; we are adopting a style of speech which allows us colorfully to deny a proposition (that expressed by the antecedent of the conditional) without uttering the words "not", "it is not the case that", or "I deny that", etc.

Apart from the just-mentioned curious use of a conditional sentence, there do not seem to be any other sorts of examples in which the use of the "if . . . then . . . " connective is *purely* truth-functional — examples in which that connective is used to express the (truth-functional) concept of material conditionality and that concept alone. For as philosophers of language have often pointed out, sentences of the form "If P then Q" usually express much *more* than a mere truth-functional relation. Usually such sentences assert or presuppose more of a connection between P and Q than that which holds when it is not the case both that P is true and Q false. For instance, the connection may be the logical relation of implication, as is expressed by the sentence

(5.38) "If the Queen's husband has children, then he is someone's father."

(We might call this a *logical* conditional. The proposition expressed by a logical conditional is true if and only if the proposition expressed by the antecedent of that conditional *logically implies* the proposition expressed by the consequent.) Or, the connection may be a causal one, as it is in the case of the sentence

(5.39) "If the vacuum cleaner motor short-circuits, the fuse in the electrical box in the basement will blow."

(We might call this a *causal* conditional. The proposition expressed by a causal conditional is true if and only if the proposition expressed by the antecedent of that conditional *causally implies* the proposition expressed by the consequent.) Or, the connection may be the sort of connection which involves explicit or implicit statistical correlations as in the case of the sentence,

(5.40) "If there are six plates on the table, then there are six persons expected for dinner."

(We might call this a *stochastic* or *statistical* conditional. The proposition expressed by a stochastic conditional is true if and only if the proposition expressed by the antecedent of that conditional *probabilifies* (i.e., raises the probability of) the proposition expressed by the consequent.)

Each one of these sentences, *(5.38)*, *(5.39)*, and *(5.40)*, being of the form, "If P then Q", is a conditional sentence, but the connections asserted between the propositions expressed by their respective antecedents and their respective consequents are *stronger* than the purely truth-functional relation of material conditionality.

A puzzle arises. If virtually none of the conditional sentences we utter in ordinary discourse are to be construed as material conditionals, why, then, have logicians been concerned to define the relation of material conditionality in their conceptual notation? For we must admit that it seems hardly likely that logic should be much concerned with propositions of the sort expressed by *(5.37)*. Much could be written by way of an answer. But for present purposes three points will have to suffice.

In the first place, it is important to point out that it is a *necessary* condition for the truth of any proposition which is expressed by a conditional sentence — of any sort whatever, including the non-truth-functional ones — that it should not be the case that the proposition expressed by the antecedent of that sentence be true while the proposition expressed by the consequent be false. But this is just to say that no proposition expressed by any sort of conditional sentence is true unless the proposition expressed by the corresponding material conditional sentence is true. This fact can be put to advantage. Suppose we have a non-truth-functional sentence such as *(5.39)*, and we are intent on discovering the truth-value of the proposition it expresses. The specification of the truth-conditions of non-truth-functional sentences is very much more difficult than of truth-functional ones, and *(5.39)* is no exception. To say precisely under what conditions *(5.39)* expresses a truth and precisely under what conditions it expresses a falsehood is no easy matter and has been an object of perennial interest and investigation. Clearly the truth-conditions of *(5.39)* cannot be the same as the truth-conditions of the corresponding material conditional: *(5.39)* need not express a true proposition even though its antecedent and consequent both express true propositions. For example, we can imagine a situation in which the vacuum cleaner motor did short-circuit and the fuse did blow and yet the proposition expressed by *(5.39)* is false; the circuit for the vacuum cleaner does not pass through the fuse box in the basement; the fuse's blowing was the result of a 'coincidence'; it was not *caused* by the vacuum cleaner's malfunction. Under these circumstances, the proposition expressed by *(5.39)* would be false, even though the corresponding material conditional would express a truth. In sum, then, the truth-conditions for non-truth-functional conditionals differ from the truth-conditions for the truth-functional material conditional. Nonetheless, the material conditional has a role to play when it comes to ascertaining the truth-value (as opposed to the truth-conditions) of the proposition expressed by *(5.39)*. For this much we may confidently assert: if the *material* conditional which corresponds to *(5.39)* expresses a false proposition, that is, if the antecedent of *(5.39)* expresses a true proposition, and the consequent of *(5.39)* expresses a false proposition, then *(5.39)* expresses a false proposition. This result is perfectly general, and we may summarize by saying that the *falsity*-conditions of the material conditional constitute part of the truth-conditions (i.e., truth-value conditions) of *every* conditional.

In the second place, there are other occasions when it is useful to render certain conditionals as material ones. For example, we have said earlier that arguments are deductively valid if their premises imply their conclusions. Another way of putting this is to say that an argument is deductively valid if (and only if) a material conditional sentence, whose antecedent is the conjunction of all the premises of that argument and whose consequent is the conclusion of that argument, expresses a proposition which is necessarily true. (Later in this chapter [section 4] we will have more to say about this point, and will actually ascertain the deductive validity of some arguments by means of constructing a material conditional sentence and by looking to see what the modal status is of the proposition expressed by that sentence.)

In the third place, our conceptual notation for the material conditional lends itself to supplementation by symbolic notations which capture some of those 'extra' elements of meaning which characterize non-truth-functional conditionals. For instance, as we shall see when we consider Modal Propositional Logic in the next chapter, the non-truth-functional, modal relation of implication which holds between the proposition expressed by the antecedent and that expressed by the consequent in a logical conditional can be captured in our symbolic notation by supplementing the notation for the truth-functional material conditional in this way: "$\Box(P \supset Q)$". Similarly, the non-truth-functional relation which holds between the proposition expressed by the antecedent and that expressed by the consequent in a causal conditional may be expressed in an expanded notation in this way: "$\boxed{c}(P \supset Q)$". Here "$\boxed{c}$" is to be read as "It is causally necessary that . . . " or as "In all possible worlds in which the same causal laws hold as in the actual world, it is true that . . . "

In sum, then, there is ample reason for logicians to be interested in defining and using such a notion as material conditionality, even though this particular relation is only rarely asserted in ordinary discourse to hold between two propositions. It is, for the most part, a technical notion which plays an important and basic role in logic; in particular in the analysis of all conditionals, truth-functional and non-truth-functional alike.

Nonetheless — in spite of its genuine utility — we ought not to lose sight of the *peculiar* nature of the relation of material conditionality. Unfortunately, some logic books incautiously refer to the relation symbolized by "$\supset$" as the relation of "material *implication*". The trouble with this description is that it has misled countless people into supposing that, where a proposition $P \supset Q$ is true, there must be some connection between the antecedent, P, and the consequent, Q, akin to that which holds when P really does imply Q, (i.e., when P logically implies Q). But this supposition leads to apparent paradox. It can easily be seen, by attending to the truth-conditions for $P \supset Q$ (as captured in table *(5.h)*), that when P is false then no matter whether Q is true or false the material conditional $P \supset Q$, will be true [see rows (3) and (4)]; and again that where Q is true then no matter whether P is true or false the material conditional, $P \supset Q$ will be true [see rows (1) and (3)]. Give "$\supset$" the description "material implication" and these truth-conditions generate the so-called "paradoxes of material implication": that a false proposition materially implies any proposition whatever, and that a true proposition is materially implied by any proposition whatever. We should have to say accordingly that a false proposition such as that Scotch whisky is nonalcoholic materially implies any and every proposition that one cares to think of — that Harding is still president of the U.S., that he is not still president of the U.S., and so on. Similarly, we should have to say that a true proposition such as that potatoes contain starch is materially implied by any and every proposition that one cares to think of — that Aristotle was a teacher of Alexander the Great, that he wasn't, and so on.

These consequences seem paradoxical because, on the one hand, they accord with our understanding of the truth-conditions for so-called material implication (and so seem to be true), while, on the other hand, they do not accord with our understanding of what the word "implication" ordinarily means (and so seem to be false). Of course, there is no real paradox here at all. We can avoid puzzlement either by constantly reminding ourselves that the term "implication", as it occurs in the description "material implication", must be stripped of all its usual associations, or (more simply and preferably) by avoiding the term "implication" altogether in this context and choosing to speak instead of "the relation of material conditionality". Likewise, instead of reading "$P \supset Q$" as "P materially implies Q" we may, if we wish, read it as "P materially conditionalizes Q". We have chosen the latter course. The only connection between the relation of material conditionality and the relation of implication properly so-called lies in the fact, observed a moment ago, that the relation of material conditionality will hold between P and Q in each and every possible world (i.e., $\Box(P \supset Q)$ will be true) just when the relation of implication holds between P and Q (i.e., when P implies Q). But the relation of material conditionality is *not* the relation of (logical) implication, and ought to be carefully and deliberately distinguished from it.

The uses of "if and only if"

Sometimes we want to assert not only that a proposition P isn't true without a proposition Q being true but also (conversely) that a proposition Q isn't true without a proposition P being true. One way of saying this in English would be to utter a sentence of the form "P if and only if Q". We shall call any sentence of this latter form a *biconditional.* An example (albeit a non-truth-functional one) is:

(5.41) "The motion voted on at the last meeting was passed legally if and only if at least eight members in good standing voted for it."

A biconditional sentence, then, is a compound sentence formed out of two simpler sentences by means of the dyadic sentence-connective "if and only if" (often abbreviated to "iff").

Biconditionals have many of the attributes that conditionals have. True, it makes no sense to speak of the antecedent and consequent of a biconditional, but in other respects there are obvious parallels. Like conditionals, biconditionals may be used to express simply a truth-functional relation or may be used to express any of several non-truth-functional relationships, e.g., logical, causal, or stochastic.

We shall call the truth-functional 'core' of any use of a biconditional sentence "the relation of *material biconditionality*" and will symbolize it in our conceptual notation by "$\equiv$" (to be called *triple bar*). The truth-conditions for the relation of material biconditionality may be set out as follows:

P	Q	$P \equiv Q$
T	T	T
T	F	F
F	T	F
F	F	T

TABLE *(5.j)*

Table *(5.j)* allows us to introduce a rule for the depiction of the relation of material biconditionality on a worlds-diagram:

Represent the relation of material biconditionality obtaining between two propositions by a bracket spanning *both* the area representing those possible worlds, if any, in which both propositions are true *and* the area representing those possible worlds, if any, in which both propositions are false.

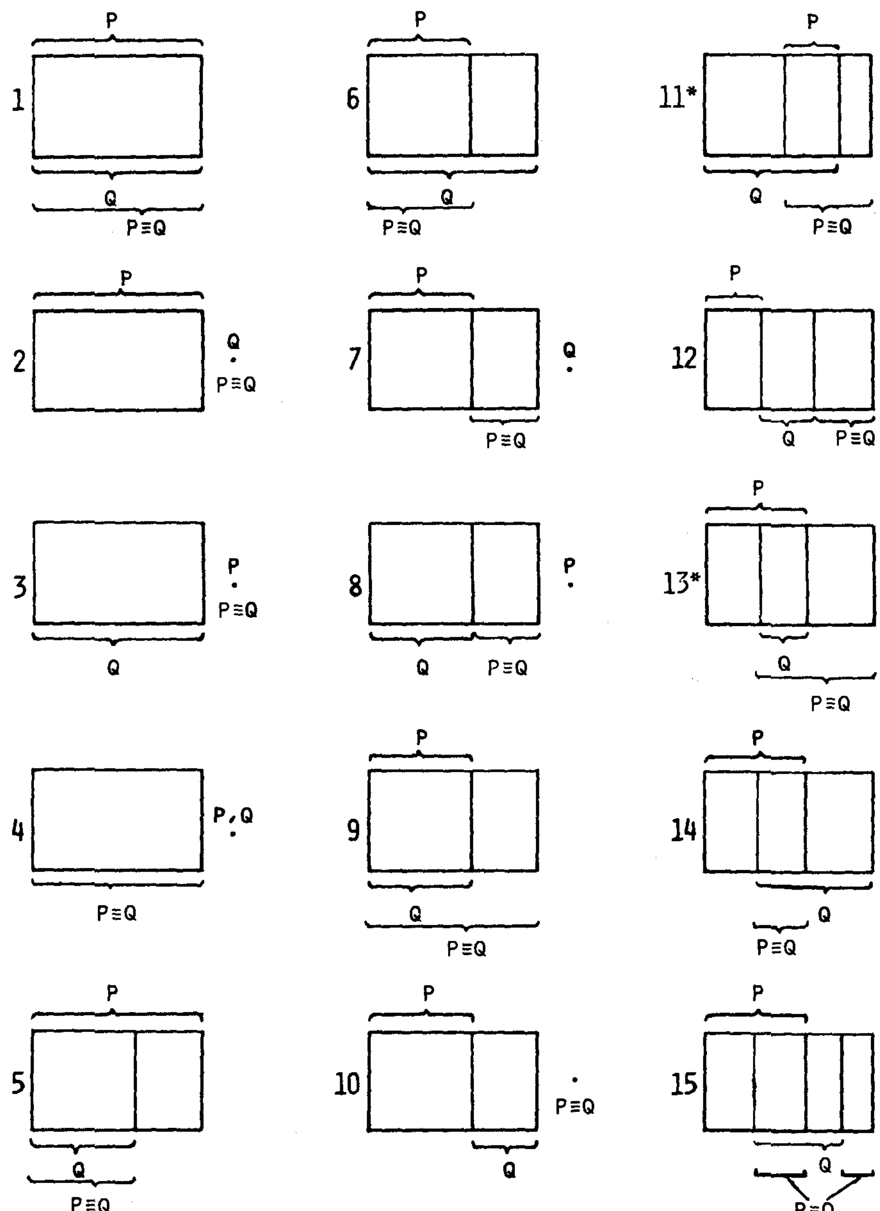


FIGURE *(5.k)*

* See footnote for figure *(5.i)*.

Figure *(5.k)* shows that the relation of material biconditionality holds between two propositions in some possible world unless those propositions are contradictories of one another (see diagrams 2, 3, and 10.) That is to say, unless two propositions are contradictories of one another, there will always be some possible world in which they are both true and/or both false, and hence there will be some possible world in which the relation of material biconditionality holds between them. (Note that the relation of material biconditionality will hold in some possible world for propositions which are contraries of one another. Not all cases of inconsistency preclude the relation of material biconditionality holding.)

Examples of purely truth-functional uses of the sentence connective "if and only if" are at least as rare, and odd, as those of purely truth-functional uses of the sentence connective "if . . . then . . . " But examples of non-truth-functional uses are easy to find. And such uses are of the same diverse sorts as are the non-truth-functional uses of "if . . . then . . . "

The connective "if and only if" is being used to express a logical biconditional in the sentence

(5.42) "Today is the day after Monday if and only if today is the day before Wednesday."

Here the connective "if and only if" is not being used *merely* to assert that the two propositions, (1) that today is the day after Monday and (2) that today is the day before Wednesday, have the same truth-value in the actual world. It is being used to express something stronger: namely, that in all possible worlds the two propositions have matching truth-values. In a word, what is being asserted is that the two propositions are logically equivalent.

Similarly, as was the case with the connective "if . . . then . . . ", the connective "if and only if" may be used to express a causal relation, to express what we might call a "causal biconditional".

(5.43) "This object will continue to move in a straight line at a fixed velocity if and only if no external force is applied to it."

Again, here the connective is not being used merely to assert that the two propositions, (1) that this object will continue to move in a straight line, and (2) that no external force is applied to this object, have the same truth-value in the actual world. Something more is being asserted than just this truth-functional minimum. What more is being asserted is that in all possible worlds in which the same causal laws hold as hold in the actual world these two propositions have matching truth-values.

There are, of course, many other kinds of non-truth-functional uses of the connective "if and only if" — uses in which the relation between the propositions expressed by the connected sentences is stronger than that of material biconditionality. It is unnecessary for us to describe such uses exhaustively and, of course, we couldn't do so even if we were to try. It suffices, for our purposes, that we recognize their existence and understand the reasons why logicians, despite the overwhelming preponderance of non-truth-functional uses in everyday discourse, have tended to concentrate in their conceptual notation — until comparatively recently — on the purely truth-functional uses. The reasons parallel those given in our discussion of material conditionality.

In the first place, by virtue of the fact that it is a logically necessary (although not, of course, a sufficient) condition of the truth of a proposition expressed by non-truth-functional biconditional that the corresponding material biconditional should express a truth, it follows that if the material biconditional expresses a falsehood, then the original non-truth-functional biconditional also expresses a falsehood; i.e., the falsity of the proposition expressed by a material biconditional is a (logically) sufficient condition of the falsity of the original proposition expressed by the non-truth-functional biconditional.

Secondly, the relation of material biconditionality, like the relation of material conditionality, is truth-preserving and falsity-retributive. But unlike the relation of material conditionality, it is in addition truth-retributive and falsity-preserving. By virtue of these facts, we can easily determine the truth-value of one of two propositions which stand in the relation of material biconditionality if we are antecedently given that the relation does hold and are given the truth-value of the other proposition: if one is true, we can validly infer that the other is also; and if either is false, we can validly infer that the other is also. For purposes of making *these* sorts of inferences any 'extra', non-truth-functional, elements of meaning may safely be ignored.

Thirdly, where need arises, we can always supplement the notation for the material biconditional by other symbolic devices such as "$\Box$" and "$\boxed{c}$" so that, for example, a logical biconditional can be rendered by writing a sentence of the form "$\Box(P \equiv Q)$ ". The need for such symbolic supplements to the basic notation for material biconditionality arises, for instance, when we want to record the fact (previously noted) that it is a (logically) necessary, but not a sufficient, condition for the truth of a logical biconditional that the corresponding material biconditional should be true. We can record this fact by saying that a proposition, expressible by a sentence of the form "$\Box(P \equiv Q)$" implies every proposition expressible by a sentence of the form "$(P \equiv Q)$", but not vice versa. This is a logical fact, entitling us to make certain inferences, which cannot be recorded symbolically without the explicit recognition, in symbols, of the non-truth-functional element of meanings which a logical biconditional has 'over and above' its purely truth-functional core.

Not surprisingly, there is still a further respect in which our discussion of material biconditionality parallels our discussion of material conditionality. We saw that the latter relation has sometimes been referred to misleadingly by the name "material implication". In much the same sort of way, the relation of material biconditionality has sometimes been referred to misleadingly by the name "material equivalence" — and with the same sort of apparent air of paradox. Read "$\equiv$" as "is materially equivalent to" and one is forced to conclude that any two true propositions are materially equivalent and that any two false propositions are materially equivalent. But, one is inclined to object, "equivalence" is too strong a description for the relation which holds, e.g., between the true proposition that Socrates was a teacher of Plato and the true proposition that Vancouver is the largest city in British Columbia, or again between the false proposition that $2 + 2 = 5$ and the false proposition that painting is a recently developed art form. The air of paradox may be removed, this time, either by putting the emphasis on the word "material" as it occurs in the expression "material equivalence", or by choosing to speak of the relation of material biconditionality. We have chosen the latter course as less likely to mislead. But whichever manner of speaking is adopted, the important point to bear in mind is this: it is a sufficient condition of the relation of material biconditionality holding that two propositions be logically equivalent to one another; but the converse does *not* hold. That two propositions stand in the relation of material biconditionality (or material equivalence, if one prefers) does not suffice to ensure that they also stand in the relation of logical equivalence. The two propositions, (1) that Socrates was a teacher of Plato, and (2) that Vancouver is the largest city in British Columbia, have matching truth-values (in the actual world) — they are true — and hence stand in the relation of material biconditionality. But they certainly are not logically equivalent.

Appendix: truth-tables for wffs containing three or more letters

For cases where we wish to construct a truth-table for a compound sentence with three propositional symbols we shall require a truth-table with eight rows; for a case where there are four propositional symbols, sixteen rows. More generally, where n is the number of propositional symbols occurring, we shall require 2^n rows in our truth-table.

We adopt the following *convention* for the construction of these various rows. Let m equal the number of required rows ($m = 2^n$). We begin in the column to the immediate left of the double vertical line and alternate "T"s and "F"s until we have written down m of them. We then move one column to the left and again write down a column of "T"s and "F"s, only this time we write down two "T"s at a time, then two "F"s, etc., until (again) we have m of them. If still more columns remain to be filled in, we proceed to the left to the next column and proceed to alternate "T"s and "F"s in groups of four. We keep repeating this procedure, in each column, doubling the size of the group occurring in the immediate column to the right, until we have finished filling in the left-hand side of the truth-table. In following this mechanical procedure we will succeed in constructing a table such that the various *rows* represent every possible combination for "T" and "F". The top row will consist entirely of "T"s; the bottom row, entirely of "F"s; and every *other* combination will occur in some intermediate row.

3. EVALUATING COMPOUND SENTENCES

Truth-functional compound sentences do not, of course, bear truth-values: no sentences do, whether they are simple or compound, truth-functional or not. Only the propositions expressed by sentences bear truth-values. Nonetheless there is a sense in which it is proper to speak of the "evaluation" of sentences. As we have seen, the truth-values of propositions expressed by truth-functional compound sentences are logically determined by the truth-values of the propositions which are expressed by the sentences which are the arguments of the truth-functional operators in those sentences. Evaluating a sentence consists in a procedure for ascertaining the truth-value of the proposition expressed by a truth-functional compound sentence given truth-value assignments for the propositions expressed by its sentential components.

Each of the examples of truth-functional compound sentences considered in the previous section featured only *one* sentential operator and at most *two* sentential arguments — one argument in the case of the monadic operator "$\sim$", and two arguments in the cases of the dyadic operators "$\cdot$", "$\vee$", "$\supset$", and "$\equiv$". It is time now to look at techniques for evaluating well-formed compound sentences which might feature any arbitrary number of truth-functional operators.

Although in ordinary speech and in casual writing, we have little occasion to produce sentences with more than just a few operators in them, the special concerns of logic require that we be able to construct and evaluate compound sentences of any degree whatever of complexity, short of an infinite degree of complexity. That is, we must be able to construct and to evaluate (at least in principle if not in practice) any truth-functional compound sentence having any finite number of truth-functional operators.

The Rules for Well-formedness allow us to *construct* sentences of any degree of complexity whatever. But how shall we *evaluate* intricate compound sentences? How might we evaluate a sentence such as "$\sim\sim A$" in which there are two operators; and how might we evaluate a still more complicated sentence such as "$(A \supset \sim B) \cdot (\sim A \supset B)$" in which there are five operators?

To answer this question we shall have to see how the truth-tables of the previous section might be *used*, and this requires that we make a distinction between sentence-variables and sentence-constants.

The "P"s and "Q"s which were featured in our truth-tables for negation, conjunction, disjunction, material conditionality, and material biconditionality, as arguments of the operators, "$\sim$", "$\cdot$", "$\vee$", "$\supset$" and "$\equiv$" respectively, were *sentence-variables*. They stood indiscriminately for any proposition-expressing sentences whatever. But in addition to these kinds of symbols, we shall also want our conceptual notation to contain symbols which stand for *specific* sentences, and not — as variables do — for sentences in general. These symbols we shall call *sentential-constants* since they have a constant, fixed, or specific interpretation. We shall use capital letters from the beginning of the English alphabet — "A", "B", "C", "D", etc. — as our symbols for sentential-constants, and will reserve capital letters from the end of the alphabet — "P" through "Z" — as our symbols for sentential-variables.[10] Finally we add that any wff containing a sentential-variable is to be called a

10. All capital letters of the English alphabet are to be considered wffs, and hence the rules of the construction of wffs containing sentential-constants are just those already given.

sentence-form, while any wff containing only sentential-constants or containing only sentential-constants and sentence-forming operators, is to be called (simply) a *sentence.*

To see how we might use the truth-tables of the previous section to evaluate truth-functional compound sentences containing any number of operators, we must view the sentential-constants in sentences as substitution-instances of the sentential-variables (i.e., the "P"s and "Q"s) featured on those tables. If the truth-values of the propositions expressed by the sentential-constants in a truth-functional sentence are given, then — by referring to the truth-tables for the various truth-functional operators — we may evaluate the whole sentence by means of a step-by-step procedure beginning with the simplest sentential components of that sentence, evaluating then the next more complex components of that sentence, repeating the procedure — evaluating ever more complex components — until the entire sentence has been evaluated.

Consider some examples. Let us start, as it were "from scratch", with some sentences in a natural language such as English.

Example 1:

A believer and an atheist are arguing. The believer begins by enunciating the proposition that God exists. She says

(5.44) "God exists."

A little later, after advancing some of the standard arguments for atheism, the atheist concludes

(5.45) "God doesn't exist."

The believer makes the immediate rejoinder:

(5.46) "That's not the case"

and goes on to say what she thinks is wrong with the atheist's case.

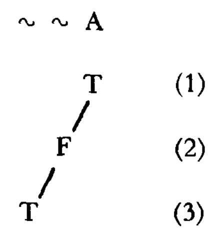
Here it is evident that *(5.46)* is to be construed as expressing the negation of the proposition expressed by *(5.45)*, and that *(5.45)* is to be construed as expressing the negation of the proposition expressed by *(5.44)*. Adopting, now, our conceptual notation for sentential-constants, we may symbolize each of these three sentences respectively as

(5.44a) "A"

(5.45a) "∼A"

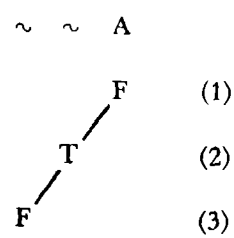
(5.46a) "∼∼A"

Now since negation is a truth-functional operation, it follows that the truth-value of the proposition expressed by "∼∼A" is a function of the proposition expressed by "∼A", and that the truth-value of the proposition expressed by "∼A" is, in turn, a function ot the truth-value of the proposition expressed by "A". If, then, we could presume the truth-value of the proposition expressed by "A", it would be an easy matter to evaluate both the sentences "∼A" and "∼∼A", and thereby to ascertain the truth-values of the propositions expressed by these sentences. Without committing ourselves to claiming that "A" does in fact express a truth, let us consider the consequences of hypothesizing its truth. To do so, we simply *assign* "T" to the sentence "A". By treating "A" as a substitution-instance of "P" in the truth-table for negation (p. 251), we can infer that the sentence expressing the negation of A, viz., "∼A", is to bear the evaluation "F"; and then, as a further step, by treating "∼A" in turn as itself a substitution-instance of "P" in the truth-table for negation, we can infer that the sentence expressing the negation of ∼A, viz., "∼ ∼A", is to bear the evaluation "T". All of these steps may be combined on a single "evaluation tree".



Here step (1) records our initial assignment of "T" to "A"; step (2) records the consequential assignment, made by reference to the truth-table for negation, of "F" to "∼A" (see row 1 in table *(5.a)*); and step (3) records the consequential assignment, made once more by reference to the truth-table for negation, of "T" to "∼ ∼A" (see row 2 in table *(5.a)*).

If, on the other hand, we had chosen as our initial assignment "F" to "A", it is an easy matter to see that we would have generated instead the following evaluation tree:



Example 2:

A partygoer says:

(5.47) "If I am out of town this weekend I won't be able to make your party. Otherwise I'll be there."

Here it is evident enough that what the partygoer has asserted might be expressed less colloquially and more perspicuously by saying:

(5.48) "If I am out of town this weekend then it is not the case that I'll be at your party. If it is not the case that I am out of town this weekend then I'll be at your party"

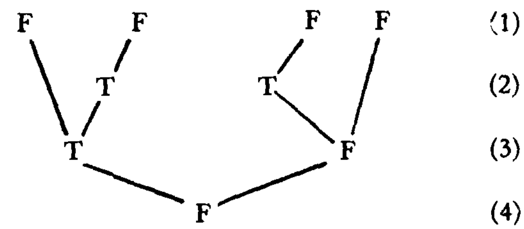
and that this might be expressed even more perspicuously in our conceptual notation as:

(5.49) "$(B \supset \sim C) \cdot (\sim B \supset C)$"

(with obvious readings for the constants "B" and "C").

Suppose now that we were given certain truth-values for the propositions expressed by the truth-functionally simple sentences "B" and "C". Suppose, for instance, that both propositions are false, i.e., that it is false that the partygoer is out of town on the weekend and false that he attends the party. Then we can evaluate the compound sentence which expresses the partygoer's claim by a number of simple steps which may be recorded thus:

$$(B \supset \sim C) \cdot (\sim B \supset C)$$



Step (1) records the initial assignment of "F" to each of "B" and "C". Step (2) records the consequential assignment of "T" to the sentences expressing the negations of B and C. Step (3) records the assignment, on the one hand, of "T" to "(B ⊃ ~ C)" (see row 3 of the truth-table for "⊃", p. 264), and, on the other hand, of "F" to "(~ B ⊃ C)" (see row 2 of the same truth-table). The final step, (4), records the assignment of "F" to the conjunction of (B ⊃ ~ C) and (~ B ⊃ C) (see row 2 of the truth-table for "·", p. 253). As we can see, each of these evaluations, after the initial assignment, is made by reference to the appropriate truth-table for the logical operator concerned.

Example 3:

Finally, by way of illustrating the technique of evaluating extended compound sentences, let us consider a sentence which is unlikely to be uttered in ordinary conversation:

(5.50) "In view of the facts that not only will there not be a downhill race today if the rain doesn't stop and the fog doesn't clear but also that there will not be a giant slalom tomorrow if the course doesn't harden overnight, the World Cup skiers will have no opportunity to gain points unless the rain stops and the fog clears or the course hardens overnight."

Given the information that it is true that the rain stops, that the fog clears, and that the course hardens overnight, but false that the downhill race is held today, that the giant slalom is held tomorrow, and that the World Cup skiers have no opportunity to gain points, what is the truth-value of the proposition expressed by sentence *(5.50)*? To ascertain the truth-value of the proposition expressed by this extended compound sentence we need only render that sentence in the conceptual notation of symbolic logic and proceed to evaluate it. Having expressed *(5.50)* in our conceptual notation, and having written below each sentence-constant a "T" or an "F" according as that constant expresses a true or a false proposition, we may then proceed to evaluate — by reference to the truth-tables — the consequential assignments for ever larger components of that sentence.

We assign various sentential constants as follows:

We let "A" = "There will be a downhill race today";

"B" = "The rain will stop";

"C" = "The fog will clear";

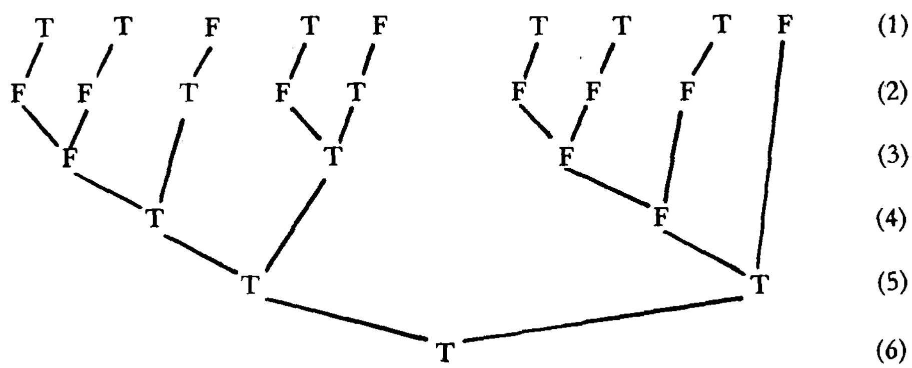
"D" = "There will be a giant slalom tomorrow";

"E" = "The course will harden overnight"; and

"F" = "The World Cup skiers will have no opportunity to gain points".

Using these sentential constants we may express and evaluate *(5.50)* thus:

$$(((\sim B \cdot \sim C) \supset \sim A) \cdot (\sim E \supset \sim D)) \supset (((\sim B \cdot \sim C) \vee \sim E) \supset F)$$



By taking recourse to the symbolism and truth-tables of formal logic we have been able to determine in a purely mechanical way, given the truth-values of the propositions expressed by "A" through "F", what the truth-value is of the proposition expressed by *(5.50)*.

This is no mean accomplishment, for it is unlikely that many of us could have done the exercise wholly in our heads. By having the means to 'break an evaluation down' into a series of completely mechanical steps, we are in a position to be able to evaluate sentences of any finite degree of complexity, whether they are sentences of ordinary conversation or the rather longer, more complex, sentences generated by the various special concerns of logic (e.g., in the testing of the validity of arguments — a matter which we shall begin to investigate shortly).

A note on two senses of "determined"

We have seen that each of the sentential operators "it is not the case that", "and", "or", "if . . . then", and "if and only if" admits of truth-functional uses — uses in which each generates compound sentences out of simpler ones in such a way that the truth-values of the propositions expressed by the compound sentences are *determined by* or are a *function of* the truth-values of propositions expressed by their simpler sentential components. In saying that the truth-values of the propositions expressed by truth-functional sentences are thus determined, we are, of course, making a purely *logical* point. We are saying, for instance, that what *makes* a proposition expressed by a compound sentence of the form "$\sim$ P" true are just those conditions which account for the falsity of the proposition expressed by the simpler sentence "P", and that what *makes* a proposition expressed by a compound sentence of the form "$\sim$P" false are just those conditions which account for the truth of the proposition expressed by the simpler sentence "P"; we are saying that what *makes* a proposition expressed by a compound sentence of the form "P∨Q" false are just those conditions which account for the falsity of both "P" and "Q"; and so on. The logical point we are making holds independently of whether anyone ever comes to know the truth-value of the propositions expressed by these compound sentences by coming to know the truth-values of the propositions expressed by their simpler sentential components.

But there is another sense in which we can speak of the truth-values of propositions expressed by compound sentences in Truth-functional Propositional Logic being "determined". We may speak of the truth-values of these propositions being determined, in the sense of being ascertained, by *us* on the basis of our knowledge of the truth-values of the propositions expressed by their simpler sentential components. In saying that their truth-values may be thus determined we are, of course, making an epistemic point.

The epistemic and logical points just made are, of course, connected. It is only insofar as the truth-values of the propositions expressed by compound sentences we are considering are, so to speak, *logically* detemined by the truth-values of the propositions expressed by their simpler sentential components that we can determine, *epistemically,* what their truth-values are, given initial assignments of truth-values to the propositions expressed by their simpler sentential components. How these initial assignments are made is, of course, another story. Sometimes it is on the basis of experience: we know what value-assignment to make experientially. Sometimes it is on the basis of reason or analytical thinking: we know what value-assignment to make ratiocinatively. And sometimes it is on the basis of mere supposition: we neither know experientially nor know ratiocinatively what the truth-values of these simple sentential components happen to be, but merely assume or suppose them to be such and such or so and so. But in whatever way these initial value-assignments are made, it is clear that the *consequential* assignments that we make for the propositions expressed by compound sentences of which these simple sentences are the components can be made ratiocinatively, and hence in a purely a priori way. Although the initial truth-value assignments may be made experientially or even empirically, the consequential assignments in a truth-functional propositional logic may be made a priori.[11]

EXERCISES

On the assumption that "A", "B", and "C" are each to be assigned "T", and that "D" and "E" are each to be assigned "F", evaluate each of the following.

1. $(A \cdot D) \supset (D \vee (B \cdot E))$

2. $A \equiv (\sim A \cdot B)$

3. $(C \vee B) \supset (C \vee (B \cdot A))$

4. $B \cdot (\sim A \supset B)$

5. $E \vee (D \cdot \sim (A \supset C))$

11. Recall, however, that knowledge gained by inference from experientially known truths is to be counted as experiential knowledge. When we say that consequential assignments may be made a priori, we are not claiming that the resultant knowledge is itself a priori. Whether or not it is a priori is a question whose answer depends upon whether or not it is possible to arrive at that same item of knowledge without any appeal to experience.

4. ELEMENTARY TRUTH-TABLE TECHNIQUES FOR REVEALING MODAL STATUS AND MODAL RELATIONS[12]

Modal status

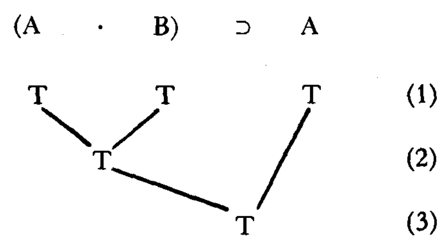
So far we have seen how the method of evaluating a truth-functional sentence may serve to reveal the truth-value of the proposition expressed by that sentence. But the real importance for Truth-functional Propositional Logic of the technique of sentential evaluation lies elsewhere. The technique assumes far greater importance when it is extended to encompass not just an evaluation for one particular assignment of "T"s and "F"s to the sentential components in a complex sentence, but a series of evaluations for every possible assignment of "T"s and "F"s to the sentential components. As a matter of fact we have already done one such complete evaluation in the previous section when we evaluated the sentence "∼ ∼A" first with "T" having been assigned to "A" and then subsequently with "F" having been assigned to "A". In that instance nothing particularly remarkable ensued. But there are other cases in which giving an exhaustive series of evaluations may serve to reveal various modal attributes of the propositions expressed. Perhaps this is best explained by beginning with an example.

Suppose we start with the sentence

(5.51) "(A · B) ⊃ A"

Sentence *(5.51)* contains three sentential-constant *tokens* representing two sentential-constant *types*. In order to determine how many distinct assignments are possible for the sentential-constants in a sentence, we must count the types represented, not the number of tokens of those types occurring. In this instance the relevant number is two. The formula for ascertaining the number of distinct initial assignments, N, which can be made is simply, $N = 2^n$, where "n" represents the total number of sentential-constant types represented. Thus there are 2^2, i.e., *four* distinct initial assignments which might be made for *(5.51)*.

Rather than completing each evaluation in a tree-fashion as we did in the previous section, we will now write out each evaluation on the very same line as the one on which we make the initial assignment. In effect we simply compress the tree onto a single horizontal line. Thus instead of writing out the first evaluation of *(5.51)* in a tree-fashion such as



we will now write it out in this way:

(A	·	B)	⊃	A
T	T	T	T	T
(1)	(2)	(1)	(3)	(1)

12. Advanced truth-table techniques will be introduced in section 5.

where the numbers across the bottom correspond, as before, to the order in which the steps are performed.

Doing the evaluation on a single horizontal line allows us to perform many evaluations on a single truth-table. Indeed, we can do all possible evaluations on one truth-table. We need only set up the truth-table in the manner described earlier in section 2.

In the present instance we have

A	B	(A	·	B)	⊃	A
T	T	T	T	T	T	T
T	F	T	F	F	T	T
F	T	F	F	T	T	F
F	F	F	F	F	T	F
		(1)	(2)	(1)	(3)	(1)

TABLE *(5.l)*

Of course it may be that one or more of these rows represents a set of impossible worlds. This latter possibility arises from the fact that the initial assignments (i.e., the left-hand columns) have been made in a purely *mechanical* fashion with no regard being paid to which proposition "A" and "B" are being used to express. For example, suppose that "A" and "B" are two sentences which express logically equivalent propositions, then both the second (i.e., "T" and "F") and third assignment (i.e., "F" and "T") represent sets of *impossible* worlds. After all, there are no possible worlds in which two logically equivalent propositions have different truth-values.

Does the fact that some rows in a mechanically constructed truth-table may represent sets of impossible worlds undermine the method we are describing? Hardly. For even if some of the rows in a complete truth-table evaluation represent impossible worlds, the *remaining* rows will still represent an exhaustive classification of all *possible* worlds. Provided we do not assume that every row of a truth-table necessarily represents a set of possible worlds, but only that all of them together represent all possible worlds (and perhaps some impossible ones as well), we will be in a position to draw valid inferences from such truth-tables.[13]

Let us pay particular attention to the last column evaluated in table *(5.l)*, viz., column (3). It is a column consisting wholly of "T"s. What is the significance of this? It is simple: the proposition which is expressed by the sentence "(A · B) ⊃ A" is true in every possible world; it is, simply, a necessary truth. (Note that this conclusion follows even if some of the rows of table *(5.l)* happen to represent sets of impossible worlds. No matter, for the remaining rows represent all possible worlds.)

What we have here, then, is a case in which an exhaustive evaluation of a truth-functional compound sentence has revealed that the proposition expressed by that sentence is a necessary truth. There is no possible world in which that proposition is false. By a purely mechanical exercise we have

13. If there *are* any rows in a given truth-table which represent sets of impossible worlds, their elimination will put us in a position to draw *additional* information from that table. In section 5 we will explore ways of eliminating these rows and the consequences of so doing.

been able to learn in this instance that a particular proposition is necessarily true. In short, we have here a method to aid us in attempting, epistemically, to determine modal status.

Let us now consider as a second example, the sentence

(5.52) " $\sim(\sim(A \cdot C) \vee (B \supset A))$".

Its truth-table is:

A	B	C	∼	(∼	(A	·	C)	∨	(B	⊃	A))
T	T	T	F	F	T	T	T	T	T	T	T
T	T	F	F	T	T	F	F	T	T	T	T
T	F	T	F	F	T	T	T	T	F	T	T
T	F	F	F	T	T	F	F	T	F	T	T
F	T	T	F	T	F	F	T	T	T	F	F
F	T	F	F	T	F	F	F	T	T	F	F
F	F	T	F	T	F	F	T	T	F	T	F
F	F	F	F	T	F	F	F	T	F	T	F
			(5)	(3)	(1)	(2)	(1)	(4)	(1)	(2)	(1)

TABLE *(5.m)*

Looking at the last column evaluated in table *(5.m)*, viz., column (5), we can see immediately that sentence *(5.52)* expresses a necessary falsehood, a proposition which has the same truth-value in all possible worlds: falsity. Once again in a purely mechanical fashion we have been able epistemically to determine the modal status of a proposition expressed by a particular sentence.

How powerful is this method? It has definite limitations. It yields results only of a certain kind and only in certain circumstances. This method can never be used to demonstrate that the proposition expressed by a truth-functional compound sentence is *contingent*. This is surprising, for it is easy to think that if the final column of an exhaustive evaluation is not either all "T"s or all "F"s but is instead some combination of the two, then it would follow that the proposition expressed is contingent. But this does not follow. Suppose we have the sentence

(5.53) "All squares have four sides and all brothers are male."

Expressed in the notation of Truth-functional Propositional Logic, this sentence might properly be translated simply as

(5.53a) "$F \cdot M$"

A complete truth-table evaluation of this latter sentence would yield

F	M	F	•	M
T	T	T	T	T
T	F	T	F	F
F	T	F	F	T
F	F	F	F	F
		(1)	(2)	(1)

TABLE *(5.n)*

Here, the last column to be evaluated, viz., (2), contains both "T"s and "F"s. Yet if we were to conclude that sentences *(5.53)* and *(5.53a)* express a contingent proposition, we would be wrong. The proposition expressed by these two sentences is noncontingent, and more particularly is noncontingently true.

What the method can, and cannot, show may be summarized thus:

1. If the final column in a complete truth-table evaluation of a compound sentence consists wholly of "T"s one may validly infer that the proposition expressed by that sentence is necessarily true.
2. If the final column in a complete truth-table evaluation of a compound sentence consists wholly of "F"s one may validly infer that the proposition expressed by that sentence is necessarily false.
3. If, however, the final column in a complete truth-table evaluation of a compound sentence consists of both "T"s and "F"s one is *not* entitled to infer that the proposition expressed is contingent.[14] As a test for contingency, this method is *inconclusive*.

This last point is so important, yet so often overlooked, that we can hardly emphasize it enough. The failure to take proper cognizance of it has led many persons to hold distorted views of the logical enterprise. It immediately follows from point 3 that a sentence may express a necessary truth or a necessary falsity even though a truth-tabular evaluation does not reveal it to be true in all possible worlds or to be false in all possible worlds.

The difficulty with truth-tabular methods of determining modal status is that they assign initial evaluations in a mechanical fashion and do not distinguish between assignments which designate impossible worlds and assignments which designate possible ones. (E.g., in table *(5.n)*, all of rows (2), (3), and (4) represent impossible worlds.) In short, being expressible by a sentence having a certain

14. Later, in section 5, we will explore methods to supplement the method of truth-table evaluation to make it more powerful so that it can be used as an adjunct to making an epistemic evaluation of contingency.

kind of truth-tabular evaluation (viz., a final column consisting wholly of "T"s or wholly of "F"s) is a *sufficient but not a necessary* condition for a proposition's being noncontingent. Ipso facto, being expressible by a sentence having a certain kind of truth-tabular evaluation (viz., a final column consisting of both "T"s and "F"s) is a *necessary but not a sufficient* condition for a proposition's being contingent.

EXERCISES

Part A

Translate each of the following sentences into conceptual notation using the sentential-constants specified. Then construct a truth-tabular evaluation for each translated sentence, and in each case tell what, if anything, the evaluation reveals about the modal status of the proposition expressed.

1. *"If John and Martha are late, then John or Betty is late."*

   Let "J" = "John is late"
   "M" = "Martha is late"
   "B" = "Betty is late"

2. *"There are fewer than two hundred stars or it is not the case that there are fewer than two hundred stars."*

   Let "F" = "There are fewer than two hundred stars."

3. *"There are fewer than two hundred stars or there are two hundred or more stars."*

   Let "F" = "There are fewer than two hundred stars."
   "E" = "There are (exactly) two hundred stars."
   "M" = "There are more than two hundred stars."

4. *"It is raining and it is not raining."*

   Let "G" = "It is raining."

5. *"If the pressure falls, it will either rain or snow."*

   Let "F" = "The pressure falls."
   "J" = "It will rain."
   "K" = "It will snow."

6. *"There are fewer than ten persons here and there are more than twenty persons here."*

   Let "C" = "There are fewer than ten persons here."
   "D" = "There are more than twenty persons here."

7. *"If there are ten persons here, then there are ten or eleven persons here."*

Let "E" = "There are ten persons here."
"F" = "There are eleven persons here."

8. *"If there are ten persons here, then there are at least six persons here."*

Let "E" = "There are ten persons here."
"I" = "There are at least six persons here."

9. *"If* a *is a square, then* a *is a square."*

*Let "A" = "*a *is a square."*

10. *"If* a *is a square, then* a *has four sides."*

*Let "A" = "*a *is a square."*
*"F" = "*a *has four sides."*

Part B

11. *For each case above in which the truth-tabular evaluation failed to reveal the modal status of the proposition expressed, say what the modal status is of that proposition.*

12. *What is the modal relation obtaining between the propositions expressed in exercises 2 and 3 above?*

* * * * *

Modal relations

By evaluating two truth-functional sentences together on one truth-table it is sometimes possible to ascertain mechanically the modal relation obtaining between the propositions those two sentences express.

Suppose for example that we were to evaluate the following two sentences together:

(5.54) "Today is Sunday and I slept late"

and

(5.55) "Today is Sunday or Monday."

We would begin by translating these into the conceptual notation of Truth-functional Propositional Logic, e.g.,

(5.54a) "A · L"

and

(5.55a) "A∨M"

To evaluate both these wffs on a single truth-table we will require 2^3 rows.

A	L	M	A	·	L	:	A	∨	M
T	T	T	T	T	T	:	T	T	T
T	T	F	T	T	T	:	T	T	F
T	F	T	T	F	F	:	T	T	T
T	F	F	T	F	F	:	T	T	F
F	T	T	F	F	T	:	F	T	T
F	T	F	F	F	T	:	F	F	F
F	F	T	F	F	F	:	F	T	T
F	F	F	F	F	F	:	F	F	F
			(1)	(2)	(1)		(1)	(2)	(1)

TABLE *(5.o)*

A comparison of the final column filled in under "A · L" with the final column filled in under "A∨M" is very revealing.

	A·L	A∨M
Row 1	T	T
Row 2	T	T
Row 3	F	T
Row 4	F	T
Row 5	F	T
Row 6	F	F
Row 7	F	T
Row 8	F	F
	(2)	(2)

FIGURE *(5.p)*

We note that there is no row in which "T" has been assigned to "A · L" and in which "F" has been assigned to "A∨M". Simply, this means that there is no possible world in which the proposition expressed by "A · L" is true and the proposition expressed by "A∨M" is false. But that this is so tells us that the first of these two propositions *implies* the second.

The method utilized in this example is perfectly general and may be stated in the following rule:

> If, in the truth-tabular evaluation of two sentences, it is found that there is no row of that table in which a sentence, α, has been assigned "T" and a sentence, β, has been assigned "F", one may validly infer that the proposition expressed by α *implies* the proposition expressed by β.

Again it is important to realize that this rule, like the rules of the previous section, states a sufficient condition but not a necessary one. Two propositions may stand in the relation of implication even though a truth-tabular evaluation of the sentences expressing those propositions fails to reveal it. One need only consider the two sentences

(5.56) "Sylvia bought a new car"

and

(5.57) "Someone bought a new car"

to see that this is so. Using "B" for *(5.56)* and "C" for *(5.57)*, the truth-tabular evaluation is:

B	C	B	:	C
T	T	T	:	T
T	F	T	:	F
F	T	F	:	T
F	F	F	:	F
		(1)		(1)

TABLE *(5.q)*

It is easy to see that this table fails to reveal what we already know to be the relation between the propositions expressed by "B" [or *(5.56)*] and "C" [or *(5.57)*], viz., implication.

Just as a truth-tabular evaluation may serve to reveal that two propositions stand in the relation of implication, it may also serve to reveal that two propositions stand in the modal relation of *equivalence* or the modal relation of *inconsistency*.

> If, in the truth-tabular evaluation of two sentences, it is found that in each row of the table these sentences have been assigned matching evaluations (i.e., both have been assigned "T" or both have been assigned "F"), one may validly infer that the propositions expressed by the two sentences are *logically equivalent* to one another.

> If, in the truth-tabular evaluation of two sentences, it is found that there is no row in which both sentences have been assigned "T", one may validly infer that the propositions expressed by the two sentences are *logically inconsistent* with one another.

It is easy to provide illustrative cases of these rules. Let us begin with the case of equivalence. Consider the two sentences

(5.58) "A"

and

(5.59) "(A · ∼B)∨(A · B)".

The truth-table evaluation is:

A	B	A	:	(A	·	∼	B)	∨	(A	·	B)
T	T	T	:	T	F	F	T	T	T	T	T
T	F	T	:	T	T	T	F	T	T	F	F
F	T	F	:	F	F	F	T	F	F	F	T
F	F	F	:	F	F	T	F	F	F	F	F
		(1)		(1)	(3)	(2)	(1)	(4)	(1)	(2)	(1)

TABLE *(5.r)*

A comparison of the column appearing under "A" with the final column appearing under "(A · ∼B)∨(A · B)" reveals that the columns are identical. Such *sentences* will be said to be truth-functionally equivalent.

Truth-functionally equivalent sentences, it is clear, express propositions which have matching truth-values in all possible worlds, i.e., truth-functionally equivalent sentences express propositions which are logically equivalent to one another.

Of course two propositions may be logically equivalent even though a truth-tabular evaluation of the sentences expressing those propositions fails to reveal that they are. A case in point would be the two sentences

(5.60) "Iron is heavier than copper"

and

(5.61) "Copper is lighter than iron."

Expressed in the notation of Truth-functional Propositional Logic these two sentences might become respectively "I" and "C". A truth-tabular evaluation of the two sentence-constants "I" and "C" would not assign matching evaluations for every row of the table and hence would fail to reveal what we already know (by other means) about the two propositions expressed, viz., that they are logically equivalent.

Now let us turn to an illustration of the application of the rule for inconsistency. Suppose we take as our example

(5.62) "A ≡ B"

and

(5.63) " ∼(∼A∨B)".

The truth-tabular evaluation is

A	B	A	≡	B	∼	(∼	A	∨	B)
T	T		T		F				
T	F		F		T				
F	T		F		F				
F	F		T		F				
		(1)	(2)	(1)	(4)	(2)	(1)	(3)	(1)

TABLE *(5.s)*

When we compare the final columns filled in under each compound sentence we find that there is no row in which both sentences have been assigned "T". Thus, in this case, where no row assigns "T" to both the first and second sentence, we may be assured that the propositions expressed by these two sentences are not both true in any possible world, i.e., that they are inconsistent with one another.

Two propositions may be inconsistent with one another even though a truth-tabular evaluation of the sentences expressing them fails to reveal it. An example is the following:

(5.64) "Something is square"

and

(5.65) "Nothing is square."

A mechanical truth-tabular evaluation of sentence-constants (e.g., "E" and "N") representing these sentences will assign "T", on the first row of the truth-table, to both of these constants. Hence the table will fail to show what we already know by other means, viz., that these two sentences express contradictory, and ipso facto, inconsistent propositions.

In the case of the modal relation of consistency, we find that truth-tabular methods have the same *inconclusiveness* as they were found to have in the case of our trying to use them to determine that a proposition has the modal status of contingency.

The fact that a truth-table cannot, in general, be used to reveal that two propositions stand in the relation of consistency has, once again, to do with the manner in which truth-tables are constructed. What *would* reveal that two propositions were consistent would be the existence of at least one row on a truth-table which (1) assigns "T" to both of the sentences expressing those propositions and (2) represents a set of possible worlds. The stumbling block here is the second condition, the condition which requires that one of the rows assigning "T" and "T" to the two sentences respectively be a row which represents a set of *possible* worlds. For the trouble is that in constructing truth-tables mechanically we have no way of determining from the truth-table itself which rows represent sets of *possible* worlds and which represent sets of *impossible* worlds. Provided we are looking, e.g., in the case of implication and equivalence, for the *non*-existence of a certain kind of assignment, it makes no difference whether the rows represent possible or impossible worlds. But when, e.g., in the case of consistency, we are looking for a dual assignment of "T" in a row representing a set of possible worlds, the truth-tabular method fails us.

In short, it is a necessary condition for validly inferring that two propositions are consistent that there be no truth-tabular evaluation of any sentences expressing those propositions which reveals them to be inconsistent. But this latter is by no means a sufficient condition.

EXERCISES

Part A

Translate each of the following pairs of sentences into conceptual notation using the sentential constants specified. Then construct truth-tabular evaluations for each pair of sentences, and in each case tell what, if anything, the evaluation reveals about the modal relations obtaining between the two propositions expressed.

1. *"If I overslept, then I was late for work" and "If I was late for work, then I overslept"*

   Let "O" = "I overslept"
   "L" = "I was late for work"

2. *"If everyone was late, then someone was late" and "Everyone was late"*

   Let "E" = "Everyone was late"
   "B" = "Someone was late"

3. *"John has been taking lessons from his father; and he can pass his driver's test if and only if he has been taking lessons from his father" and "John can pass his driver's test"*

   Let "J" = "John has been taking lessons from his father"
   "C" = "John can pass his driver's test"

4. *"There are fewer than two hundred stars" and "It is not the case that there are fewer than two hundred stars"*

   Let "H" = "There are fewer than two hundred stars"

5. *"There are fewer than two hundred stars" and "There are two hundred or more stars"*

   Let "H" = "There are fewer than two hundred stars"
   "E" = "There are (exactly) two hundred stars"
   "M" = "There are more than two hundred stars"

6. *"*a *is a green square tray" and "*a *is a square tray"*

   *Let "B" = "*a *is green tray"*
   *"A" = "*a *is a square tray"*

7. *"*a *is a square" and "*a *has four sides"*

*Let "A" = "*a *is a square"*
*"H" = "*a *has four sides"*

8. *"Today is Tuesday" and "It is earlier in the week than Wednesday and later in the week than Monday"*

Let "D" = "Today is Tuesday"
"E" = "It is earlier in the week than Wednesday"
"L" = "It is later in the week than Monday"

9. *"Diane and Efrem love chocolate ice cream" and "Efrem and Diane love chocolate ice cream"*

Let "D" = "Diane loves chocolate ice cream"
"E" = "Efrem loves chocolate ice cream"

10. *"Everything is square" and "Everything is square or not everything is square"*

Let "E" = "Everything is square"

Part B

11. *Are there any cases above in which the two propositions stand in the relation of implication but for which the truth-tabular evaluation fails to reveal that relation? Which, if any, are they?*

12. *Are there any cases above in which the two propositions stand in the relation of equivalence but for which the truth-tabular evaluation fails to reveal that relation? Which, if any, are they?*

13. *Are there any cases above in which the two propositions stand in the relation of inconsistency but for which the truth-tabular evaluation fails to reveal that relation? Which, if any, are they?*

* * * * *

Deductive validity

In chapter 1, we defined "deductive validity" in terms of "implication". Elaborating a bit, we may now offer the following definition:

> "An argument A consisting of a premise-set S and a conclusion C is deductively valid" $=_{df}$ "The premise-set S (or alternatively the conjunction of all the propositions of S) of argument A implies the conclusion C".

In short, a necessary and sufficient condition of an argument's being deductively valid is that the premises of that argument imply the conclusion. Thus to the extent that truth-tabular methods can reveal that the relation of implication holds between two propositions (or proposition-sets), to that extent it can reveal that an argument is deductively valid.

The most obvious way of using truth-tables in an attempt to ascertain whether an argument is deductively valid is simply to conjoin all the premises and then to evaluate together on one truth-table both this compound sentence and the sentence expressing the conclusion. If no row on the truth-table assigns "T" to the first sentence and "F" to the second, we may validly infer that the argument is deductively valid.

Consider the following argument:

(5.66)

Premises:
- If the seeds were planted in March and it rained throughout April, the flowers bloomed in June.
- The seeds were planted in March but it is not the case that the flowers bloomed in June.

Conclusion:
- It is not the case that it rained throughout April.

Translating this argument into conceptual notation, using fairly obvious interpretations for our sentential-constants, gives us:

(5.66a)

$$\begin{array}{l} (M \cdot A) \supset J \\ \underline{M \cdot \sim J} \\ \therefore \sim A \end{array}$$

Conjoining the sentences of the premises as our next step gives us:

(5.67) "$((M \cdot A) \supset J) \cdot (M \cdot \sim J)$"[15]

The truth-table for *(5.67)* is:

A	J	M	((M	·	A)	⊃	J)	·	(M	·	∼	J)	⁞	∼	A
T	T	T						F					⁞	F	
T	T	F						F					⁞	F	
T	F	T						F					⁞	F	
T	F	F						F					⁞	F	
F	T	T						F					⁞	T	
F	T	F						F					⁞	T	
F	F	T						T					⁞	T	
F	F	F						F					⁞	T	
			(1)	(2)	(1)	(3)	(1)	(4)	(1)	(3)	(2)	(1)		(2)	(1)

TABLE *(5.t)*

15. It should now be obvious why we said earlier, in section 3, that we require in our logic the means to be

By comparing the final column filled in under "$((M \cdot A) \supset J) \cdot (M \cdot \sim J)$" with the final column filled in under "$\sim A$", we can see that there is no row on the truth-table which assigns "T" to the first of these sentences and "F" to the second. Thus we may validly infer that the proposition expressed by the former sentence implies the proposition expressed by the latter. But since this is so, then we may likewise infer that the original argument, from which the two compound sentences evaluated were derived, is itself deductively valid. Here, then, is an instance in which we have been able to demonstrate in a purely mechanical fashion that a certain argument, viz., *(5.66a)*, is deductively valid.[16]

In the example which we have just worked through, the results of the truth-tabular test were positive: the test revealed that the argument is deductively valid. But suppose we were to try a truth-tabular test for deductive validity in the case of some other argument and were to find that the test failed, i.e., that at least one row of the truth-table assigned "T" to the sentence formed by conjoining all the sentences expressing the premises of the argument and it assigned "F" to the sentence expressing the conclusion of the argument. Under such circumstances are we entitled to infer that the argument is *deductively invalid,* i.e., that its premises do *not* imply its conclusion? The answer is: No. And the reason parallels exactly the reason we gave earlier for saying that truth-tabular methods cannot in general be used to show that one proposition does not imply another: the row of the truth-table which assigns "T" to the first sentence and "F" to the second may represent, not a set of possible worlds, but a set of impossible worlds.

If, then, in an attempt to ascertain whether an argument is deductively valid, a truth-tabular test yields some row which assigns "T" to the sentence expressing the conjunction of the premises and an "F" to that sentence expressing the conclusion of the argument, we are not entitled to infer that the argument is deductively invalid. The test is simply *inconclusive;* and other, more sophisticated ways of determining deductive validity and invalidity, i.e., logical methods which embody a deeper conceptual analysis, will have to be adopted.

Another, but no more powerful, way to use truth-tables in an attempt to ascertain whether a given argument is deductively valid is to capitalize on the fact that "implication" (and hence "deductive validity") may be defined in terms of (1) the relation of material conditionality and (2) truth in all possible worlds.

> "P implies Q" $=_{df}$ "The relation of material conditionality holds between P and Q in all possible worlds."

In symbols, this same definition may be expressed thus:

$$\text{“}P \rightarrow Q\text{”} =_{df} \text{“}\Box(P \supset Q)\text{”}$$

These definitions suggest, then, a second way to use truth-tables to ascertain validity. Because of the interdefinability of "deductive validity" and "implication" it will suffice to show that an argument is deductively valid if we show that a material conditional sentence whose antecedent expresses the conjunction of the premises of that argument and whose consequent expresses the conclusion of the argument, expresses a proposition which is true in all possible worlds.

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able to evaulate well-formed sentences of any arbitrary length. Although we do not often *utter* sentences which contain more than five or six operators, in the testing of arguments for validity, we *manufacture* sentences which may be very long indeed and which may contain a great many operators. This will certainly be the case when the argument itself contains several premises.

16. Note, however, that the step which gets us from *(5.66)* to *(5.66a)*, i.e., the step in which we translate the original argument as stated in English into the conceptual notation of Logic, is *not* a mechanical procedure.

We will use the same example as above. This time, however, instead of evaluating the *two* sentences, "((M · A) ⊃ J) · (M · ∼J)" and "∼A", we shall evaluate the *one* material conditional sentence "(((M · A) ⊃ J) · (M · ∼J)) ⊃ ∼A":

A	J	M	(((M · A) ⊃ J) · (M · ∼ J)) ⊃ ∼ A
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	T
F	T	F	T
F	F	T	T
F	F	F	T
			(5)

TABLE *(5.u)*

Here, the final column, consisting as it does entirely of "T"s, filled in under the material conditional sentence, reveals that the proposition expressed by that sentence is true in all possible worlds. Since that proposition is true in all possible worlds, the proposition expressed by the antecedent of that material conditional sentence *implies* the proposition expressed by the consequent of that same sentence. But insofar as these two propositions are just those propositions expressed by the premise-set and conclusion respectively of the original argument, we have succeeded in showing that that argument is deductively valid.

Which of these two ways of using truth-tables a person adopts in an attempt to ascertain deductive validity is purely a matter of personal taste. There is nothing as regards their efficacy to recommend one over the other. They will always yield identical results: any argument which the one reveals to be valid, the other will also; any argument which the one fails to reveal to be deductively valid, the other will also.

Even though these two ways of using truth-tables in an attempt to ascertain the deductive validity of arguments do not, and cannot, differ in their results, a word of caution ought to be sounded concerning the latter. Because the latter technique involves constructing and subsequently evaluating a material conditional, some persons have been misled into thinking that the relation one is seeking to establish when one is looking to see whether an argument is deductively valid is the relation of material conditionality. This is a totally unwarranted inference, but an all-too-common one. True enough, the latter technique, as we have described it and as it is put into practice here and in countless other books as well, does utilize a material conditional sentence. But this is not to say that we are looking to see whether simply the relation of material conditionality holds between the premises and conclusion of an argument. Rather we are looking to see whether the relation of material conditionality holds between the premises and conclusion *in all possible worlds* (not just in the actual world). For when the relation of material conditionality holds in all possible worlds between two propositions, those propositions stand in the modal relation of implication. Failure to understand this point has deceived many students of logic into thinking that deductive validity is not a *modal* property of arguments. But of course it is. Deductive validity cannot be defined simply in terms of truth and falsity and ipso facto cannot be defined in terms of the relation of material conditionality simpliciter. For an argument to be deductively valid there must be *no possible world* in which all its premises are true and its conclusion false. That is to say, for an argument to be deductively valid the relation of material conditionality must hold between its premise-set and conclusion not just in the actual world but in all possible worlds.

EXERCISES

Translate each of the following arguments into conceptual notation using the sentential constants specified. Then construct truth-tabular tests for validity, and in each case tell whether that test reveals the argument to be deductively valid.

Let "A" = "The turntable is grounded"
"B" = "The hum persists"
"C" = "The amplifier is grounded"
"D" = "Diane is older than Efrem"
"E" = "Efrem is older than Martin"
"F" = "Diane is older than Martin"

1. *If the turntable is grounded and the amplifier is grounded, then it is not the case that the hum persists. But the hum persists. Therefore, either it is not the case that the amplifier is grounded or it is not the case that the turntable is grounded.*

2. *If either the turntable or the amplifier is grounded, then it is not the case that the hum persists. It is not the case that the hum persists. Therefore, the amplifier is grounded.*

3. *Diane is older than Efrem. Efrem is older than Martin. Therefore, Diane is older than Martin.*

4. *If Diane is older than Efrem, then Efrem is older than Martin. If Efrem is older than Martin, then Diane is older than Efrem. Therefore, if it is not the case that Efrem is older than Martin, then it is not the case that Diane is older than Efrem.*

5. ADVANCED TRUTH-TABLE TECHNIQUES

Corrected truth-tables

In the previous section we have seen how the mechanical construction of truth-tables leads to certain restrictions on their interpretation, e.g., they cannot be used to demonstrate the *contingency* of a proposition expressed by a truth-functional compound sentence; they cannot be used to demonstrate the *consistency* of two propositions expressed by truth-functional compound sentences; and (in general) they cannot be used to demonstrate the deductive *invalidity* of an argument expressed by truth-functional compound sentences.

There is, however, a way to supplement Truth-functional Propositional Logic in such a way as to remove these restrictions. In effect, it involves stepping outside of the purely mechanical techniques of that Logic and supplementing them with the results of nonformal conceptual analysis. What the method amounts to is striking out all those rows on a truth-table which represent a set of impossible worlds.

Let us see how the method works. Suppose we were to start with the following two sentences

Let "G" = "There are fewer than four apples in the basket",

"H" = "There are more than ten apples in the basket",

and were to ask, "What is the logical relation obtaining between the propositions expressed by '∼(G · H)' and '∼G · H'; in particular, are these propositions consistent or inconsistent?" If we proceed to construct a truth-table in the standard way, we will discover that that table is *inconclusive:*

G	H	∼(G · H)	∼G · H
T	T	F	F
T	F	T	F
F	T	T	T
F	F	T	F

TABLE *(5.v)*

The trouble here, of course, lies in the third row. The presence of a "T" under "∼(G · H)" and a "T" under "∼G · H" would indicate the consistency of the propositions expressed by these two sentences if we could be assured that this row represents a set of possible worlds and not a set of impossible worlds. But as truth-tables are standardly constructed, no such inference may validly be made.

However, such an inference can be made if truth-tables are constructed in a nonstandard way: in a way which insures that every row of the table represents a non-empty set of possible worlds. To do this we systematically pass down through the assignments made on the left-hand side of the double vertical line and ask of each of these assignments whether it represents a set of possible worlds or a set of impossible worlds for the particular propositions expressed by the simple sentential-constants appearing at the top of the columns. When we do this, we step outside the techniques of ordinary truth-functional logic; for in doing this we are performing nonformal analysis on the propositions expressed by those simple sentential-constants.

In the present case we begin with row 1, asking ourselves whether the assignment of "T" to both "G" and "H" represents a set of possible worlds. The answer our conceptual analysis gives us is: No. By analyzing the concepts of *more than, fewer than, four, ten,* etc., which figure in these propositions we are able to ascertain analytically that there is no possible world in which it is true that there are fewer than four apples in the basket and in which it is true that there are more than ten. So we strike out row 1. By similar sorts of analysis, we also ascertain that the other three rows do represent sets of possible worlds, and consequently we allow them to remain. The truth-table which results from this process, we call, simply, a *corrected* truth-table.

G	H	∼(G · H)	∼G · H
~~T~~	~~T~~	~~F~~	~~F~~
T	F	T	F
F	T	T	T
F	F	T	F

TABLE *(5.w)*

From this corrected truth-table we *can* obtain the information we desire. Row 3 has survived the striking-out process, and hence the propositions expressed by "∼(G · H)" and "∼G · H", respectively, are *consistent* with one another: there *is* a possible world in which these propositions are both true.

The method is perfectly general. Having performed a nonformal analysis of the modal relations obtaining between propositions expressed by *simple* sentences, we are then in a position to ascertain *mechanically* (by means of a corrected truth-table) the full range of modal attributes (including contingency, consistency, and deductive invalidity) which might be exemplified by propositions expressed by truth-functional *compound* sentences.

EXERCISES

1. (a) By constructing a corrected truth-table, ascertain the modal status of the proposition expressed by "I ⊃ (K ⊃ J)" where

"I" = "There are fewer than nine apples in the basket",
"J" = "There are more than three apples in the basket", and
"K" = "There are (exactly) five apples in the basket".

(b) Which 4 of the 8 rows of the truth-table had to be struck out in order to construct the corrected truth-table?

2. *Using the same interpretation for "I", and "J", and "K" as in question 1, determine the modal status of the proposition expressed by "$(\sim I \cdot \sim J) \supset K$".*

3. *Similarly, determine the modal status of the proposition expressed by "$K \supset (I \cdot J)$".*

4. *Using standard truth-table techniques, viz., those outlined in section 4, what can one determine about the validity of the following argument?*

$$\begin{array}{l} I \supset (J \vee K) \\ \underline{\sim K \qquad\quad} \\ \therefore \quad (I \vee J) \end{array}$$

5. *Using the nonstandard techniques of this section, and the interpretation of "I", "J", and "K" as given in question 1, what can one learn about the validity of the argument cited in question 4?*

6. *Again, using the same interpretation for "I", "J", and "K", what does a corrected truth-table tell us about the validity of the following argument?*

$$\begin{array}{l} \underline{I \cdot J} \\ \therefore K \end{array}$$

* * * * *

Reduced truth-tables

We saw in chapter 1 that if we arbitrarily choose any two propositions whatever, the two propositions must stand to one another in exactly one of the fifteen relationships depicted in the worlds-diagrams of figure *(1.i)*. This result holds even if those propositions happen to be expressed by truth-functional compound sentences containing several sentential constants.

As it turns out, there is a simple, indeed mechanical, way to get from a corrected truth-table for a pair of propositions to the one worlds-diagram among the fifteen which uniquely depicts that relationship. The method utilizes what we shall call a *reduced* truth-table.

In order to construct a reduced truth-table, one begins with a corrected truth-table and then focuses attention on the right-hand side of the double vertical line. One simply deletes all but one of those rows which happen to be alike. For example, suppose we begin with the two sentences

"A" = "There are fewer than 100 stars" and
"B" = "There are more than 1500 stars".

An infinity of compound sentences is constructible out of "A" and "B". Of this infinity, we choose two as examples, viz., "$A \equiv B$" and "$A \vee B$", and ask of these, "Which one of the fifteen worlds-diagrams depicts the modal relationship obtaining between the propositions expressed by these two compound sentences?" To answer this question we begin by constructing a corrected truth-table.

Since the propositions expressed by "A" and "B" are contraries, there is no possible world in which both are true. We must strike out the first row of the truth-table.

A	B	A ≡ B	A ∨ B
T	T	T	T
T	F	F	T
F	T	F	T
F	F	T	F

TABLE *(5.x)*

Next we construct the reduced truth-table. To do this we look only at the right-hand side of the table and delete all but one of any set of rows which happen to be identical. As we can see, the second and third rows are identical: they both assign "F" to "A≡B", and "T" to "A∨B". Thus we strike out the third row. The *reduced* truth-table is thus

A ≡ B	A ∨ B
F	T
T	F

TABLE *(5.y)*

The reduced truth-table tells us two things: (1) that there are some possible worlds in which A≡B is false and A∨B is true; and (2) that there are some possible worlds in which A≡B is true and A∨B is false. By careful inspection of the fifteen worlds-diagrams for two propositions we can see that only one diagram, the tenth, depicts such a relationship. And the tenth diagram, we may recall, represents the case of contradiction holding between two contingent propositions.

There is, however, even an easier way to get from the reduced truth-table to the appropriate worlds-diagram.

A moment's reflection tells us that there is, after all, a maximum number of rows possible in a reduced truth-table: no reduced truth-table can have more than four rows, viz.,

T–T
T–F
F–T
F–F

(Any other row would have to be 'reduced' to one of these.) This fact puts an upper limit on the number of different reduced truth-tables. The only possible reduced truth-tables are those whose rows are some subset of these four. And the maximum number of ways of selecting from among four things is fifteen.[17] The fifteen possible reduced truth-tables may all be listed:

1. T–T

2. T–F

3. F–T

4. F–F

5. T–T
T–F

6. T–T
F–T

7. T–F
F–F

8. F–T
F–F

9. T–T
F–F

10. T–F
F–T

11. T–T
F–T
F–F

12. T–F
F–T
F–F

13. T–T
T–F
F–F

14. T–F
T–T
F–T

15. T–F
T–T
F–T
F–F

FIGURE *(5.z)*

Each of these fifteen reduced truth-tables corresponds to a unique worlds-diagram. And each of these reduced truth-tables may heuristically be regarded as a 'code' or 'tabular description' of its corresponding worlds-diagram. Each of these reduced truth-tables, these *codes* so to speak, is reproduced in figure *(5.aa)* (p. 300) alongside its respective worlds-diagram and again on the very diagram itself. For example, consider reduced truth-table 11:

T–T
F–T
F–F.

We can see how this code is mapped directly onto worlds-diagram 11. The first segment of that diagram represents those possible worlds in which P is true and Q is true (T–T); the second segment, those possible worlds in which P is false and Q is true (F–T); and the third segment, those possible worlds in which P is false and Q is false (F–F).

In sum, to find the modal relation obtaining between any two propositions expressed by truth-functional compound sentences: (1) construct a corrected truth-table for those compound sentences, (2) then proceed to construct a reduced truth-table, and (3) finally match the reduced

17. The formula for the number of ways, k, of selecting from among n things is, $k = 2^n - 1$. In the present instance, $n = 4$.

truth-table to one code among the worlds-diagrams. That diagram represents the modal relation obtaining between the two propositions expressed by the compound sentences.

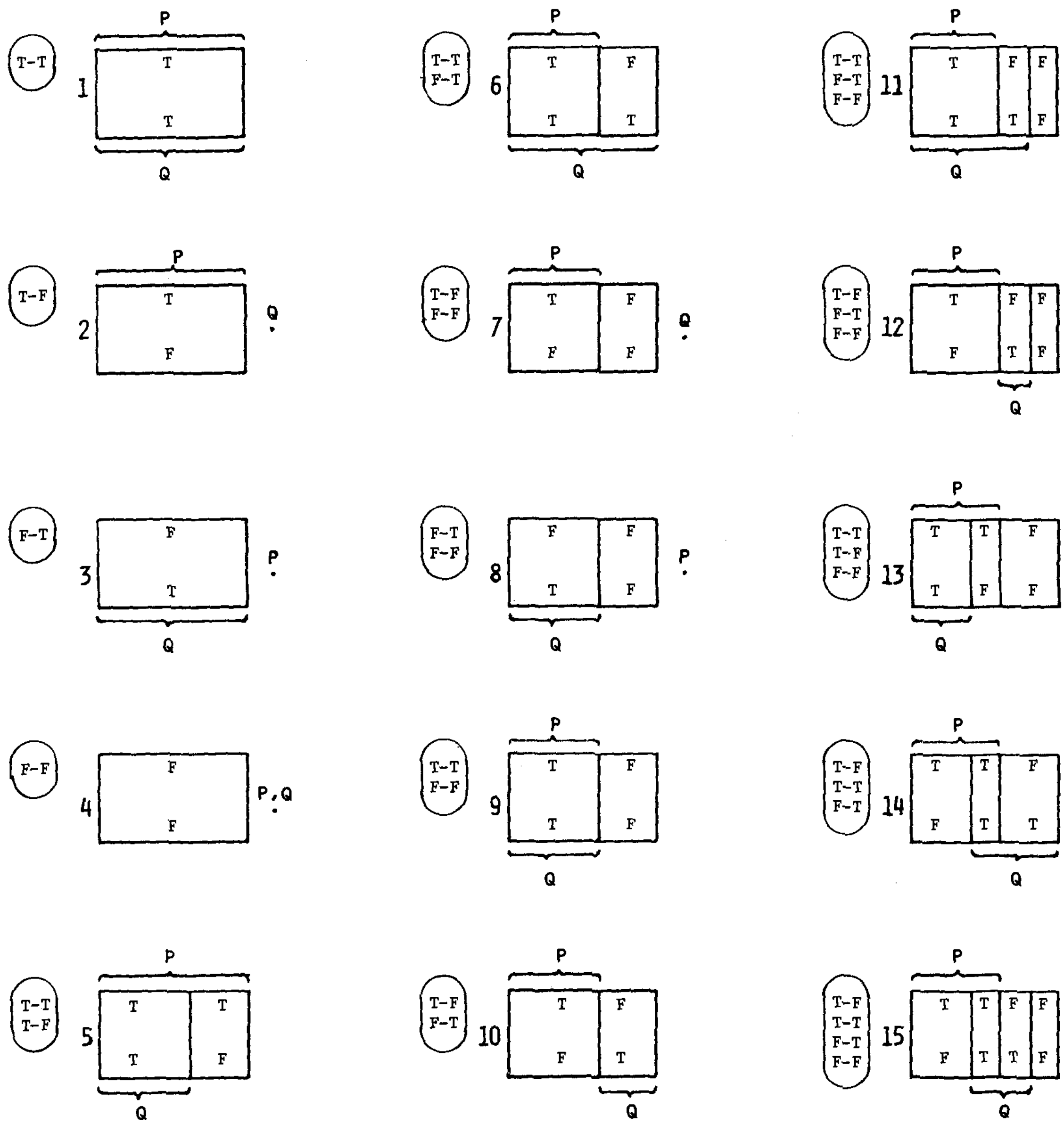


FIGURE *(5.aa)*

EXERCISES

Part A

1. *Let "A" = "Sylvia has (exactly) one sister"*
   "B" = "Joseph has (exactly) four sisters"
   "C" = "Joseph has twice as many sisters as Sylvia".

   Using reduced truth-tables determine which worlds-diagram depicts the modal relation obtaining between the proposition expressed by "$(B \cdot C) \supset A$" and the proposition expressed by "$(B \cdot C)$".

2. *Let "M" = "Al is taller than Bill"*
   "N" = "Al is older than Bill"
   "O" = "Al is the same height as Bill".

   Using reduced truth-tables determine in each case which worlds-diagram depicts the modal relation obtaining between the propositions expressed by the following pairs of compound sentences:

   a. "$(M \cdot N)$" *and* "$(N \supset O)$"

   b. "$(M \vee N)$" *and* "$\sim O$"

   c. "$(M \vee O)$" *and* "$(M \cdot O)$"

   d. "O" *and* "$((M \cdot N) \vee (N \supset \sim O))$".

Part B

3. *(For mathematically adept students.) Show how to derive the formula cited in chapter 1, p. 57, for the number of worlds-diagrams required to depict all the ways of arranging* n *propositions on a worlds-diagram. (Hint: how many reduced truth-tables are there in the case of three arbitarily chosen propositions?)*

6. THE CONCEPT OF FORM

Sentences and sentential forms in a logic

We have already introduced the distinction between sentences and sentential forms, or more specifically, the distinction between sentences in Truth-functional Propositional Logic and sentence-forms in that same Logic.

A sentence (in Truth-functional Propositional Logic), we have said, is a well-formed formula which contains only sentence-*constants* (i.e., the letters "A" through "O" of the English alphabet) or only sentence-constants and truth-functional operators. A sentence-form, on the other hand, in

Truth-functional Propositional Logic, is a well-formed formula which contains at least one sentence-*variable,* i.e., one of the letters "P" through "Z".

Sentence-forms, unlike sentences, do *not* express propositions. What, then, is their role? Why do logicians concern themselves not only with sentences but with sentence-forms as well? What is the relationship between sentences and sentence-forms? Let us turn to these questions.

The relationship between sentences and sentence-forms

Sentences *instantiate* sentence-forms. Indeed, a given sentence may instantiate *several* sentential forms within a formal logic. Consider, for example, the sentence

> *(5.68)* "It is not the case that Henry brought the typewriter, and if Mary did, then she has put it out of sight."

Rendered in the conceptual notation of Truth-functional Propositional Logic, this sentence becomes,

> *(5.68a)* "$\sim A \cdot (B \supset C)$".

This latter sentence has (or instantiates) the forms:[18]

> (F1) "$\sim P \cdot (Q \supset R)$"
>
> (F2) "$P \cdot (Q \supset R)$"
>
> (F3) "$P \cdot Q$"
>
> (F4) "$\sim P \cdot Q$".

And — like *all* other sentences — it also instantiates the form

> (F5) "P".

(F1) through (F5) are said to be forms of the sentence *(5.68a)* because it is possible to instantiate (which is to say, to find substitution-instances of) the variables appearing in these forms so as to generate the sentence *(5.68a)*. Thus, for example, if in (F1), we were to substitute the sentential constant "A" for the sentential variable "P", "B" for "Q", and "C" for "R", we would produce the sentence "$\sim A \cdot (B \supset C)$". Similarly if, in (F3), we were to substitute "$\sim A$" for "P" and "$(B \supset C)$" for "Q", then, again, we would produce the sentence *(5.68a)*.

By way of contrast, consider the form

> (F6) "$P \cdot \sim Q$".

Is (F6) a form of *(5.68a)?* It turns out that it is not. There are no substitutions we can make for "P" and for "Q" in (F6) so as to yield the sentence *(5.68a)*. There are, of course, substitutions we can make which will yield a sentence which expresses the same *proposition* which is expressed by

18. In writing the forms of sentences we adopt the convention of selecting the capital letters beginning with "P" *in alphabetical order.*

(5.68a), but this is not the same as generating the very *sentence (5.68a)*. For example, we can substitute "∼A" for "P" in (F6), and "(B · ∼C)" for "Q", to yield

(5.69) "∼A · ∼(B · ∼C)",

which is truth-functionally equivalent to *(5.68a)*. Yet since (F6) cannot be instantiated to yield *(5.68a)*, (F6) is *not* a form of *(5.68a)*.

In Truth-functional Propositional Logic we may define the form of a sentence in this way:

> A sentence, S, will be said to have the form, F, if F is a well-formed formula and if the sentence S can be generated from F by substituting sentences (or sentential constants) for the sentential variables occurring in F.[19]

It is sometimes convenient to regard the forms of sentences as capsule descriptions of the logical structure of those sentences. Thus, for example, in saying of (F3) [viz., "P · Q"] that it is a form of *(5.68a)*, we may be taken to be saying that *(5.68a)* is a conjunction of two sentences. Equally well, in saying that (F2) [viz., "P · (Q ⊃ R)"] is a form of *(5.68a)* we may be taken to be saying that *(5.68a)* is a conjunction whose second conjunct is a material conditional. Both these descriptions are true of *(5.68a)*. *(5.68a)* being a fairly complex compound sentence, has *several* descriptions of its logical structure which are each true. This is not surprising since most things allow several descriptions each of which is true. For example, this page (1) is made of paper; (2) is made of white paper; (3) is rectangular; (4) has printing on it; (5) has black printing; etc., etc. Each of these descriptions is true of this page.

A common manner of speech sometimes fosters the belief that sentences typically have but a single form. Very often in reporting that such-and-such is a form of the sentence so-and-so, we express ourselves thus: "The sentence so-and-so has *the* form such-and-such." Speaking in this way, using the definite article "the", we find it easy to slip into also saying, "The form of the sentence so-and-so is such-and-such." But this is a mistake. We ought not to infer the latter from the former any more than we ought to infer the false claim that *the* attribute of a ripe banana is the attribute, yellowness, from the true claim that ripe bananas have *the* attribute of being yellow. The definite article, "the", cannot be shuttled about willy-nilly in this way. Although the sentence *(5.68a)* has the form (F2), it does not follow that (F2) is the (single) form of *(5.68a)*. In addition to saying that *(5.68a)* has the form (F2), we can also say that *(5.68a)* has the form (F1), that it has the form (F3), etc.

However, not *all* sentences instantiate more than one form in a specific logic. Although many do, not all do. Consider the two sentences

(5.70) "D"

(5.71) "∼D".

19. According to this definition, the formula "∼A · (P ⊃ Q)", where "A" is a constant and "P" and "Q" are variables, would turn out to be a form of "∼A · (B ⊃ C)". For some purposes it is helpful to regard such "mixed" formulae as forms. For example, in some cases it might be helpful to regard the sentence, "If today is Monday, then the shipment is overdue", as being of the form, "If today is Monday, then P", where "P" represents a sentential variable. However, we will have no need of such 'mixed' forms anywhere in these introductory studies, and will hereinafter usually mean by "form" a *non*-mixed formula of the sort described in the definition.

The sentence *(5.70)* instantiates exactly *one* form in Truth-functional Propositional Logic. The single form of *(5.70)* is

(F5) "P"

while the two forms of *(5.71)* are

(F5) "P"

(F7) "∼ P".

The sentence *(5.70)* can be generated only from (F5) by substituting "D" for "P"; the sentence *(5.71)* can be generated either from (F5) by substituting "∼ D" for "P", or from (F7) by substituting "D" for "P". But note, the sentence *(5.70)*, viz., "D", *cannot* be generated from (F7). Although we may substitute "∼ D" for "P" in (F7) to yield

(5.72) "∼ ∼ D",

which is of course truth-functionally equivalent to *(5.70)*, *(5.72)* is not the same *sentence* as *(5.70)*.

Any sentence which contains even a single truth-functional sentential operator will have more than one form in Truth-functional Propositional Logic. And as the number of such operators increases, the number of forms instantiated will increase as a complicated exponential function.

If there is more than one form of a sentence these forms may be arranged in order of "length". The length of a formula is determined by counting the number of letters occurring in it, the number (if any) of operators and the number (if any) of punctuation marks, viz., "(" and ")". Thus the length of (F1) is 8; while the length of (F4) is only 4. The forms of *(5.68a)* may be arranged in ascending order of length:

(F5)	"P"	(length 1)
(F3)	"P · Q"	(length 3)
(F4)	"∼ P · Q"	(length 4)
(F2)	"P · (Q ⊃ R)"	(length 7)
(F1)	"∼ P · (Q ⊃ R)"	(length 8)

Sometimes, two or more forms of a sentence may be of equal length and will have to be assigned to the same slot in the ordering thus giving rise to what is called a 'weak' ordering, not a 'complete' ordering as above.[20] For example, consider the sentence

(5.73) "(∼ A · ∼ B) ∨ (C ⊃ C)".

Its forms, arranged in order are:

20. For an informal discussion of the concept of order and of three kinds of orders, see Abraham Kaplan, *The Conduct of Inquiry,* San Francisco, Chandler, 1964, pp. 178–183.

	"P"			(length 1)
	"P ∨ Q"			(length 3)
	"(P · Q) ∨ R"	"P ∨ (Q ⊃ Q)"		(length 7)
	"(∼P · Q) ∨ R"			(length 8)
	"(∼P · ∼Q) ∨ R"			(length 9)
	"(P · Q) ∨ (R ⊃ R)"	"(P · Q) ∨ (R ⊃ S)"		(length 11)
"(∼P · Q) ∨ (R ⊃ R)"	"(P · ∼Q) ∨ (R ⊃ R)"	"(∼P · Q) ∨ (R ⊃ S)"	"(P · ∼Q) ∨ (R ⊃ S)"	(length 12)
	"(∼P · ∼Q) ∨ (R ⊃ R)"	"(∼P · ∼Q) ∨(R ⊃ S)"		(length 13)

FIGURE *(5.bb)*

To what extent are we able to say of two forms that one is more (or less) specific than another? Although we cannot determine the relative specificity for every pair of arbitrarily selected forms, we can often do so for some pairs. Consider, for example, the two forms,

(F2) "P · (Q ⊃ R)"

(F3) "P · Q".

Intuitively we should want to say that (F2) is more specific than (F3). More generally, one form may be said to be *more specific* than another when every sentence which can be generated from the former can also be generated from the latter, but not conversely. Or, putting this another way, one form will be said more specific than another if all the sentences which can be generated from the former comprise a proper subset of the sentences which can be generated from the latter. Thus (F2) is more specific than (F3). Every sentence which can be generated from (F2) can also be generated from (F3), but (F3) may be used to generate sentences not generable from (F2); for example "A · (B ∨ ∼C)".

When two forms [e.g., "P ∨ (Q ⊃ Q)" and "(∼P · Q) ∨ R",] generate sets of sentences such that neither set is a subset of the other, then the two forms are *not* comparable as to specificity — neither can be said to be more, or less, specific than the other.

The so-called *specific form* in a particular logic of a sentence is simply its *most* specific form, or that one form which is more specific than any other form of that sentence. In cases where there is a single form of greatest length, that form turns out to be the most specific form of the sentence and hence *the* specific form of the sentence. Thus (F1), viz., "∼P · (Q ⊃ R)", will be said to be the specific form of "∼A · (B ⊃ C)". (F1) is the single longest form of *(5.68a)*. In cases where two or more forms are of equal length and longer than all other forms, the one having the *fewest* number of *different* sentential variable types turns out to be the specific form. Thus although both "(∼P · ∼Q) ∨ (R ⊃ R)" and "(∼P · ∼Q) ∨ (R ⊃ S)" have the same length and together comprise the longest forms of "(∼A · ∼B) ∨ (C ⊃ C)", the former represents only three *different* sentential variable types, (viz., "P", "Q", and "R") while the latter represents four (viz., "P", "Q", "R", and "S"). Hence the former, and not the latter, is the specific form of "(∼A · ∼B) ∨ (C ⊃ C)".

The specific form in a particular logic of a sentence represents the deepest conceptual analysis possible of that sentence in that logic. Note how the specific form of the sentence, "(∼A · ∼B) ∨ (C ⊃ C)", viz., "(∼P · ∼Q) ∨ (R ⊃ R)", contains, as it were, *more* information than the form "(∼P · ∼Q) ∨ (R ⊃ S)". The former tells us something more than the latter, namely, that one of the four component sentences in the compound sentence occurs twice. The latter form also tells us that the compound sentence has four simple sentential components, but neglects to tell us that two of these sentences are tokens of the same type.

There are only three possible relationships between the class of forms of one sentence and the class of forms of another: (1) the two classes coincide; (2) the two classes overlap but do not coincide; and (3) one of the two classes is totally contained within the other but not conversely. It is logically impossible that the classes of the forms of two sentences should be disjoint, for every sentence has among its forms a form shared by every other sentence, viz., the form "P".

It is in the facts that two or more sentences may share the same form and that each and every form represents an infinity of sentence-types, that the real importance in logic of the study of sentential-forms resides. For to the extent that certain properties of sentential-forms may be correlated with the modal attributes of the propositions expressed by sentences instancing those forms, we may determine the modal attributes not just of this or that proposition but of an infinity of propositions. By attending to the forms of sentences there is the potential for us to learn (some of the) modal attributes of all of the propositions in each of many infinite classes.

EXERCISES

1. *Which forms are shared by the sentence, "$A \vee (B \cdot C)$" and "$(B \vee C) \vee (B \cdot C)$"? What is the specific form of each of these sentences? Are the specific forms identical?*

2. *Arrange all the forms of "$(A \vee \sim B) \supset (A \equiv \sim B)$" in order of length.*

7. EVALUATING SENTENCE-FORMS

Sentence-forms, as we have seen, are well-formed formulae, and as such they may be evaluated on truth-tables in exactly the same sort of way that sentences may be evaluated. However, the *interpretation* of the completed truth-table is not quite so straightforward as in the case of sentences, for sentence-forms do not, of course, express propositions. Thus we must spend a little time pursuing the matter of just what an evaluation of a sentence-form can tell us about the sentences which may instantiate that form.

The validity of sentence-forms

In addition to using the term "validity" in the context of assessing arguments and inferences, logicians also use the term with a *different* meaning. Logicians often use the term "validity" *generically* to designate a family of three properties which may be ascribed to sentence-forms.

A sentence-form will be said to be *valid* (in a particular logic) if all of its instantiations express necessary truths; it will be said to be *contravalid* if all of its instantiations express necessary falsehoods; and will be said to be *indeterminate* if it is not either valid or contravalid.[21]

21. This threefold division of sentence-forms, along with the names "valid", "contravalid", and "indeterminate", was introduced into philosophy by Rudolf Carnap in 1934 in his *Logische Syntax der Sprache*. (This book has been translated by A. Smeaton as *The Logical Syntax of Language*, London, Routledge & Kegan Paul, 1937. See pp. 173–4.)

Applying these concepts to Truth-functional Propositional Logic, we may say that a sentence-form in Truth-functional Propositional Logic is *valid* if the final column of a truth-tabular evaluation of that form contains only "T"s; is *contravalid* if the final column contains only "F"s; and is *indeterminate* if it is not either valid or contravalid, i.e., if the final column contains both "T"s and "F"s.

Valid sentence-forms in Truth-functional Propositional Logic are often called "tautological", and sentences which instantiate such forms, "tautologies". But note, a sentence-form might be valid even though it is not tautological, e.g., it might be a sentence-form in Modal Propositional Logic. Being tautological is thus a special case of a formula's being valid.

If, in Truth-functional Propositional Logic, *any* form whatever of a sentence is either valid (tautological) or contravalid, then the specific form of that sentence will also be valid or contravalid respectively. And if the specific form of a sentence is valid, then that sentence expresses a necessary truth; and if the specific form of a sentence is contravalid, then that sentence expresses a necessary falsehood. Thus finding *any* form of a sentence to be valid or contravalid is a sufficient condition for our knowing that that sentence expresses a necessary truth or necessary falsehood respectively.[22]

Having an indeterminate form is a necessary, but not a sufficient, condition for a sentence's expressing a contingency. This is a point which is often obscured in some books by their use of the term "contingency" to name indeterminate sentence-forms. This is regrettable, since it is clear that a sentence having an indeterminate form need not express a contingency. Consider again the sentence *(5.53)*, which expresses a necessary truth:

(5.53) "All squares have four sides and all brothers are male."

Translated into the notation of Truth-functional Propositional Logic, this sentence, as we have seen, becomes

(5.53a) "F · M"

whose specific form is

(F3) "P · Q".

(F3) clearly is an indeterminate form, not a valid one. In light of this, we state the following corollaries to our definitions:

i) Every sentence which has a valid form expresses a necessary truth.

ii) Every sentence which has a contravalid form expresses a necessary falsehood.

But we do *not* say that every sentence which has an indeterminate form expresses a contingency. Instead, the correct statement of the third corollary is:

iii) Only those sentences (but even then, not all of them) whose specific forms are indeterminate express contingencies.

Note that *every* sentence has at least one indeterminate form, viz., "P".

22. It follows immediately that it is logically impossible that one sentence should number among its forms both valid and contravalid ones.

EXERCISES

1. *For each form below, determine whether it is valid, contravalid, or indeterminate.*

(a) "$P \vee \sim P$"
(b) "$P \cdot \sim P$"
(c) "$P \cdot \sim Q$"
(d) "$P \supset P$"
(e) "$(P \cdot Q) \supset P$"
(f) "$\sim (P \supset P)$"
(g) "$P \supset (P \vee R)$"
(h) "$P \vee (P \supset Q)$"
(i) "$(P \supset Q) \vee (Q \supset P)$"
(j) "$(P \supset Q) \vee (R \supset P)$"
(k) "$P \cdot (Q \cdot \sim Q)$"
(l) "$(P \vee \sim P) \supset (Q \cdot \sim Q)$"
(m) "$P \supset (P \supset Q)$"
(n) "$(P \vee (Q \supset Q)$"
(o) "$(P \vee R) \supset P$"

2. (a) *For each of the valid forms above, find an instantiation of that form which expresses a necessary truth.*

   (b) *For each of the contravalid forms above, find an instantiation of that form which expresses a necessary falsehood.*

   (c) *For each of the indeterminate forms above, (i) find an instantiation of that form which expresses a necessary truth; (ii) find an instantiation of that form which expresses a necessary falsehood; and (iii) find an instantiation of that form which expresses a contingency.*

3. *Find all the forms of the sentence, "$A \supset (A \vee B)$". What is the specific form of this sentence? Are any of its forms valid?*

4. *Which of the forms of* (5.73) *[see figure* (5.bb)*] are valid? Which are indeterminate?*

* * * * *

Modal relations

Just as we may evaluate two sentences on one truth-table, so we may also evaluate two sentence-forms. What inferences may we validly make in light of the sorts of truth-tables we might thereby construct? The answers parallel so closely those already given for sentences that we may proceed to state them directly:

Implication

> If, in the truth-tabular evaluation of two sentence-forms, it is found that there is no row of that table in which the first sentence-form has been assigned "T" and the second sentence-form has been assigned "F", one may validly infer — on the assumption of the constancy of substitution for the various sentence-variables in the two forms — that any proposition expressible by a sentence of the first form *implies* any and every proposition expressible by a sentence of the second form.

Equivalence

If, in the truth-tabular evaluation of two sentence-forms, it is found that in each row of the truth-table these sentence-forms have been assigned matching evaluations, one may validly infer — on the assumption of constancy of substitution for the various sentence-variables in the two forms — that any proposition expressible by a sentence of the first form is *logically equivalent* to any proposition expressible by a sentence of the second form.

Inconsistency

If, in the truth-tabular evaluation of two sentence-forms, it is found that there is no row in which both sentence-forms have been assigned "T", one may validly infer — on the assumption of constancy of substitution for the various sentence-variables in the two forms — that any two propositions expressible by sentences of these forms are *logically inconsistent* with one another.

(There is, of course, as we should expect, no rule in this series for the modal relation of *consistency*.)

EXERCISES

1. *For each pair of sentential forms below, determine whether the members of the pair are truth-functionally equivalent, i.e., are such that their instantiations express logically equivalent propositions.*

   a. "$P \supset Q$" *and* "$\sim (P \cdot \sim Q)$"

   b. "$P \supset Q$" *and* "$\sim P \vee Q$"

   c. "$P \supset (Q \supset R)$" *and* "$(P \supset Q) \supset R$"

   d. "$P \supset Q$" *and* "$(P \cdot R) \supset (Q \cdot R)$" *(Hint: use one 8-row truth-table)*

   e. "$\sim (P \supset Q)$" *and* "$(P \supset \sim Q)$"

2. *The operator vee-bar (i.e.,"$\underline{\vee}$") may be defined contextually in terms of tilde, dot, and vel:*

   "$P \underline{\vee} Q$" $=_{df}$ "$(P \vee Q) \cdot \sim (P \cdot Q)$".

   Show that the definiendum and the definiens have identical truth-conditions, i.e., that they are truth-functionally equivalent.

3. *Demonstrate also the truth-functional equivalence of* "$P \underline{\vee} Q$" *and* "$P \equiv \sim Q$".

4. *Since the truth-table for* "$P \supset Q$" *and* "$P \vee Q$" *lacks the combination F–F, it is impossible to find*

instantiations of these wffs which express propositions which stand in modal relations whose codes (see section 5, pp. 297–301) contain F–F, i.e., nos. 4, 7, 8, 9, 11, 12, 13, and 15.

Find those modal relations (using the numerical labels of figure (5.aa)*) which* cannot *obtain between two propositions having, respectively, the forms "P · ∼Q" and "P≡Q".*

5. *Which of the 15 relations depicted in figure* (5.aa) can *obtain among the possible instantiations of the two forms "P ⊃ Q" and "Q ⊃ P"?*

* * * * *

Argument-forms and deductive validity

An *argument-form* is, simply, an ordered set of sentence-forms which may be instantiated by a set of sentences expressing an argument.

For the sake of simplicity, however, we shall adopt a somewhat more restricted notion of an argument-form. It will suit our purposes if we construe an argument-form as a material conditional sentence-form whose antecedent is a conjunction of sentence-forms which may be instantiated by sentences expressing the premises of an argument and whose consequent is a sentence-form which may be instantiated by a sentence expressing the conclusion of an argument. Thus the argument

$$\begin{array}{ll} (5.74) & A \cdot \sim B \\ & \underline{A \supset C} \\ \therefore & C \equiv \sim B \end{array}$$

has among its forms

(F8) "(P · Q) ⊃ R",

(F9) "((P · Q) · R) ⊃ S",

(F10) "((P · ∼Q) · R) ⊃ (S ≡ T)",

(F11) "((P · ∼Q) · (P ⊃ R)) ⊃ (R ≡ ∼Q)", etc.

Argument-forms may be evaluated on truth-tables. If an argument-form is found to be valid, then any set of sentences instancing that form, if used to express an argument, will express a deductively *valid* argument. In general most sets of sentences expressing arguments (for example *(5.74)* above) will instantiate several argument-forms. If any of these forms is valid, then the specific form (defined in parallel fashion to "the specific form of a sentence") will likewise be valid. This is why logicians, in their quest for ascertaining deductive validity by using truth-tabular methods for evaluating argument-forms, will usually attend only to the specific form of an argument. For if any lesser form is valid, the specific form will be also, and thus they can immediately learn that no lesser form is valid if they learn that the specific form is not valid.

Although there is, as we can see, an important relationship between an argument's *form* being valid, and that *argument* itself being valid, these two senses of "valid" are conceptually distinct. We should take some pains to distinguish them.

Let us remind ourselves of the very definitions of these terms. An argument is said to be valid (or more exactly, deductively valid) if its premises imply its conclusion. An argument-form, on the other hand, is said to be valid if all of its instantiations expresses deductively valid arguments.

If an argument-form is valid then we are guaranteed that every set of sentences instantiating that form expresses an argument which is deductively valid, i.e., having a valid *form* is a sufficient condition for an argument's being deductively valid. But it is not a necessary condition.

Arguments come in only two varieties: the deductively valid and the deductively invalid. Argument-forms come in three: the valid, the contravalid, and the indeterminate. All too easily, many writers in logic have argued that if an argument-form is not valid, then any argument having that form is (deductively) invalid. But this is a mistake. Only if an argument's form is *contravalid* may one validly infer, from an examination of its form, that a given argument is deductively invalid. If an argument-form is *indeterminate* (which is one of the two ways in which an argument-form may be nonvalid), nothing may be inferred from that about the deductive validity or deductive invalidity of any argument expressed by a set of sentences instantiating that form. The simple fact of the matter is that any argument-form which is indeterminate will have instantiations which express deductively valid arguments and will have instantiations which express deductively invalid arguments.

EXERCISES

Which of the following argument-forms are valid, which contravalid, and which indeterminate?

a. $\begin{array}{l} \underline{P \supset Q} \\ \therefore P \supset (P \cdot Q) \end{array}$

b. $\begin{array}{l} P \supset Q \\ \underline{\sim P} \\ \therefore \sim Q \end{array}$

c. $\begin{array}{l} \underline{P \supset P} \\ \therefore Q \cdot \sim Q \end{array}$

d. $\begin{array}{l} P \supset Q \\ \underline{Q \supset R} \\ \therefore P \supset R \end{array}$

e. $\begin{array}{l} P \supset Q \\ \underline{P \supset \sim Q} \\ \therefore \sim P \end{array}$

f. $\begin{array}{l} \underline{P} \\ \therefore P \vee Q \end{array}$

g. $\begin{array}{l} \underline{P \supset Q} \\ \therefore (P \vee R) \supset Q \end{array}$

h. $\begin{array}{l} \underline{P \supset Q} \\ \therefore P \supset (Q \vee R) \end{array}$

i. $\begin{array}{l} \underline{P} \\ \therefore \sim P \end{array}$

j. $\begin{array}{l} \underline{(P \supset Q) \vee (Q \supset R)} \\ \therefore \sim (P \supset Q) \cdot \sim (Q \supset P) \end{array}$

8. FORM IN A NATURAL LANGUAGE

Consider the following sentence:

(5.75) "Today is Monday or today is other than the day after Sunday."

Anyone who understands the disjunct, "today is other than the day after Sunday" knows that this means the same as "today is not Monday" which is to say that *(5.75)* expresses the very same proposition as

(5.76) "Today is Monday or today is not Monday."

Now the question arises: When we wish to render *(5.75)* in our conceptual notation, shall we render it as

(5.77) "A∨B",

where "A" = "today is Monday", and "B" = "today is other than the day after Sunday"; or, using the same meaning for "A", shall we render *(5.75)* as

(5.78) "A ∨ ∼A"?

If we render *(5.75)* as *(5.77)* we shall not be able to show in Truth-functional Propositional Logic what we already know, viz., that *(5.75)* expresses a necessary truth.[23] If, however, we render *(5.75)* as *(5.78)*, we will be able to show that *(5.75)* expresses a necessary truth.

Does it follow from these considerations that the 'real' form of the English sentence *(5.75)* is "P ∨ ∼P", (i.e., the specific form of *(5.78)*), or does it follow that the 'real' form of *(5.75)* is "P∨Q" (i.e., the specific form of *(5.77)*)?

We would like to suggest that the question just posed is improper, that it has a false presupposition. Sentences in natural languages, e.g., English, French, German, etc., do not have a single or 'real' form any more than do sentences expressed in the conceptual notation of a formalized language. Typically, sentences in natural languages, like sentences in formalized languages, have several forms.

Even more to the point, however, is the question whether in speaking of the forms of a natural-language sentence, we are using the term "form" in the *same* sense in which we use it in talking of the forms of sentences which occur in our conceptual notation. This is not an easy question. Note, for example, that when we defined "form" above (p. 303), we, in effect, defined, not "form" *simpliciter,* but rather, "form in Truth-functional Propositional Logic". That is, we defined a sense of "form" which was relativized to a given language. It is not at all clear that we can cogently define a single concept of "form" which will apply to *every* language.

In our studies we do not need to ask what are the forms of the English sentence, *(5.75)*, viz., "Today is Monday or today is other than the day after Sunday." Our purposes will be satisfied if we ask instead, whether we should render *(5.75)*, whatever its forms in English might be decided to be, into our conceptual notation as *(5.77)* whose specific form is "P∨Q", or as *(5.78)* whose specific form is "P ∨ ∼P".

Our answer to this last question cannot be categorical; that is, we cannot say simply that one rendering is the right one and the other the wrong. Our answer must be conditional. We note that the specific form of "A ∨ ∼A", viz., "P ∨ ∼P", is *more specific* (in the sense previously explained) than is the specific form of "A∨B", viz., "P∨Q". It follows, then, that *(5.78)* represents a deeper conceptual analysis of the original English sentence, *(5.75)*, than does *(5.77)*. But this does not mean that *(5.78)* is *the* correct translation of *(5.75)*, nor even that *(5.78)* is 'more correct' than *(5.77)*. There is no question of right or wrong, correctness or incorrectness, in choosing between these two alternatives. Which rendering we choose will depend on the degree of nonformal analysis we perform on *(5.75)* and wish to capture in our conceptual notation. We can give only conditional answers. *If* we wish simply to record the fact that *(5.75)* is a grammatical disjunction of two different sentences,

23. Recall that the method of constructing a corrected truth-table (which would be able to show that *(5.77)* expresses a necessary truth) is a nonformal method lying outside Truth-functional Propositional Logic.

we will render it as "A∨B", and as a result will be unable to show in Truth-functional Propositional Logic that *(5.75)* expresses a necessary truth. *If,* however, we wish to record the fact that the two grammatical disjuncts of *(5.75)* express propositions which are contradictories of one another, we will render it as "A ∨ ∼A", and as a result will be able to show in Truth-functional Propositional Logic, (i.e., without recourse to corrected truth-tables) that *(5.75)* expresses a necessary truth.

There are no (known) mechanical procedures for translating from a natural language into formalized ones such as those of Truth-functional Propositional Logic: the translations come about through our understanding and analysis of the original sentences. But we would re-emphasize that analysis comes in degrees: it can be carried on quite superficially or to a greater depth and with corresponding differences in the various translations one might give for a single sentence.

The route from a sentence in a natural language to a sentence expressed in the conceptual notation of a formalized, or reconstructed language, is through nonformal logic, i.e., conceptual analysis.

Admittedly, each time we translate a sentence from a natural language into a formal one we are faced with the possibility that we may perform a nonformal analysis which is not as deep as is possible and hence we may lose certain information in the process. We have just seen such a case in the instance of sentence *(5.75)*. Does this ever-present possibility of losing information call into doubt the wisdom and efficacy of examining sentences formally?

Not in the slightest. Because formal methods have (for the most part) only a positive role to play, the possibility of losing information as one translates from a natural language into a formal one is not insidious. If the translation and subsequent formal analysis reveal something about the modal attributes of the corresponding proposition (or propositions), well and good: we are that much further ahead in our researches. But if the translation loses something important — as it might if, for example, we were to translate "Today is Monday or today is other than the day after Sunday" as "A∨B" — we are none the worse off for having tried a formal analysis. At the worst we have only wasted a bit of time; we have *not* got the *wrong* answer. In finding that "A∨B" does not have a tautologous or contravalid form we are not entitled to infer that the original English sentence expresses a contingency. Keeping our wits about us, we will recognize that in such a circumstance we are entitled to infer *nothing* about the modal status of that proposition.

In short, we adopt formal methods because in some instances they bring success; when they fail to bring success, they simply yield nothing. They do not yield wrong answers. It may be put this way: by adopting formal analyses we have a great deal to gain and nothing (except a bit of time) to lose.

9. WORLDS-DIAGRAMS AS A DECISION PROCEDURE FOR TRUTH-FUNCTIONAL PROPOSITIONAL LOGIC

It is obvious that truth-tables provide a simple as well as *effective decision procedure* for truth-functional propositional logic. That is to say, they provide a mechanical procedure for determining the validity of any formula in that logic.

Worlds-diagrams, too, may be so used. As compared with truth-table methods they prove to be extremely cumbersome for truth-functional logic. Nevertheless we introduce them here in order to prepare the way for their use as a decision procedure in modal logic where the disparity in cumbersomeness is not so marked and the worlds-diagrams method has the virtue of intuitive transparency.

Though somewhat cumbersome, the technique of using worlds-diagrams as a decision procedure is easy to comprehend. Suppose we have a formula instancing just one variable type, "P", e.g., "(P∨P)", "(P⊃P)", "P∨(P⊃ ∼P)". In such a case we would need only the three worlds-diagrams for monadic modal properties.

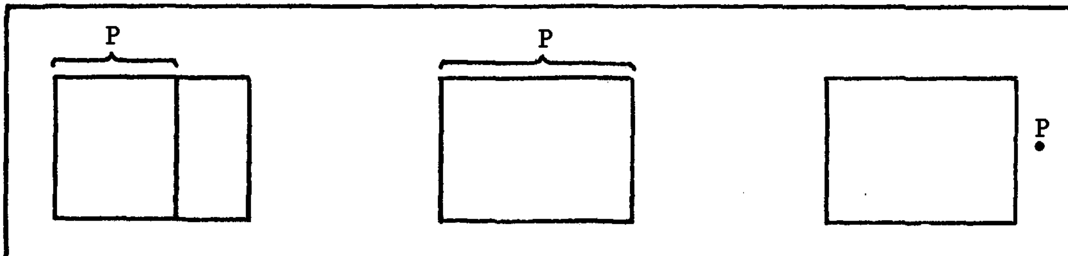


FIGURE *(5.cc)*

We use the rules for depiction of the various truth-functional operators in order to represent the chosen formula on each of these three worlds-diagrams. We start with the sentential components of least complexity and build up through those of greater complexity until we have depicted the whole formula. For example, in the case of the formula "P∨(P ⊃ ∼P)" we first place "P" on each of the worlds-diagrams in the set; then, in accord with the rule for placing the negation of a proposition on a worlds-diagram (section 2, p. 252), we add brackets labelled with "∼P"; at the third stage, in accord with the rule for adding brackets for material conditionals (p. 264), we add "(P⊃∼P)" to the diagrams; and finally, in accord with the rule for depicting the disjunction of two propositions (p. 257), we place the entire formula "P∨(P⊃∼P)" on each of the diagrams. The completed process, with each of the intermediate stages shown, looks like this:

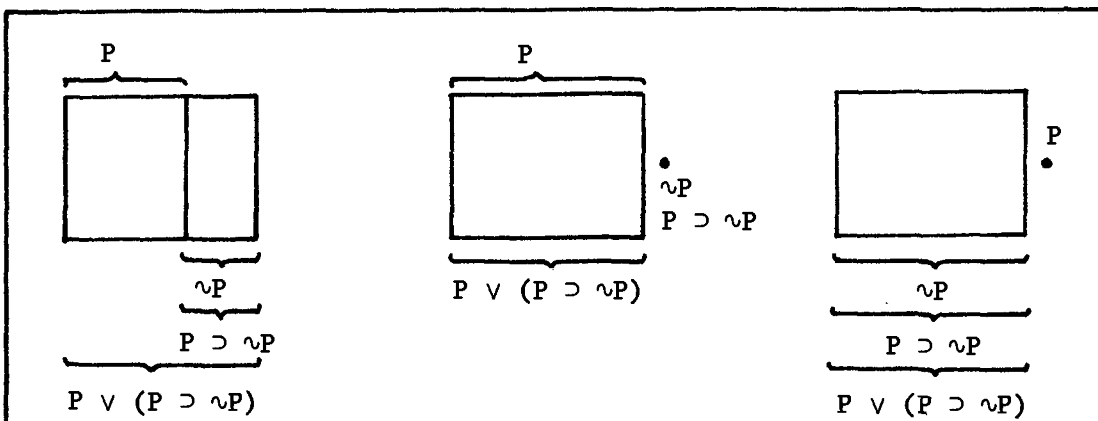


FIGURE *(5.dd)*

By inspection we can see that the formula "P ∨ (P ⊃ ∼P)" is *valid* since in every case its bracket spans all possible worlds, that is to say that every possible substitution-instance of this formula is necessarily true. A formula will be contravalid if its bracket in no case spans any possible world, and will be indeterminate if it is neither valid nor contravalid.

In the case of a formula instancing two variable types, "P" and "Q", e.g.,"P ⊃ (P ∨ Q)", "((P ⊃ Q) · ∼P) ⊃ ∼Q" and "(P ∨ ∼P) ⊃ (Q · ∼Q)", we need fifteen worlds-diagrams. Following the same procedure as above, we are then able to prove that the first of these formulae is valid. the second indeterminate, and the third contravalid.

EXERCISES

Use worlds-diagrams to ascertain the validity of each of the following formulae.

1. $(P \supset Q) \vee (Q \supset P)$

2. $(P \vee Q) \supset P$

3. $(P \supset Q) \supset (P \supset (P \cdot Q))$

4. $(P \cdot Q) \cdot (Q \supset \sim P)$

10. A SHORTCUT FORMAL METHOD: REDUCTIO AD ABSURDUM TESTS

The method of exhaustively evaluating a wff on a truth-table and the method of depicting a wff on each diagram in a complete set of worlds-diagrams are perfectly general: they may be applied to any and every truth-functional wff to determine whether that wff is tautological, contravalid, or indeterminate. But it is clear that these methods are very cumbersome: the sizes of the required truth-tables and the size of the set of required worlds-diagrams increase exponentially with the number of propositional variable-types instanced in the formula. If as few as five propositional variable-types are instanced in one formula, we would require a truth-table of 32 rows, or a set of 4,294,967,295 worlds-diagrams; for a formula containing propositional variables of six different types, a truth-table of 64 rows; etc. Fortunately there are available other methods which sometimes yield results in far fewer computational steps.

Certain, but not all, wffs lend themselves well to a method of testing which is called "the reductio ad absurdum" or often, more simply, "the reductio test".

The general strategy in a reductio test is to try to prove that a certain sentential (or argument) form is tautological (or contravalid) by showing that the assumption that its instantiations may be used to express falsehoods (or truths respectively) leads to contradiction. For example, one might use a reductio form of proof to establish that the formula "$P \vee \sim P$" is tautological by showing that to assign "F" to this formula leads to contradiction. Then, employing the principle that every contradictory of a self-contradictory proposition is necessarily true, we may validly infer that any proposition expressed by a sentence instantiating the original formula must be necessarily true.

Example 1: Show that "$P \vee \sim P$" is tautological.

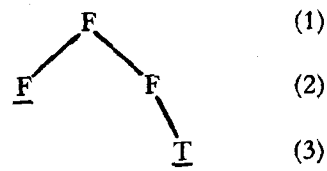
Since we wish to show that a stipulated formula is tautological, we begin by making the assumption that the formula may be instantiated by a sentence expressing a false proposition. To show this assumption, we simply assign "F" to the formula "$P \vee \sim P$":

$$P \vee \sim P$$

$$F \qquad (1)$$

From this point on, the subsequent assigments are all determined. The only way for a disjunction to be false is for both disjuncts to be false (see below, step 2). And the only way for a negation to be false is for the negated proposition to be true; thus at step 3 we must assign "T" to the second occurrence of "P". But "P" has already been assigned "F" at step 2. To complete the proof we underline the two inconsistent assignments.

$$P \vee \sim P$$

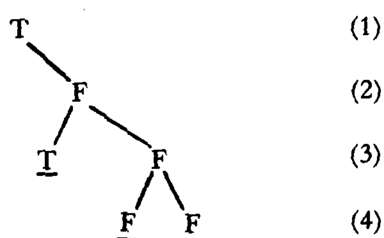


What precisely does this proof tell us? It tells us that in any world in which "P ∨ ~ P" is assigned the value "F", "P" must be assigned both the value "T" and the value "F". In short, we learn that any proposition expressed by a sentence having the form "P ∨ ~ P" may be false only in a world in which some proposition is both true and false, i.e., in an *impossible* world. But if the only sort of world in which a proposition is false is an impossible world, then clearly, in every *possible* world that proposition is true, which is just to say that that proposition is necessarily true. Hence any and every proposition expressed by a sentence of the form "P ∨ ~ P" is necessarily true. In other words, "P ∨ ~ P" is tautological.

Example 2: Show that "~ (P ⊃ (P ∨ Q))" is contravalid.

To show that a stipulated formula is contravalid, we begin by making the assumption that at least one of its instantiations may express a proposition which is true. If the formula *is* contravalid, we should be able to show that a sentence instantiating this form expresses a proposition which is true only in an impossible world, i.e., expresses a proposition which is *false* in every possible world. Indeed, this is precisely what we do show:

$$\sim [P \supset (P \vee Q)]$$



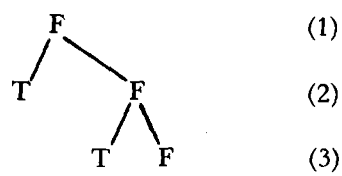
Example 3: Determine whether "P ⊃ (P · Q)" is tautological, contravalid, or indeterminate.

Given the presumption that we do not know whether "P ⊃ (P · Q)" is tautological, contravalid, or indeterminate, our strategy will be, first, to ascertain whether it is tautological. If it is not, then, secondly, we will determine if it is contravalid. If it is neither of these, then we may validly infer that it is indeterminate. We begin, then, by trying to prove it to be tautological. To do this, we look to see whether a proposition expressed by a sentence of this form is false in any possible world. If there is such a possible world, the sentence form is not tautological.

Stage 1: Is there a possible world in which "P ⊃ (P · Q)", without contradiction, may be assigned "F"?

We begin by assigning "F" to this formula. Thereafter all the subsequent assignments are logically determined, and by inspection we can see that no sentential formula is assigned both a "T" and an "F". Hence there is a possible world in which the truth-value of the proposition expressed by a sentence having this form is falsity, and hence the form is *not* tautological.

P ⊃ (P · Q)



Stage 2: Is there a possible world in which "P ⊃ (P · Q)" may, without contradiction, be assigned "T"?

At this second stage we begin by assigning "T" to the formula. This time, however, we find that not all the subsequent assignments are logically determined.

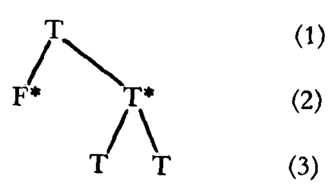
P ⊃ (P · Q)

T (1)

Having assigned "T" to the material conditional, what are we to assign to the antecedent and consequent of that conditional? Any of three *different* assignments are consistent with having assigned "T" to the conditional: we may assign "T" to both antecedent and to consequent; "F" to both antecedent and consequent; or "F" to the antecedent and "T" to the consequent. Which of these three possible assignments are we to choose?

At this point we must rely on a bit of guesswork and insight. The one assignment to be avoided in this case is the last one. Let us see how it leads to inconclusive results. Suppose we were to assign "F" (at step 2) to the antecedent and "T" to the consequent. Thereafter, at step 3, the assignments would, again, be fully logically determined and we would have an inconsistent assignment.

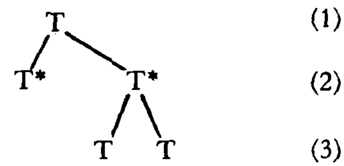
P ⊃ (P · Q)



What may we validly infer from this diagram? May we, for example, infer that the *only* sort of world in which "P ⊃ (P · Q)" may be assigned "T" is an impossible one? The answer is: No. And the reason is that we have not shown that there is no possible world in which "P ⊃ (P · Q)" may be consistently assigned "T". The route by which we arrived at the inconsistent assignment was not fully logically determined at every step: there was a 'choice'-point in step 2. (It will be helpful to mark assignments at choice-points with an asterisk.) Only if every possible choice were to lead to inconsistent assignments could we validly infer that the initial assignment of "T" (or "F") was a logically impossible one. At this stage (having examined only one of three choices), the results are inconclusive. So now we must examine the consequences of making a different assignment at the choice-point in step 2.

This time let us try assigning "T" to both antecedent and consequent. Again, thereafter the assignments are fully logically determined.

P ⊃ (P · Q)



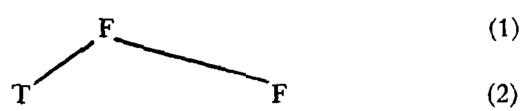
Can anything of significance for our problem be validly inferred from this diagram? Yes. We may validly infer that there *is* a possible world in which the formula "P ⊃ (P · Q)" may consistently be assigned "T", and this may be inferred from the diagram. It follows then that the form "P ⊃ (P · Q)" is not contravalid.

Together these two stages demonstrate that the form "P ⊃ (P · Q)" is *neither* tautological *nor* contravalid. It follows, then, that it is indeterminate.

Example 4: Show that "(P ⊃ Q) ⊃ [(R ⊃ P) ⊃ (R ⊃ Q)]" is tautological.[24]

What is interesting about this example is that it includes what at first appears to be a choice-point, but turns out not to include one after all. If we apply the reductio method to this formula we will generate the following:

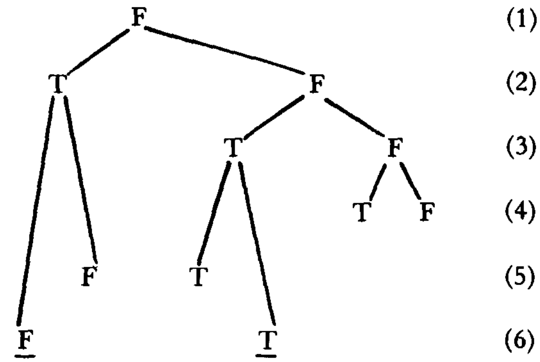
(P ⊃ Q) ⊃ [(R ⊃ P) ⊃ (R ⊃ Q)]



At step 2 we have written down a "T" under a material conditional. Normally, doing this would generate a choice-point; the subsequent assignments of "T"s and "F"s to the antecedent and consequent of that conditional could, as we have seen in example 3, be any of three kinds. But this is not a normal case. For if we hold off making a trial assignment at step 3, and proceed instead to examine the consequences of having placed an "F" also in step 2, we shall eventually find that the assignments appropriate for the "P" and "Q" in "(P ⊃ Q)" are logically determined.

24. In lengthy formulae — such as the present case — it aids readability to write some of the matching pairs of parentheses as "[" and "]".

(P ⊃ Q) ⊃ [(R ⊃ P) ⊃ (R ⊃ Q)]



At step 2 we were forced to assign "F" to "[(R ⊃ P) ⊃ (R ⊃ Q]". This, in turn, forces us in step 3 to assign "T" to "(R ⊃ P)" and "F" to "(R ⊃ Q)". Once again, as in step 2, we have assigned "T" to a material conditional. So, again, our strategy will be to postpone making any further assignments to the antecedent and consequent of this conditional in the hope that later events will dictate the appropriate choice. Therefore we turn our attention to the "F" which we have assigned at step 3 to "(R ⊃ Q)". This of course necessitates that we assign "T" to "R" and "F" to "Q". This we do at step 4. But having assigned "T" to "R" and "F" to "Q", we are now in a position to return to the earlier apparent choice-points. At step 5, we write "F" under "Q" in "(P ⊃ Q)", and "T" under "R" in "(R ⊃ P)". This now brings us to step 6 where, again, the assignments are logically determined. Having assigned "T" to "(P ⊃ Q)" and "F" to "Q", we are forced to assign "F" to "P" in that formula; and having assigned "T" to "(R ⊃ P)" and "T" to "R", we are forced to assign "T" to "P" in "(R ⊃ P)". Finally, looking across the assignments in step 6, we can see that we have assigned both "F" and "T" to the same sentence-variable type, and hence may validly conclude that the original assignment (in step 1) was an impossible one. Since that assignment was "F", the formula is tautological.

The map of assignments in this example is fairly lengthy. As one acquires proficiency in the construction of reductio proofs one will doubtless prefer to compress the proof into a single horizontal row. The order of the steps, then, will be written along the bottom. Thus the above proof may be compressed to appear this way:

(P	⊃	Q)	⊃	[(R	⊃	P)	⊃	(R	⊃	Q)]
F̲	T	F	F	T	T	T̲	F	T	F	F
(6)	(2)	(5)	(1)	(5)	(3)	(6)	(2)	(4)	(3)	(4)

Example 5: Show that "(P ∨ ∼P) · (Q ⊃ Q)" is tautological.

In accordance with the standard procedure we begin by assigning "F" to the formula. Immediately we generate a choice-point:

(P ∨ ∼P) · (Q ⊃ Q)
F
(1)

There are three different assignments which can be made to the two conjuncts of the conjunction consistent with the assignment "F" having been made to the conjunction: (a) "T" and "F" respectively; (b) "F" and "T" respectively; and (c) "F" and "F" respectively. Only if we examine all of these, by constructing additional diagrams, and find that each one leads to an inconsistent assignment may we validly infer that the original formula is tautological. As it turns out, however, two of these three assignments, the first and the second, themselves generate further choice-points. The first, in assigning "T" to a disjunction, generates three further possibilities for assignment; and the second in assigning "T" to a material conditional, also generates three further possibilities for subsequent assignments.

Clearly in such a case, the attraction of the method of reductio ad absurdum is dissipated. In this instance, the earlier truth-tabular method turns out to be the shorter one, and hence the preferred one.

Summary

The reductio ad absurdum method works in a fashion somewhat the reverse of that of the method of evaluation. In the method of evaluation, one begins with assignments of "T"s and "F"s to simple formulae and proceeds in a strictly determined fashion to assign "T"s and "F"s to longer formulae of which the original formulae are components. In the reductio method, the order of assignments is reversed. One begins by assigning "T" or "F" to a truth-functional compound sentence, and then proceeds to see what assignments of "T"s and "F"s to components of that formulae are consistent with that initial assignment. But in this latter case, one often finds that the assignments subsequent to the first are not all determined. If we assign "T" to a material conditional, we may generate a three-pronged choice-point; similarily if we assign "T" to a disjunction, or "F" to a conjunction. And no matter what we assign to a material biconditional, we may generate a two-pronged choice point. Clearly, then, the reductio ad absurdum method has its distinct limitations. The method is at its best when it is used to establish the tautologousness of a formula most of whose dyadic operators are hooks, or to establish the contravalidity of a formula most of whose dyadic operators are dots.

Whenever, in using a reductio, we run across a choice-point, we have to make various 'trial' assignments to see whether, on our initial assignment, we can find any subsequent assignment which is consistent. If it is possible, then there is nothing self-contradictory about our initial assignment. Only if it is impossible in every case whatever to make consistent assignments (and this means that we must examine every possible choice at every choice-point), may we validly conclude that the initial assignment was an impossible one for the formula.

In spite of its limitations, the reductio method is one much favored among logicians. For in many cases its application results in very short proofs. Then, too, it is often very useful in testing arguments for validity. Since showing that the form of an argument is tautological suffices for showing that argument to be deductively valid, the reductio method is often called upon by logicians to demonstrate the validity of arguments. Of course the same limitations as were just mentioned prevail in this latter context as well; but so do the same advantages.[25]

25. Truth-tables, worlds-diagrams, and reductio ad absurdum tests do not exhaust the logician's arsenal of techniques for ascertaining modal status, modal relations, validity of forms, deductive validity, etc. In addition, logicians make use of the method of deduction, both in Axiomatic Systems for the valid formulae of Truth-functional Propositional Logic and in so-called Natural Deduction. In this book, however, we do not develop the method of deduction any further than it was developed in chapter 4.

EXERCISES

1. *Use the Reductio ad Absurdum method to prove the following formulae to be tautologies.*

   (a) $P \supset (Q \supset P)$

   (b) $P \supset (P \vee Q)$

   (c) $(P \cdot Q) \supset P$

   (d) $(P \cdot Q) \supset (P \vee R)$

   (e) $[(P \supset Q) \cdot (Q \supset R)] \supset (P \supset R)$

2. *Use the Reductio ad Absurdum method to prove the following formulae to be contravalid.*

   (f) $P \cdot \sim P$

   (g) $\sim (P \supset P)$

   (h) $P \cdot (P \supset \sim P)$

3. *Use the Reductio ad Absurdum method to prove whether the following formulae are tautologous, contravalid or indeterminate.*

   (i) $(P \vee Q) \supset P$

   (j) $(P \supset Q) \supset (\sim P \supset \sim Q)$

4. *Use the Reductio ad Absurdum method to prove whether the following argument-forms are valid, contravalid, or indeterminate.*

   (k) $P \supset Q$
   $\underline{Q \supset \sim P}$
   $\therefore \sim P$

   (l) $P \supset (Q \vee R)$
   $\underline{Q}$
   $\therefore P \supset \sim R$

   (m) $\underline{P \supset P}$
   $\therefore Q \cdot \sim Q$

   (n) $P \supset (Q \vee R)$
   $\underline{Q}$
   $\therefore P \supset R$

   (o) $\underline{P}$
   $\therefore \sim P \supset P$