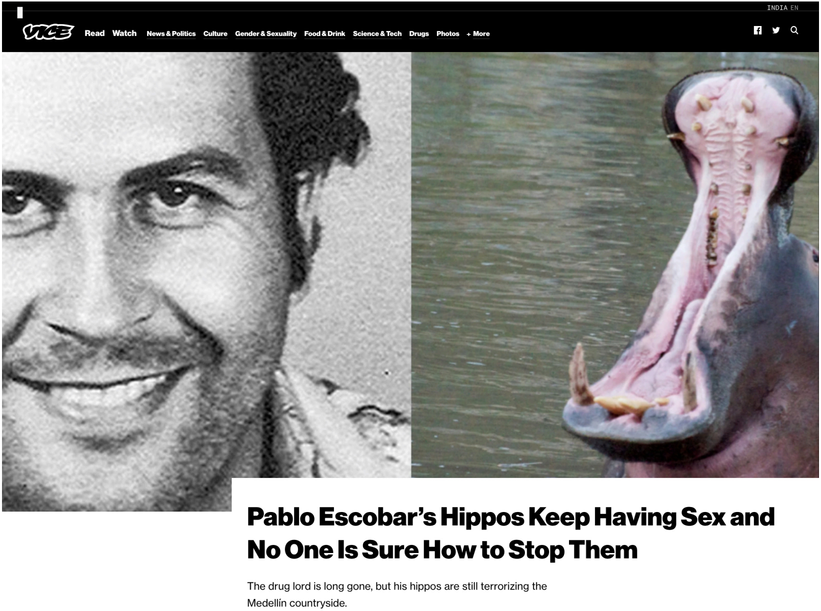
Exponential Growth

January 25, 2021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Pablo Escobar's Hippos Keep Having Sex and No One Is Sure How to Stop Them
The drug lord is long gone, but his hippos are still terrorizing the Medellín countryside.

When drug kingpin Pablo Escobar was killed by the Colombian National Police in 1993, he left a vast and bloody legacy in his wake. The Medellín Cartel boss is regarded as one of the most prolific criminals in history, and is notorious for having built a cocaine-fueled empire on the bodies of thousands of murdered individuals.

But El Patrón is also remembered by more than 50 hippopotamuses (Hippopotamus amphibius) that currently roam free near his palatial estate, Hacienda Nápoles. Escobar's captive hippos were never meant for the rivers and estuaries of northern Colombia, yet since his death they've behaved as wild animals are wont to: by vigorously breeding and multiplying, slowly establishing themselves as the largest invasive species in the world.

Today, it appears their troublesome reign is nowhere near ending because no one really knows how to stop them.

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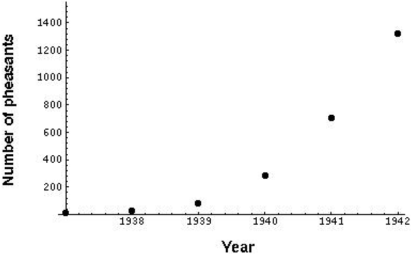

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How you analyze a model and the behaviour of the model both can depend on whether time is discrete or continuous.

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We'll start with the exponential model.

Exponential Growth in *Discrete* Time

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If we assume that each individual produces, on average, b offspring, and has a probability d of dying

how many individuals will die?

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This is a **recursion equation** for exponential growth.

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This yields the **general solution**:

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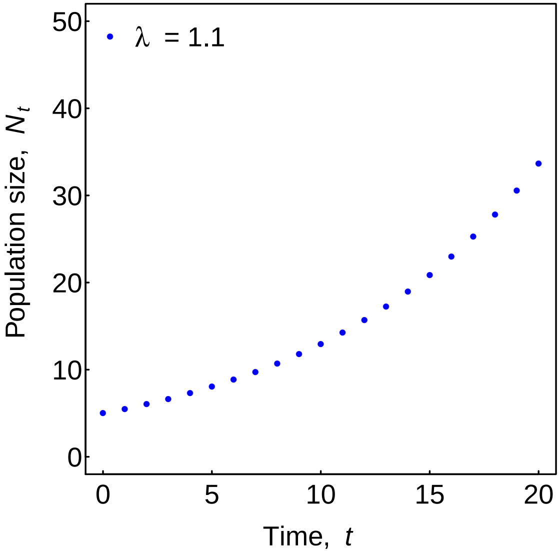

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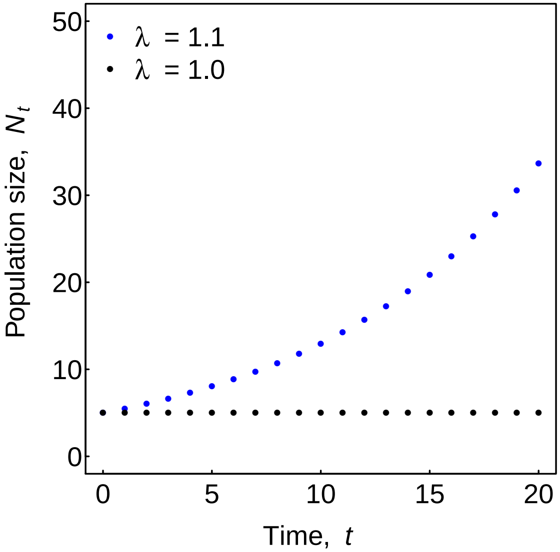

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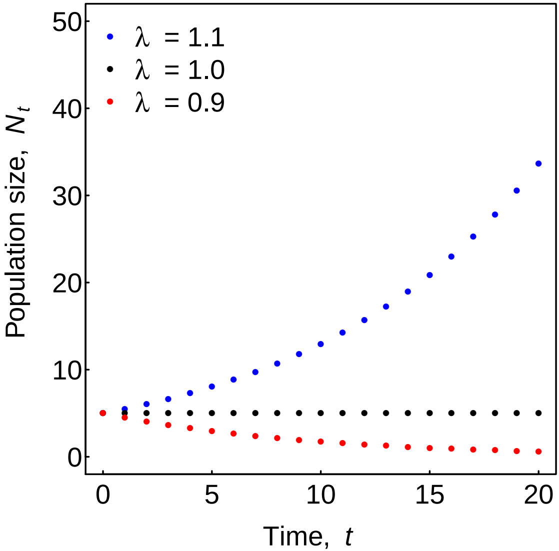

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With this equation and our data, we can calculate 5 estimates of λ:
$\lambda_1 = 5/1 = 5$
$\lambda_2 = 35/5 = 7$
$\lambda_3 = 80/35 = 2.29$
$\lambda_4 = 326/80 = 4.075$
$\lambda_5 = 1956/326 = 6$
The average of these, 4.87, gives us an estimate of λ.

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Now, substituting $t = 0$, we can solve for C_2.

$$N[0] = e^{r0}C_2 = C_2$$

and thus we have

$$N[t] = e^{rt}N[0]$$

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In fact, Lack observed that "the figures suggest that the increase was slowing down and was about to cease, but at this point the island was occupied by the military and many of the birds shot."