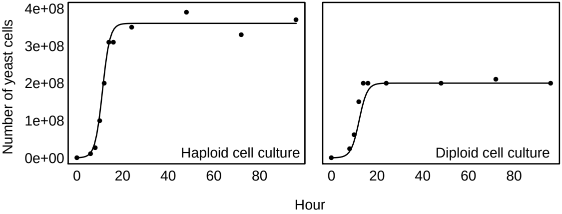
Logistic Growth

January 28, 2021

Although populations may initially experience exponential growth, resources eventually become depleted and competition becomes more severe.

This suggests that we must change the assumption that each individual will have the same number of offspring on average, regardless of the population size.

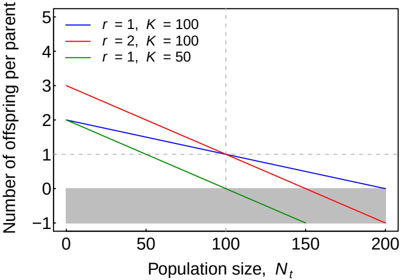


Logistic Growth in *Discrete* Time

The expected number of offspring may depend on population size in any number of ways.

The logistic equation assumes that the expected number of offspring decreases linearly with population size:

$$\text{Expected number of offspring per parent } = 1 + r\left(1 - \frac{N[t]}{K}\right)$$



A parent has, on average, one offspring if $r = 0$ or if $(1 - N[t]/K) = 0$.
The latter holds when $N[t] = K$.

r is known as the **intrinsic rate of growth**, because it measures whether the population tends to grow ($r > 0$) or shrink ($r < 0$).

K is known as the **carrying capacity**, because it measures the population size at which the population produces exactly enough offspring to just replace itself.

The population size in the next generation is the expected number of offspring per parent times the total number of parents:

$$N[t+1] = \left(1 + r\left(1 - \frac{N[t]}{K}\right)\right) N[t]$$

This is the **recursion equation** describing the change in population size from generation to generation under the logistic model.

This **recursion** is:

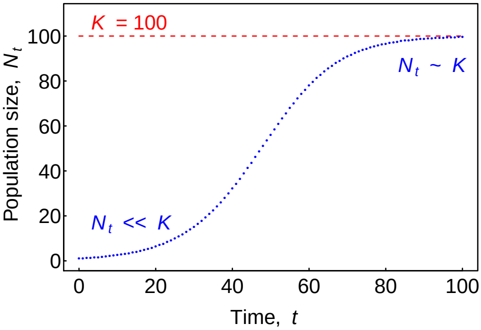
- a function of two parameters (r, K)
- a non-linear function of one variable $(N[t])$
  Non-linear means that there is a term in the equation where the variable is taken to some power other than 1.

The change in population size from one generation to the next, ΔN, is given by

$$\Delta N = N[t+1] - N[t] = r\left(1 - \frac{N[t]}{K}\right) N[t]$$

When will the population grow in size?

$$N[t+1] = \left(1 + r\left(1 - \frac{N[t]}{K}\right)\right) N[t]$$



Logistic Growth in *Continuous* Time

The model of logistic growth in continuous time follows from the assumption that each individual reproduces at a **rate** that decreases as a linear function of the population size.

Rate of reproduction per parent = $r(1 - N[t]/K)$.

If there are $N[t]$ individuals in the population at time t, then the rate of change of the population size will be:

$$\frac{dN[t]}{dt} = r\left(1 - \frac{N[t]}{K}\right)N[t]$$

This is the **differential equation** describing the rate of change in population size in the logistic model.

Connection

The logistic equation reduces to the exponential equation under certain circumstances.

If K equals infinity, $N[t]/K$ equals zero and population growth will follow the equation for exponential growth.

If the population size, $N[t]$, is much smaller than the carrying capacity, K, then $N[t]/K$ is small. In this case, the population will grow nearly exponentially (until the it is no longer much smaller than K).

We can use the definition of the derivative to show that the continuous and discrete time versions of the logistic are equivalent to each other as long as r is small.

Mathematical aside: Definition of the derivative:

$$\frac{df}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta f}{\Delta t}$$

According to the discrete time model, as long as r is small, the population will change (approximately) by:

$$\Delta N = r\Delta t \left(1 - \frac{N[t]}{K}\right) N[t]$$

over the time interval Δt near generation t as long as Δt is not too great.

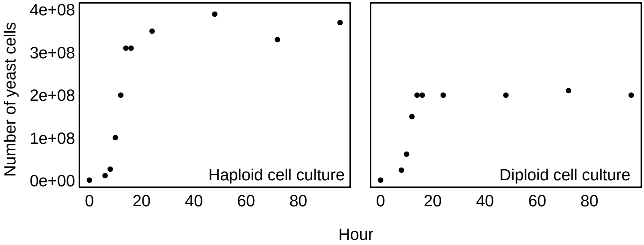
Using the definition of the derivative, we can use this equation to define the derivative of the population size with respect to time in the discrete model:

$$\frac{dN}{dt} = \lim_{\Delta t \to 0} \frac{\Delta N}{\Delta t} = r\left(1 - \frac{N[t]}{K}\right) N[t]$$

This is the same equation that we derived assuming a continuous time model.

If r is large, however, then the population can change in size rapidly over short periods of time and the discrete and continuous time models give **very different** answers.

Dr. Otto cultured haploid and diploid populations of *Saccharomyces cereviseae*. She observed the following growth for the two types of cells:



Although the population grows nearly exponentially at first, growth decreases as the population size increases (density-dependent growth is observed).

The equilibrium population size (K) is larger for the haploid cells, but do haploid and diploid cells have different intrinsic rates of increase (r)?

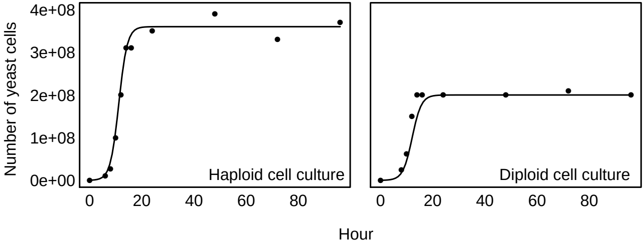
By fitting the logistic equation to the data:

haploid: $r = 0.62$, $K = 3.6 * 10^8$

diploid: $r = 0.57$, $K = 2.0 * 10^8$

The r do not differ greatly (and are not significantly different for this data).

With these parameter estimates, the logistic model nicely fits the data:



Note: This may be a bit misleading, as such excellent model fits are not often observed outside of the laboratory with organisms other than microorganisms.