Introduction to Matrix Algebra

January 29, 2021

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Let's start by re-examining the model we just looked at tracking the number of birds on each of two islands. Rather than $N_1[t]$ and $N_2[t]$, here we'll use x_t and y_t.

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Under these definitions, the number of birds on each island in the next generation will equal:

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These equations are linear functions of the variables (i.e., they contain only constant multiples of x and y and nothing more complicated such as x^2 or e^x).

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Linear systems of equations like these can also be written in matrix form:

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \end{bmatrix} = \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix} \begin{bmatrix} x_t \\ y_t \end{bmatrix}$$

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$$\text{vector} = \text{matrix} \quad \text{vector}$$
$$\vec{v}_{t+1} = \mathrm{M} \quad \vec{v}_t$$

That is, the vector representing the number of birds on each island is written as the product of a matrix times the vector in the previous time step.

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First we have to review some basics of linear algebra.

A **column vector** has elements arranged one on top of another, e.g.,

$$\begin{bmatrix} 5 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 5 \\ 9 \\ 7 \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

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A **row vector** has elements arranged left to right, e.g.,

$$[5, 2], [1, 5, 9, 7], [x, y], [x, y, z], [x_1, x_2, \cdots, x_n]$$

The number of elements in the vector indicates its **dimension**. For instance, the $[x, y]$ coordinates drawn on a plane are in 2-dimensions:

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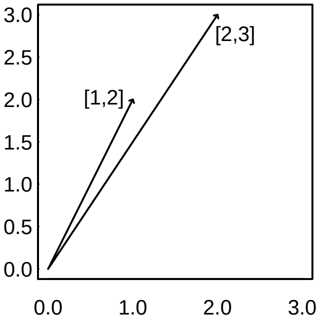

An $m \times n$ **matrix** has m rows and n columns, e.g.,

$$\begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1n} \\ x_{21} & x_{22} & \cdots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{m1} & x_{m2} & \cdots & x_{mn} \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} 75 & 67 \\ 66 & 34 \\ 12 & 14 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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We will write matrices in boldface (e.g., A) and vectors with an arrow on top (e.g., $\vec{x}$).

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Qualification: The vectors (or matrices) added together must have the same dimension.

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$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax + by + cz \\ dx + ey + fz \\ gx + hy + iz \end{bmatrix}$$

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On the other hand, matrix multiplication does satisfy the following laws:
(AB)C = A(BC) (associative law)
A(B + C) = AB + AC (distributive law)
(A + B)C = AC + BC (distributive law)
α(AB) = (αA)B = A(αB) (scalar multiplication)

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The **determinant** of a 2×2 matrix is:

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The **determinant** of an $n \times n$ matrix is obtained by taking the first row and multiplying the first element of the first row by the determinant of the matrix created by deleting the first row and first column *minus* the second element of the first row times the determinant of the matrix created by deleting the first row and second column *plus* the third element... and so on.

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$$\left|\mathrm{M}\right| = \sum_{j=1}^{n} (-1)^{j+1} x_{1j} \left|\mathrm{M}_{1j}\right|$$

(M_{1j} is the matrix M with the first row deleted and the j^{th} column deleted.)

A square $m \times m$ matrix A is **invertible** if it may be multiplied by another matrix to get the identity matrix. We call this second matrix the **inverse** of the first:

$$\mathsf{A}\mathsf{A}^{-1} = \mathbf{I} = \mathsf{A}^{-1}\mathsf{A}$$

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There are rules to find the inverse of a matrix (when it is invertible), but for a 2×2 matrix, you can just use the following:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{det}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$$

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To find the eigenvalues of a matrix, we can rearrange the above equation, using the distributive law for matrix multiplication:

$$\mathrm{M}\vec{v} - \lambda\vec{v} = (\mathrm{M} - \lambda\mathrm{I})\vec{v} = \vec{0}$$

where I is the *identity matrix*, and $\vec{0}$ is a vector of zeros.

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For example, in the $n = 2$ case,

$$(\mathrm{M} - \lambda \mathrm{I}) = \begin{pmatrix} m_{11} - \lambda & m_{12} \\ m_{21} & m_{22} - \lambda \end{pmatrix}$$

so that

$$|\mathrm{M} - \lambda \mathrm{I}| = (m_{11} - \lambda)(m_{22} - \lambda) - m_{21} m_{12} = \lambda^2 - (m_{11} + m_{22})\lambda + (m_{11} m_{22} - m_{21} m_{12}) = 0$$

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The two roots can be found using the quadratic formula:

$$\lambda_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and

$$\lambda_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

are the two eigenvalues.

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Any non-zero vector $\vec{v}$ which satisfies this equation is an eigenvector for eigenvalue λ_1.

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The first row of the matrix multiplication on the left hand side yields

$$m_{11} * v_1 + m_{12} * v_2 = \lambda_1 * v_1$$

Any non-zero vector $\vec{v}$ which satisfies this equation is an eigenvector for eigenvalue λ_1.

You could also use the second row of the matrix multiplication (and sometimes you have to).

For an eigenvalue, λ, there are an infinite number of possible eigenvectors.

If $\vec{v}$ is an eigenvector of the matrix M corresponding to the eigenvalue λ, it must satisfy:

$$M\vec{v} = \lambda\vec{v}$$

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To find an eigenvector for eigenvalue λ_2, simply repeat with λ_2.

Worked Example:

Let's find the eigenvalue(s) for the matrix:

$$M = \begin{bmatrix} 1 & 1 \\ -5 & 0 \end{bmatrix}$$

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$$\begin{aligned} Det(M - \lambda I) &= 0 \\ Det\left(\begin{bmatrix} 1-\lambda & 1 \\ -5 & 0-\lambda \end{bmatrix}\right) &= 0 \\ (1-\lambda)(0-\lambda) - (1)(-5) &= 0 \\ \lambda^2 - \lambda + 5 &= 0 \\ (\lambda+3)(\lambda-2) &= 0 \end{aligned} \tag{1}$$

and so the eigenvalues are $\lambda_1 = -3$, $\lambda_2 = 2$.

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This gives us two equations:

$$v_1 + v_2 = -3v_1$$

and

$$-5v_1 = -3v_2$$

The first tells us that

$$v_2 = -4v_1$$

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and, finally, let's check that we haven't made any mistakes...

Find the eigenvalue(s) and corresponding eigenvectors for the matrix:

$$M = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

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Do the vectors $v_1 = [1, 0]$ and $v_2 = [3, 0]$ form a basis for $\mathbb{R}^2$. Why or why not?

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Consider the recursion equations for any model that describes the change in state of a population from one generation ($x_i[t]$) to the next ($x_i[t+1]$). To make it easier to write, we will use x_i to denote the variables in the current generation and x_i' to denote the variables in the next generation.

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If there are n variables, then there will be n equations:

$$\begin{aligned} x_1' &= m_{11}x_1 + m_{12}x_2 + \cdots + m_{1n}x_n \\ x_2' &= m_{21}x_1 + m_{22}x_2 + \cdots + m_{2n}x_n \\ &\vdots \\ x_n' &= m_{n1}x_1 + m_{n2}x_2 + \cdots + m_{nn}x_n \end{aligned}$$

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E.g., in a predator-prey model, $n = 2$, because we have to track both the number of predators and the number of prey.

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Because the equations are linear, we can also write these equations in matrix form:

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Denoting the matrix by M and the vector of x_i by $\vec{x}$, we can then write this equation as

$$\vec{x}' = \mathrm{M}\vec{x}$$

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To find out where the population will be at some generation t (described by the vector $\vec{x}[t]$), we can use the equation $\vec{x}' = M\vec{x}$ over and over again:

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This will enable us to determine what happens to the vector, $\vec{x}$ over time.

If we let D denote the diagonal matrix whose diagonal elements are the eigenvalues of M and A denote the matrix whose columns are the eigenvectors of M (placed in the same order as the eigenvalues of D), then $M = ADA^{-1}$.

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The great thing about this is that, because D is a diagonal matrix, D^t is easy to calculate.

Specifically:

$$\mathrm{D}^{\mathrm{t}} = \begin{pmatrix} \lambda_1^t & 0 & \cdots & 0 \\ 0 & \lambda_2^t & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \lambda_n^t \end{pmatrix}$$

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$$\vec{x}[t] = M^t \vec{x}[0] = AD^t A^{-1} \vec{x}[0]$$

This above method is referred to as a "change of basis", because we changed our vantage point from the original M matrix to the D matrix using the A matrix and changed it back using A^{-1}.

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In fact, we can back-transform to get the general solution in the original basis from the general solution in the new basis.

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For example, you can determine what the stable equilibria are by determining where the population will tend as time goes to infinity.