MEDSC 2013

PROCEEDINGS OF THE
8TH ANNUAL MATHEMATICS EDUCATION
DOCTORAL STUDENTS CONFERENCE

Faculty of Education
Simon Fraser University

TABLE OF CONTENT

2013 PROGRAMME: MATHEMATICS EDUCATION DOCTORAL STUDENTS CONFERENCE

8:40 – 8:55	**Welcome and Coffee**	
	EBD 7500b	**EBD 7500f**
9:00 – 9:35	**Oi-Lam Ng** Shearing and IPads: Exploring Geometry with Touchscreen and a Dynamic Geometry Environment	**Lyla Aslalim** Using the Patterns-of-Participation Approach to Understand High School Mathematics Teachers' Classroom Practice in Saudi Arabia
9:35 – 10:10	**Ike Udevi-Aruevoru** The Relationship Between Motivation, Achievement Goals, Achievement Variables, And Mathematics Achievement	**Gaya Jayakody** Effects of Inconsistent Definitions: Definition of Continuity
10:10 – 10:20	**Break**	
10:20 – 10:55	**Veda Roodal Persad** Sofya Kovalevskaya: Mathematics as Fantasy	**Masomeh Jamshid Nejad** Influence Of An Educator's Rich Mathematics Knowledge On Developing Prospective Teachers' Professional Knowledge
11:00 – 12:00	**Plenary Speaker: Beth Herbel-Eisenmann**	
12:00 – 12:15	**Plenary Q & A**	
12:15 – 1:15	**Lunch**	
1:15 – 1:50	**Kevin Wells** Students' Use of Gesture and Posture Mimicry in the Mathematics Classroom	**Natasa Sirotic** Transformation in Mathematics Teaching Practice: A Case Study in Teacher Noticing
1:50 – 2:25	**Sean Chorney** Using Two Frameworks to Analyze Two Geometric Tasks: Semiotic Mediation and New Materialism	**Minnie Liu** The Process of Rudimentary and Complex Mathematical Tasks

2:25 – 3:05	**Marta Venturini** Designing Tasks In A Dge For Student Assessment	**Simin Sadat Chavoshi Jolfaee** Subjective Probability as Presented in BC Curriculum and Teacher Preparation Textbooks
3:05 – 3:15	**Break**	
3:15 – 3:50	**Peter Lee** Understanding Mathematical Learning Disabilities (MLD): Definitions, Cognitive Characteristics, And Subtypes	**Krishna Subedi** Mathematical Abstraction and Teaching Challenges: Viewing Teaching Activity through the Lens of Reducing Abstraction in Teaching (RAiT)
3:50 – 4:25	**Harpreet Kaur** Young Children's Thinking about Various Types of Triangles using Dynamic Geometry Tasks	**Melania Alvarez** Use of a Phenomenology Theory Perspective to Study Teachers' Engagement in the Reading of Manuals during a Professional Development Session.
4:25 – 5:00	**George Ekol** Exploring Statistical Variability With Sketchpad-Designed Graphs	**Chanakya Wijeratne** Confronting Infinity Via Painter's Paradox (Skype presentation)

PLENARY SPEAKER

From Talking the Talk to Walking the Walk: An Exploration of Data Using Tools from SFL

Beth Herbel-Eisenmann

In this talk, I share some theoretical and analytic tools from systemic functional linguistics (SFL) that I have used in my recent research in order to engage participants in exploring these tools with data. The data are transcripts that originate from a study group focused on mathematics classroom discourse with 9 secondary mathematics teachers. The particular idea we'll explore is how the study group talks about the "mathematics register" (Halliday, 1978) across the year. We'll explore the themes and shifts in the ways the teachers collectively made sense of the mathematics register, focusing on particular discourse practices they use and how they put words in relationship to one another.

ABSTRACTS:

USING THE PATTERNS-OF-PARTICIPATION APPROACH TO UNDERSTAND HIGH SCHOOL MATHEMATICS TEACHERS' CLASSROOM PRACTICE IN SAUDI ARABIA

Lyla Alsalim

My research goal is to gain a better understanding of how high school mathematics teachers in Saudi Arabia are coping with recent education reform including how their practices are changing in response to the changes that are happening in the education system in general, and specifically, to the introduction of the new mathematics textbooks. These reform initiatives call for more research in order to understand the role of the mathematics teachers' classroom practices. In this paper, patterns-of-participation theory serves as a lens to interpret and understand Saudi high school mathematics teachers' practices during the current reform movement and the role the new textbooks play in influencing teachers practice. The data presented is about Nora, an experienced, high school mathematics teacher.

CONFRONTING INFINITY VIA PAINTER'S PARADOX

Chanakya Wijeratne

In mathematics education research paradoxes of infinity have been used in the investigation of conceptions of infinity. In this study the Painter's paradox is used to investigate how undergraduate students studying Calculus understand infinity. This study contributes to research on the use of paradoxes in mathematics education as a research tool and to research on understanding infinity.

YOUNG CHILDREN'S THINKING ABOUT VARIOUS TYPES OF TRIANGLES USING DYNAMIC GEOMETRY TASKS

Harpreet Kaur

This paper will show preliminary results of how young children (age 7-8, grade 2/3) can exploit the potential of dynamic geometry environments to identify, classify and *define different types of triangles (scalene, isosceles, equilateral). This study is based on three-lesson classroom intervention, during which the children worked both in a whole classroom setting in which they could interact directly with Sketchpad on an interactive whiteboard as well as individual/pair work with paper-and-pencil. This paper reports on first lesson only. Using Sfard's (2008) communicational approach, we extend the work of Battista (2008) to show how students developed a reified discourse on various types of triangles. Further, we show how the students used a dynamic language to describe the behaviour of various triangles in terms of invariances about how the sides and angles of these triangles would change under dragging.*

EFFECTS OF INCONSISTENT DEFINITIONS: DEFINITION OF CONTINUITY

Gaya Jayakody

This paper reports on a problematic situation that arises through certain definitions involving the concept of continuity. The paper mainly focuses on bringing out these problems in the context of textbooks and other mathematical resources and in addition gives an instance to elaborate how this problem could create tensions and conflicts in students' thinking processes. Sfard's commognitve framework is used in the analysis of a student's work on continuity. A potential remedy for this problem is presented in conclusion.

EXPLORING CONSTRUCTS OF STATISTICAL VARIABILITY USING DYNAMIC GEOMETRY.

George Ekol

The qualitative study reported in this paper examines university statistics students' aggregate reasoning with data using a semiotic mediation perspective. I focus on students' understanding of the links among distribution, the mean and standard deviation in a data set. Participants in the study explored these concepts using a dynamic mathematics sketch designed in Sketchpad. Findings suggest that the dynamic sketch mediated the meaning of statistical variability during and after participants' interactions with the sketch. However, some participants also showed mixed reasoning—a combination of elements of aggregate reasoning with some textbook

procedures—after they stopped using the dynamic sketch. Some implications for post-secondary statistics curriculum are suggested.

USING TWO FRAMEWORKS TO ANALYZE A GEOMETRIC TASK: SEMIOTIC MEDIATION AND NEW MATERIALISM

Sean Chorney

This study looks at a geometry situation on The Geometer's Sketchpad and analyses the activity using two different theoretical frameworks. Semiotic mediation which emerges from the Vygotskian school, and the more organic and emergent framework of Barad and Ingold. The purpose is to identify what is highlighted by using different tools of analysis. The finding is that the outcome of observation is dependent upon the framework; the results are completely different with little commonality.

USE OF PHENOMENOLOGY THEORY PERSPECTIVE TO STUDY TEACHERS' ENGAGEMENT IN THE READING OF MANUALS DURING A PROFESSIONAL DEVELOPMENT SESSION.

Melania Alvarez

The purpose of this phenomenological study is to provide a description of a professional development session which main purpose was to engage teachers to read and discuss the manual of a math program recently introduced in their school.

PLAYING NUMBERS ON TOUCHCOUNTS

Vajiheh(Mina) Sedaghat Jou

This paper explores how young children build meaning through communicative, touch-based activities involving talk, gesture and body engagement working with TouchCounts (Educational iPad app). The main goal of this paper is to show the *impact of touch-based interactions and finger counting on the development of children's perception and motor understanding of numbers. In this study, Nemirovsky's perceptuomotor integration approach theoretical framework revealed strong value of digital touch-based interaction and mathematics embodied in emergent numerical expertise by making and objectifying numbers.*

TRANSFORMATION IN MATHEMATICS TEACHING PRACTICE: A CASE STUDY IN TEACHER NOTICING

Natasa Sirotic

This case study focuses on the professional growth of an elementary teacher who participated in a practice based professional development initiative centred on the goal of creating a culture of mathematical thinking in the classroom. Through sustained inquiry, reflective practice, and collaboration with colleagues, which was focused around the examination of the impact of mathematical lessons on the student thinking, learning and understanding, a gradual but significant transformation was achieved which transferred into her daily instructional practice.

SHEARING AND IPADS: EXPLORING GEOMETRY WITH DYNAMIC GEOMETRY AND TOUCHSCREEN TECHNOLOGY

Oi-Lam Ng

In this paper, we report on two lessons for teaching junior high school students to the idea of shearing in a dynamic geometry environment. Through a classroom-based intervention involving the active use of a class set of touchscreen tablet devices, we analyse students' evolving discourse about area. The touchscreen tablet technology seemed to have supported the ways in which students talk about shearing as a temporal and continuous process and made the idea more tangible. We highlight the specific roles of the teacher and digital technology in supporting the process of semiotic mediation through which the students learned about shearing.

SOLVING RUDIMENTARY AND COMPLEX MATHEMATICAL TASKS

Minnie Liu

Tasks are used in mathematics education for a variety of purposes – from delivering course material to developing students' mathematical thinking skills. In this article, I present research on a type of task specifically designed to foster students' ability to flexibly apply their existing mathematical knowledge and skills in problem solving situations. In particular, I look at students' problem solving processes when working collaboratively on such tasks. Results indicate that while the processes of solving these tasks are similar to those of modeling tasks, differences also exist.

SOFYA KOVALEVSKAYA: MATHEMATICS AS FANTASY

Veda Roodal Persad

What do accounts by and about mathematicians of their involvement with mathematics tell us about the nature of the discipline and the attendant demands, costs, and rewards? Working from an autobiographical sketch and biographies of the first woman in the world to achieve a doctorate of mathematics, Sofya Kovalevskaya (1850-1891), and using the Lacanian notion of desire, I examine the forces that shape and influence engagement with mathematics. I contend that involvement with mathematics is impelled, fuelled, and sustained by desire.

DESIGNING STUDENT ASSESSMENT TASKS IN A DYNAMIC GEOMETRY ENVIRONMENT

Marta Venturini

This paper explores the design of assessment tasks involving the use of Dynamic Geometry Environments (DGEs). I adapt the work of Laborde and the results of Sinclair, which focus on the design of DGE tasks, to the context of formative assessment. I provide an initial framework, along with illustrative examples, for different types of DGE-based assessment tasks that can be used in the classroom but also to study technology-based teacher practices. This research develops new directions in finding how to design suitable tasks for student mathematical assessment in a DGE.

WHO SHOULD TEACH PROSPECTIVE MATHEMATICS TEACHERS: MATHEMATICIANS WITH EDUCATION BACKGROUND OR ONLY EDUCATION EXPERTS?

Masomeh Jamshid Nejad

Teachers' knowledge plays a vital role in developing students' academic achievements. Therefore, training prospective teachers for the purpose of developing their professional knowledge considers as one of the most important issues for teacher trainers. The question raises here is who should train prospective mathematics teachers, a mathematician who is an educator or an expert in education? In this research study, a mathematician who has experience of teaching prospective teachers had been interviewed to investigate how he tended to develop prospective teachers' professional knowledge.

MATHEMATICAL ABSTRACTION AND TEACHING CHALLENGES: TEACHING ACTIVITY THROUGH THE LENS OF REDUCING ABSTRACTION IN TEACHING (RAIT)

Krishna Subedi

Mathematics is an abstract subject. One of the most important challenges for mathematics teachers therefore involves the task of dealing with mathematical abstraction and figure out ways of translating them into understandable ideas for their students. By analysing teaching episodes through the lens of reducing abstraction in teaching (RAiT), this paper explores the notion of mathematical abstraction and illustrates various strategies and tendencies of teachers dealing with mathematical abstraction.

THE RELATIONSHIP BETWEEN MOTIVATION, ACHIEVEMENT GOALS, ACHIEVEMENT VARIABLES AND MATHEMATICS ACHIEVEMENT

Ike Udevi-Aruevoru

The purpose of this study is to investigate the relationship between motivation, achievement goals, and achievement variables on academic achievement. Using the six variables from the motivation questionnaire (Glynn, et. al, 2009) and five other

variables associated with academic achievement namely, academic achievement emotions – hope and pride, academic interest, academic achievement goal, and the importance of mathematics to future career goal, this study will use multiple regression analysis and also, path analysis to determine which of these variables accounts for the most variance in student's academic achievement.

SUBJECTIVE PROBABILITY AS PRESENTED IN BC CURRICULUM AND TEACHER PREPARATION TEXTBOOKS

Simin Chavoshi Jolfaee

Abstract: Today, unlike the curriculum documents, among the practitioners of probability, subjective probability (Bayesian methods) is a known term. The Bayesian probabilistic models have been receiving considerable attention over the last few decades from the users of probability, i.e. scientists and engineers. In many fields including computer science, Biostatistics, cognitive science, medicine, and meteorology using Bayesian models are common practice. In this paper I have tried to identify and discuss various ways in which the subjective probability is implicated in K-12 mathematics education with respect to documents such as the BC mathematics education curriculum, and the mathematics education research literature.

UNDERSTANDING MATHEMATICAL LEARNING DISABILITIES (MLD): DEFINITIONS AND COGNITIVE CHARACTERISTICS

Peter Lee

While the amount of research on reading disabilities far exceeds that of mathematical learning disabilities (MLD), research on MLD over the past decade has shown significant growth. This paper examines some of this emerging research. More specifically, this paper reveals the challenges of defining MLD and examines the cognitive deficits that characterize children with MLD.

USING THE PATTERNS-OF-PARTICIPATION APPROACH TO UNDERSTAND HIGH SCHOOL MATHEMATICS TEACHERS' CLASSROOM PRACTICE IN SAUDI ARABIA

Lyla Alsalim
Simon Fraser University

My research goal is to gain a better understanding of how high school mathematics teachers in Saudi Arabia are coping with recent education reform including how their practices are changing in response to the changes that are happening in the education system in general, and specifically, to the introduction of the new mathematics textbooks. These reform initiatives call for more research in order to understand the role of the mathematics teachers' classroom practices. In this paper, patterns-of-participation theory serves as a lens to *interpret and understand Saudi high school mathematics teachers' practices during the current reform movement and the role the new textbooks play in influencing teachers practice. The data presented is about Nora, an experienced, high school mathematics teacher.*

INTRODUCTION

Teaching is generally considered a complex practice that involves the constant and dynamic interaction between the teacher, the students and the subject matter. One of the main goals of most education reform initiatives has been to change teachers' classroom practices. In the past, educators viewed changing the curriculum as an endeavour to change the content of instruction more than the teacher's classroom practices. However, most recent reform curricula focus on highlighting teacher practices that promote and evoke students' understanding of mathematics alongside the changes in content (Tirosh & Graeber, 2003). Changes to a teacher's role that are included in the education reform movement call for more research in order to understand and theorise teachers' classroom practices.

The Saudi Arabian education system has undergone major changes in the past decade. Government agencies involved in education have introduced new policies, standards, programs, and curriculum with the expectation that teachers incorporate the changes seamlessly, without consideration of existing beliefs and practices. This reform movement has motivated me to study change in teachers' practices. My research goal is to gain a better understanding of how high school mathematics teachers in Saudi Arabia are coping with recent education reform including how their practices are changing in response to the changes that are happening in the education system in general, and specifically, to the introduction of the new mathematics textbooks. These reform initiatives call for more research in order to understand the role of the mathematics teachers' classroom practices. Patterns-of-participation approach will serve as a lens to interpret and understand Saudi high school mathematics teachers' practices during the current reform movement and the role the new textbooks play in influencing teachers practice.

Textbooks in mathematics classroom:

For a long time, school mathematics has been associated with textbooks and curriculum material (Remillard, 2005). According to *Trends in International Mathematics and Science Study* (TIMSS), textbooks and documents such as exercise resources for use in classrooms as teaching aids, remain important elements in mathematics classrooms in many countries. Textbooks play an important role in shaping the curriculum experiences of mathematics (TIMSS 2011; Valverde, Bianchi, Wolfe, Schmidt & Houang, 2002). This fact is especially true in Saudi Arabian high schools. They provide teachers with a basic outline for thinking about what mathematics should be taught, when, and how. In 2010, the Ministry of Education introduced new mathematics textbooks, the primary, and sometimes only, resource for teachers. The Ministry sees this initiative as a major step towards creating change in teaching practices

In Saudi Arabia, one of the major reform initiatives directly addresses existing mathematics curriculum. In 2010, the Ministry of Education introduced new mathematics textbooks, the primary, and sometimes only, resource for teachers. The Ministry sees this initiative as a major step towards creating change in teaching practices. The new approved mathematics textbooks in Saudi Arabia are based on the curricula published by McGraw Hill Education learning company. According to the ministry of education in Saudi Arabia, the new mathematics curriculum aims to (a)help students to develop higher-order mathematics thinking skills, (b) develop ways of mastering these skills, (c)construct a strong conceptual foundation in mathematics that enable students to apply their knowledge, (d) make connections between related mathematical concepts and between mathematics and the real world, and(e) apply mathematics logically to solve problems from daily life (ministry of education website).

In Saudi Arabia textbooks have official status clearly reflecting official curriculum. This textbook series is distributed for free by the Ministry of Education to be a classroom resource; each student would have his or her own textbook. The accompanying teacher's guides tell teachers how to use the textbooks, lesson by lesson. The new textbook were gradually introduced in 2010 starting by grads one, four, seven and ten; by last yeas all grads in Saudi Arabia have been introduced to the new textbook. Each grade has two textbooks; semester one and semester two textbooks. In high school, every textbook for every semester has four chapters; every chapter is divided into lessons. All the six textbooks for high school have the same introduction which includes the same objectives of the textbooks that mentioned in the ministry of education website.

Traditionally, curriculum materials or textbooks have been a center agent of policies to regulate mathematics practice in ways that parallel instruction with the reform perspective (Remillard, 2005). Textbooks are often the main resource for students and teachers in the classroom, offering the everyday materials of lessons and guiding the activities teachers and students do. As a result, educational policy makers use textbooks as an essential means to decide what students learn. Textbooks are an essential part of curriculum materials which are used for directing students' acquisition of certain culturally appreciated concepts, procedures, intellectual dispositions, and ways of thinking (Battista & Clements, 2000).

While effective student learning is one expected outcome of textbook use, the development of teachers' techniques and practice is an additional desired outcome. Researchers have only recently started to shed the light on the impact of curriculum materials on teachers and how teachers use them (Davis and Krajcik, 2005; Remillard, Herbel-Eisenmann, & Lloyd, 2009.). The focus of how teachers interact with and use curriculum materials has not been always considered significant to studying curriculum. Historically, research about school curricula relied mainly on examining the textbooks to restructure the contents of classroom practice (Love & Pimm, 1996). Reform efforts in mathematics education are the product of curriculum development supported by standards adopted by the National Council of Mathematics Teachers (NCTM, 2000). Teachers face the demand of applying new curriculum materials, and adopt new conceptual and pedagogical approaches to teach new standards-based curriculum. Standard-based curriculum requires students to answer questions with high levels of cognitive demands that emphasize conceptual understanding and connection of many mathematical ideas rather than traditional procedural skills. As a result, Remillard (2005) calls for more research in order to understand teachers' use of the reform based textbooks.
Some studies suggest that using new curriculum materials do not necessarily lead to changes in teacher practices. Manouchehri and Goodman (2000) observed teachers' reactions to the implementation of new standard-based mathematics textbooks and found that changes in teachers' practices do not occur as a consequence to new textbooks and other materials. They came to the conclusion that teachers do not change their teaching practices merely from interaction with new materials.

There are a variety of factors that influence the ways teachers interact with curriculum materials such as "their beliefs about mathematics teaching and learning, their beliefs about the role of curriculum materials, their strategies and practices around the use of curriculum materials, and their capacity to competently use curriculum materials to enact particular forms of instruction” (Choppin,2011, p. 343). Remillard (1999) states teachers assign meaning to new textbooks based on the relationship between their beliefs and the features of the material; this meaning fits within the context of their teaching as a whole.. Drake and Sherin (2006) also indicate that teachers’ beliefs are essential to understanding their curriculum use and adaptation. “Teachers’ narrative identities as learners and teachers of mathematics frame the ways in which they use and adapt a reform-oriented mathematics curriculum” (p. 154).

THEORETICAL FRAMEWORK

Skott (2010, 2011, & 2013) introduced PoP as a promising framework which provides coherent and dynamic theoretical understandings of mathematics teacher practices. Skott’s (2009, 2010) main motivation in developing this framework was to overcome the conceptual and methodological problems of belief research. In his later work, Skott (2011, 2013) extended the use of the framework to include what is traditionally referred to as knowledge and identity in research on mathematics teachers.

The challenges and complexity associated with beliefs research has led some researchers, such as Skott (2009, 2010, 2011, and 2013) and Gate (2006), to call for more social approaches to beliefs research. Gate (2006) indicates that there is a need to take a social approach when studying teacher belief systems because it will shift focus from cognitive constructs. A change toward sociological constructs will balance existing views about the nature and genesis of beliefs. Skott also supports this view indicating that taking a context – practice approach by adopting patterns-of-participation framework provides more coherent and dynamic understandings of teaching practices. Furthermore, it will help in resolving some of the conceptual and methodological problems of a belief–practice approach while maintaining an interest in the meta-issues that constitute the field of beliefs. The PoP framework challenges dominant traditional belief research by questioning the very notion of beliefs and its acquisitionist theoretical foundation (Skott, 2010).

PoP framework elaborates on the view that teachers’ practices in classrooms are not simple expressions of their desire and personal resources; it also views their practices as adaptations to social conditions in which they work. As noted by Skott (2013), “teacher contributes to classroom interaction by re-engaging in other past and present practices, possibly reinterpreting and transforming them in the process” (p. 548). The framework presents a useful tool to understand teachers’ position for emerging classroom practices that takes into account the multiple perspectives of student learning in educational research.

PoP is a theoretical framework developed in line with other several social approaches to research in mathematics education. It aims to develop a more coherent understanding of the teacher's role for learning and life in mathematics classrooms. This alternative framework emphasizes the emergent nature of classroom practices. To a considerable degree, PoP adopts participationism as a metaphor for human functioning more than mainstream belief research. Therefore, PoP draws on the work of participationism researchers, specifically Vygotsky, Lave & Wenger, and Sfard.

Skott initially developed the patterns-of-participation framework in relation to teachers' beliefs. However, in order to develop a more coherent approach to understand teachers' practices, Skott(2013) extended the framework to include knowledge and identity. Skott (2013) notes that research on teachers has mainly focused on studying three relatively distinct domains: teachers' knowledge, beliefs, and identity. This leads to some incoherence that negatively influences the understanding of the teachers' role in classrooms. Skott presents POP as a coherent, participatory framework that is capable of dealing with matters usually faced in the distinct fields of teachers' knowledge, beliefs, and identity.

In classrooms, students and teachers interact in several simultaneous practices. Some of these practices are directly related to the teaching and learning of mathematics while others are not. Some of them are discourse related an explicit verbal feature, while others are not. And some of them relate to communities that are not actually present in the classroom or at the school. Understanding the teachers' role in the classroom entails understanding the complex relationship between these simultaneous practices. PoP is a promising framework which aims to understand the complexity of teachers' practices in classroom.

In research of mathematics education, there is, to some extent, an unexpected disengagement between research on teachers' beliefs, knowledge, and identity. This disconnect in research hinders the development of coherent understandings of the teachers' role for classroom practice and for student learning. Researchers could use PoP as a coherent, participatory framework that has the potential to address issues usually faced with in the distinct fields of teachers' knowledge, beliefs, and identity. However, PoP does not connect the analyses of teachers' knowledge, beliefs, and identity by regulating the use of theoretical views across the acquisition–participation part. As an alternative, it employs a participatory approach and looks for patterns in individual teacher's participation in different social practices. Therefore, PoP is a theory that could enrich research approach in mathematics education, especially the one that is interested in understanding and theorising mathematics teaching.

METHODOLOGY

The data presented are about Nora, an experienced, high school mathematics teacher. Nora has 13 years experience teaching mathematics in public and private middle and high school in Saudi Arabia. She graduated from university with Bachelor of Science with specialization in mathematics. She does not have a degree in education which means she has never taken any courses in education. After she graduated from university, she started teaching in private school. She worked in the private school for seven years where she taught different grades of elementary, middle and high school. After that she got an offer to tech in public high school. She has been teaching in the public high school for six years. When the new textbooks introduced three years ago, Nora was teaching grade ten, therefore, she was among teachers who used the textbooks the first year when the books introduced. During last year, Nora taught grads 10 and 11 with 23 lessons per week.

I conducted a semi-structured 60 minutes interview with Nora during the last summer. I invited Nora to reflect on her experiences with mathematics and its teaching and learning during her 13 years of experience. During the interview, I asked Nora to express her view about recent reform movement in Saudi Arabia. I also asked Nora to reflect on her experience teaching mathematics using the old and new textbooks. Interview was audio recorded and transcribed.

Discussion and conclusion:

Being a teacher in an era of educational reform:

Nora indicted her deep personal commitment to current educational reform in Saudi Arabia. She believes that the pace of educational reform has been increasing at the global level and Saudi Arabia has to join the global movement of education reform. She emphasizes that the need to be reasonable and fair when we talk about recent reform efforts. “Reform is one of the controversial topics among people who are interested in educational issues in Saudi Arabia. I see that the government is making a noticeable effort to provide quality education in public school. But we have to admit the changing is difficult and complicated “.

In the interviews, Nora indicates that success of reform movements depend, at least in part, on the degree of match between teachers’ perceptions of the teaching act and their role as teachers, and the demands of the reform movement. She states that “creating a positive change starts with creating a motivated teacher”. Although she believes her view of the role of teachers put great pressure on teachers, she indicates that not all teachers are able to carry out the interventions. “Most teachers work in schools with very bad conditions, classrooms are full with more than 40 students in every class ...most teachers do not have the opportunity to join any professional development programs... and most of the available professional development programs are not adequate”.

New mathematics textbooks impact:

Nora expressed that before the introduction of the new textbooks she was very enthusiastic. She believes that the new textbooks are generally better than the old textbooks. She believes that the new textbook supports student learning and create more positive environment in the classrooms. The new textbooks provide more opportunities for students' engagement and participation. However, Nora indicated that she feels isolated and unsupported in her use of the new curriculum materials. She feels that she needs more time and training to become familiar with the new textbook's content. She states, "very often I have questions about the textbook, but I don't know where I can't find answers". She complains that the ministry of education did not put teachers' preparation to the use of the new textbooks into account. She indicates that the other resource she has other than the textbook is her communication with other mathematics teachers in her school. The conversation Nora has almost every day with other teachers provides support and rich resource for Nora's practice.

Nora commented about her teaching using the new textbooks; "Although I feel that the new textbook could offer better learning experience to the student.., I am not sure if I am using it effectively". She indicated that the textbooks motivated her to reflect on her own teaching practice. She believes that using the new textbooks allows her to spend more time in the classroom listening to students' explanation. However, Nora indicated that some of the activities presented in the textbook do not make sense. She feels that the textbook structure the lessons in a certain way which does not work all the time.

What does it mean to do mathematics

During the interview, Nora discussed the issues of classroom culture around what it means to "do math". She believes that there is a common culture in school mathematics which view doing math as sitting quietly at a desk, completing a worksheet, using the textbook as a resource and turning in the completed assignment prior to class ending. The new textbooks in Nora's view challenge this old lasting culture. The new textbooks encourage the use of collaborative learning and group work. However, it takes time for teachers and student to manage these activities. .

Nora commented on the textbook presentation of situational problems which are connected with real life situations. She believes that the textbook surely make some positive transformations compared to old textbooks which simply delivered mathematical concepts in a very isolated manner. She also indicated that making the connection is not always easy. Students sometimes cannot see the connection since many of the questions are not realistic enough so students would be able to relate themselves to the questions.

Nora's practice as a high school mathematics teacher reveals patterns of distress and sense of obligation to improve her practice. Nora uses the new textbook as a tool for self-directed professional development. She endorses students' mathematical narratives. She emphasizes both concepts and procedures understanding by encouraging her students to share their understandings and discuss how the content relates to everyday life. However, to have a better understanding of Nora's practice as a mathematics teacher more data is needed. The use of multiple open interviews in combination with observations of classroom and staff-room interactions may allow some understanding of the meaning of teachers in classroom.

References

Drake, C., & Sherin, M. G. (2006). Practicing change: Curriculum adaptation and teacher narrative in the context of mathematics education reform. Curriculum Inquiry, 36(2), 153-187.

Tirosh, D., & Graeber, A. (2003). Challenging and changing mathematics teaching classroom practices. In A. Bishop, M. Clements, C. Kietel, J. Kilpatrick, & F. Leung (Eds.), *Second international handbook of mathematics education* (pp. 643–688). Dordrecht, the Netherlands: Kluwer Academic Publishers.

McLeod, D. B., & McLeod, S. H. (2002). Synthesis – Beliefs and mathematics education: Implications for learning, teaching and research. In G. C. Leder, E. Pehkonen, & G. Törner (Eds.), *Beliefs: Ahidden variable in mathematics education?* (pp. 115–126). Dordrecht: Kluwer Academic Publishers.

Skott, J.(2009). Contextualising the notion of 'belief enactment'. *journal of mathematics teacher education. (*12)27–46.

Skott, J. (2013). Understanding the role of the teacher in emerging classroom practices: searching for patterns of participation. *ZDM Mathematics Education.* 45:547–559

Skott, J., Moeskær Larsen, D. & Hellsten Østergaard, C. (2011). From beliefs to patterns of participation – shifting the research perspective on teachers. *Nordic Studies in Mathematics Education*, 16 (1-2), 29–55.

Skott, J. (2011). Beliefs vs. patterns of participation – towards coherence in understanding the role the teacher. Paper presented at *MAVI 17*

Remillard, J. T. (1999). Curriculum materials in mathematics education reform: A framework for examining teachers' curriculum development. Curriculum Inquiry, 29(3), 315–342.

Remillard, J. T. (2005). Examining key concepts in research on teachers' use of mathematics curricula. Review of Educational Research, 75(2), 211–246.

Remillard, J. T. (2009). Considering what we know about the relationship between teachers and curriculum material. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), Mathematics teachers at work: Connecting curriculum materials and classroom instruction (pp.85-92). New York: Routledge.

Remillard, J. T., Herbel-Eisenmann , B. A., & Lloyd, G. M. (2009). Teachers' use of curriculum materials: An emerging field. In J. T. Remillard, B. A. Herbel-Eisenmann & G. M. Lloyd (Eds.), Mathematics teachers at work: Connecting curriculum materials and classroom instruction (pp. 3-14). New York: Routledge.

Valverde, G.A., Bianchi, L.J., Wolfe, R.G., Schmidt, W.H., Houang, R.T.(2003). According to the Book: Using TIMSS to Investigate the Translation of Policy into Practice Through the World of textbooks, Kluwer, Dordrec

CONFRONTING INFINITY VIA PAINTER'S PARADOX

Chanakya Wijeratne

Department of Mathematics, University of Colombo

In mathematics education research paradoxes of infinity have been used in the investigation of conceptions of infinity. In this study the Painter's paradox is used to investigate how undergraduate students studying Calculus understand infinity. This study contributes to research on the use of paradoxes in mathematics education as a research tool and to research on understanding infinity.

Introduction

Gabriel is an archangel, as the Bible tells us, who "used a horn to announce news that was sometimes heartening (e.g. the birth of Christ in Luke l) and sometimes fatalistic (e.g. Armageddon in Revelation 8-11)" (Fleron, 1991, p.1). The surface of revolution formed by rotating the curve $y = \frac{1}{x}$ for $x \geq 1$ about the *x*-axis is known as the Gabriel's horn (Stewart, 2012). This surface and the resulting solid were discovered and studied by Evangelista Torricelli in 1641. It is not clear why this surface came to be known as Gabriel's horn but if we go by what the bible says about Gabriel and his horn then we have to ask whether the discovery of this surface by Torricelli was good news or bad news for mathematics. The answer depends on the time period in which this question is asked and our views of infinity.

At the time of the discovery of Gabriel's horn in the 17th century the term infinity referred to unending processes. The Greek mathematicians of antiquity, up to the time of Aristotle, used the term *apeiron* to refer to such processes as [endless] counting, successively halving a linear segment (as in Zeno's paradoxes), and evaluating of an area by exhaustion. In the 4th century BC Aristotle explained the idea of infinity as an [endless] process (Kim et. al., 2012). He introduced the dichotomy of potential infinity and actual infinity as a means of dealing with paradoxes of the infinite that he believed could be resolved by refuting the existence of actual infinity. One can think of potential infinity as a process, which at every instant of time within a certain time interval is finite. Actual infinity describes a completed entity that encompasses what was potential. Aristotle's potential/actual dichotomy dominated and influenced conceptions of infinity for centuries. Kant (1724–1804), for example, believed that we are finite beings in an infinite world. Therefore we cannot conceive the whole but only the partial and finite. Even more contemporary thinkers such as Poincaré (1854–1912) held largely Aristotelian views (Dubinsky et. al., 2005a). Then in 1851 Bolzano's work *The Paradoxes of Infinity* was a serious attempt to introduce infinity into mathematics as an object of study. Though Bolzano considered one-to-one correspondence as a criterion for comparing infinite sets he selected the part-whole relationship as the criterion. Then Cantor put forth a theory of infinity [infinite sets] using one-to-one correspondence as a criterion for comparing infinite sets. This was a turning point in the history of infinity as it established actual infinity as a mathematical object in mathematics (Moreno and Waldegg, 1991).

Mancosu and Vailati (1991) point out that Gabriel's horn is a solid that is consistent with the definition of a solid given in Euclid; a solid is that which has length, breadth, and depth. But it was counterintuitive to the early seventeenth-century geometer as it is not bounded on every side. Torricelli showed that the surface area of this solid is infinite but its volume is finite. The most striking feature of Gabriel's horn is that, although finite in volume, it is actually, not just potentially, infinite in length.

> The result seemed so counterintuitive and astonishing that at first some of the leading mathematicians thought it impossible; even eighty years later Bernard de Fontenelle commented, "One apparently expected, and should have expected, to find [Torricelli's solid] infinite" in volume. (p. 1)

So was the discovery of this solid by Torricelli in 1641 good or bad news for mathematics? According to Hilbert this heralded the very good news of actual infinity having a prominent place in mathematics three centuries later: "No one shall expel us from the paradise [theory of infinite sets] which Cantor created for us" (George Cantor; quoted in Kline, 1972, p. 1003), but not according to Poincaré: "Later generations will regard Mengenlehre [Cantor's theory of infinite sets] as a disease from which one has recovered" (Henri Poincaré; quoted in ibid.).

In mathematics education research paradoxes have been used as a lens on student learning. Movshovitz-Hadar and Hadass (1990 & 1991) investigated the role mathematical paradoxes can play in the pre-service education of high school mathematics teachers. They concluded that "a paradox puts the learner in an intellectually unbearable situation. The impulse to resolve the paradox is a powerful motivator for change of knowledge frameworks. For instance, a student who possesses a procedural understanding may experience a transition to the stage of relational understanding" (Movshovitz-Hadar and Hadass, 1991, p. 88). Sriraman (2008) used the Russell's paradox in a 3-year study with 120 pre-service elementary teachers and studied their emotions, voices and struggles as they tried to unravel the paradox. Mamolo and Zazkis (2008) used the Hilbert's Grand Hotel paradox and the Ping-Pong Ball Conundrum to explore the naive and emerging conceptions of infinity of two groups of university students with different mathematical backgrounds. Nunez (1994) used Zeno's paradox, the Dichotomy, in a progressive manner to investigate how the idea of infinity in the small emerges in the minds of students aged 8, 10, 12, and 14. Nunez (1994) concluded that conceptions of infinity in the large and infinity in the small are very different, especially for young learners. For example, though 8 years olds can conceive the notion of endless in their consensual world they cannot see "infinity in the small".

Painter's Paradox

The inner surface of the Gabriel's horn is infinite; therefore an infinite amount of paint is needed to paint the inner surface. But the volume of the horn is finite (π), so the inner surface can be painted by pouring a π amount of paint into the horn and then emptying it.

The volume of the horn is given by $\int_1^{\infty} \pi(\frac{1}{x})^2\,dx$ where $\pi(\frac{1}{x^2})$ is the cross sectional area of the horn perpendicular to the x-axis. This improper integral evaluates to π. The surface area of the horn is given by $\int_1^{\infty} 2\pi(\frac{1}{x})\sqrt{1+(-\frac{1}{x})^2}\,dx$ and this improper integral diverges.

Theoretical considerations

Several theoretical frameworks informed our investigation. We use APOS theory (Dubinsky, Weller, McDonald & Brown, 2005), reducing abstraction by Hazzan (1999), and platonic and contextual distinction by Chernoff (2011).

APOS theory (Dubinsky et. al., 2005) suggests that an individual deals with a mathematical situation by using the mental mechanisms interiorization and encapsulation to build the cognitive structures actions, processes, objects and schemas that are applied to the situation. A mathematical concept begins to form as one applies actions on objects to obtain other objects. As an individual repeats and reflects on the action, it may be interiorized into a mental process. A process is a mental structure that can perform the same operation as the action being interiorized in the mind of the individual that allows him or her to imagine performing the transformation without having to execute each step explicitly. If the individual can act on the process and transform it explicitly in his or her imagination, then the individual has encapsulated the process into a cognitive object. A mathematical topic may involve many actions, processes and objects that need to be organized and linked into a coherent framework called schema (Dubinsky et al., 2005).

APOS theory clarifies the distinction between the potential infinity and the actual infinity. Potential infinity is the conception of the infinite as a process. This process is constructed by beginning with the first few steps (e.g., 1, 2, 3 in constructing the set N of natural numbers), which is an action conception. Repeating these steps (e.g., by adding 1 repeatedly) ad infinitum requires the interiorization of that action to a process. Actual infinity is the mental object obtained through encapsulation of that process (Dubinsky et al., 2005). Researchers have acknowledged the difficulty in transitioning from process to objects conceptions. Sfard suggested that this phenomenon "seems inherently so difficult that at certain levels it may remain practically out of reach for certain students" (p. 1). Moreover, Dubinsky, Arnon, & Weller (2013) noted "the difficulty with this progression [from process to object] may be particularly strong for infinite processes" (p. 251). Focusing on this difficulty, Dubinsky et al. (2013) proposed the construct of 'totality', which is a stage in between the stages process and object. They also proposed the mechanism *detemporalization* "by which an individual moves from thinking of a process as a sequence of continuing steps to being able to imagine these steps all at once" (ibid.) to progress from process to totality. They observed that students who did not make the transition from process to object were often 'stuck' at the totality stage.

Another theoretical framework we use is reducing abstraction by Hazzan (1999). According to Hazzan (1999) there are three ways in which abstraction level can be interpreted: (1) Abstraction level as the quality of the relationships between the object of thought and the thinking person, (2) Abstraction level as reflection of the process–object duality, and (3) Abstraction level as the degree of complexity of the mathematical concept. As infinity is a complex process, it can be accessed by a learner at a lower level of abstraction that is required by a particular task.

We also rely on the theoretical constructs introduced by Chernoff (2011) in distinguishing between platonic and contextualized situations. Chernoff distinguished between platonic and contextualized sequences in the relative likelihood tasks in probability. A platonic sequence is characterized by its idealism:

> For example, a sequence of coin flips derived from an ideal experiment (where an infinitely thin coin, which has the same probability of success as failure, is tossed repeatedly in perfect, independent, identical trials) would represent a platonic sequence (p. 4).

But, a contextualized sequence is characterized by its pragmatism. For example, "the sequence of six numbers obtained when buying a (North American) lottery ticket (e.g., 4, 8, 15, 16, 23, 42)" (p. 4) would represent a contextualized sequence.

The StudyParticipants and the setting

Participants in our study were 12 undergraduate students taking Calculus classes at Simon Fraser University. All of them volunteered to take part in this study outside their course work at the university. At the time of the study they all had done volume and surface integrals. They used the textbook *Calculus* by James Stewart (seventh edition). Gabriel's horn is mentioned once in the book in a problem on page 574 under the section on area of a surface of revolution. In this problem they have to show that the surface area of the Gabriel's horn is infinite. They show that the volume of the Gabriel's horn is finite in a previous problem on page 552 under the section on improper integrals. They were presented the Painter's paradox with the following detailed mathematical justifications of computing the volume of the Gabriel's horn and showing that its surface area is infinite, and asked to respond in writing what they thought of the paradox:

The volume of the Gabriel's horn is given by

$$\int_1^\infty \pi(\tfrac{1}{x})^2\,dx = \lim_{a\to\infty}\int_1^a \pi(\tfrac{1}{x})^2 dx = \lim_{a\to\infty}(\pi\int_1^a \tfrac{1}{x^2}dx) = \lim_{a\to\infty}(\pi\left[-\tfrac{1}{x}\right]_1^a) = \lim_{a\to\infty}(\pi\left[1-\tfrac{1}{a}\right]) = \pi\ .$$

And its surface area is given by $\lim_{a\to\infty}\int_1^a 2\pi(\tfrac{1}{x})(\sqrt{1+(-\tfrac{1}{x^2})^2}\,dx$. Now,

$$\lim_{a\to\infty}\int_1^a 2\pi(\tfrac{1}{x})(\sqrt{1+(-\tfrac{1}{x^2})^2}\,dx \geq \lim_{a\to\infty}\int_1^a 2\pi(\tfrac{1}{x})dx = \lim_{a\to\infty}2\pi[\ln x]_1^a = \lim_{a\to\infty}2\pi[\ln a - \ln 1] = \infty.$$

Later they were interviewed by the author and audio recorded. In this study we focus on two of the participants, Bruce and Bryan, because of their detailed responses to the written tasks and the page limitations of this paper.

Results and analysis

Mancosu and Vailati (1991) say that the Gabriel's horn "is not merely potentially but actually infinite in length" (p. 57). But as "it is well known, potential infinity subsists within mathematics as the *modus operandi* which constitutes the operatory nucleus of standard calculus" (Moreno and Waldegg, 1991, p. 213). So how do Calculus students visualize Gabriel's horn?

Let us look at the calculation of the volume of the Gabriel's horn given above. What kind of thought process is associated with the above calculation? First the volume of the horn truncated by the plane perpendicular to the *x*-axis containing the point $(a,0)$ is computed. This volume is $\pi(1-\tfrac{1}{a})$. When one repeats this action for increasing values of *a* and reflects on it, this action of computing the volume of the truncated Gabriel's horn can be interiorized into a process. For each increasing value of *a* there corresponds a truncated Gabriel's horn with volume $\pi(1-\tfrac{1}{a})$. When one evaluates the limit of $\pi(1-\tfrac{1}{a})$ as *a* tends to ∞ one sees the entire process as a totality. Further with more reflection, one may encapsulate the above process into the cognitive object of Gabriel's horn with volume π.

Gabriel's horn is a platonic object. It is formed by rotating a breathless infinitely long curve. To resolve Painter's paradox one needs to decontextualize it from its apparent real life context. With real paint the horn cannot be painted by pouring a π amount of paint into the horn and then emptying it as the paint will not travel when the cross sectional area of the horn perpendicular to the *x*-axis becomes too small for the paint molecules. But if we can consider paint which can be *thinned infinitely*, let's call such paint 'ideal paint', then the inner surface area of the horn can be painted in the above manner by disregarding the time it takes to fill the horn and assuming that ideal paint sticks to the surface. In this case the paint coat does not have a uniform thickness, and the thickness goes to zero fast enough for the amount used to be finite, this is the very same reason that guarantees that Gabriel's horn has a finite volume. To paint an infinite surface area with a uniform thickness an infinite amount of paint is needed no matter how small the thickness is.

All of the participants seemed experiencing a cognitive conflict in dealing with an infinitely long solid with a finite volume. Like the seventeenth century mathematicians they reacted in disbelief. For example Bruce said "It goes against something that really is intuitive. Like in our minds we all know that if something has a finite volume usually it has a finite surface area" What he added further is indicative his cognitive conflict:

> If the math is right which we are told that it is then … because often times arrive at a paradox so from my experience like dealing with stuff from Physics I know … I am not too foreign to the idea of something being one thing and at the same time another thing.

Bryan seemed to be in a similar conflict. He wrote that this paradoxical situation is correct and that it reminded him of the paradoxical situation that arises in the summation of an infinite geometric series. For him a geometric series summing to a finite sum is a paradox. Again, this seems to be due to the counter intuitive aspect of infinitely many terms adding up to a finite number instead of infinity and this resonates with the ideas of Bruce.

But to make sense of a finite volume having an infinite surface area Bryan reduced abstraction level of it by connecting it to familiar Silly Putty he used to play with. "What I most often do is roll the Silly Putty into a long cylindrical roll of Silly Putty and continue to do so until I had a very thin roll of Silly Putty" he wrote. He explained how a finite volume can have an infinite surface area:

> using a finite amount of silly putty, we could theoretically roll the silly putty to an infinitely thin thickness, and length (infinite surface area) […] Therefore, a finite amount of material can have an infinite surface area.

By repeatedly thinning a fixed finite amount of Silly Putty its surface area can be made as large as possible. But extending this to seeing that a finite volume can have an infinite surface area requires one to see this repeatedly thinning process as a totality. Bryan seemed to unravel the paradox. In the interview conducted immediately after the written task he explained that as "surface area is increasing infinitely large however it is approaching zero" a finite amount of paint would be sufficient. But he added "it is not one billion square meters of wall that we are trying to paint". When he was explained that the surface area is infinite means that it is bigger than any finite number he doubted that the horn can be painted with a finite amount of paint. He acknowledged that the mathematics is correct. But he added that it does not make sense: "**that is impossible!** […] if you were to paint this room and it is growing infinitely large you know you have only 3.14 litres of paint it does not make sense". The bold headed *that* *is impossible* was in disbelief. Bryan seemed to be in an intellectually unbearable situation in spite of his earlier observation that a finite amount of material can have an infinite surface area.

Bruce defined two kinds paint, one that occupies volume and one that covers surface area, in an attempt to resolve the paradox focussing on the dimensional difference between surface area and volume. Then he realized that some finite things can occupy infinite surface area in the same way Bryan did:

> let's say I have a cube let's say I squash with some kind of really strong plate so that no matter how thick it is it can always become thinner then the surface area **yeah** it would be finite volume but it would spread over an infinite surface area if you think of painting that way [...] what is thickness of an area it would have to be infinitely thin

But finally Bruce seemed to have made a connection between his observation "some finite things can occupy infinite surface area" and painting the infinite surface area of the horn with a finite amount of paint. The above segment in the transcript happened at the very end of the half an hour interview and the bold headed *yeah* was emphatic.

Both Bruce and Bryan used the words 'infinitely thin' in explaining how a finite volume can have infinite surface area, perhaps, experiencing the notion of infinitely small.

Conclusion

Paradoxes involving infinity provide a window to infinity. The cognitive conflict elicited by a paradox is difficult for a learner to resolve. Resolving this cognitive conflict requires the learner to make a cognitive leap from the intuitive to the formal or from the real world to the mathematical realm. Our participants struggled with the infinitely long Gabriel's horn having a finite volume π. But they are used to calculating volumes and surface areas through limit processes and familiar with facts such as that the area under a curve over an infinite interval could be finite. For example, one of the results they are familiar with is that $\int_1^\infty \frac{1}{x^p}dx$ is convergent if $p > 1$. $\int_1^\infty \frac{1}{x^p}dx$ is the area under the curve $y = \frac{1}{x^p}$ for $x \geq 1$. Yet, out of the twelve participants only two could intuitively see that the infinitely long Gabriel's horn can have a finite volume.

In resolving Painter's paradox one has to confront infinity with a finite attribute which goes against the Aristotelian dictum that there is no proportion between the finite and the infinite. Also, one has to decontextualize it from its apparent real life context. This turned out to be difficult for some participants. One participant said the horn can never be filled as it will take forever to fill. Another said that it cannot be painted because at the atomic level you cannot see the surface. Our study shows that the Painter's paradox can be used as a research tool to reveal the nature of conceptions of infinity including conceptions of infinitely small.

References

Chernoff, E. (2011). Investigating relative likelihood comparisons of multinomial contextual sequences. *Proceedings of CERME 7*.

Dubinsky, E., Arnon, I., & Weller, K. (2013). Preservice Teachers’ Understanding of the Relation Between a Fraction or Integer and its Decimal Expansion: The Case of and 1. *Canadian Journal of Science, Mathematics and Technology Education, 13*(3), 232-258.

Dubinsky, E., K. Weller, M.A. McDonald, & A. Brown. (2005a). Some historical issues and paradoxes regarding the concept of infinity: An APOS-based analysis: Part 1. *Educational Studies in Mathematics*, *58*, 335-359.

Fleron, J. F. (1999). Gabriel's wedding cake. *College Mathematics Journal*, *30*, 35-38.

Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics, 40*(1), 71–90.

Kim, D. J., Ferrini-Mundy, J., & Sfard, A. (2012). How does language impact the learning of mathematics? Comparison of English and Korean speaking university students’ discourses on infinity. *International Journal of Educational Research*, *51*, 86-108.

Kline, M. (1990). Mathematical thought from ancient to modern times (Vol. 3). Oxford University Press.

Luis, E., Moreno, A., & Waldegg, G. (1991). The conceptual evolution of actual mathematical infinity. *Educational Studies in Mathematics*, *22*(3), 211-231.

Mamolo, A. & Zazkis, R. (2008). Paradoxes as a window to infinity. *Research in Mathematics Education, 10*(2), 167-182.

Mancosu, P., & Vailati, E. (1991). Torricelli's infinitely long solid and its philosophical reception in the seventeenth century. *Isis*, *82*(1), 50-70.

Movshovitz-Hadar, N., and Hadass, R. (1990). Preservice education of math teachers using paradoxes. *Educational Studies in Mathematics*, *21*, 265 87.

Movshovitz-Hadar, N., and Hadass, R. (1991). More about mathematical paradoxes in preservice teacher education. *Teaching & Teacher Education. 7*(I). 79-92.

Nunez, R. (1994). Subdivision and small infinities: Zeno, paradoxes and cognition. Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education, Lisbon, 3, 368-375.

Sfard, A. (1991). On the dual nature of mathematical conceptions: reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36.

Sriraman, B. (2009). Mathematical paradoxes as pathways into beliefs and polymathy: an experimental inquiry. *ZDM Mathematics Education* (*41*), 29–38.

Stewart, J. (2011). *Single variable calculus: Early transcendentals*. Brooks/Cole Publishing Company.

YOUNG CHILDREN'S THINKING ABOUT VARIOUS TYPES OF TRIANGLES USING DYNAMIC GEOMETRY TASKS

Harpreet Kaur

Simon Fraser University

This paper will show preliminary results of how young children (age 7-8, grade 2/3) can exploit the potential of dynamic geometry environments to identify, classify and define different types of triangles (scalene, isosceles, equilateral). This study is based on three-lesson classroom intervention, during which the children worked both in a whole classroom setting in which they could interact directly with Sketchpad on an interactive whiteboard as well as individual/pair work with paper-and-pencil. This paper reports on first lesson only. Using Sfard's (2008) communicational approach, we extend the work of Battista (2008) to show how students developed a reified discourse on various types of triangles. Further, we show how the students used a dynamic language to describe the behaviour of various triangles in terms of invariances about how the sides and angles of these triangles would change under dragging.

INTRODUCTION

Research shows that a majority of students in North America have an inadequate understanding of geometric concepts and poorly developed skills in geometric reasoning (e.g., Battista, 2009; Clements and Battista 1992). The primary cause for this poor performance is both *what* and *how* geometry is taught (Clements & Battista, 1992). Most North American curricula consist of a collection of superficially covered topics with no systematic support for students' progression to higher levels of geometric thinking. For example, in BC curriculum, children are expected to describe, construct and compare 2-D shapes including triangles, squares, rectangles and circles in grade 2, but the basic geometrical concepts like symmetry, angles etc. are introduced in grade 4 and 5 respectively. Children are taught to construct and compare triangles, including scalene, isosceles, equilateral in grade 6.

We have been investigating many geometry-related concepts at early grades, using DGEs, including shape identification, parallel lines, symmetry and angles (Sinclair, Moss & Jones, 2010; Sinclair & Kaur, 2011, Kaur & Sinclair, 2012) and reported about children's readiness and ability to develop these concepts. Further we aim to conduct studies to see if early development of these concepts helps students in developing better reasoning about various geometric shapes and their properties. In this paper, we report on an exploratory study conducted with a split class of grade 2/3 children (ages 7-8) working with various types of triangle sketches using *The Geometer's Sketchpad*. The focus of this paper is to study what kind of dynamic language children use, what kind of geometric properties they attend to, what kind of reasoning they provide to describe the behaviour of different types of triangles and to discuss the specific mediating role of the use of the software on this thinking.

CHILDREN'S UNDERSTANDING OF CLASSIFICATION OF SHAPES

Many researchers have shown that students are not very successful at identifying non-prototypical triangles (Clements and Battista (1992). Research also indicates that children prefer to rely on a visual prototype rather than a verbal definition when classifying and identifying shapes (Gal & Linchevski, 2010). Specifically, when a child holds both a verbal definition and a visual prototype for a given geometric concept, the child often calls upon the visual prototype rather than, or in spite of, the verbal definition when assigning class membership. For example, Fischbein and Nachlieli (1998) note that although students could give the correct definition of a parallelogram, many relied on the visual prototype instead of applying their definition when identifying shapes. de Villiers (1994) suggests that classifying is closely related to defining (and vice versa) and classifications can be hierarchical (by using inclusive definitions, such as a trapezium or trapezoid is a quadrilateral with at least one pair of sides parallel – which means that a parallelogram is a special form of trapezium) or partitional (by using exclusive definitions, such as a trapezium is a quadrilateral with only one pair of sides parallel, which excludes parallelograms from being classified as a special form of trapezium) (p11-12). In general, in mathematics, inclusive definitions are preferred. A number of studies have reported about students' problems with the hierarchical classification of quadrilaterals (Fuys, Geddes & Tischer, 1988; Clements & Battista (1992), Jones (2000)). Battista (2008) designed *Shape Makers* computer microworld that provides students with screen manipulable shape-making objects and helps to promote the development of mental models that student can use for reasoning about geometric shapes. For instance, the computer Parallelogram Maker can be used to make any desired parallelogram that fits on the computer screen, no matter what its shape, size, or orientation–but only parallelograms. On the similar lines, to facilitate the exploration of the concept of various types of triangles we developed Triangle ShapeMaker in Sketchpad for the present study.

THEORETICAL PERSPECTIVE

In previous research, we have found Sfard's (2008) 'commognition' approach is suitable for analysing the geometric learning of students interacting with DGEs (see Sinclair, Moss & Jones, 2010; Sinclair & Kaur, 2011). For Sfard, thinking is a type of discursive activity. Sfard's approach is based on a participationist vision of learning, in which learning mathematics involves initiation into the well-defined discourse of the mathematical community. The mathematical discourse has four characteristic features: word use (vocabulary), visual mediators (the visual means with which the communication is mediated), routines (the *meta-discursive rules* that navigate the flow of communication) and narratives (any text that can be accepted as true such as axioms, definitions and theorems in mathematics). Learning geometry can thus be defined as the process through which a learner changes her ways of communicating through these four characteristic features. We are particularly interested in investigating how the students might move between different word uses and to examine the informal language they use to talk about different triangles.

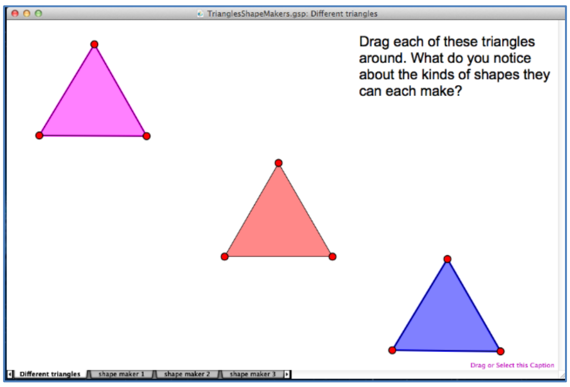
EXPLORING THE UNDERSTANDING OF DIFFERENT TYPES OF TRIANGLES

Participants and data collection

This teaching experiment is part of a larger project that involves the study of children's geometric thinking in the primary grades. We worked with grade 2/3 split classroom children from a University Lab pre-K-6 school in an urban middle SES district. There were 24 children in the class from diverse ethnic backgrounds and with a wide range of academic abilities. We worked with the children on weekly basis on a variety of geometric concepts for seven months. Three lessons were conducted on the topic of triangles. Each lesson lasted approximately 40 minutes and was conducted with the children seated on a carpet in front of a screen. Two researchers, and the classroom teacher, were present for each lesson. My research associate took the role of the teacher and conducted the lessons. Lessons were videotaped and transcribed. This paper is focused on the first lesson on types of triangles. The children had previous lessons involving *Sketchpad*, where they had worked with symmetry and angles, but they had never received formal instruction related to types of triangles.

Dynamic triangle sketches

On the lines of work of Battista (2008), we developed the Triangle ShapeMakers sketches (fig. 1(a) and 1(b)) for different types of triangles (Scalene, isosceles, equilateral triangles, right triangle). Pink triangle is scalene, red triangle is equilateral, blue triangle is isosceles, and green triangle is right triangle by construction.



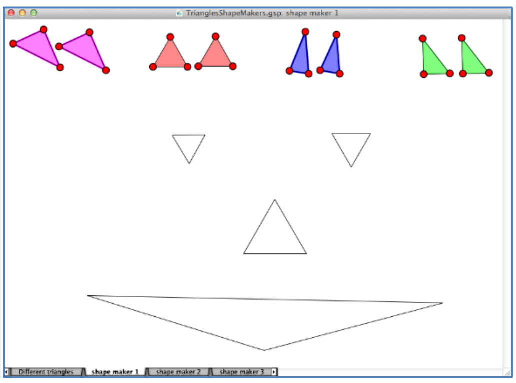


Fig. 1(a): DifferentTriangles sketch   Fig. 1(b): Triangle ShapeMaker sketch

Sketch shown in fig. 1(a) is termed as DifferentTriangles sketch and students were asked to explore the sameness and differences in the various coloured triangles. Sketch shown in 1 (b) is called Triangle ShapeMaker and students were asked to explore which coloured triangles could fit in the given empty triangles. In this paper, I will report only on students' work with first sketch. They were asked to drag the vertices and report on the behaviour of different triangles of first sketch

Working with DifferentTriangles sketch

Students were presented with the DifferentTriangles sketch (figure 1 a). In the beginning all the three shapes were static on the screen and were appearing exactly same from a geometric point of view. Teacher asked about the ways in which the triangles are different. Students talked about difference in colors, thickness of the sides of the triangles. And then teacher asked about what's same in the three shapes.

Neriya: They are all triangles.

Teacher: They are all triangles. Very good. Sometimes certain things are obvious but we have to say them anyway. Viktoria, what else is the same?

Viktoria: They all have the three same angles.

Teacher: They all have the three same angles. Wow, so you are noticing these angles right here. They all are the same in all of these triangles. Very nice. The angles are all the same. Anything else that you notice?

Nick: They all have three sides and three edges.

Neriya talked about all the three shapes being triangles. Viktoria's use of words "They all have the three same angles" shows that she noticed the similarity of angles in all the three triangles. It is worth noting that she is noticing the angles in the static shapes. Nick noticed the same number of sides and edges in the triangles. The above excerpt shows that children's description of sameness in static shapes in figure 1(a) was based on visual similarity and it moved from holistic (shape as a whole) similarity to the similarity among various parts of the triangles. Then, teacher appoints the three students to drag the three colored triangles pink, red and blue respectively and call them as drivers. Teacher asks the rest of the students to act as detectives and they have to describe what kinds of triangles can be made, what can change and what stays the same in each of the triangles while dragging.

Comparing the dragging of scalene and equilateral triangle

First a girl Neriya drags the pink (scalene) triangle into various sizes and orientations i.e. skinny and long, small and big triangles. Then Adil drags the red (equilateral) triangle and teacher asks if he can make it long and skinny. Observing the dragging patterns some students say no.

Teacher: Can I see some hands up? Why can't Adil make it long and skinny? What is matter with this triangle that makes it difficult for him to make it long and skinny? (Pointing to one student) how come you can't make it long and skinny?

Evan: Because it's different than that and I think it can only go by a perfect triangle

Teacher: It can only… what do you mean by a perfect triangle? What's a perfect triangle? (*Okay Adil, thank you very much taking marker from him*). What's a perfect triangle? What do you mean? What's Special about this triangle here? Why is he calling it perfect? Rafaela?

Rafaela: Because the other triangle can move at a point but this one can move bigger or smaller differently

Students started to notice the changes in red triangle as Adil dragged one of the vertices. Evan's use of words "it's different than that" shows that he started to notice the differences between red and pink triangle that appeared same in the beginning. In the statement "It can only go by a perfect triangle", the use of words 'go', 'only by perfect triangle' shows Evan's attention to the restrictive type of movement depicted by the red (equilateral) triangle upon dragging. Further in Rafaela's description "other triangle can move at a point" shows that she is paying attention to the particular kind of movement regularity depicted by the scalene triangle, where dragging one vertex don't move the other two vertices. The use of words "this one can move bigger or smaller differently" shows that she is noticing the different kind of size changing behaviour of equilateral as compared to scalene triangle. Thus, Evan's routine of comparison seems geometrical whereas Rafaela's comparing routine is based on the different movement regularity of the triangles, but both the routines are emerged as a result of potential of the dynamic environment.

Teacher: Why are you calling this a perfect triangle? Why do you think is he calling this a perfect triangle? What is that perfect about it? That is different about this triangle (some noises)… Jack?

Jack: Everything moves with it except one point.

Teacher: Dragging the red triangle) even when it is getting bigger and smaller, is there anything that staying the same as I make it bigger and smaller? Neriya?

Neriya: The angles

Teacher: The angles are staying the same. Good. Anything else that is staying the same? (Pointing at one student) you are going to say angles as well or something else? Yeah.

Explaining about the behaviour of perfect triangle, Jack's use of words "everything moves with it except one point" shows that the he is shifting his attention from holistic movement regularity to partitive movement regularity and thus shifting his attention to consider interrelationships between different parts of the triangle. This kind of reasoning is associated with the use of the tool. As students noticed the change happening in the perfect triangle as a result of dragging, teacher asked about if there is anything that is staying the same. One girl Neriya suggested that ‘angles’ are staying the same under dragging. Jack abstracted a tool based movement regularity of equilateral triangle, but Neriya conceptualised the movement regularity with the complete precision by expressing it in terms of a traditional geometric property.

Exploring the overlap of isosceles triangle on a fixed scalene triangle

Teacher then gives a challenge to the students and asks them to figure out if they can overlap the blue (isosceles) triangle over the pink (scalene) triangle without touching the pink triangle. Rafaela first matches the two vertices of blue triangle to one side of the pink triangle and then try to drag it in the upward diagonal direction (see fig. 2) and concludes that she can’t overlap blue triangle over pink one.

Fig. 2

Teacher: You think you can’t? How come you can’t?

Rafaela: Because I think if I move that one, that one also moves

Teacher: If you move this one here, this blue one, the other blue vertex moves too right? Actually if you move this vertex, it moves too. How come you can’t do it? How come you can’t put the blue one on the pink one?

Dan: Because the blue one can only move symmetrical

Rafaela reasoning “I think if I move that one, that one also moves” shows that the she is paying attention to the partitive movement regularity between different parts of the triangle. She is noticing what other changes occur in the shape while dragging one part of it. While Dan’s use of words “it can only move symmetrical” shows that he is abstracting the dynamic movement regularity with precision by expressing it in terms of a geometric property of symmetry. The systematic dragging of the vertex of one of the longer sides of isosceles (blue) triangle by Rafaela acted as a visual mediator and helped Dan to see the property of symmetry. The teacher asked the other students’ if they agree with Dan’s statement. There was mixed response, then teacher asked Jordan about his thinking.

Teacher: Okay, Jordan what do you want to say?

Jordan: So, the one

Teacher: Shh… Listen to Jordan because it would be hard to agree or disagree if you don't know what he is saying (*to the students who are making noises*)

Jordan: So, this one wherever you move it, then this one moves with this, so when you move, it will go that way (*stretching his arms upwards along the two longer sides (fig. 3*)

Fig. 3

Jordan's use of words "wherever you move it, then this one moves with this" and "so when you move, it will go that way" along with the stretching arms gesture shows that he is also talking about the simultaneous change in two arms of blue triangle. Teacher termed Dan's argument as symmetry argument and Jordan's argument (moving in same way) as stretching argument. Most of the students agreed with these arguments. Then teacher asked about the two sides of blue triangle. As the students already noticed the same angles in red (equilateral) triangle, upon teacher's prompt they noticed the sides stay same in the red triangle. As students identified the properties of different sides staying same or changing length, teacher introduced the special names equilateral, isosceles and scalene for the red, blue and pink triangles respectively.

Conclusion

The discussion of above excerpts and preliminary analysis shows that during the exploration with dynamic sketches students' routines moved from self-invented informal spatial structuring to formal powerful geometric structuring. In case of all the challenges given by the teacher, students' reasoning started with describing the movement patterns like "Everything moves with it except one point", " If I move that one, that one also moves", "wherever you move it, then this one moves with this" and then eventually moved to formal properties "angles are staying same", "moves symmetrical". Dragging the vertices acted as a visual mediator and helped students to develop the routine of looking at movement regularity and eventually shifting towards formal geometrical properties. This episode also provides initial evidence that the teaching of concepts like symmetry and angles in early years can lead to whole set of new possibilities of geometric reasoning about shape and space for young children.

References

Battista, M.T. (2009). Highlights of Research on Learning School Geometry. In (Eds.) T.V Craine and R. Rubenstein. *Understanding geometry for a changing world: Seventy-first yearbook* (pp 91 -108). Reston, VA: NCTM.

Battista, M.T. (2008). Development of shapemakers geometry microworld. In Blume, G.W., Heid, M.K.(Eds.), *Research on Technology and the Teaching and Learning of*

Mathematics Volume 2: Cases and Perspectives, 131-156, USA-Information Age Publishing Inc.

Clements, D. H., & Battista, M.T. (1992). Geometry and spatial reasoning. In D.A. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 420-464). New York, NY: MacMillan.

de Villiers, M. (1994). The role and function of a hierarchical classification of quadrilaterals, *For the Learning of Mathematics*, 14(1), 11–18.

Fischbein, E., & Nachlieli, T.(1998). Concepts and figures in geometrical reasoning, *International Journal of Science Education*, 20(10), 1193- 1211.

Fuys, D., Geddes, D., and Tischer, R. (1988). *The van Hiele Model of thinking in geometry among adolescents*, Reston, VA: NCTM.

Gal, H. & Linchevski, L. (2010). To see or not to see: analyzing difficulties in geometry from the perspective of visual perception. *Education Studies in Mathematics*, 74, 163-183.

Jones, K. (2000). Providing a foundation for deductive reasoning: students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics*, 44(55–85)

Kaur, H. and Sinclair, N. (2012). Young children's understanding of angle in a dynamic geometry environment. In Van Zoest, L. R., Lo, J.-J., & Kratky, J. L. (Eds.). (2012). *Proceedings of the 34th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education (pp.1081-1089).* Kalamazoo, MI: Western Michigan University.

Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses and mathematizing.* Cambridge, UK: Cambridge University Press.

Sinclair, N., Moss, J. and Jones, K. (2010). Developing geometric discourse using DGS in K-3. In Pinto M. M. F., Kawasaki T.F (Eds.). *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 4, pp. 185-192). Belo Horizonte, Brazil: PME.

Sinclair, N. & Kaur, H. (2011). Young children's understanding of reflectional symmetry in a dynamic geometry environment. In Ubuz, B. (Ed.) *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education*, (Vol.4, pp.193-200). Ankara, Turkey: PME

EFFECTS OF INCONSISTENT DEFINITIONS: DEFINITION OF CONTINUITY

Gaya Jayakody

This paper reports on a problematic situation that arises through certain definitions involving the concept of continuity. The paper mainly focuses on bringing out these problems in the context of textbooks and other mathematical resources and in addition gives an instance to elaborate how this problem could create tensions and conflicts in students' thinking processes. Sfard's commognitve framework is used in the analysis of a student's work on continuity. A potential remedy for this problem is presented in conclusion.

Continuity: two definitions

The concept of continuity is an important concept in calculus and analysis. It is usually introduced in introductory calculus courses for students who specialize in mathematics as well as for students who do not specialize in mathematics but take calculus as part of their program requirements or their general interest.

The role of definitions and its importance have been attended by many mathematics education researchers (Vinner, 1991; Edwards & Ward, 2008, Robinson, 1962). It is not at all unusual in the field of mathematics, be it among textbooks, mathematicians or teachers, to use different definitions for the same mathematical concept. Often these differences are superficial and nuanced. And more importantly, even if these definitions are superficially different, they are equivalent and consistent. They represent the 'same' concept and for the most part imply the same set of properties of that concept. For instance, the definition of a 'function' reads "A rule that assigns to each element in a set A one and only one element in a set B" in the text book "Applied Calculus" by Tan, Menz and Ashlock (2011), while it is presented in Wikipedia as "a set of ordered pairs where each first element only occurs once". There is also the practice of introducing concepts using simpler versions of definitions at lower levels and then progressively advancing to more rigorous definitions at higher levels. For instance, if we look at the same concept of a 'function', it is often introduced metaphorically at lower levels as a machine that takes an input x and returns a unique output $f(x)$.

Similarly, the concept of continuity is described and defined to suit different audiences in different levels: the intuitive description, informal definition, formal limit definition and the more rigorous epsilon-delta definition are examples.

The aim of this paper firstly, is to point out two problematic situations that arise through certain definitions, or the lack of certain definitions, involving the concept of continuity. These two issues are intertwined and one may see them as two manifestations of the same problem. The two problems are, the usage of the phrase ‘continuous function’ despite the lack of an explicit definition for a ‘continuous function’ and the mathematically inharmonious and inconsistent ramifications resulting from some of such definitions when they *are* explicit but incoherent. These concerns can be seen to be present in textbooks, mathematical websites, or even arguably within classroom instruction and discourse.

Secondly, a small piece of data is presented to serve as an example to elaborate how this situation can affect student’s thinking. Sfard’s discursive framework is used to analyze data. For the purpose of this paper, only the relevant part of the framework is briefly described which is used in the short analysis of data. Finally, in conclusion, acknowledging the importance of rigorous definitions in teaching and learning mathematics, I offer a potential remedy for the ‘problem of continuity’.

Theoretical framework

Sfard (2008) unifies thinking and communication as commognition. In the commognitive framework thinking is conceptualized as an individualized version of interpersonal communication. With the visioning of Mathematics as a discourse it is claimed to be an autopoetic system that creates the objects of its study. Hence mathematical objects are discursive objects and students personally construct these mathematical objects which can be represented as ‘realization trees’. A realization tree shows the different realizations of a particular signifier where a signifier is a word or symbol that acts as a noun in the mathematical discourse. A realization is a perceptually accessible thing so that narratives about the signifier can be translated into narratives about the realization. The main concern in this paper, definitions, belongs to the category of endorsed narratives which are sequences of utterances framed as descriptions of objects or relations between objects.

Problem 1: Inconsistent definitions

In the context of an introductory calculus course, and also in many other common resources, the definitions used for continuity related concepts are the limit definitions. There are two different limit definitions (that are labeled as D1 and D2 for reference in this paper) used for “continuity at a point” (and accordingly “discontinuity at a point”) on which the other related concepts of continuity can be based on. Below are the two definitions.

D1 (e.g., Stewart, 2012; Tan, Menz & Ashlock 2011; Khuri, 2003; Neuhauser, 2010; Lial et al., 2011)

A function f is said to be continuous at *c* if,

1. $f(x)$ is defined at $x = c$
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x)$ is equal to $f(c)$

f is discontinuous if any of the above conditions are not satisfied. [1]

D2 (e.g., Stahl, 2011; Binmore, 1982; Brown, 1963; Strang, 1991; Begle & Williams, 1954)

A function f is said to be continuous at $x = c$ in its domain if,

$$\lim_{x \to c} f(x) = f(c)$$

And f is discontinuous at $x = c$ in its domain if,

$$\lim_{x \to c} f(x) \neq f(c)$$

The deciding factor that makes a definition consistent with either D1 or D2 is treatment of a point at which the function is not defined. According to D1, a function that is not defined at a point is discontinuous at that point, while according to D2 the question of continuity or discontinuity shouldn't arise. Therefore the difference between D1 and D2 lies more in the way 'discontinuity' (at a point) is defined. It is, however, not the intention of this paper to go to mathematical lengths to investigate the accuracy or falsehood of these definitions, but to attend to and elaborate on the discrepancies and consequences of them. The problems discussed in this paper are hinged on these two definitions.

Problem 2: Absence of a definition for 'a continuous function'.

I have examined several dozen of resources (textbooks, websites, mathematical dictionaries) seeking a definition for a 'continuous function'. In most of the resources such a definition was not explicitly stated. However, the phrase 'continuous function' is loosely used in many places.

The topic of continuity starts off, in many textbooks and websites, with the definition of 'continuity *at a point*' (Stewart, 2012; Lial, Greenwell, & Ritchey, 2011; Begle, 1954; Begle & Williams, 1972). This definition is the leading definition and other related extensions to the concept of continuity of a function, each of which has its own

[1] In some resources, (e.g., Mathworld.Wolfram) this part of the definition is implicit.

definition may follow (continuity on an interval, a discontinuity/a discontinuity at a point, types of discontinuities- removable, jump, infinite-, one-sided continuities).

However, the heart of the second problem is that these definitions of continuity/discontinuity at a point are not followed by the definition of a *continuous function* (e.g.: Neuhauser , 2011, Stewart, 2012; Khuri, 2003). This situation leaves room for students, if not explained by the instructor, to intentionally or unintentionally 'construct' a meaning for "continuous function". Instinctively it is likely that this will be interpreted as "continuous everywhere" which yet again is problematic. 'Everywhere' - where? Everywhere can mean 'on the real number line'. In fact, some sources present this interpretation and this definition is consistent with D1:

> A function that is continuous on (-∞, +∞) is said to be continuous everywhere, or simply continuous. (Anton, 1995, p.105)
>
> A function is a continuous function if it is continuous at every real number. (Mathematics Harvey Mudd Collage, n.d.)

However, 'everywhere' can also be interpreted as in every point of the function domain, which is consistent with D2 (e.g.: Strang, 1991; Bogley & Robson, 1996). Therefore this situation holds the potential to lead students to construct their own meaning for a 'continuous function', which could be in discord with the intended definition.

Student confusions: an example

It was found in the first stage of a current study that university first year students have difficulties in determining whether a function is continuous or not when there is a discontinuity on an interval in particular. What follows is an example taken from the on-going second stage of the study, that shows how the discrepancy between D1 and D2 affects students when they are engaged in a mathematical task. In a questionnaire, student 'J', a university first year student who takes an introductory Calculus course, was first given a questionnaire where she was asked to give the definitions for "continuity at a point" (which she had learnt in the course) and "continuous function" (which she was not taught in the course) and then was given 6 functions in their graphical form to be identified as continuous or discontinuous. Then she was interviewed one on one to discuss her responses.

A realization tree for 'a continuous function' for J was constructed based on her responses to the questionnaire and her utterances in the interview. Among other realizations, it was found that, J had the following two realizations for a continuous function.

A: For every point *c* in its domain, *f(c)* is defined and $\lim_{x \to c} f(x) = f(c)$ [this is in accordance with D2]

B: A function that does not have a hole or an asymptote [this is in accordance with D1]

Following (Table 2) is an interpretive elaboration of an excerpt from the interview with J that illustrates this tension between these two realizations (a word that is stressed is indicated by bold letters). The first four graphs, which are discussed in the excerpt, are given in Table 1. Note that the domain for graph D was specified.

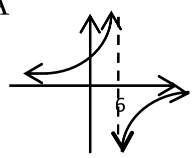
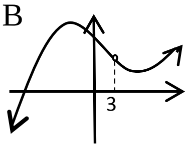
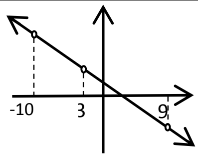
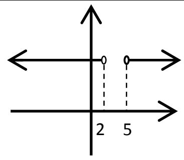
A	B	C	D
			 Domain = (-∞,2) ∪ (5,∞)

Table 1: The first four graphs in the questionnaire

Utterance no.	Who said	What is said	What is done	Interpretative elaboration
118	G	Umm, so here you **refrain** from saying that it is..	Pointing to graph D	G is pointing out that even though J has clearly classified graphs A, B and C as “not continuous”, she refrained from classifying graph d as “not continuous” but just stating the “discontinuities”.
119		Here you said no, no, no	Pointing to graphs A, B & C	
120		But here you are just saying ‘there is a discontinuity’	Pointing back to graph D	
121	J	Yeah		
122	G	At *x* equals 2 **and** *x* equals 5		G stresses on ‘and’ because J did not classify the whole interval from 2-5 as a discontinuity but only 2 **and** 5.
123	J	Yeah I wasn’t		J admits that she avoided this classification in graph D

124	G	Can you explain that to me?		G actually is seeking for two explanations, why she could not come to a decision whether function d was continuous or not, and why she only specified points 2 and 5 as discontinuities. However, in G's question it is not clear that she is expecting two explanations.
125	J	I wasn't sure; I did this question for like three minutes…		J specifying three minutes for graph D implies that she took *more* time for it than she took to do each of the graphs A, B and C. By saying she took 3 minutes and admitting she wasn't sure of this she's implying that it was challenging to her.
126	G	Ohh		
127	J	Because…and then I went back and looked at the definition and I saw that it was like within the domain that it's given..	Pointing to the definition which is realization A	This is the first graph for which she refers to the definition. She did the first three without referring to the definition. And now, she pays attention to the domain because now the domain is "given". And the definition mentions about the domain.
128	G	Hmmm?		
129	J	And then I was like.. oh but there is like a.. open circles…it should be…..		J says "but there is like a.. open circles". She uses 'but' because to her open circles, as in graphs b and c, mean 'discontinuities' (B) but she's trying to say that they are not in the domain (A). In other words, this utterance could be reworded as "the function should be continuous according to the definition **but** since there are open circles the function has discontinuities"
130	J	…	Thinking for 1 second	The pauses taken to think shows how much she is struggling to decide because there is a battle between two realizations she has for continuity.

131		there is a…there is no.. umm..		The two utterances "there is a" and "there is no" that take place adjacently clearly shows the conflicting conclusions about continuity of function d resulted through the two different realizations.
132			Does a thinking gesture	
133		… I don't know because it's not con… like within the domain.. it's not a square bracket		J wants to say that the function is not continuous ('not con…') but she is stuck because the two points 2 and 5 are not in the domain ('not a square bracket')
134	G	Yeah		
135	J	So it's not…	Pauses	J really wants to say it is not continuous and this shows that for her, realization B is stronger than A.
136	G	So		
137	J	I don't really know		J is utterly confused and gives up. She doesn't seem to be aware that the confusion stems from two different realizations.

Table 2: Interpretive elaboration for Jennifer's utterances from 118 to 137 that elaborates a conflict between the realizations A and B for continuity

The tension between the two realizations A and B which are based on the two inconsistent definitions D1 and D2 is clearly visible in J's utterances. She had learnt D1 as the definition for 'continuity at a point' in her class. She did not have any problem in deciding the continuity of the first three graphs as these were familiar graphs to her that she had often come across in the class. And as she admitted in the interview she did not refer to the definition in deciding whether they were continuous or not. This was an immediate realization (B) for the signifier ‘continuous function’ for her that included familiar features that she had seen in functions that were not ‘continuous’; holes and asymptotes. The unfamiliarity of the graph D, one with a discontinuity on an interval, pushed her towards the realization A which is the definition she had taken from a website which is an endorsed narrative for a continuous function that is consistent with D2. The realization procedure for A, however, which was not an immediate one, required her to analyze the domain. At this point, J was torn between the two realizations as the two realizations would take her to different conclusions about the continuity of the graph which resulted in a constant conflict in her utterances. This is a commognitive conflict between two of her own realizations for the signifier ‘continuous funciton’. (However, Sfard reserves the term commognitive conflict for encountering “between two interlocuters who use the same signifiers in different ways or perform mathematical tasks according to different rules” (Sfard, 2008, p. 161).)

As discussed in this paper, while there are inconsistencies in the way the continuity of a function at a point is defined there is both ambiguity and inconsistency in explaining, let alone defining, what ‘a continuous function’ is. Apparently, the problem is not a fresh one. It is interesting to see Gilbert Strang pointing out in 1991, that “it is amazing but true that the definition of "continuous function" is still debated” (p. 87). Over twenty years later, here we are, still grappling with a continuing problem about the definition of ‘a continuous function’!

Table 3 is an attempt to show how D1 and D2, the two leading limit definitions for ‘continuity at a point’, may consistently build and derive the definition for a ‘continuous function’.

	D1	D2
Continuity at a point	A function f is said to be continuous at point *c* if, 1. $f(x)$ is defined at $x = c$ 2. $\lim_{x\to c} f(x)$ exists. 3. $\lim_{x\to c} f(x)$ is equal to $f(c)$	Let c be a point in the domain of the function f. Then f is continuous at c if, $\lim_{x\to c} f(x) = f(c)$
Discontinuity at a point	If any of these three conditions fails, the function is *discontinuous* at $x = c$	f is discontinuous at point c in its domain if, $\lim_{x\to c} f(x) \neq f(c)$

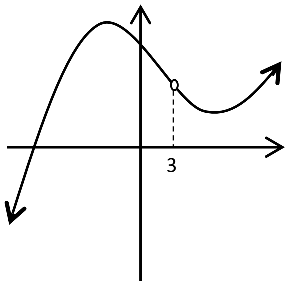
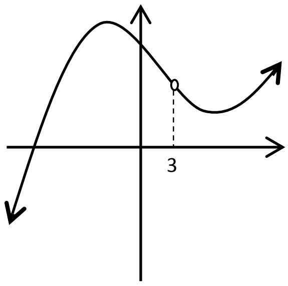
Continuous function	A function is a continuous function if it is continuous at *every real number*.	A function is a continuous function if it is continuous at every point in *its domain*.
Example to illustrate the difference	 Function is not defined at 3. Therefore there is a discontinuity at 3. The function is not a continuous function because it has a discontinuity at 3.	 Function is not defined at 3. Therefore, the question of continuity or discontinuity does not arise at 3. The function is a continuous function because it is continuous at every point in its domain.
Effect on a point in a rational function where the function is not defined	Is called a 'removable discontinuity'	Is called a '(removable) singularity' [2]

Table 3 : Consistent definitions of continuity

[2] "If $f(x) \to \infty$ as $x \to x_0$, where $x_0 \in [a,b]$, then x_0 is said to be a singularity of $f(x)$." (Khuri, 2003, p. 225).
"The term *removable discontinuity* is sometimes used by abuse of terminology for cases in which the limits in both directions exist and are equal, while the function is undefined at the point. This use is abusive because continuity and discontinuity of a function are concepts defined only for points in the function's domain. Such a point not in the domain is properly named a removable singularity." ("Classification of discontinuities", 2013)

This table is very far from '*solving'* the problematic situation pointed out in the paper. However, I believe that part of the solution lies in identifying the problematic situations, which I attempted to do in this paper. The table outlines the difference between D1 and D2 separating and placing them in two different contexts, in the hope that it will help learners to identify the context they are working on.

In conclusion, I believe, apart from being aware of these problems that exist in electronic as well as in print resources that teachers and learners should have a clear picture of the issue and its roots so that they will at least be able to deal with 'continuity' problems according to the particular chosen definition. The study also gives evidence to the problematic situations that students are led to due to implied definitions that are not explicitly stated or taught. Hence, perhaps more importantly, what this study suggests in particular is that we also need to make a shift in our choices from a mathematical one to a pedagogical one when it comes to choosing definitions and making decisions about the kind of discourse we model in the classroom.

References

Anton, H. (1995). *Calculus with analytic geometry (5th ed.)*. New York, NY: Wiley and Sons, Inc.

Binmore, K. (1982). *Mathematical Analysis (2nd ed.)*. Great Britain: Cambridge University Press.

Brown, D. (1963). *A Course of Mathematical Analysis Part II*. New York: The Macmillan Company.

Begle, E. (1954). *Introductory Calculus with Analytic Geometry*. USA: Holt, Rinehart and Winston.

Begle, E., Williams, L. (1972), *Calculus (2nd ed)*. New York: Holt, Rinehart and Winston, Inc.

Bogley, W., Robson, R. (1996). *Calculus Quest: The definition of continuity*. Retrieved from http://oregonstate.edu/instruct/mth251/cq/Stage4/Lesson/continuity.html

Classification of Discontinuities. (2013). In *Wikipedia*. Retrieved April 5, 2013, from http://en.wikipedia.org/wiki/Classification_of_discontinuities

Khuri, A. (2003). *Advanced Calculus with Applications in Statistics* (2nd ed.). Hoboken, New Jersey: Wiley & Sons, Inc. http://www.atilim.edu.tr/~math153/000cal04.pdf

Lial, M., Greenwell, R., Ritchey, N. (2011). Calculus with Applications (9th ed.). New York. Pearson.

Mathematics Harvey Mudd Collage, (n.d.). HMC Mathematics online tutorial: Continuity. Retrieved from http://www.math.hmc.edu/calculus/tutorials/continuity/

Neuhauser, C. (2010), Calculus for Biology & Medicine (3rd ed). Pearson Education. Inc.

Robinson, R. (1962). *Definitions*. London: Oxford University Press. (Reprinted by Frome, 1954, United Kingdom: D.R. Hillman & Sons).

Sfard, A. (2008). Thinking as communicating: Human development, the growth of discourses, and mathematizing. New York: Cambridge University Press.

Strang, G. (1991). Calculus. USA: Wellesley-Cambridge Press.

Stewart, J. (2012). Calculus-Early transcendentals (7th ed). USA: Brooks/Cole.

Tan. S., Menz, P., Ashlock, D. (2011), Applied Calculus for the managerial, life, and social sciences. Nelson Education Ltd.

Vinner, S. (1991). The role of Definitions in the Teaching and Learning of Mathematics. In Tall D. (Ed.), *Advanced Mathematical Thinking* (pp. 65-81). Dordrecht, Kluwer Academic.

EXPLORING CONSTRUCTS OF STATISTICAL VARIABILITY USING DYNAMIC GEOMETRY.

George Ekol

Simon Fraser University

The qualitative study reported in this paper examines university statistics students' aggregate reasoning with data using a semiotic mediation perspective. I focus on students' understanding of the links among distribution, the mean and standard deviation in a data set. Participants in the study explored these concepts using a dynamic mathematics sketch designed in Sketchpad. Findings suggest that the dynamic sketch mediated the meaning of statistical variability during and after participants' interactions with the sketch. However, some participants also showed mixed reasoning—a combination of elements of aggregate reasoning with some textbook procedures—after they stopped using the dynamic sketch. Some implications for post-secondary statistics curriculum are suggested.

INTRODUCTION

The concept variability applies across introductory statistics curriculum, including topics such as regression analysis, sampling distributions and tests of hypotheses. Understanding the connections among the measures of variability is therefore very important for students. This study takes statistical variability as how spread out or clustered the distribution of a data set is. Studies (see Garfield & Ben-Zvi, 2008) reveal that while students can calculate measures of variability such as the range, variance and standard deviation in a data set, they struggle with the meaning of those measures and rarely can connect them with other concepts that they learn in statistics. Building on the work of delMas and Liu (2005), I designed a dynamic mathematics sketch comprising of draggable data points with which students could explore connections among distribution, mean and standard deviation in a data set. Using the theory of semiotic mediation (elaborated by Falcade, Laborde & Mariotti, 2007), I analyse the students' thinking about the changing signs on the sketch, for instance, the change in the size of a geometrical square in relation to the meaning of variance and standard deviation in a data set. I also use Wild and Pfannkuch's (1999) elements of statistical thinking, in particular, aggregate reasoning with data. According to these authors, statistical thinking are the ways statisticians (and mathematicians) solve tasks, including being able to understand the underlying principles behind statistical processes. The purpose of this study is to explore post-secondary students' thinking about variability through the notions of distribution, the mean, and standard deviation before, during and after the students have interacted with the dynamic sketch.

THEORETICAL FRAME

As stated above, the study draws on two theoretical perspectives: The first perspective concerns students' reasoning with data (Wild & Pfannkuch, 1999; Konold & Higgins, 2003). Konold and Higgins propose four different ways that students may view data, from general to more elaborate views about data. They are: i) thinking about data as *pointers*—focusing on the general things such data collection processes; ii) seeing data as *cases*—focusing on the identity of individual data values such as the largest value in a data set; iii) data as *classifiers*—paying attention to the frequency of particular data values, for example, how many are above a certain fixed value and how many are below that value; and iv) data as *aggregate*—focusing on the overall characteristics of a data set including its shape, centre and the spread of values from the centre. The above researchers agree that aggregate consideration of data promotes students' statistical reasoning and thinking.

The second theoretical perspective concerns semiotic mediation—using tools and signs for learning concepts. According to Vygotsky (1978, elaborated by others e.g., Falcade, Laborde & Mariotti, 2007), artefacts—physical or non-physical tools—produce signs that can be used to mediate meanings of abstract mathematical concepts. Sketchpad's Dragging tool was used to solve a mathematical task. The signs produced in the process of solving the task mediated the concept of statistical variability. This study explores the following research questions: i) What do students think about the measures of variability such distribution, mean and standard deviation in a data set presented in a static environment?; and ii) What do students think about the measures of variability during and after interacting with the dynamic mathematics sketch introduced above?

DYNAMIC MATHEMATICS SKETCH

The dynamic mathematics sketch (*DyMS*) was designed to offer learners an environment where they can drag data points on the number line using the Dragging tool and observe how the distribution, mean and standard deviation of data vary. The term "mobile" is used in connection with data points because they can be freely dragged on the horizontal axis to the left or right of the mean. In Figure 1a, six numerical data at points A, B, C, D, E and F are shown. For the purpose of this discussion, the data values are assumed to be drawn from a normally distributed population. A data point can be dragged on the horizontal axis by selecting it with a computer mouse pointer, holding the left button of the mouse down and dragging the point along the horizontal axis with the Dragging tool. As a data point is dragged horizontally to the left or to the right side of the centre, the numerical scales for the mean and standard deviation shown on the sketch change with the position of the data point on the horizontal axis.

In Figure 1a, the six $(n=6)$ data points are positioned at some distance d_i ($i=1,...,6$) away from the mean. The perpendicular line passing through the mean, called the "mean-line", provides a physical and more visual tracking of the mean as data points are dragged along the horizontal axis. A data point i at distance d_i from the mean-line forms a geometrical square with area d_i^2. Note that the mean-line and the horizontal axis touch two sides of each of the six squares. Lastly, the squares are named using their respective letters, for example square F is formed by the horizontal distance from the mean to point F, and the mean-line.

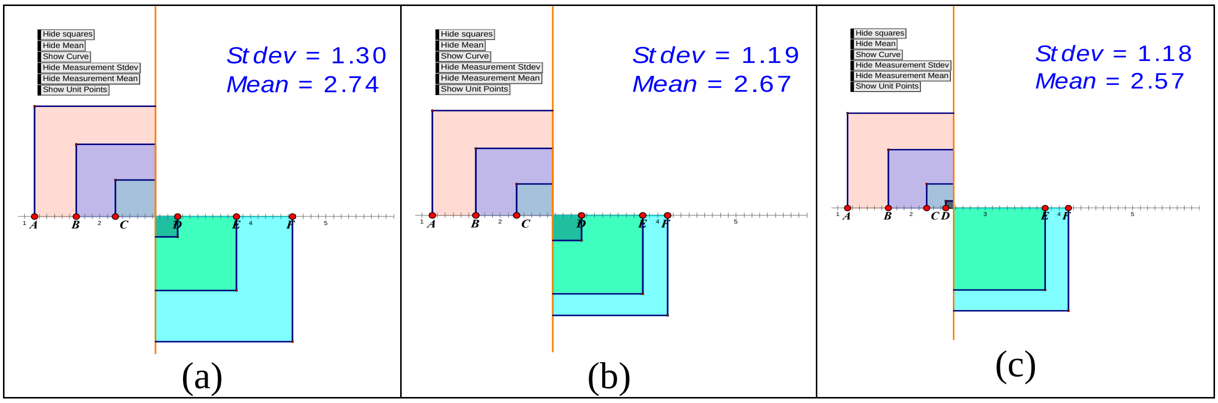


Figure 1: (a) Before dragging a data point on the horizontal axis; (b) Data point **F** on the far right is dragged slightly to the left; (c) Data point D closer to the centre on the right is dragged across the mean-line to the left side.

The sum of the area of all the six squares (not shown in Fig. 1) gives the magnitude of the sample variance $s^2 = (1/(n-1))\sum_{i=1}^{i=6} d_i^2 = (1/(n-1))\sum_{i=1}^{i=6}(x_i - \bar{x})^2$ from which the sample standard deviation $s = [(1/(n-1))\sum_{i=1}^{i=6} d_i^2]^{1/2}$ is the length of the resulting square. If the six data points are dragged far away from the mean-line, then the resulting square will be large, and the magnitudes of variance and standard deviation will both be large. The area of square was used to provide a sign for standard deviation and to show how it varies as data values vary on the horizontal axis. For instance, in Figure 1(b), point F on the far right of the mean-line has been dragged slightly to the left side toward the mean-line, reducing the magnitude of standard deviation from 1.30 to 1.19. In Figure 1c, point D on the right side of the mean-line has been dragged across the mean-line to the left side and the magnitude of standard deviation reduces further from 1.19 to 1.18. In general, the sketches in Figure 1 provide two results: i) that standard deviation decreases as the square areas decrease and it increases with increase in the square areas; ii) variability in the data distribution decreases as data points are dragged toward the mean-line and increases as points are dragged away from the mean-line.

Figure 2 provides a summary of the variability of the mean and standard deviation when dragging the data points on the horizontal line relative to the mean-line. Dragging a point to the right side away from the mean-line increases the magnitudes of both standard deviation and the mean whereas dragging the same point toward the mean-line from the right side decreases both parameters. If a point is dragged on the left side away from the mean-line, the standard deviation increases whereas the mean decreases. Lastly, selecting all the six data points and dragging them on the horizontal axis to the left or right of the mean-line does not change the magnitude of standard deviation; it just shifts the mean to the left or right.

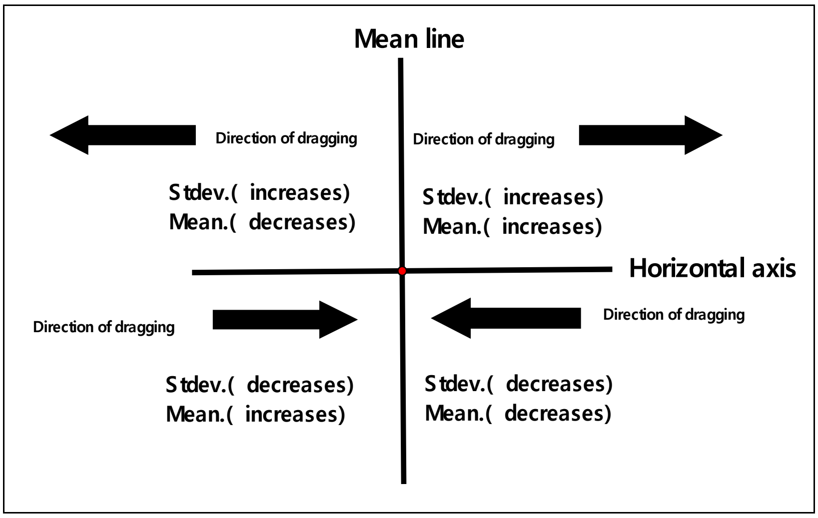


Figure 2: Changes in magnitude of standard deviation and the mean with the direction of dragging data points on the horizontal axis relative to the mean-line**.**

METHODOLOGY

The study took place in a North American University where five undergraduate students (F=3, M=2) enrolled in a 13-week introductory statistics course were interviewed. By the time of the interviews, participants in the study had covered all the topics that they needed including describing distributions with graphs; describing distribution with numbers; basic probability theory; the normal distribution curve and sampling distributions. The students were taught by instructors from the Department of Statistics. They also had drop-in tutorials five days in a week for the whole semester. I believe that participants took the interviews having adequate background information. However, my aim was not to assess how much participants had learned in their statistics classes. Rather, as stated above, this study aimed at exploring introductory students thinking about the meaning of variability before, during and after using the *DyMS* sketch.

The *DyMS* sketch is designed with the "hide" and "show" buttons so that the numerical scales for the mean and the standard deviation can be "hidden" from participants during prediction stages and "shown" to them during checking predictions. I adapted the "predict, justify, and check prediction" (PJC) methodological approach from delMas and Liu's (2005) study as well from a review of Garfield and Ben-Zvi's (2008) book.

The PJC approach asks participants to predict changes in the sketch as data points are dragged, briefly justify their predictions and then they check predictions using the Dragging tool. During checking prediction, the buttons for the numerical scales of the mean and standard deviation are turned on so that participants can confirm or refute their claims and hypotheses. This approach was chosen because it engages participants in the tasks and encourages them to use their own constructions of meanings rather than, for example, re-stating procedures from text books.

One-on-one task-based interviews (Ginsburg 1981; diSessa, 2007) were used to collect data. The method was chosen because I wanted some rich data from students' reflections about variability without and with the *DyMS* sketch that such interviews can generate. Each interview session lasted roughly 35 minutes and was divided into three segments. In the first segment (10 minutes), the first author asked participants in the study about the terms distribution, mean and standard deviation. This question was asked to enable a comparison of the participants' thinking about the measures of variability before and after they interacted with the dynamic sketch. Participants used the second segment (15 minutes) to predict, justify and check their predictions with the sketch. In the third and last segment, with the computer shut down, participants were asked to reflect on the term standard deviation. I expected them to provide an aggregate view of standard deviation after interaction with the sketch in that standard deviation measures the spread of data from the centre or from the mean. In fact in this segment, participants were not asked to reflect separately on distribution and mean as it was the case at the beginning of the interviews because I assumed that they would reason in aggregate—include the mean and the distribution of data in their statements.

RESULTS

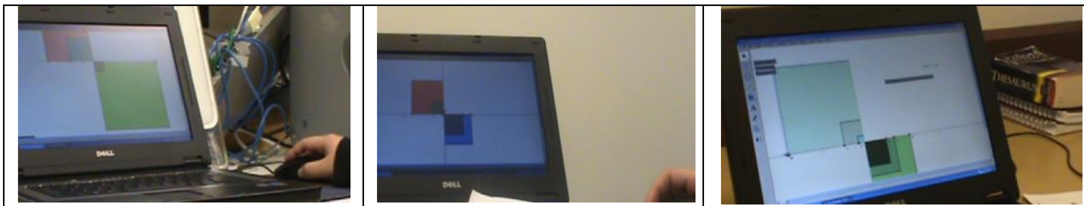
Data for three participants, Boris, Halen and Maya, were chosen because they are representative of the group. I present the data in two parts: The first part presents participants' thinking about the terms distribution, mean and standard deviation in the static environment, that is, before they used the sketch. The second part presents participants' interactions with the *DyMS* sketch in the computer environment, followed by their reflections about standard deviation at the end, with the computer shut down so that the dynamic sketch was no longer in use.

Thinking about distribution, mean and standard deviation without the *DyMS* sketch.

The first author (GE) asked Boris, Halen, and Maya "What do you think about" the terms "distribution, mean standard deviation?" Boris described distribution as the "observed frequency of some data." Maya said distribution is "how things are distributed [...] evenly or randomly" and for Halen, "the first thing" about distribution would be "the normal distribution." About the mean, Boris said the mean is the "average" and he did not say more. According to Maya, the "mean is the answer to a formula where we add up all the values in a particular data set and divide by the number of values that are there, so mean is like a number [...] calculated out.” Halen seemed to share Maya’s thinking about the mean. She explained that “if you have a couple of numbers, you add them all up and then you divide by how many numbers there are, you get like the average or the mean.” Lastly, on standard deviation, Boris stated that it “measures variation of data from the mean”, whereas Halen sketched the standard normal curve and explained that “standard deviation […] is similar to deviation […] from the mean [...] one standard deviation is sixty eight percent […].” Maya stated that “[…] I see standard deviation in graphs […] there is like one, two […], then there is negative one, negative two […]. You can calculate standard deviation.”

Interactions with the *DyMS* sketch and reflections at the end.

Using the “predict, justify, and check” framework, GE asked Boris, Halen and Maya each to “predict how the squares [Fig. 1a] would vary if you drag any of the data points on the horizontal axis.” I expected participants to describe variation in the size of the squares with the magnitudes of standard deviation as data points were dragged on the horizontal axis. I used Figure 2 as a framework to track predictions but did not show to participants before predictions. Boris predicted and explained that, “if you move [data points] away from the centre, the square is getting bigger and bigger because the square is the distance from the centre. Later, as he checked his prediction by dragging data point F to the far right (Fig. 3a), he said, “yeah, yeah, they [the squares] are getting bigger, the boxes are just scales; they are actually just scales to see how the relative [distances] are.” For Boris, the size of the square seemed to be used as a sign for how far a data point was from the mean. Although Boris did not directly say that his “scales” were measuring the spread of data from the centre, his statement suggests such a meaning. After his task, with the computer closed, the GE asked Boris, “What do you think about standard deviation?” He was silent for about half a minute before he replied saying if all the “data points” were selected and dragged on the horizontal line “to the left or right” of the mean-line, that movement “just shifts the mean, but it won’t change the standard deviation.”



(a) (b) (c)

Figure 3: (a) Boris drags point F farther away to the right as he checks his prediction; (b) Halen drags data point A to the left side of mean-line; (c) Maya drags data point B farther to the left side of the mean-line.

Halen predicted that if a data point was dragged outward to the left of the mean-line, the square “[...] will become a bit narrow […] like skinnier” To her, the square seemed to be stretching into a rectangle, in that case the square was not functioning as a sign for the size of standard deviation. But we know that the shape of the square does not change with the magnitude of standard deviation. Halen checked her prediction by dragging a data point C slightly to the left side of mean-line (Fig. 3b) and as she noticed how the squares varied, she said, “Oh, I thought they [squares] would all go together [...] if I did this [drags a point to the left side] [...] but oh […] so.” Halen’s statement suggests that she noticed the signs in the squares, which signs were different than what she “thought.” She also seemed surprised at what she saw. However, Halen did not connect the signs in the squares with the change in the magnitude of standard deviation as Boris did. At the end, when asked “what do think about standard deviation?” Halen said, “If you change standard deviation, the mean is going to change. When I was looking at the graphs, I didn’t realize that.” Lastly, Maya correctly predicted and explained that “when I move the point to the left, the square will increase [...] because the farther away the point is from the centre, then the greater area it has” Although Maya’s prediction is much clearer than Halen’s, she did not directly link the changing area of the square with the magnitude of standard deviation. But as she checked on the sketch (Fig.3c) while dragging a point to the left side of the mean-line, she was able to confirm that “standard deviation increases [as] […] the mean decreases.” After her task on the computer, GE closed the computer and asked Maya, “What do you think about standard deviation?” Maya was silent for a moment and then she said, “Standard deviation is certain point away from the centre of a population […]. I have a picture of a normal distribution divided into sections which are called standard deviation […]. There is a formula, I forgot, but it’s like standard deviation equals the square root of variance.”

DISCUSSION

In the static environment, participants’ thinking about the terms distribution, mean, and standard did not show clear connections among the terms. For example, the mean was more likely to be used as a pointer (Konold & Higgins, 2003) to the “average” of data [Boris, Halen & Maya] whereas standard deviation suggested a pointer to the normal distribution curve [Halen & Maya]. Overall, participants’ thinking in the static environment did not show strikingly different results from what has been reported in the literature (e.g., Garfield and Ben-Zvi 2008; delMas &Liu, 2005). Garfield and Ben-Zvi report that “while students can learn how to compute formal measures of variability” such as the range, variance and standard deviation, “they rarely understand what the summary statistics represent […] and do not understand their connection to other statistical concepts” (p. 205).

When solving tasks on the *DyMS*, Boris and Maya correctly predicted the change in the area of the squares with the dragging of data points on the horizontal axis. Halen struggled in her predictions, particularly linking the area of the square with the movement of data points. But after checking her prediction, she "realized" that "if you change standard deviation, the mean is going to change too." Halen's statement ignores the fact that standard deviation may remain unchanged when the mean changes, a result that only Boris was able to obtain from the sketch. Nevertheless, Halen's statement also shows more dynamic thinking about standard deviation, which is different than in the static environment. The dynamic thinking was possibly evoked by Halen's interaction with the dynamic sketch, particularly the Dragging tool. The numerical scales for standard deviation and the mean enabled participants to check their predictions by dragging points on the horizontal axis and linking the signs in the squares with the magnitude of standard deviation and the mean. Maya, for instance was able to notice that by dragging a data point to the left side of the mean-line "standard deviation increases […] [as] the mean decreases."

Furthermore, Boris' statement that dragging all the data points to the left or right "just shifts the mean, but it won't change the standard deviation" was an important result that provides a clear example of aggregate reasoning with data (Wild & Pfannkuch, 1999). His statement suggests that the signs (e.g., the numerical scales of the mean and the changing area of the squares) enabled him notice the functional connections between standard deviation and the distribution of data on the horizontal axis. I argue that the Dragging tool of *Sketchpad* was used as an instrument of semiotic mediation for the meaning of variability (Falcade, Laborde & Mariotti, 2007) of data.

CONCLUSION

Although my study findings may be restricted to a small sample of participants, and I can also add possible limitations imposed by the vagaries of an exploratory study such as asking students to "predict" change based on a new sketch to them, I believe that the study nevertheless provides some evidence that the dynamic sketch engaged participants and moved them away from focusing on textbook definitions and procedures at the beginning, to reasoning about the features of variability more qualitatively and in their own words at the end. The dynamic, physical, visual and tactile properties of the sketch seemed to mediate participants' dynamic thinking of statistical variability. However, with the computer closed, Halen and in particular Maya, showed some "mixed reasoning"—aggregate reasoning combined with some textbook procedures. Only Boris showed more complete aggregate reasoning with data. It may be that Maya and Halen needed a longer time with the sketch than was possible in the interview. I propose that students are allowed more time with dynamic learning tools, e.g., in the Labs or Workshops, facilitated by their instructors.

References

delMas, R. C., & Liu, Y. (2005). Exploring students' conceptions of the standard deviation. *Statistics Education Research Journal, 4*(1), 55–82.

diSessa, A. A. (2007): An Interactional Analysis of Clinical Interviewing. *Cognition and Instruction*, *25*:4, 523-565.

Falcade, R., Laborde, C., & Mariotti, M.A. (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics*, *66,* 317–3.

Garfield, J., & Ben-Zvi, D. (2008). Developing students' statistical seasoning: Connecting research and teaching practice, New York: Springer.

Ginsberg, H. (1981). The Clinical Interview in Psychological Research on Mathematical Thinking: Aims, Rationales, Techniques. *For the Learning of Mathematics*, *1*(3), 4-11.

Konold, C., & Higgins, T. (2003). Reasoning about data. In J. Kilpatrick, W. G. Martin & D.E. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 193–215). Reston, VA: National Council of Teachers of Mathematics.

Garfield, J., & Ben-Zvi, D. (2008). Developing students' statistical seasoning: Connecting research and teaching practice, New York: Springer.

Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes.* Cambridge, MA: Harvard University Press.

Wild, C. J., & Pfannkuch, M. (1999). Statistical thinking in empirical enquiry. *International Statistical Review*, *67*(3), 223–265.

USING TWO FRAMEWORKS TO ANALYZE A GEOMETRIC TASK: SEMIOTIC MEDIATION AND NEW MATERIALISM

Sean Chorney

Simon Fraser University

This study looks at a geometry situation on The Geometer's Sketchpad and analyses the activity using two different theoretical frameworks. Semiotic mediation which emerges from the Vygotskian school, and the more organic and emergent framework of Barad and Ingold. The purpose is to identify what is highlighted by using different tools of analysis. The finding is that the outcome of observation is dependent upon the framework; the results are completely different with little commonality.

INTRODUCTION

Learning has typically been attributed to people. In the field of education there has been and still is a profound focus on the individual. Have they learned the material? Do they understand? Can they use this new knowledge in innovative ways? There is a common perception that knowledge is transmitted and stored in the mind. And to be accessed later when needed. This has been the sole focus of mass education. Preparing the individual with knowledge needed for society and for themselves. Education has become like an assembly line, where the students come to school and teachers tell them things they should know and students leave with that information. This is, in a very crude form, the practice of learning in education.

This study looks at two frameworks to look at the process of learning. Semiotic mediation which has emerged from Vygotsky's (1978) work and also, what I will term, new materialistic perspectives of Barad (2007) and Ingold (2013). Both of these frameworks can be applied to the school culture where students participate in activities. Both framework value the process of activity and do not focus on the finality of an event. Particularly for Barad and Ingold whereby process never ends; there is always a further becoming.

The question for this study lies more in a description of events. How do the process of events in a mathematical activity look when looking through two lenses?

THEORETICAL FRAMEWORK

The two frameworks of semiotic mediation and new materialism is outlined respectively:

SEMIOTIC MEDIATION

Semiotic mediation emerges from two generations of activity theory. Generation one includes Vygotsky (1978) and his central tenet of mediation as well as the move from the inter to the intra. Generation 2 involves Leontiev (1978) who describes the essential aspect of object/motive in activity. Semiotics are signs, these signs according to Vygotsky include language, signs, as well as material objects. So semiotic mediation involves the working with cultural tools and internalizes the function of these tools.

The object and the tool present a double stimuli. The psychological tool is the mental functioning that is involved in moving from seeing the object as a stimulus in which behaviour is directed to realizing the "psychological operations… necessary for the solution of the problem" (Vygotsky, 1930, p. 1). Vygotsky calls this psychological development "the instrumental act." There is an external action and an analogous function occurs in the mental component of the mind. In a mathematical learning situation one might consider that mathematical meanings are potentially contained in the artefacts. If one were using a compass to draw a circle, the compass is used externally to draw the circle, this would be the external aspect to the activity but when the compass is used to find points that satisfy a given relationship it is internal (Bartolini, Bussi & Mariotti, 2008).

Task choice is important. The a priori analysis of the activity is an essential component for the teacher. What is to be mastered by the student must be orchestrated by the teacher. Planning and design are major considerations in semiotic mediation. "…object/motive …gives sense to (learning) action" (p. 111), Roth and Radford (2011) goes to argue that no object/motive means no sense to the activity. Leontiev (1978) introduced the idea of motive to activity in second-generation activity theory. The object/motive is the highest level of the activity. In a mathematical activity, an example of an object/motive would be the overall mathematical practice that the teacher has outlined. For example, identifying the relationship between a circle's tangent and its radius.

NEW MATERIALISM

Karen Barad (2007), a philosopher and physicist, adopts a post humanist perspective in analyzing Neils Bohr's theories which emerged out of nuclear physics. But she goes further than Pickering (1995) or ANT theorists in that she challenges the distinction of things a priori. She challenges the boundaries drawn by traditional human knowledge and how it constructs our perception of differentiating things from one another. For example, we recognize a human as a distinct creature that is unified, whole and that can move from context to context without changing. We abstract this being, provide it with an identity, contemplate its abilities, but according to Barad this an assumption and she challenges the way we identify things and the way we construct our reality. Part of Barad's post-humanism is that she refuses "to take the distinction between human and non-human for granted" (p. 32). The human is only defined within the context of activity.

Barad describes that agency emerges from activity and in reflection of that activity, we choose, arbitrarily she claims, where the focus of attention is going to be placed and how boundaries are to be drawn. One of the challenges some of the post-humanist writers have had is to refer to activity by way of verbs without necessary reference to nouns. Ingold (2013) describes a problem with our language in that when we say the wind blows it is not the wind that is blowing but the blowing is the wind. We abstract the idea of wind, we reify it and treat it like an object causing us to deal with things as opposed to process. Benjamin Whorf, a linguist, writes:

> We dissect nature along lines laid down by our native languages. The categories and types that we isolate from the world of phenomena we do not find there because they stare every observer in the face; on the contrary, the world is presented in a kaleidoscopic flux of impressions which has to be organized by our minds – and this means largely by the linguistic systems in our minds. We cut nature up organize it into concepts, and ascribe significances as we do, largely because we are parties to an agreement to organize it in this way (p. 45)

Ingold, an anthropologist, is similar to Barad in that he challenges the idea of predefined states. While Barad challenges the "belief that beings exist as individuals with inherent attributes, anterior to their representation is a metaphysical presupposition" (p. 46) which is similar to Caruso's claim described earlier, Ingold offers an alternative when he describes his field of anthropology as "the study of human becomings as they unfold within the weave of the world" (p. 9). With both writers there is an attempt to undo the belief that individuals are unified, whole beings before becoming or before acting. This supports Caruso's argument that if one were to accept the human as the "unmoved mover" (p. 65), the one who can change, or alter her world, this would lead to a belief that the self is an ontologically robust entity. Pickering also refers to such a sentiment by arguing that "there is no especially informative pattern to be discovered about what changes and what does not…. We are stuck with the emergent posthumanism of the mangle" (p. 207).

Ingold criticizes the use of agency arguing that because materials have been treated as inert and dead for so long there have been to revitalized with agency. But I suggest that his phrases such as "interlaced meshwork" and "relational matrix" are very similar to Barad's "assemblage of human and non-human agencies" (p. 103) In both cases the writers are arguing that distinction and finality should not precede interaction and that we, as humans, are defined in what we do with what we have.

The reliance upon the mind as the principle focus of education is being challenged. What is the mind? Does it really store information? Can we think of it as an organic computer? The question that emerges from such inquiries is why are we only looking at the learner? What about the resources? What about the context? The activity? The tool? This study challenges such anthropocentric perspectives and puts the centrality of the human aside. New perspectives of materiality have acted similarly. If one is to view the human as a thing, we might focus on a classroom activity as things interacting with other things. The human is a construct. It is a self imposed attribution of special qualities, separating itself from other things. But this construct is self-referential. That is, humans claim to be special and different but it is a claim couched in the fact that we can make such statements because we are special and different. This circularity is problematic because it assumes too much. The construct is faulty and challenging the assumptions implicit within the construct is an important component to a study in education.

If the human is then to be set off with the other things one must realize that alongside of the human goes the construct of thinking, understanding, intending. Although these words may still be used, the place or the space from where they emerge are different. The mind of the human is not necessarily the place from where intention begins. This study positions itself alongside of Sinclair and de Freietas (in press) in understanding human constructs such as creativity, ingenuity, intentions to be context and or tool dependent. That is, qualities or capabilities are not existing independent of context. They depend on things, one is not creative, they are creative with paint brush.

Tool is not a conduit or a medium or a mediator or something beyond, something less concrete occurring in the head. The tool is the thing that the human touches, engages, moves, while the tool is the thing that determines how the human touches, engages and moves it. While some may call this a dialectic interaction, I argue that an interaction is often analyzed based on the coming together of predefined elements. Autonomous things, defined, detailed in affordances but instead of once again looking at things with attributes one might be inclined to look at the new ways in which things interact which define or declare the new affordances. The difficulty with affordances is that for many it defines

Wolfe writes: “Fundamentally a prosthetic creature that has coevolved with various forms of technicity and materiality, forms that are radically “not-human” and yet have nevertheless made the human what it is” (p. xxv).

This coevolving is not complete. In schools, students are growing into and becoming practicing cultural active participants. But they are not alone. Material resources, including tools and practices, are as present as the human.

METHODOLOGY

The materiality in this study is manifested in a technological tool. Papert (1993) describes this as an expressive technology and it is a way to see how technological and social processes interact. The agency of computers is particularly interesting, given its range of expressive possibilities and feedback. It presents an environment where students can make choices giving them the freedom for expression and exercise their agency.

This study also embraces the idea that observation is a form of participation. I suggest that the act of teaching, researching and observing are interrelated and entangled agencies of the same process. There is no distancing one-self to evaluate the act of teaching and/or learning. The researcher is to be identified in action, not in an analysis of action, post priori.

Working under the assumption that student, teacher and researcher are participants are not only entangled with mathematics learning but also each relation is entangled with each other. As well that these distinctions are not pre defined nor understood a priori but become and emerge in activity. I will study student engaging with tools, considering their activity, their constraints, demands, as well as analyze the similar relation between teacher-researcher and observation and action.

PARTICIPANTS, DATA AND ANALYSIS

The data collection took place in a Vancouver high school with some students who had been working in an environment using The Geometer's Sketchpad (GSP). Data was collected by means of a software capturing software, SMRecorder as well as videotaping.

The first task given to the students was in the form of a black box sketch. Initially two points were visible and when the student dragged either point the other point would move in a deterministic path. The points were related by a mathematical relationship. The problem posed was to identify the relationship, either in words, or as an equation, between the two points. This activity was chosen for it was a challenging but accessible problem that related to previous curricular work on the topic of transformations. Initially these points, A and B, were visible on the screen. It was also written on the screen that it was recommended to have the points dragged trace their trajectory so as to have an image of paths travelled. Joanne and Emily worked in class for approximately 20 minutes on this black box sketch. Near the end of their exploration Joanne drew a wavy line on the screen as shown in Fig1, Fig 2.

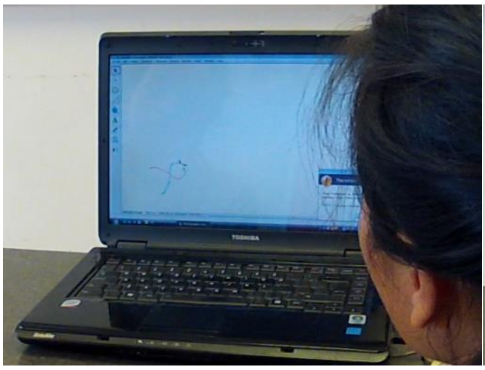


Fig. 1

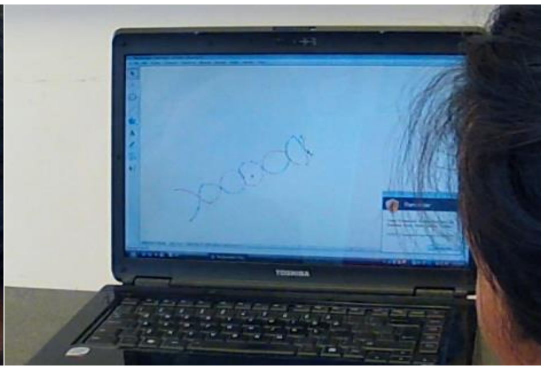


Fig. 2

Using a framework of semiotic mediation, one comes to the following description. Joanne has the goal to convince herself and her friend that there is an invisible reflection line. She moves toward that goal by drawing a curved wave back and forth across the implicit line. She had previously stated that she knows the line is there but her partner did not "see" it and was still not sure of its existence nor of its meaning. Joanne drew the wavy line. But the intended object/motive to activity is not to draw a wavy line but to internalize the notion of what a reflection is about. As Joanne is drawing, crossing line over line, weaving across the reflection line she is analogously internalizing the idea of equidistance from the reflect line. The property of reflection was the motive/object and her external representation indicates this is so. The repeated weave across the line is a method in "finding" the line. This is a particular instance of the general object/motive materializes here in theory form. The constructed curve exists as the relation of reflection properties and it subsequently exists in consciousness.

From a framework of Ingold's meshwork, one of the first things to identify is that the student and the computer are becoming together. The initial acts of the student just before they draw the wave is not a formed a priori, the act is the forming and becoming of two entities. But it is challenging to think this way because it is not two entities. As soon as we refer to two entities, it establishes a construction of two things. Barad uses the term intra-action to indicate that it is not two things interacting but a construction of one entity where the human and the computer software are within each other. It is a level below inter; here we have two things that are only defined within each other. Without each other there is nothing to comment within this context. Two things are becoming. So what does that tell us? First of all, the computer software has agency, it acts and that act reconfigures the human. If we cast aside the belief in intention and recognize that it is within the movement of the hand, repetitively moving back and forth across the reflection line, that is the mathematics. There is no interiorization. The mathematics is the pattern of movement because that is what is traced out on the screen. If we were able to move out and objectively look at the situation (which we are not able to) we might see, if we choose, two things (human and computer) engaging in an activity, the output is the trace on the screen.

Ingold draws upon a story of a person cutting wood to describe his understanding on activity and interaction. He describes that initial acts of the person with the wood are to "guide work and not strictly determine it" (p. 53). And that the "initial plan is not inside the head, it involves all kinds of decisions" (p. 55). He also explains that the "hand (on the saw) is in what it makes or does, not in what it is, that the human hand comes into its own" (p.55).

CONCLUSION

Both semiotic mediation and new materialism look at process of activity. The subtleties of activity are essential in analysis. A single situation was analysed using both framework so as to get a glimpse into what emerges when the framework changes as well as to challenges anthropocentric assumptions. Although semiotic mediation is a valuable and thoroughly enriching framework, the assumption of interiorization is questionable. A new materialism perspective is more focused on the external but observation is a form of participation and an agentic decision so it is questionable as well. But the overall analysis of the repetitive, organic move of the hand to construct a wavy line outlining a mathematical relationship is an enriching as we desire.

REFERENCES

Barad, K. (2007). *Meeting the Universe Halfway: Quantum Physics and the Entanglement of Matter and Meaning*. Duke University Press.

Bartolini Bussi, M. G., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom: artifacts and signs after a Vygotskian perspective. In L. D. English (Ed.), Bussi, M. B., Jones, G. A., Lesh, R. A., Sriraman, B. (Assoc. Eds.), Handbook of international research in mathematics education. New York: Routledge.

Caruso, G. D. (2012). Free Will and Consciousness: A Determinist Account of the Illusion of Free Will. Lexington Books: Lantham, Boulder.

de Freitas, E., & Sinclair, N. (in press). New materialist ontologies in mathematics education: The body in/of mathematics. *Educational Studies in Mathematics*.

Ingold, T. (2013). Being Alive: Essays on Movement, Knowledge and Description. Routledge: Taylor and Francis Group, London and New York.

Leontiev, A. N. (1978). Activity, consciousness, and personality. Hillsdale: Prentice-Hall.

Papert, S. (1993). *Mindstorms: Children, Computers, and Powerful Ideas*. Basic Books, New York.

Pickering, A. (1995). The Mangle of Practice: Time, agency, and Science. Chicago: The university of Chicago Press.

Roth, W.R. & Radford, L. (2011). A Cultural-Historical Perspective on Mathematics Teaching and Learning. Sense Publishers: Rotterdan, Boston, Taipei.

Vygotsky, L. S. (1930). “The Instrumental Method in Psychology”. Text of a talk given in 1930 at the Krupskaya Academy of Communist Education. Retrieved January17th,2011. http://www.marxists.org/archive/vygotsky/works/1930/instrumental.htm

Vygotsky, L. S. (1978). Mind in society: The development of higher psychological processes. Cambridge, MA: Harvard University Press.

Wolfe, C. (2010). What is Posthumanism? University of Minnesota Press: Minneapolis, London.

USE OF PHENOMENOLOGY THEORY PERSPECTIVE TO STUDY TEACHERS' ENGAGEMENT IN THE READING OF MANUALS DURING A PROFESSIONAL DEVELOPMENT SESSION.

Melania Alvarez

Simon Fraser University

The purpose of this phenomenological study is to provide a description of a professional development session which main purpose was to engage teachers to read and discuss the manual of a math program recently introduced in their school.

INTRODUCTION

A new curriculum or program many times requires changes in the focus of the practice as well as tending to a change in how we elicit conceptual thinking and understanding in students. Manuals are seen as one of the main artifacts that would engender change in teaching practices when a new program is implemented. Current teacher's manuals/guides usually provide a variety of classroom activities for teachers to act out while teaching/introducing a mathematical idea; however researchers are finding out that in spite of how well written or how thorough the guides are, many teachers use the materials in ways not planned by the developers of these programs (Collopy, 2003; Remillard & Bryans, 2004).

The purpose of this phenomenological study is to provide a description of a session whose main purpose was the introduction, reading and discussion of some lesson in the manual of the math program that was being implemented. I will describe different attempts on my part to better engage teachers in the process of reading the manuals, discussing the use of the materials, the lessons, and the mathematical ideas contained in the lessons.

I will reflect on the "lived experience"of a professional development session by describing teachers behaviors, use of materials, level of activity interaction with resources, and so on.

FRAMEWORK:

This is phenomenological reflection on the behavior of teachers while learning and how through engagement they show their motivation. Motivation is a key factor for learning that is self-regulated. "Participants create meaning as they engage themselves in challenging learning activities. In engagement, the learners are active and might be searching, evaluating, constructing, creating, or organizing some kind of learning material into new or better ideas, memories, skills, values, feelings, solutions, decisions" (Wlodkowski 1999:44)

In order to analyze this process I am using a modified version of Remillard's (2012) model, that she uses to analyze various ways curriculum developers apply to make their manuals and textbooks materials attractive to teachers. Remillard's model is inspired by Ellsworth's book *Teaching Position* based on analysis of film studies where assumptions about the audience background influences the structure of the film's narrative in order to maximize their interaction or attention to the film; we find that the main goal is to position the audience in a way where interaction is possible and furthermore this is something that the audience wants to do of their own volition

This model consists of four main parts:

Mode of address: are ways of positioning an audience that are needed to initiate an interaction. In this case to position the teacher in a place where she can actively engage the resources of the program the way their developers envisioned it.

Forms of address: artifacts to carry out a "goal-directed activity", in this case the resources used by the instructor and that the teachers are expected to use.

Modes of engagement: in this case I will expand Remillard's model to include not only an analysis of engagement trough text forms, but to describe teachers' engagement with the materials during the session.

Forms of Engagement: how teachers engage and appropriate different aspects of the program, the ways the teachers re-source the resource, where I will use Adler's "conceptualization of 'resource' as both a noun and a verb; as a verb "re-source" will connote to source again or differently, and "source" will denote origin (Adler 2000).

The theoretical framework that guided this research is a phenomenological reflection on the behavior of teachers while learning, and the change in teachers through learning in the social context of professional development sessions, -- how new knowledge is incorporated into an existing schema and how can teachers participate in a discourse that extends their knowledge and system of beliefs if the professional development is effective in addressing its goals.

METHODOLOGY:

Several schools I have worked with have implemented this program, and it has always puzzled me how a substantial number of teachers who use it never even looked at the manual but they used the books and workbooks in a very traditional way. By traditional way, I mean providing students with examples of how to do a specific calculation, teaching them the rules and procedures and supplementing from other sources because this program does not provided the great number of math exercises that other programs provide. This is mainly the rule and not the exception, research done at schools in North America showed that most teachers' practices are

procedural and that in fact teachers usually teach as they were taught as students and there was very little change on their part to teach in a way that will develop the kind of learning and understanding that we would like to see in students as new curriculum reforms require a deeper content and pedagogical knowledge (Hiebert, et al., 2005; Stigler & Hiebert, 2004; Stigler, Fernandez, & Yoshida, 1996).

The participants in this study are the teachers involve in the professional development session and I, the professional developer. The professional developer is a participant because her reflections on teachers' behaviors will be part of the study.

My planned mode of address for the session was the following: During this session I asked teachers in the group to first tell me how they would start the year in their math class. I did this because we were going to look at the first 8 lessons in the year and I wanted for them to compare what they have been doing with what this program featured. I expected that there would be a discussion and after it I was to provide teachers with the first eight lessons in the textbook and workbook without the benefit of the manual, and teachers were asked how they would teach with those resources. Afterwards the manual was introduced with the goal of furthering discussions regarding the teaching of the eight lessons. I planned to ask teachers about additional ideas that were included in the manual, and how useful these ideas and activities were for their practice as they taught the concepts students found in the textbook and workbook. This was the plan and in the following section I will give a short description of what happened.

It is important to point out that in this particular math program the textbooks and workbooks are geared towards students. The manuals are different in that for most grades, except for kindergarten, the teachers will not see a reproduction of the textbook and workbook pages in the manual. The manual presents the following information as they introduce a new concept: objectives, notes which inform teachers what ideas students have learned in the past which will help them introduce the new concept, how to introduce a new concept by using students previous knowledge (many times with activities that are not in the students materials), how to use the book in class to elicit students discussion and workbook for practice and assessment. I explained all of these details to all the teachers at a school wide meeting that took place before the session I describe below.

PRELIMINARY RESULTS

This is the description of a 75 minute session with 4 grade 1 teachers. I will call the teachers Mandy, Erika, Elisa and Monika (not their real names). The goal of this session was to introduce them to the first 8 lessons in the program and for them to look at the manual to discuss any changes in their practice. The focus for the researcher was to find out how much they would look at the manual and to observe and participate (if necessary) in the discussion that would follow.

First I explained to the teachers that I was there as a coach, not a person who would tell them what to do, and also I explained the goal of the session, which was to look at a portion of the manual, textbook and workbook carefully and to discuss the use and the ideas contained in these resources.

I started by asking: How do you usually start your year in math? The teachers immediately started to provide a list, most of them started mentioning concepts like: patterns, number facts, number formation, looking at the calendar, number songs, number families, counting by twos, fives and tens, that they will use the power of ten cards to get them counting. They did not explain how they taught all this, they mentioned that kids like patterns and that is why they like to start with this concept and some of them also mentioned that they also will mix several of these ideas in one session to make it more interesting.

I asked how they will introduce the numerals and their representation. Most of them responded that most kids had a very solid background from kindergarten and that they remembered how to write numbers, though one teacher (Monika) mentioned that some kids still have trouble writing their numbers. The three other teachers at this point they felt confident that a review in numeral formation was not really needed.

Afterwards I provided copies of the first 15 pages in the book and the textbook without the manual. I asked them to take 10 minutes to look at the materials, and to think about how they would teach what was contained in those pages and that we would be having a discussion about this afterwards.

Within a few seconds Mandy was the first to make a comment:

> "Oh they are counting backwards and forwards all at the same time, that is kind of neat, I like that. Personally I don't think I do a lot of counting backwards in the beginning."

Erika and Elisa thought that what was presented would be too easy for the kids and that parents would ask why they were not doing something more difficult or challenging.

I explained that it was about making sure that they understood concepts that perhaps they were familiar with in kindergarten, but that it was important to make sure.

Instead of commenting on how they would teach the material presented, teachers wanted to know how long it would take to teach those materials using this program and what to do with kids who would for any reason fall behind or with those who would be bored by such easy concepts. They asked if there were any additional resources for them. I mentioned some additional resources, they wrote this information down.

Since no comments were being made regarding how they would approach the given materials, I gave them the first 8 lessons in the manual and told them to take at least ten minutes to look them over and we could have a discussion afterwards about the activities, and what was different and the same about what they have been currently doing in their practice. I also pointed out that some of the questions that they were having would be answered by the manual, like objectives, and lesson timing.

The part of the manual which I presented to them touched on the following: Mental image for digit 0 to 9, count within 10, match different representations of a number within 10, compare two numbers within ten, count from 0 to 10, count backwards from 0 to 10, arrange numbers 0 to 10 in order and applications.

Within three minutes Mandy made a comment that started the conversation.

Mandy:

> “Oh, this is, it is interesting. There is, there is a lot more explanation and talking and things and other concepts that are brought in, not just the numbeeeersssss. …Like when you read this, it is different, than just you saying this is what two looks like. It's more, it's more, …just… the teaching part of it is different than...”

Elisa :

> “(is) get the amount right with the symbol. They put together form with the symbol and what it actually means”

Mandy: “yeaaah.”

I personally felt that that what Elisa pointed out was not what Mandy meant, however this conversation was interrupted by Erika, who was asking about cards that could be needed for the lessons.

I told them where they could get them or that they could make them themselves and I immediately brought the conversation back to the manual, by pointing out how children have a difficult time remembering how to write some numbers. I was hoping to try to bring back Mandy’s train of thought. A thorough conversation followed about number formation, how to allow kids to familiarize themselves with the numeral symbols and to have fun doing it. To create a display with objects that look like a specific numeral could be helpful to help them remember. For example the number two looks like a swan and number 8 like a snow man. From here the conversation moved to a discussion about our numerical system and the use of only ten numerals and how the kids make sense of it and becoming really familiar with numbers is so important. Mandy and Elisa were leading the conversation.

I thought that the conversation was thought provoking, and somewhat connected to the ideas presented in the manual and let it flow for a few minutes. After a while, I asked them to continue looking at the manual and the next lessons.

Erika made a comment: I worry that this is so teacher focused.

At this point I realized that all the teachers were focused on the textbook and they would not look at the manual which is full of activities which are not teacher focused.

This was the pattern followed the rest of the time.

I wanted them to read the manual so I had to bring back the conversation to the lesson and activities. The only way I could do it was by reading the activities from the manual and asking about particular points that I thought were relevant to the lesson. Usually a short conversation would follow pertaining to the mathematical ideas in the activities contained in the manual and then it would deviate to a discussion of other math concepts, or clarifications about mathematical notation.

Also, the mathematical discussion was many times interrupted by questions which would indicate what was important for teachers. Those questions were about logistic use of the materials which in most cases had been clarified before, or about additional resources, or about possible problems that they could encounter with students or parents. Teachers will not instinctively look at the manual, they worry much more about the book and textbook even though only two out of the eight lessons that we reviewed used any of the materials in the textbook and workbook.

Here is a sample of the questions that usually would stop the conversation about the activities that we were analyzing and take us into discussing other things like additional resources, or problems with students or parents.

Erika: " is any of this material ready to use on a smart board?"
Monika: "who is going to scan the textbook to use on the smart board?"
Erika: "I can scan the pages, but I want to get rid of those lines and some of the pictures."
Elisa: "are there homework problems?"
Monika: "is that book behind you also part of the resources that we can use?"
Erika: "When do we finish with book 1A and start 1B?"
Eliza: "How long will it take to teach all these lessons? Would you say that will be about a week and a half?"

There were other questions or affirmations about lessons or ways of dealing with particular concepts that teachers wanted to continue as before, and it was reassuring for them to realize that many of their ideas would only enrich the process.
Elisa: "I like to start with patterns, frankly when we are reviewing writing numbers sometimes it becomes too boring with letter formation."

Erika: “I will still do my calendar from the beginning of the year.”
Mandy: “I use tally marks for counting. Can I still do that or just count using dots like they are doing in the lesson about counting?” (this question was asked when we were reviewing arranging number from 0 to 10)

And there were the comments which showed that in the end there was a possibly a shift:
Mandy: “Kids have enormous problems with word problems so spending more t ime with all this in the beginning will help them.”

And at the end of the session Erika changed her mind about the lessons being to teacher focused.
Erika: “yes, they are (the lessons) very interactive.”

Conclusion

Of the teachers participating in this session, I had the sense that Mandy was the one who was looking more carefully at the mathematical content, context and sequence, her comments are very telling. However many times her train of thought was interrupted by questions from other teachers.

There were comments like the one about the tally marks which came about two lessons after we discussed counting. This is an important point to make given that many times we are learning about an activity, and only afterwards when we have some time to reflect, we are able to see how new ideas can be in conflict or enrich what we are doing. These questions showed that teachers were reflecting on their practice, comparing with previous practices, and they were engaged.

Before this session I had given a general overview to all the teachers in this school about the use of the manuals and textbooks and workbooks. Still many of them needed to review some of the things I had already explained.

Some of the questions which according to me “interrupted” the process also tell about how teachers plan to re-source the resources. For example they planned to use smart boards instead of blackboard/white board which is not recommended by the program and there was also a discussion about they wanted to modify some of the pages in the textbook to use on the smart board. A teacher wanted to add tally counting, and another wanted to include patterns early on in the year, earlier than the program prescribed.

The teachers were open to having discussions about math ideas, but classroom management was one of their main concerns.

What was consistent was the subtle resistance to take a long and careful look at the manual and its activities. As I mentioned there were times when I had to read the activity and then ask specific questions to make it possible to discuss what was in the manual, otherwise the conversation would have move to classroom management and resources.

Was this mode of address successful? I think that it was, in that teachers were engaged, there was a lot of discussion about mathematical ideas that unfortunately I am not able to describe in this paper and also it allowed me to see a reality about the use of manuals. The truth is that if you ask around most people don't read manuals. Why are we using manuals as one of the main artifacts for curriculum change?

References:

Adler , J. (2000) Conceptualising Resources as a Theme for Teacher Education, *Journal of Mathematics Teacher Education*, (3), 205 – 224.

Collopy, R. (2003). Curriculum materials as a professional development tool: How a mathematics textbook affected two teachers' learning. *The Elementary School Journal*, 103(3), 287–311.

Hiebert, J., Stigler, J. W., Jacobs, J. K., Givvin, K. B. Garnier, H., Smith, M., et al., (2005). Mathematics Teaching in the United States Today (and Tomorrow): Results From the TIMSS 1999 Video Study. *Educational Evaluation and Policy Analysis*, 27(2), 111-132.

Remillard, J. T., & Bryans, M. B. (2004). Teachers' orientations toward mathematics curriculum materials: Implications for teacher learning. *Journal for Research in Mathematics Education*, 35(5), 352–388.

Remillard, J. (2012) Modes of Engagement: Understanding Teachers' Transactions with Mathematics Curriculum Resources. From Text to 'Lived' ResourcesMathematics Curriculum Materials and Teacher Development Series: Mathematics Teacher Education, Vol. 7 Gueudet, Ghislaine; Pepin, Birgit; Trouche, Luc (Eds.) 1st Edition., 2012

Stigler, J. W., Fernandez, C., and Yoshida, M. (1996). Cultures of mathematics instruction in Japanese and American elementary classrooms. In T. Rohlen and G. Le Tendre (Eds.), *Teaching and learning in Japan*. New York: Cambridge University Press. Pp. 213-247.

Stigler, J.W. & Hiebert, J. (2004, February). Improving mathematics teaching. *Educational Leadership*, 61(5), 12-17.

Wlodkowski, R. (1999).Fostering Motivation in Professional Development Programs. New Directions for. *Adult and Continuing Education*. 98, 39 – 47.

PLAYING NUMBERS ON TOUCHCOUNTS

Vajiheh(Mina) Sedaghat Jou

Simon Fraser University

This paper explores how young children build meaning through communicative, touch-based activities involving talk, gesture and body engagement working with TouchCounts (Educational iPad app). The main goal of this paper is to show the impact of touch-based interactions and finger counting on the development of children's perception and motor understanding of numbers. In this study, Nemirovsky's perceptuomotor integration approach theoretical framework revealed strong value of digital touch-based interaction and mathematics embodied in emergent numerical expertise by making and objectifying numbers.

Mathematical knowledge can be learned, based on previous constructed schema, by the formation of a real-world-grounded chain of cognitive metaphors that at each stage provide an understanding of the new, in terms of what is already familiar (Von Glasersfeld, 1995). Later, this mathematical knowledge expands in a similar way that we learn to play musical instruments such as a guitar. First, just by following the rules with little comprehension, then, with practice and finally archiving the level of play where meaning and expertise emerge (Nemirovsky, Kelton, & Rhodehamel, 2013).

NUMBERS AND FINGERS: NEURO-FUNCTIONAL AND CULTURAL LINK

There is a neuro-functional link between fingers and number processing. For instance, Butterworth (1999, 2005) has hypothesized that numerical representations, and processes are supported by three abilities: *Subitizing*: the innate ability to recognize small number of spots without counting; *Fine motor ability*: such as finger tapping; and *Finger gnosis*: the ability to mentally represent one's fingers. He writes: "Without the ability to attach number of representations to the neural representations of fingers and hands in their normal locations, the numbers themselves will never have a normal representation in the brain" (pp. 249-250, Butterworth, 1999). Butterworth argues that via our fingers, we construct concrete and abstract representations of numbers, number words, and number symbols. In addition, he believes finger gnosis is intrinsically linked to numerical representations as in all cultures, fingers represent numerosities with different pattern. Moreover, fingers are always available, and they can also be used as an aid in calculations, and therefore, they can work as a bridge between concrete and abstract representations of the notions of quantity and operations. Accordingly, using certain fingers for counting or constructing numbers can be related to method of counting by a five-finger system across different cultures (Butterworth, 1999).

In Penner-Wilger's (2007) study, those three component abilities (Subitizing, Fine motor ability and Finger gnosis) were found as a significant unique predictor of number system knowledge, which in turn was related to calculation skill. Noël has obtained consistent results in this regard (2005), and she has also demonstrated how consistent use of fingers positively affects the formation of number sense and thus also the development of calculation skills (Gracia-Bafalluy & Noël, 2008). Along this line, other researchers have suggested that finger-based counting may facilitate the establishment of number practices (Andres, Seron, & Olivier, 2007).

It is possible that these results could be explained through the hypothesis that part of the neural circuitry supporting finger gnosis is also part of the neural circuitry supporting certain mathematical abilities. As a case in point, a functional circuit originally evolved for finger representation has since been redeployed in support of magnitude representation, and now serves both functions (Anderson, 2007). However this discussion goes beyond the scope of this paper.

Based on the evidence from the fields of psychology, neuroscience and mathematics education, it can be claimed that use of fingers and appropriate gestures in early arithmetic facilitated by digital technology will help to develop number sense and mental calculation strategies. In particular, number representation with finger symbols is related to the nonverbal-symbolical form of representation (Moeller et al., 2011). In addition, Butterworth in his book "The Mathematical Brain" notes that developmental and cross-cultural studies have shown that children use their fingers early in life while learning basic arithmetic operations and the conventional sequence of counting words, fingers contribute to:

"(a) giving an iconic representation of numbers (b) keeping track of the number words uttered while counting up or down at the numerable chain level (c) prompting the understanding that every symbolic number is a sum and/or a multiple of 10 (the base 10 numerical system) and that 10 is equal to 2 × 5 (the sub-base 5 system), (d) sustaining the induction of the one-to-one correspondence principle by helping children to coordinate the act of tagging the object with saying the number word, and (e) sustaining the assimilation of the stable-order principle by sup- porting the emergence of a routine to link fingers to objects in a sequential culture-specific stable or- der." (Butterworth, 1999, as cited in Crollen, Mahe, Collignon, & Seron, 2011, p. 526).

Multi-touch technology supports the substantial role of body engagement and senses in education. It opens new windows up to the realm of teaching and learning. Having indicated the important role of body and fingers for numerosity, we postulate that using fingers to create numbers when it is supported by auditory and visual modes of perception will support and augment cardinal and ordinal understandings of numbers. TouchCountas an educational iPad application, benefits form multi-touch features/gestures of iPad. It provides users to manipulate numbers in a digital space and offers visual and vocal provisions in two sub-applications, namely Counting and Adding worlds. In this paper, both interviews conducted in Counting World (Crollen et al., 2011; Sinclair & Sedaghat Jou, 2013).

THEORETICAL FRAMEWORK

In our point of view, learning is situated in practice. This approach is participationistic and accounts for sociocultural aspects of learning, which have been widely used over the past years since Vygotsky's time. From an embodied cognition perspective, concepts may be formed in the minds of individual learners through social practice and activities or through the individuals' bodily-based metaphors. Vergnaud (2009), addresses five principles necessary for mathematical numerical abilities identified by Gelman and Gallistel (1978). He argues that understanding cardinality implies more than knowing that the last number-word of the counting sequence applied to a set of objects represents the numerosity of the set. Understanding cardinality also means being capable of using numbers and operations, and in particular, being able to use strategies like "counting on."

Various theoretical views, such as instrumentalism, sociocultural theory, semiotic mediation, and Nermirovsky's perceptuomotor integration approach (Nemirovsky, Kelton, Rhodehamel, 2013) emphasise the role of body in mathematical practice. These theories foc about us specifically on ways that mathematical expertise develops through a "systematic interpenetration of perceptual and motor aspects of playing mathematical instruments" (Nemivosky et al., 2013). On the other hand, there is still an ongoing debate on whether children should be encouraged to use fingers in early arithmetic.

Nemirovsky et al. (2013) in relating their theory to sociocultural factors suggest that, "while perceptuomotor integration constitutes a transformation that is experienced by an individual, it is (a) shaped by relatively local social interaction and relatively global cultural factors and (b) socially consequential because one's degree of instrumental fluency has bearing on one's membership to various social groups" (p. 381).This approach shares many similarities with the emerging body of work in mathematics education that moves away from a mentalist focus on structures and schemas toward a description of lived experiences in which learners' activities are at once bodily, emotional and interpersonal (Radford, 2008). The perceptuomotor integration approach refers to achievement of intertwined perceptual and motor aspects of tool fluency. It assumes that mathematical thinking is centrally constituted by bodily activity, which may be more or less overt, and that mathematical learning occurs through a transformation in the lived bodily engagement of a learner in a particular mathematical practice. This approach takes a strong stance toward embodiment, seeing it not just as a precursor or underpinning of mathematical thinking, thereby further promoting a mind/body dualism. Instead, mathematics learning entails transformations in the lived body experience, not just at the primary school age when children interact with physical manipulatives, but for learners of all ages.

In this paper, we are exploring how particular aspects of numerical abilities can be developed through TouchCounts and the impact of touch-based interactions in the development of children's perception and motor understanding of numbers.

METHODOLOGY

This study is part of an ongoing project of Dr. Sinclair's "Tangible Mathematics Learning" in Canada. After reviewing related literature and launching the first version of TouchCounts, the first stage of the study involves an exploratory pilot study to see how young children use TouchCounts, what they are interested in doing with the tool ("app") and how they learn with it. During one academic year, several sessions lasting between 20 to 50 minutes were conducted in a kindergarten in Northern Canada, on a monthly basis. The study took place in the participants' regular day care time. The setting for two focal episodes from these sessions of concern to us here was one of the classrooms, where children (aged 3 to 6) could freely join or disjoin the event.

RESULTS AND DISCUSSION

About eight students are sitting around the interviewer in the classroom. Rodrigo and Mike are singing a song and laughing while they are sitting in the left side of the interviewer. Rodrigo has already played with the app, although for Mike TouchCounts is totally new. [3] Interviewer asks all children to sit in crisscross position. TouchCounts is running on the "Counting World" while gravity and bar-line functions are off. Square brackets include descriptive commentary of nonverbal participations in transcriptions.

First Episode: Counting World, Without Gravity And Bar Line

20 I: OK, can you show us how to make four all at once?[Interviewer to Amanda who had already played with app]

21 [Amanda taps on screen by her four right fingers.] (Figure 1-a)

22 iPad: Four

23 I Very nice. [Interviewer to the next child, Sarah] OK. You wanna take a turn?

Child looks at interview's eyes and smiles.

[3] For details on TouchCounts please see (Sinclair & Sedaghat Jou, 2013)

24 I Can you count up to five?

Sarah bends on screen and taps on iPad. She makes numbers from one to five in a line pattern (Figure 1-b).

25 iPad one, two, three, four, five.

Sarah looks at interviewer and smiles.

Figure 1: Making 3 and 4 all at once

26 I Can you make three all at once? “three” all at once. [Interviewer stresses on three]

Sarah makes three using her index, middle and ring fingers while her palm also touches the screen.

27 iPad Four

Another child press reset key and Sarah tries again.

28 iPad Three (Figure 1-c)

28 I Good job!

Then, interviewer asks Kate to make four, all at once. Nonetheless, she makes four one by one using her index finger.

29 iPad One, two, three, Four (Figure 1-d).

29 I Do you hear one, two, three, four. Do you think you can make four all at once?

Mike is new in the classroom and had not a chance to play with the app before. He was sitting beside the interviewer and laughing with Rodrigo until this moment. Suddenly he looks at interviewer and says

40 Mike Like this? [Shows his four fingers to interviewer](Figure1-e)

51 I Like that? [Meanwhile Kate makes a four after watching Mike hand gesture and doing the same] (Figure 1-f).

Children actively participate in this scenario, some act as newcomers and some “old timers” [experts] (Lave & Wenger, 1991). Looking at the figures 1-d,1-e, and1-f; illustrates how this community comforts Kate’s emergence of instrumental and mathematical expertise (as well as others in a following episode). At this point, responses to the interviewer’s questions are more gestural than vocal (20-24). This episode indicates a high level of body engagement emphasizing numbers and corresponding fingers. The children are actively engaged in processes that are far from just being mental, with the task of exploring [making/doing] numbers in a sensuous manner (Radford, 2012).

We view this episode as an illustration of the relatively early stages of perceptuomotor learning. At this moment, Kate and Mike’s attempt to make/show four all at once is primarily motor, consisting of Kate’s manipulation on the iPad. Mike’s action is also predominantly perceptual, consisting of representing numbers by fingers (by Mike) following by creating them on the iPad (by Kate).

Second Episode: Counting world, with Bar line

In this world, numbers fall down unless child put them on the bar line. Mike put fourteen on the bar line successfully.

130 Mike Next time I'll put 19 on the bar [he smiles, presses reset and adjusts the bar line. Then, he continues tapping by his right index finger on the screen]

133 iPad 1, 2 ,3, 4, 5, 6,..10, 12 [Mike raises head, smiles and look at interviewer. It seems he noticed that he skipped 11]

134 iPad 13, 14,..18(falls), 19 (on the bar)

135 I WOW [smiles and expresses her wonder], I didn't think you could do it. [pauses. Looks into Mike's eyes]. You did it. Do you think you could get to the nineteen a little faster? How you can get there a little faster?

Mike taps by his index finger on screen *faster* to reach to nineteen. Interviewer repeats her question in a different way:

140 I I wanna see if you can get to 20 with using more than one finger.

Figure 2: Mike counts by five to reach 20 faster using 5 fingers.

Mike starts uses his five right fingers (Figure 2) following by counting one by one.

150 Ipad 5,10,15…27,28,29,…31,32.

151 Mike That's how old is my daddy. [Smiles while rolling on floor]

Comparing these two episodes, where Mike was involved, reveals the emergence of Mike's expertise that has developed gradually as his body engagements/actions integrate relatively more precise, faster and rhythmic. For example, faster finger movements, using more than one finger at the moment (150), choosing arbitrary number by himself and taking his own way of counting. He showed a high level of shifts in bodily and mathematically engagements from using one finger to more than one or changing the bar position. Mike's trained eyes and ears, noticed that he skipped counting eleven, which is another evidence of tool fluency (133).

Also, we have seen in all the occasions as well as pervious episodes, children use the exact fingers gesture as finger counting while dealing with numbers on TouchCounts. These observations well support Butterworth's arguments.

CONCUSSION AND REMARKS

In this paper, we addressed a number of literature reviews that argue the importance of fingers counting and body engagement in early childhood for further numerosity advancement. Also, we discussed ways in which theTouchCounts app takes advantage of a multi-touch device and make those engagement confront digitally. Analysing two short episodes indicates how touch-based interactions with TouchCounts can support development of young children's numerical perception and motor understanding in general and counting in particular. Social and individual interactions where observed via children's bodily gestures and actions. We found TouchCounts has excellent potential in providing visual, vocal and tactile supports, encouraging children's finger counting and movement. Moreover, our findings reveal that children become fluent with this mathematical instrument via manipulating, picking up and creating given numbers.

Further longitudinal studies with a greater number of children and iPads, in an actual classroom setting can reveal how interacting with TouchCounts might facilitate the development of children's number sense over a long term.

REFERENCES

Andres, M., Seron, X., & Olivier, E. (2007). Contribution of hand motor circuits to counting. *Journal of Cognitive Neuroscience*, *19*(4), 563–576. Retrieved from *http://www.ncbi.nlm.nih.gov/pubmed/17381248*

Butterworth, B. (1999). The mathematical brain. London: Macmillan.

Crollen, V., Mahe, R., Collignon, O., & Seron, X. (2011). The role of vision in the development of finger-number interactions: Finger-counting and finger-montring in blind children. *Journal of experimental child psychology*, *109*(4), 525–39. doi:10.1016/j.jecp.2011.03.011

Gracia-Bafalluy, M., & Noël, M.-P. (2008). Does finger training increase young children's numerical performance? *Cortex; a journal devoted to the study of the nervous system and behavior, 44*(4), 368–75. doi:10.1016/j.cortex.2007.08.020

Lave, J., & Wenger, E. (1991). *Situated learning: Legitimate peripheral participation*. (R. Pea & J. S. Brown, Eds.)*Learning in doing* (Vol. 95, p. 138). Cambridge University Press. doi:10.2307/2804509

Nemirovsky, R., Kelton, M. L., & Rhodehamel, B. (2013). Playing Mathematical Instruments: Emerging Perceptuomotor Integration With an Interactive Mathematics Exhibit. *Journal for Research in Mathematics Education*, *44*(2), 372–415. doi:10.5951/jresematheduc.44.2.0372

Radford, L. (2012). Sensuous Cognition. Université Laurentienne, Canada. *Springer*, 1–21.

Sato, M., Cattaneo, L., Rizzolatti, G., & Gallese, V. (2007). Numbers within our hands: modulation of corticospinal excitability of hand muscles during numerical judgment. *Journal of Cognitive Neuroscience*, *19*(4), 684–693. Retrieved from http://www.ncbi.nlm.nih.gov/pubmed/17381258

Sinclair, N., & Sedaghat Jou, V. (2013). Eighth Congress of European Research in Mathematics Education (CERME 8). In *Finger Counting And Adding On A Touchscreen Device*. Ankara.

Von Glasersfeld, E. (1995). *Radical Constructivism: A Way of Knowing and Learning*. (P. Ernest, Ed.)*Studies in Mathematics Education Series* (p. 213). Falmer Press.

TRANSFORMATION IN MATHEMATICS TEACHING PRACTICE: A CASE STUDY IN TEACHER NOTICING

Natasa Sirotic
Simon Fraser University

This case study focuses on the professional growth of an elementary teacher who participated in a practice based professional development initiative centred on the goal of creating a culture of mathematical thinking in the classroom. Through sustained inquiry, reflective practice, and collaboration with colleagues, which was focused around the examination of the impact of mathematical lessons on the student thinking, learning and understanding, a gradual but significant transformation was achieved which transferred into her daily instructional practice.

INTRODUCTION

In my research I look at the professional development through teaching, and so I begin from what teachers do in the classroom. Teaching, however, is complex, situated, and integrated, as well as culturally based, and therefore believed to be difficult to change. I facilitate professional development of teachers and I research the effects of these efforts on their practice. My work in this area is based on the assumptions that teachers are the key to improving student learning, professional learning must relate directly to the classroom, and teachers should have opportunities to observe other teachers in action and collectively reflect upon their practice in order to improve upon it.

This report comes from an ongoing study of in-service teachers at an independent, co-educational, university preparatory school in British Columbia, West Coast Academy (pseudonym), who participated in a situated professional development practice (Chazan, Ben-Chaim, & Gormas, 1998) known as "lesson study". Lesson study is a well-defined process for ongoing professional development of teachers that originated in Japan over 50 years ago, but it first gained attention in North America with the publication of The Teaching Gap (Stigler & Hiebert, 1999), and has since been documented and researched across various implementations in local settings (Fernandez & Yoshida, 2004, Fernandez, 2005, Lewis, 2006). In the traditions of "lesson study" and "community of inquiry" (Jaworski, 1998), the teachers at WCA worked together as a professional learning community (Wenger, 2007), engaging in considerable shared planning, observation, and discussion of lessons, also called "research lessons". Initially, for the first two years a pilot study was implemented with one group of teachers only. At that time, the data was not being collected and the purpose was only to introduce teachers to the methods and potential benefits of lesson study. In the following school year, the actual field study and data collection took place in a school-wide implementation whereby all 16 teachers who taught mathematics in the K-12 grades voluntarily participated in this “practice based professional development” process in a research setting.

METHODOLOGY

Typically, in each lesson study cycle, a team of teachers who taught the same or neighbouring grade levels collaboratively designed a mathematics lesson to challenge a difficult topic or to learn about an aspect of students’ ways of thinking. Next, the lesson was taught by one of the teachers in the team while the other team members observed and documented students’ learning; then, the teachers collectively reflected upon, discussed and analysed the lesson and its effect on students’ learning. Typically a cycle would span over three to four weeks.

One of the main features of lesson study as a professional development process is the articulation of an overarching goal stated as an inquiry into how to build a culture of mathematical thinking in the classroom. This goal was common for all teacher teams and it acted as a compass during the entire period of the study. It remained fixed for all lessons study cycles. Teachers were not expected to depart from the usual content of their instruction; therefore, the specific student learning goals of the lessons were still entirely curriculum based. Teachers would teach what they would normally teach and when they would normally teach it, according to their programs; however, they were encouraged to select for their research lessons mathematical contents that they considered students had difficulty understanding. The planning stage involved creating or selecting the mathematical tasks, anticipating student responses, working on and considering variations of the task to gain personal mathematical experience with the content they were planning to teach, and engaging in a deep analysis of the mathematical and pedagogical affordances embedded in the task (Liljedahl, Chernoff, & Zazkis, 2007). This report draws from a single lesson enactment in a class of 15 Grade 2 students, where the specific lesson goal was to apply and extend the students’ idea of fraction using a task called “A Pattern to Colour”.

The main source of the data is from the collaboration and discussions (audio taped and transcribed), the lesson implementation (videotaped and transcribed), and the artefacts that were created during the process (lesson plan, instructional materials, and student work), and a semi-structured interview with the teacher teaching the lesson. Sam (pseudonym) was then in her 12th year of teaching elementary students. She was part of a five-member early-elementary teacher team (Grade K-3), and this particular lesson took place close to the end of the school year, so in the more mature stage of the lesson study activity, when all the teachers in the team had already taken at least one turn in teaching a research lesson in their own classroom while being observed by the other members of the team. Since the lesson was co-created, the focus was not on the teacher teaching the lesson but rather on the lesson itself and its impact on the student thinking, learning, and understanding. There was a shared understanding that the effectiveness of the post-lesson discussion would depend on the quality of observation and data collected by each participant during the implementation of the lesson. In turn, the quality of observation and subsequent systematic reflection will be affected by what teachers notice and how they notice as the lesson unfolds.

THEORETICAL FRAMEWORK

Teacher noticing as a theoretical construct has been gaining attention in the recent educational literature on professional development of teachers. Mathematics educators have used the noticing construct to understand how teachers make sense of complex classroom situations where they attend to certain things while ignoring others from the backdrop of myriad of stimuli in their work environment depending on their current instructional purpose. Professional noticing is closely linked to teachers' in-the-moment decision making (Jacobs, Lamb, & Philipp, 2010), and so expanding the spectrum of what teachers notice can lead to greater instructional choices. Teachers must also integrate what they notice with their knowledge of how students learn and with their own knowledge of the content that they are teaching.

The most extensive body of literature on this topic comes from van Es and Sherin (2002) who defined *noticing* as having the following three components:

> (a) identifying what is important or noteworthy about a classroom situation; (b) making connections between the specifics of classroom interactions and the broader principles of teaching and learning they represent; and (c) using what one knows about the context to reason about classroom events.

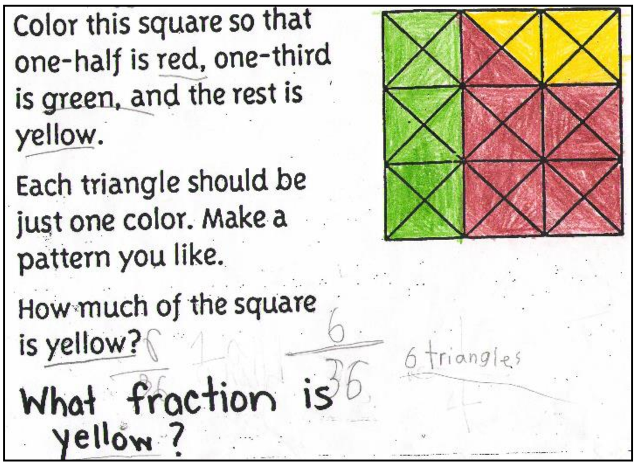
For the purpose of this report, I draw upon the specialized type of mathematics teacher noticing known in the literature as *professional noticing of children's mathematical thinking*, which is conceptualized as an expertise in three interrelated skills: (a) attending to children's strategies, (b) interpreting children's understandings, and (c) deciding how to respond on the basis of children's understanding (Jacobs et al., 2010).

> "… before the teachers respond, the three component skills of professional noticing of children's mathematical thinking – attending, interpreting, and deciding how to respond – happen in the background, almost simultaneously, as if constituting a single, integrated teaching move. Thus, our conceptualization of the construct on professional noticing of children's mathematical thinking makes explicit the three component skills but also identifies them as an integrated set that provides the foundation for teachers' responses"

These authors aimed to unpack the in-the-moment decision making in a cross-sectional study of prospective and practicing teachers who had different amounts of experience with children's mathematical thinking, in order to characterize what this expertise entails. They provided snapshots of varied levels of this expertise, and they concluded that this expertise can be learned. It is important to note that in their conceptualization of teacher noticing these researchers did not include the execution of the teacher response (because the study was based on teacher focus groups where the participants examined video clips of teaching interactions, not in the classroom) but rather focused on the intended response of the participants in their study. Other researchers also used videos of teaching situations to study professional noticing of teachers and its growth over time (Sherin and van Es, 2005; Hannah, 2012). Van Es (2011) developed a framework for learning to notice student mathematical thinking, where she articulated two central features of noticing related to *what teachers notice* and *how teachers notice*, and proposed a four level trajectory of development in these two dimensions (Baseline, Mixed, Focused, to Extended Noticing). While in all these studies teachers' noticing skills were reported to have improved over time, these studies do not report on the effects of these improvements on day to day practice. One of the studies (Star & Srtickland, 2008) on teacher noticing found that the teacher noticing expertise improved in some categories (noticing classroom features and the ways teachers manage their classroom) but not so much in others (for example noticing features of mathematical content, tasks, and communication), and the authors questioned why might teachers show improvement in their abilities to notice but still struggle to notice *important* classroom events.

DISCUSSION

In the authentic setting of her own classroom of 15 Grade 2 students, Sam set her students to work in small groups of two or three on the "A Pattern to Colour" task, while being observed by the other teachers from the team. Sam circulated the class, scaffolding student learning and encouraging students to discuss their strategies before beginning to colour the pattern. In the post lesson discussion, she made explicit the similarities and differences in children's strategies that she observed, displaying an extended level of noticing as defined in van Es' framework. Extended level of noticing in the dimension of *what teachers notice* is characterized by teacher's noticing of the particular students' strategies and then connecting these to her pedagogical moves. For example, Sam noticed a student who after colouring half of the large square proceeded



to find 1/3 of the rest. She pointed him in the right direction saying it had to be 1/3 of the entire square. This was also one of the anticipated student difficulties which teachers discussed during the planning stage of the lesson.

Figure 1: “A Pattern to Colour” problem

The idea of *how teachers notice* captures the analytic stance and the depth of analysis of what teachers notice. Sam went well beyond providing descriptive and evaluative comments by referring to specific events and interactions as evidence and she elaborated and made connections between events and principles of teaching and learning. Sam shared with the teachers in the post lesson discussion that her own approach would be to take the smallest unit, in this case the smallest isosceles right triangle, and determine that the entire diagram is made up of 36 equal units. Next by means of computation ½ of 36 = 18 of them would be shaded red, 1/3 of 36 =12 would be green, and the rest yellow. Since 36 – (18+12)=6 it means that 6/36 or 1/6 of the entire figure is to be coloured yellow. She considered this to be the most sensible and efficient solution strategy, but also recognized it as a trained response, and admitted that in the past she would have been compelled to teach her own solution process to her students right away. She no longer felt the need to impose her personal solution strategy on her students, but instead rather listened and appreciated the students’ ways of thinking about the problem, granting them the opportunity to experience the thrill of solving a difficult problem on their own. She was sensitive enough and attuned to the ways the students were approaching the task. She also recognized that dividing 36 by even such small numbers as 2 and 3 was outside of what students have experienced before and would add an unnecessary level of computational demand with which students were unfamiliar (up to this time they had only studied multiplication tables of 2, 3, and 5 up to 10, and many of them had not even grasped that yet).

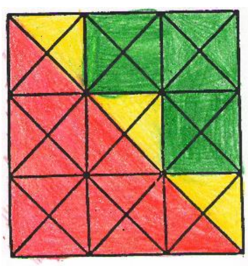
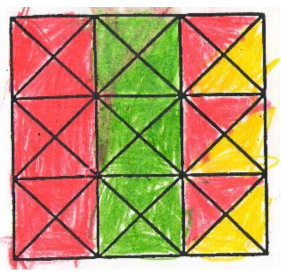
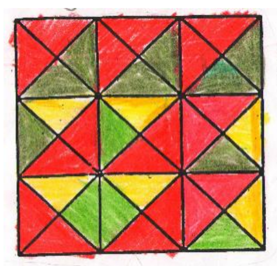
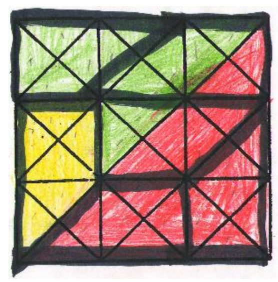
 (a)	 (b)	 (c)	 (d)

Table 1: Students’ work and Sam’s explanation

A number of student approaches emerged, and Sam was by this time skilled enough in her noticing to be able to reconstruct each group's thinking process from the observation of students' communication and work. She followed closely in terms of what units the students were paying attention to (large square, small square, half the small square, or the little triangle) and how they shifted their attention to different units as they proceeded in their solution path, and whether they were using the idea of ratio or the idea of the "part of something".

In relation to student work displayed in Figure 1, Sam reported this:

> Sam: When I asked and I said ok how did you figure that out? And it's just like she's going, "Well look, here's a 3rd. Here's another 3rd here's another 3rd. so this has got to be 3 squares. So I'm just gonna color 3 squares."

With the other four artefacts displayed in Table 1, she offered specific interpretations of the students' thinking strategies, for example:

> Sam: Student M chugs along. She looks and she says to herself, ok, 1/2 well here's 4, I'm gonna colour half of that. Here's another 4, I'm gonna color half of that. She's the one that figured that out. And notice, none of the other groups did that (commenting on student strategy c).

The most difficult part for the students was to figure out was "what fraction is yellow", which is something that Sam wanted to challenge her students with, and was not part of the original problem. She wanted the students to discover that they could form another, even less obvious unit, made up of one and a half small squares, or 6 small triangles and that in doing so, the entire square would then break up into 6 such units of equal size leading to a conclusion that 1/6 of the entire square is yellow.

> Sam: They knew this is a 3rd, 3 squares are a 3rd. So one and a half squares must be half of a 3rd. But this is interesting because it's the switching of the unit. What is a unit? Unit could be the triangles or it could be the squares.

Most students recorded 6/36 as the fraction of the square that is yellow, and one student wrote "half of a third". Sam did not pick up on the student reasoning behind this conclusion during the lesson, but had clarified it to herself afterwards, during the briefing with her colleagues. In the final stage of the lesson the students displayed their work and were invited to share their different thinking strategies, but the "half of a third" was not discussed. To conclude, Sam outlined the 6 congruent trapezoids in one of the groups' work with a thick black marker (figure (d) in Table 1) to display the idea that 6/36 can also be seen as 1/6 (and as "half of a third" as one student noticed); however, she did not explicitly mention that the parts needed to be of equal size whenever there is the idea of fraction.

Sam was both surprised and impressed with the variety of students' approaches. She had come to see her students as much more capable than she thought before. She also shared her thoughts about the changes she had made in her instruction on fractions. At the Grade 2 level, the instruction on fractions begins with recognizing simple fractions. In prior years, Sam would explain the fraction notation, show some examples, and then proceed with having students fill out many worksheets where they would be asked to respond to two kinds of questions, one to shade the given fraction of a figure such as a square or a circle, and the other to write down what fraction of the figure is shaded. Instead, she now focuses on developing the meaning of fraction, asking students to provide their own examples, and broadening the interpretations of fraction in her instruction (part of a set, part of a figure, fraction as a ratio).

> My first lesson was, seriously, I thought it was stupid. You know how the book says "recognise 1/2", well you know what? This is what a fraction is. Like, I wrote it on the board and I said this is one half. So I threw it back at them. I said, "This is one half. What do you think each of these numbers mean?" So it's like I said ok, if I ask you to share a cookie and each person gets 1/2, and they go, "we break it in half, I get this half, and the other person gets this half." And I said, "Well this is how you write down 1/2. What do you think each of these numbers mean, in 1/2?" so I wanted them to think about what each of the numbers meant.

Sam’s instructional and problem posing practices changed dramatically in comparison to her earlier ones. Instead of offering traditional single step and computational problems that involve a single answer, she ventured into posing problems that were cognitively more complex, had multiple approaches and solutions, and were open-ended and exploratory. Early on in the study, she felt compelled to adapt mathematical tasks in ways that made students' work easier by leading them through a series of steps, essentially breaking down the problem into small, easily executable computational exercises which narrowed the mathematical scope of the problem. She was also concerned about being observed by peers, and wanted to ensure that students succeeded. She experienced great tension between trying out tasks that would engage students in meaningful, authentic mathematical thinking and reasoning and risking student failure. This prompted her to start experimenting with her teaching practice and expecting more autonomy from the students.

As she continually reflected upon, “What are we trying to achieve here?” in the context of the professional development setting, and continued to ask herself, “Who is doing the thinking?”, her practices have gradually shifted. She has become less leading and less focused on avoiding students' errors. Posing problems to an authentic audience, engaging in collaborative professional development practice centred on developing student mathematical thinking and reasoning, and having opportunities to explore new kinds of problems are highlighted as important factors in promoting and supporting the reported changes.

CONCLUSION

The overarching goal of the professional development was for teachers to learn how children think about and develop understandings in particular mathematical domains and how teachers can elicit and respond to children's ideas in ways that support these understandings. This report documents how one teacher developed the expertise of attending to the subtle details in individual student strategies – details that reflected mathematically relevant differences in the understandings children bring to their problem solving. The teacher kept returning to this overarching goal when reflecting upon her practice and the practices of other teachers in the team.

Sam's professional development trajectory can be viewed as passing through three levels. It begun with a specific but distant goal - the technical concern of how to create a culture of mathematical thinking in her classroom. How to attain this specific practical goal was the dominant concern, which then moved inwards towards sensitizing herself (with the aid of colleagues) to notice situations in which alternative actions are possible, until finally she was able to change practices by choosing to act differently (Mason, 2002).

References

Chazan, D., Ben-Chaim, D., & Gormas, J. (1998). Shared teaching assignments in the service of mathematics reform: Situated professional development. *Teaching and Teacher Education,* 14(7), 687-702.

Fernandez, C. (2005). Lesson study: A means for elementary teachers to develop the knowledge of mathematics needed for reform-minded teaching? *Mathematical Thinking & Learning: An International Journal,* 7(4), 265-289.

Fernandez, C., & Yoshida, M. (2004). *Lesson Study: A Japanese Approach to Improving Mathematics Teaching and Learning.*

Hannah, W.G., (2012). Examining what in-service elementary teachers notice when presented with instances of students' mathematical thinking. In Van Zoest, L. R., Lo, J.J., & Kratky, J.L. (Eds.). *Proceedings of the 34th annual meeting of the North Amrican Chapter of the Internatioanl Group for the Psychology of Mathematics Education*. Kakamazoo, MI: Western Michigan University.

Jacobs, V. R., Lamb, L. L., & Philipp, R. A. (2010). Professional noticing of children's mathematical thinking. Journal for Research in Mathematics Education, 169-202.

Jaworski, B. (1998). Mathematics teacher research: Process, practice and the development of teaching. *Journal of Mathematics Teacher Education*, 1(1), 3-31.

Lewis, C. (2006). Lesson Study in North America: Progress and Challenges. In M. Matoba, K. A. Crawford & M. R. Sarkar Arani (Eds.) *Lesson study: International Perspective on Policy and Practice*. Beijing : Educational Science Publishing House.

Liljedahl, P., Chernoff, E., & Zazkis, R. (2007). Interweaving mathematics and pedagogy in Task Design: A tale of one task. *Journal for Mathematics Teacher Education – Special Issue on the Role and Nature of Mathematics-Related Tasks for Teacher Education*. 10(4–6), 239–249.

Mason, J. (2002). *Researching your own practice: The discipline of noticing*. New York: Routledge.

Sherin, M., & van Es, E. (2005). Using video to support teachers' ability to notice classroom interactions. *Journal of technology and teacher education*, *13*(3), 475-491.

Star, J. R., & Strickland, S. K. (2008). Learning to observe: Using video to improve preservice mathematics teachers' ability to notice. *Journal of Mathematics Teacher Education*, *11*(2), 107-125.

Stigler, J., & Hiebert, J. (1999). *The teaching gap: Best ideas from the world's teachers for improving education in the classroom.* New York: The Free Press.

van Es, E. A., & Sherin, M. G. (2002). Learning to notice: Scaffolding new teachers' interpretations of classroom interactions. *Journal of Technology and Teacher Education*, *10*(4), 571-596.

van Es, E.A. (2011). A framework for learning to notice student thinking. In M.G. Sherin, V.R. Jacobs, R.A. Philipp (Eds.) *Mathematics Teacher Noticing: Seeing Through Teachers' Eyes*. New York: Routledge.

Wenger, E. (2007). Communities of Practice: Learning, Meaning and Identity. Cambridge: University Press

SHEARING AND IPADS: EXPLORING GEOMETRY WITH DYNAMIC GEOMETRY AND TOUCHSCREEN TECHNOLOGY

Oi-Lam Ng

Simon Fraser University

In this paper, we report on two lessons for teaching junior highschool students to the idea of shearing in a dynamic geometry environment. Through a classroom-based intervention involving the active use of a class set of touchscreen tablet devices, we analyse students' evolving discourse about area. The touchscreen tablet technology seemed to have supported the ways in which students talk about shearing as a temporal and continuous process and made the idea more tangible. We highlight the specific roles of the teacher and digital technology in supporting the process of semiotic mediation through which the students learned about shearing.

INTRODUCTION

Geometry has traditionally been taught as a study of static theorems and properties about shapes and space. In their book, Sinclair, Pimm, and Skelin (2012) argued that this calculation and formula-driven approach to teaching geometry is the very root of students' struggle to grasp the "big ideas" in the learning of high school geometry, especially in terms of the measurement strand. In our experience, teachers spend little time on exploring or proving the geometric relations and invariances that support measurement formulas, and instead focus on the numerical or algebraic manipulations involves in applying these formulas. There are two main geometric approaches to working with area: decomposition/rearrangement and shearing. While the former is often used to introduce area at the elementary school level, the method of shearing—which Euclid used to prove the Pythagorean Theorem—is more rarely used (Fischman and McMurran, 2011). Shearing is a *continuous* and *temporal* geometrical transformation that preserves area and can be extended in the case of the Cavalieri's principle (Pimm and Proulx, 2008). It is difficult to exemplify through the static medium of paper, but perfectly suited for dynamic geometry environments, which produce "a seemingly limitless series of continuously-related examples, and in so doing, to represent visually the entire phase-space or configuration potential of an underlying mathematical construction" (Jackiw and Sinclair, 2009, p. 414).

The purpose of the current study is to explore the potential for dynamic geometry environments (DGEs) to mediate the learning of shearing in the junior highschool level. Through a teaching experiment involving the design of two lessons incorporating pair-work activities on students' iPads, I investigated the evolving discourse of students in terms of the way they communicated about area and shearing, and how this communication was supported by a DGE and touchscreen technology.

THEORETICAL PERSPECTIVES

As mentioned, shearing is a continuous and temporal – dynamic – geometrical transformation. For this reason, the use of DGEs evokes the idea of shearing more readily than static diagrams. In this section, we briefly describe the theory of *semiotic mediation* as a theoretical framework to explain the complex process through which a tool (such as DGE) produces signs through which meaning making occurs.

Vygotsky (1978) distinguishes the dialectical relationship between signs and tools as follows: *practical tools* are those used in the labour, whereas signs are *symbolic tools* used in the psychological operation. Externally-oriented tools may be transformed into internally-oriented ones through the process of *internalisation*. Internalisation is directed by semiotic processes and rests on a system of signs involved in the social activity, i.e. signs such as words, drawings and gestures (Wertsch and Addison Stone, 1985). These signs generated by the use of a tool, through the complex process of internalization accomplished after social interchange, may shape new meanings. Therefore, semiotic mediation is a process of meaning making through internalising the signs that are produced from an external, intrapersonal activity.

An external, goal-oriented activity such as “dragging” and “tracing” in a dynamic sketch can be internalised to shape personal meanings. The teacher’s role is to exploit such opportunities by facilitating a meaningful social exchange during the use of the corresponding tools. This perspective is shared by Falcade, Laborde, and Mariotti (2007) in their teaching experiment with high school students on functions. They suggest that the internalisation of the Dragging and Trace tools may contribute to introducing function as covariance and the notions of domain and range. They argued that the role of the teacher is crucial in this process, as she promotes of different semiotic activities related to the use of the Dragging and Trace tools, and later facilitates a class discussion in order to guide students to mediate mathematical meaning upon the activities. Moreover, signs can be interpreted at different levels, from artefact to mathematical. This means that in a classroom community or discussion, certain words, gestures and uses of visual mediators can be interpreted as instrumental to the use of the artefact or mathematical. In an educational context, the goal of the teacher is to orchestrate a transformation from artefact signs to mathematical signs in a path that students can follow. This can be accomplished when the teacher tunes with the students' semiotic resources and uses them to guide the evolution of mathematical meanings.

Within this theoretical perspective, we designed two lessons incorporating dynamic sketches with particular tools that were intended to serve as instruments for semiotic mediation in the learning of shearing and comparing area geometrically (as opposed to, for example, numerically or algebraically). In the sections below, we describe the participants involved in the study, followed by the design of the lessons and sketches that were used in the lessons.

THE TEACHING EXPERIMENT

The current teaching experiment involved two consecutive lessons taught in a secondary school in Western Canada. Each lesson lasted approximately 75 minutes and included some iPad-based activities followed by paper-and-pencil activities. Students worked in pairs and were asked to explore, through the guidance of the teacher, the dynamic sketches that were presented in the multitouch application *SketchExplorer* (Jackiw, 2011). Then, during the last 20 minutes of each lesson, all students were asked to reflect on their learning by opening the last page of the sketch to answer some assessment questions on paper. The students were encouraged to discuss in pairs, but they were asked to do the write-ups individually. One of the co-authors, who was also a teacher at the school, took on the role of a guest-teacher in an 8th grade mathematics classroom, while the other co-author observed the lessons as a visitor of the school. The regular classroom teacher was present in each lesson and helped manage various aspects of the lesson. He did not undertake any work on shearing with the students, nor did he use iPad-based activities outside of the intervention.

Design of Lessons and Sketches

The main goal of this first lesson is to introduce shearing. We chose to begin with the shearing of quadrilaterals. The teacher began by asking students to write down what “they knew about area” and to look at an introductory problem, the “problem of Eda and Azusa” (Stigler et al., 1999), in which a new border has to be drawn between Eda and Azusa’s land in such a way as to not change the amount of land each has (Figure 1a). The students were invited to think about possible ways of doing this. Then, the teacher opened a sketch with a page initially showing four quadrilaterals (Figure 1b). To the left of the page was a rectangle coloured yellow, and to its right were three parallelograms with different slants and colours. The teacher asked the students to explore this page in pairs, asking them to *compare* the base, height and area of the shapes and to explain *how* they made these comparisons. The students showed that the base and height of all the shapes were the same by dragging one parallelogram on top of another, one at a time. Furthermore, some students explained how each parallelogram could be “cut off” at one end and be “put” on the other in order to form a rectangle whose area is equivalent to the yellow one. At this point, the teacher offered the word "dissecting" as a name describing this process of comparing shapes.

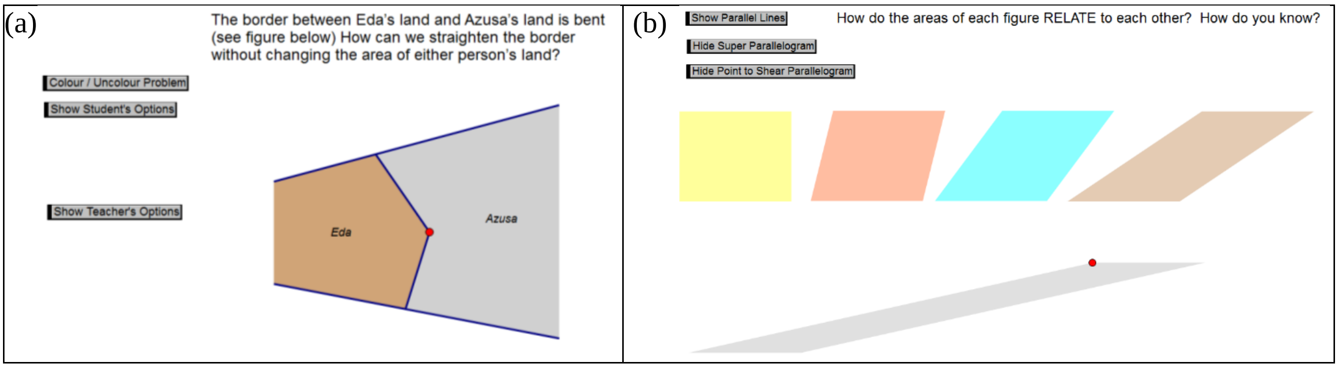


Figure 1(a). The problem of Eda and Asuza. (b) Comparing areas of parallelograms.

The teacher then pressed the "Show Super Parallelogram" button on the same page and explained that the newly shown, grey "Super Parallelogram", which has a draggable vertex, can be transformed into any configurations of parallelogram with the same base and height. In fact, the draggable vertex is constructed to be on the line parallel to the base of the parallelogram so it can move along that line only. She dragged the vertex horizontally back and forth, arrived at a very "skinny" parallelogram (see Figure 2b), and asked if the method of dissecting was feasible for comparing the area of the grey parallelogram with that of the rectangle. Upon interacting with the sketch, some students proposed that the two areas were the same, but they were not able to explain how they knew that nor to justify by the method of dissecting.

The teacher then turned to the next page in the sketch, which showed the same two shapes as on the previous page, the yellow rectangle and the grey parallelogram. The students were asked to press the "Show Triangles" button (Figure 3a) and to drag the two overlapping right triangles in order to make inferences about whether the two original shapes had the same area. The teacher emphasised that simply saying that they were or were not the same was not "enough"; she stated that in geometry, they had to *show* their *reasoning*. After a good ten minutes of interacting with the sketch, a student volunteered to explain what he did on his iPad. He moved the red triangles and the yellow rectangle directly below the composite figure to show that the areas of the two composite figures were the same (see Figure 2b). The teacher pressed the "Show Point to Shear Parallelogram" button on her sketch and dragged the movable vertex horizontally to "resize" the parallelogram. She explained that the area of the grey parallelogram continues to be the same as the area of the yellow rectangle. While she was dragging the vertex of the parallelogram, she said, "I'm shearing the parallelogram".

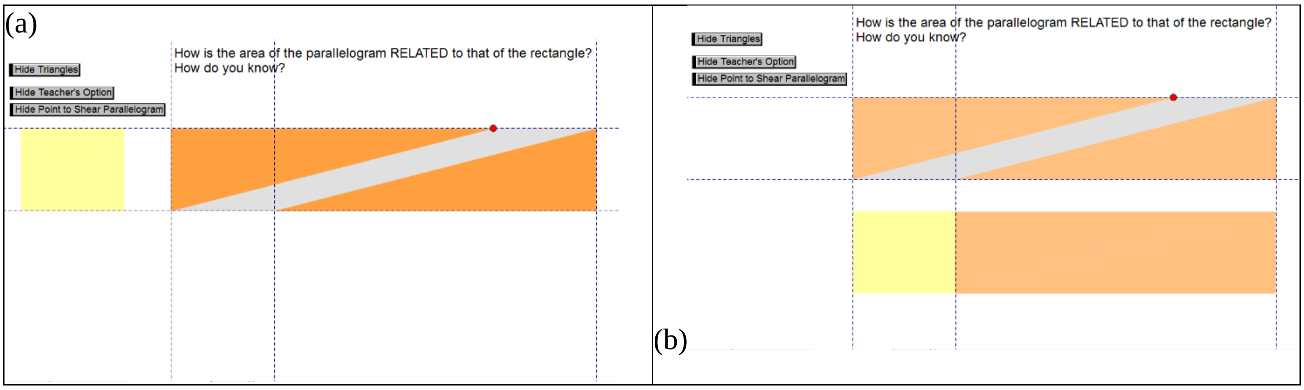


Figure 2(a) and (b). A page that was used to compare the area of the yellow rectangle and the grey parallelogram.

The assessment task at the end of this lesson consisted of four questions. Of particular interest is Question 4, which involved comparing the areas of four shapes and relating the areas using symbols "+" and "=". Since the question was presented on the iPads, all of the shapes, except for Shape C, which was a rectangle, had a draggable vertex and therefore could be sheared through onscreen dragging (see Figure 3).

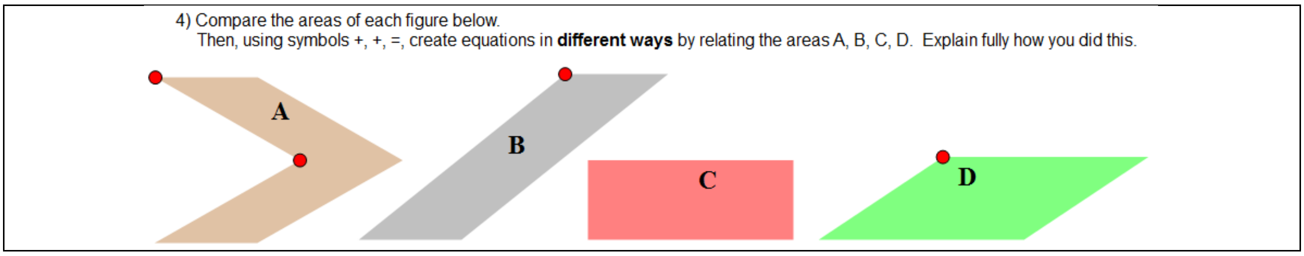


Figure 3. An assessment question at the end of Lesson 1.

Building on the previous day's work, Lesson 2 was focused on shearing triangles. Two sketches similar to those in Lesson 1 were used to introduce both dissection and shearing of triangles. The teacher then introduced four problems to the students, one of which was the "Eda and Azusa" problem from Lesson 1 to try solving on their iPads.

RESULTS OF THE TEACHING EXPERIMENT

The students were asked to write down what they knew about "area" in the beginning of Lesson 1. They were also asked to try to solve the problem of "Eda and Asuza" as an introductory problem about area. Our data show that students' initial discourse was dominated by numerical and algebraic approaches to area. This can be observed through their talk of area as a quantity upon calculations and applying some formulae, as exemplified in the typical response shown in Figure 4a. In this response, the student reflected on area as a "measurement" that is assigned a certain "unit" (cm^2, m^2, etc.) and is associated with some formulae. Note that the student had said nothing about what exactly is being measured. As predicted in the literature, we observe that only 6 out of a class of 26 students communicated the very meaning of area as the space enclosed by a 2-dimensional figure, while all students have mentioned in this part of their assessments at least one formula for computing area, such as $A = l*w$, $A = b*h$, $A = \pi r^2$ etc.

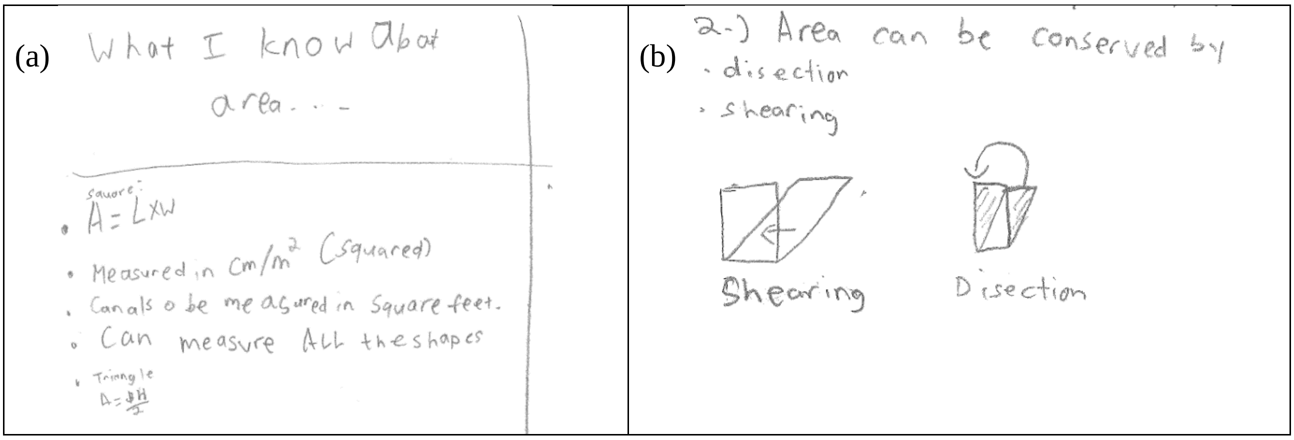


Figure 4. A student reflecting on area at the start (a) and at the end (b) of Lesson 1.

We observe an evolution in students' thinking about area at the end of Lesson 1. For example, all students communicated the idea that area can be compared, "without using numbers," by the method of dissecting and/or shearing. Furthermore, they were also able to explain what dissecting and shearing meant. In Figure 4b, the student mentioned two important concepts about area: 1) area can be conserved, and 2) shearing and dissecting conserve area. This student also justified that shearing and dissecting both conserve area with two diagrams, each with an arrow conveying the temporal transformation of one shape into another. The curved arrow conveys dissecting and moving a shape to the other side, whereas the straight arrow conveys the continuous process of shearing. Both Figures 5a and 5b were done by the same student, and the differences between the two ways of thinking about area are quite striking.

At the end of Lesson 1, the students were also asked to describe shearing, which was a new word for them. As shown in the table below, the students talked about shearing as a temporal and continuous process (see Lines 3-6) and as a method for conserving area (see Lines 7-10). Recall that the teacher never defined shearing in the lesson; instead, she introduced the idea of shearing in the act of dragging a vertex of a parallelogram. In light of students not explicitly given a mathematical definition, our finding is interesting because most students were able to describe at least two important characteristics of shearing: a *temporal* and *continuous process* that *conserves area*.

On the question, *"I think shearing means…"* [Lesson 1]			
Shearing as a temporal and continuous process		Shearing as conserving area	
3	… moving the parallelogram into the square or rectangle without dissecting.	7	… to change the look of a shape but keeping the same area.

4	… moving the shape so that the base and height match up to the other.	8	… being able to change the shape while keeping the area the same.
5	… to re-shape a shape.	9	… to change the look of any shape, but not change its area.
6	… stretching the shape.	10	… changing the shape of a parallelogram without changing its area

Lastly, there is a change in the way the students talked about the problem of Eda and Asuza between the beginning of Lesson 1 and the end of Lesson 2. When attempting to solve the problem in Lesson 1, the students found they could only partially solve the problem by drawing a straight line border so that both Eda and Asuza would "supposedly" not gain or lose any land (Figure 5a). This solution involved estimation, and did not compare the base and height of the portions of land being added or lost. In contrast, the students communicated the solution of the problem in a dynamic sense at the end of Lesson 2. Although not all students were able to provide a full solution, most students did use shearing.

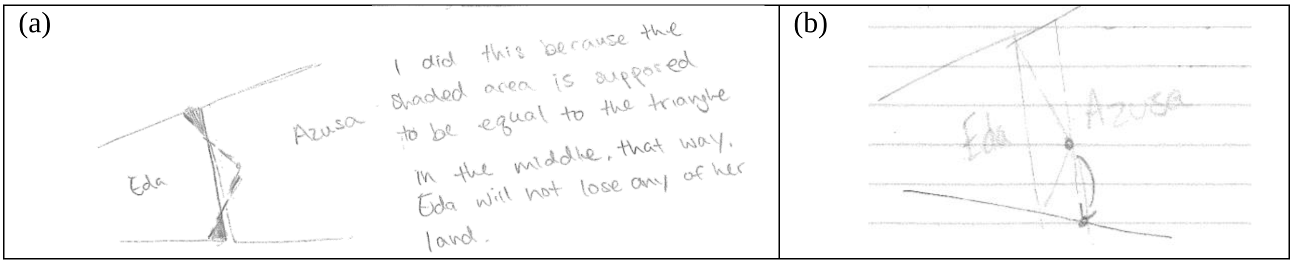


Figure 5. Solutions to the problem of Eda and Asuza in Lesson 1 (a) and in Lesson 2 (b).

As seen in Figure 5b, one student solved the problem at the end of Lesson 2 by first constructing a line in order to form a triangle and then another line parallel to the base of the triangle, and then shearing the vertex of the triangle along the parallel line until it reached the edge of the land.

DISCUSSION

The use of technology was instrumental for developing students' working definition of shearing and assisting with students' problem solving. Some students commented on the enhanced ability to drag or move objects around on the iPads to visualise the process of shearing more effectively, while others explained that the iPads made it easier for them to communicate their learning. Both of these aspects were captured in Figure 6, where a student wrote that he "*sheared* the shapes and *found* that they have the same area" and drew diagrams below his written explanation in his solution of Question 4 from Lesson 1. The use of past tense in "sheared" and "found" suggests that the student had been interacting with an iPad first before formally writing his solution down. Also, some of the vertices in his diagram were bolded exactly like they were in the sketch, further suggesting that he was referring to the shapes as seen on his iPad. Hence, it is very likely that the drawings reflected what the student saw and acted upon before and after the shapes were sheared on the iPad. In addition to students' consistent use of past tense in problems solving, we found evidence that students were making sense of shearing using the colour-coding of points as designed in the sketches. In solving the problems posed in Lesson 2, several students labelled vertices as "blue" points or "red" points in their diagrams. This shows that the way the student communicated about shearing was highly influenced by the design of the sketch.

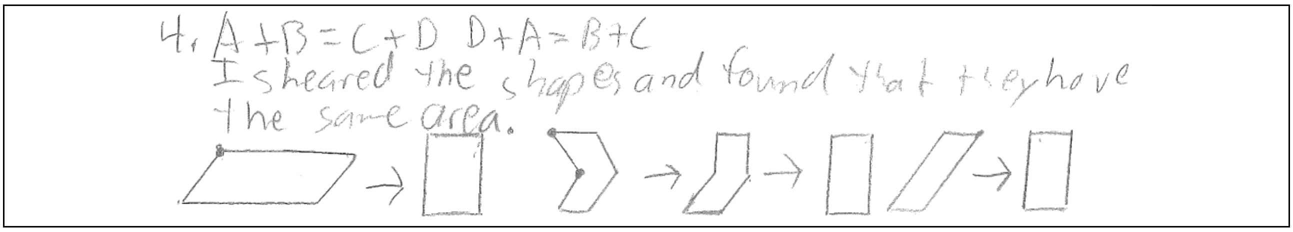


Figure 6. A student's written explanation and diagrams for Question 4 of Lesson 1.

In the lens of semiotic mediation, the colour-coding of points not only served as signs that enabled learners to make reference to the points more easily but also mediate the meaning of the respective coloured points as independent (initial vertices of a shape) and dependent (vertices after the shape is sheared) objects in the sketch. Furthermore, the dragging tool was exploited in two pages of the sketch for introducing "dissecting" and "shearing". This was exploited by dragging one shape to overlap with another to convey dissecting as well as dragging one vertex along the line parallel to the base to convey shearing. The fact that the students seemed to have drawn what looked like screenshots suggests that they have made use of the signs produced by the DGE to internalise their thinking.

In conclusion, findings from this teaching experiment were especially encouraging in terms of the students' ability to describe shearing based on their interactions with the sketches and in relation to their enriched conceptions of area. Despite having had no prior experience working with DGEs, the students easily grasped the function of the Dragging Tool for exploring the sketches for learning dissecting and shearing, which helped them communicate mathematically during the paper-and-pencil part of the lessons. In particular, students communicated effectively through words and diagrams using paper-and-pencil, even though they had been solely working with iPads during the exploratory and problem solving activities. They seemed to have interacted with the iPads in a profound way, which allowed them to communicate their ideas later by drawing what looked like screenshots of the iPads. This finding points to the possibilities for DGEs, presented on multitouch devices, to explore dynamic geometrical relationships and the compatibility of also using paper-and-pencil tasks as assessments of learning at the end of the interactions with DGEs.

Indeed, as a result of the teaching experiment, several of the students described shearing as a way of doing “area without number”. Because of the design of the teaching experiment, we were not able to see how students might connect the notion of shearing with measurement formulas, and this process of combining the geometric with the algebraic could form the basis for fruitful future research.

References

Jackiw, N. (2011). *SketchExplorer* [Computer Program]. Key Curriculum Press.

Jackiw, N. and Sinclair, N. (2009). Sounds and pictures: dynamism and dualism in dynamic geometry. *ZDM*, 41, 413-426.

Fischman, D., & McMurran, S. (2011). Shearing with Euclid. In D. Jardine, & A. Shell-Gellasch (Eds.), *Mathematical Time Capsules: Historical Modules for the Mathematics Classroom* (pp. 45-54). Washington, D. C.: Mathematical Association of America

Falcade, R., Laborde, C., & Mariotti, M. (2007). Approaching functions: Cabri tools as instruments of semiotic mediation. *Educational Studies in Mathematics, 66,* 317-333.

Pimm, D. & Proulx, J. (2008). Algebraic Formulas, Geometric Awareness and Cavalieri's Principle. *For the Learning of Mathematics, 28,* 2, 17-24.

Sinclair, N., Pimm, D., & Skelin, M. (2012). Developing Essential Understanding of Geometry for Teaching Mathematics in Grades 6–8. Essential Understanding Series. Reston, VA: NCTM.

Stigler, J. W., Gonzales, P. A., Kawanka, T., Knoll, S., & Serrano, A. (1999). The TIMSS Videotape Classroom Study (NCES 1999074). Washington, D.C.: National Center for Educational Statistics.

Vygotsky, L. S. (1978). *Mind in Society*. Cambridge, MA: Harvard University Press.

Wertsch, J. V., & Stone, C. A. (1985). The concept of internalization in Vygotsky's account of the genesis of higher mental functions. In J. V. Wertsch (Ed.), *Culture, communication and cognition: Vygotskian perspectives.* Cambridge: Cambridge University Press.

SOLVING RUDIMENTARY AND COMPLEX MATHEMATICAL TASKS

Minnie Liu

Simon Fraser University

Tasks are used in mathematics education for a variety of purposes – from delivering course material to developing students' mathematical thinking skills. In this article, I present research on a type of task specifically designed to foster students' ability to flexibly apply their existing mathematical knowledge and skills in problem solving situations. In particular, I look at students' problem solving processes when working collaboratively on such tasks. Results indicate that while the processes of solving these tasks are similar to those of modeling tasks, differences also exist.

Mathematics educators use a variety of tasks for a number of goals. These goals can range from delivering new curriculum to promoting students' mathematical thinking skills (Kaiser & Sriraman, 2006). One class of tasks used for these purposes are modeling tasks.

MODELING TASKS

Modeling tasks can be loosely defined as "authentic, complex and open problems which relate to reality" (Maaβ, 2006, p.115). These tasks demand students to do a substantial amount of mathematical modeling, or to make transitions between reality and the world of mathematics (Blum & Borromeo Ferri, 2009).

Researchers describe the processes of solving modeling tasks in terms of modeling cycles. While differences exist between researchers' descriptions of modeling cycles, a general structure exists. Modeling cycles begin with a real situation in reality. Students then create a real world model to describe the situation based on their understanding and the data provided. Afterwards, students mathematize the real world model to create a mathematical model, which is used to generate mathematical results. Finally, students validate these mathematical results by comparing them to data given in the real situation (Borromeo Ferri, 2006).

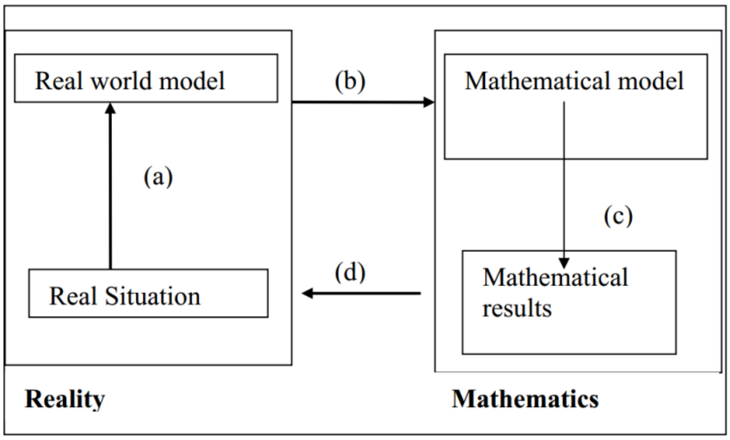


Figure 1: Modeling Cycle proposed by Kaiser (1995) and Blum (1996) (Borromeo Ferri, 2006, p.88)

Modeling tasks can be classified under the major goals for their use: (1) to deliver curriculum, (2) to become responsible and engaging citizens, (3) to develop students' critical thinking skills, and (4) to promote students' ability to use their mathematical knowledge to problem solve (Kaiser & Sriraman, 2006). The last two of these purposes can be achieved through the use of a specific subset of modeling tasks that I have come to call 'rudimentary and complex mathematical tasks' (RCMTs). These tasks are specifically designed to foster students' ability to flexibly apply their existing mathematical knowledge to problem solve.

RUDIMENTARY AND COMPLEX MATHEMATICAL TASKS (RCMTs)

Rudimentary and Complex Mathematical Tasks (RCMTs) are tasks that require the sophisticated use of rudimentary mathematics to solve complex problems. Like modeling tasks, RCMTs can be loosely defined as messy open-ended novel problem solving questions situated in reality, where the contexts of these tasks are engaging and relevant to students' lives. Students need to make sense of the situation to not only generate possible solutions, but also to present these solutions in a mathematical way. Students may also need to draw upon their non-mathematical knowledge and experiences in order to problem solve. Unlike the more common types of modeling tasks, however, RCMTs

> …require a lot of mathematical thinking without relying on a lot of mathematical knowledge. … These tasks really do allow for a great deal of mathematical activity and discussion without relying heavily on specific pre-requisite knowledge (Liljedahl, 2010).

In general, RCMTs aim to promote students' ability to flexibly use their EXISTING mathematical knowledge to problem solve, to engage students in thinking critically about real life situations in mathematical ways, to improve their communication skills by giving students the opportunity to work in teams, and to engage students in problem solving situations (Liljedahl, 2010). This study aims to investigate students' problem solving processes when working collaboratively on RCMTs using modeling cycles as a framework.

RCMTS AND MODELING TASKS

As previously discussed, RCMTs can be classified as a subset of modeling tasks due to the common goals they share, and modeling tasks can be described using modeling cycles. As such, modeling cycles are used as a framework for analysis in this study to describe students' behaviours during RCMTs. However, while RCMTs can be described as a subset of modeling tasks, RCMTs also carry some unique characteristics. For example, RCMTs focus on the flexible use of rudimentary mathematics to solve problems. On the other hand, some modeling tasks may require modelers with a strong mathematics background to use specific and academic mathematical tools to create predictive models.

This leads to another subtle difference between the function of solutions of these tasks. A general function of models is to make predictions about situations. Therefore, the solutions to modeling tasks (the models) should also be predictive in nature (Howison, 2005). On the other hand, the primary function of RCMT solutions is to provide solutions to the situation, which are not necessarily predictive in nature. The solution of a RCMT can simply be an answer to the problem.

While students with different RCMT experiences and ability levels may exhibit different behaviours during the RCMT process, a general RCMT process should exist. In order to paint a detailed picture of the RCMT process, I first investigate and describe the steps a particular group of students take to solve a RCMT, and compare these processes to Kaiser's modeling cycle pictured in figure one (Borromeo Ferri, 2006).

PARTICIPANTS AND METHODS

Although the larger study looks at the RCMTs across a wide range of grade and ability levels, for the research presented here the participants are grade 9 (age 13-14) mathematics students (n = 29) enrolled in a high school in a middle class neighborhood in western Canada. At the time of this study, these students have just over a year of RCMT experiences.

Students were assigned RCMTs to be completed in groups of two to three. All students were asked to pay attention to their thoughts and approaches used as they worked on the problem in order to help them accurately describe their RCMT processes. Data includes in class observations, field notes, and class discussions and impromptu interviews that focus on their actions taken to solve the task. The class discussions and impromptu interviews were transcribed immediately as these conversations happened.

The RCMT task chosen for this pilot study is "Trader Joe's vs. Private Joe's". The task is based on a current event in Vancouver, B.C., where a Canadian was sued for purchasing groceries from a U.S. chain grocery store at retail price and reselling these products in Canada (see http://www.theglobeandmail.com/news/national/trader-joes-lawsuit-against-bcs-pirate-joes-dismissed-by-us-judge/article14712590/ for more details). Students were asked to determine the amount of markup required for the Canadian to break even. I chose this task because it is relevant to students' lives, as many of their families shop in the U.S.. Students were given newspaper excerpts that describe the situation. The following is a summary of the task:

> Trader Joe's is a popular grocery store in the United States that is known for its good quality and low prices. The closest Trader Joe's from Vancouver is in Bellingham, W.A.. A Canadian believes that many Vancouverites are not willing to go shopping at Trader Joe's in the U.S., but there is a market for their products in Vancouver. Therefore, he purchases Trader Joe's products in the U.S. at retail price, imports the products back to Vancouver, and opens a store in Vancouver calls Private Joe's and sells Trader Joe's products. How much markup is needed for this Canadian to break even?

The mathematics required for this task are rudimentary. On the other hand, the planning for this task is complicated. Students need to estimate the cost required to run a company, including the cost of products, transportation for products, import taxes, rent, salary, etc.

Given the close association between RCMTs and modeling tasks, I will present the results of the analysis as it fits into Kaiser's modeling cycle (Borromeo Ferri, 2006) described above and exemplified with excerpts from the data.

RESULTS

Results indicate that students' RCMT process in this study parallels Kaiser's modeling cycle in general (Borromeo Ferri, 2006). Students' immediate reaction to the task was to create a real model that leads them to the solution. However, as soon as they began, they realized they haven't fully understood the question yet and went back to re-read the information provided for them.

As students reread the newspaper excerpts, they defined various terms, such as markup and break even. As they gained a better understanding of the context, they identified important information given in the newspaper excerpts and reflected on variables that might play a role in the solution. This is similar to the first step of the modeling cycle: understand the problem.

After collecting sufficient information to get started on the problem, students created a real model to represent the situation by organizing the information into categories and supplementary factors contributing to these categories. For example, under the category fuel cost, students considered fuel price, fuel efficiency, driving distance, type of vehicles, weight of driver and products, etc. as its supplementary factors. At this point, students also tried to simplify the problem by eliminating variables. Some took advantage of the ambiguities in the question and attempted to avoid or minimize cost.

S1: So do we need to consider rent?

S2: What if he owns?

Although these are ways for students to eliminate variables, none of the groups actually eliminated these variables at the end, because they were not able to support these approaches and none of their group members approved of eliminating these variables. I believe these "attempts" to eliminate variables served as ways for students to release their frustration and ways for students to become unstuck. However, further interviews are needed to clarify the reasons behind these statements students made.

As students considered the categories and their supplementary factors, they also began to represent these factors using mathematical language and calculated the cost involved in the category. This led them to the next stage in the modeling cycle – mathematization. At the beginning of mathematization, students seemed to be overwhelmed by the number of factors involved in the calculations. Therefore, they focused on specific factors and left out other ones.

S3: Are they all part-time? Or full-time?

S4: Full time. Assume three people per shift per week, so then three people times minimum wage, then…

S5: What about the cash register, and the stock…

S4: Aaargh! Leave that out for now!

As students carried on with the mathematization process, they become more comfortable with additional factors and eventually mathematized all relevant categories and its supplementary factors in their solutions.

Finally, students brought all the relevant results together and determined the final markup to report their solutions. At this final stage, students listed only the categories and excluded the factors and calculations used to determine the cost of the categories. Students presented a clean solution to the problem which does not provide a clear picture of their thinking process.

A DISCUSSION ON STUDENTS' RCMT PROCESS

In general, the RCMT processes parallel Kaiser's modeling cycle. However, while the modeling cycle is useful in providing us with a general structure of the RCMT process, it does not account for all that happened during the task. As such, further descriptions are needed to explain and exemplify the processes that are specific to RCMTs. The following highlights the differences between the two processes.

Validation of Ideas

Due to the ambiguous nature of RCMTs, students raised questions and made assumptions during the RCMT process. For example, as students attempted to represent the situation using a real model, they realized the many ambiguities in the task and raised questions regarding the situation, such as the location of the closest Trader Joe's from Vancouver, the type of vehicle the owner drives, the cost for rent and salary, etc. They then quickly realized not all the information they wanted was provided in the question.

S6: There is not enough info here…

As students became stuck with insufficient information, they attempted to find out more about the situation. However, students also know and are used to that the teacher never gives them direct answers during RCMTs. As a result, most of them used their smart phones to look up what they believed to be relevant information online, such as fuel price, fuel efficiency, border hours, etc.

S7: Is the border 24-7?

R: I dunno.

S7: I don't think so. (after using his smart phone to look up information) It's like 6 to 2 in the morning. So not 24-7…

As students looked up relevant information, they also made assumptions and clarified ambiguities based on their interpretations of the situation.

S8: I don't understand gas… So assuming he's paying for gas in Canada, and he has to drive back, I don't know… (looked up and with a loud voice) What kind of car does he have?

R: I dunno.

S8: I'm going to assume he has a truck.

The classroom was very noisy at this stage. Students constantly looked up information, raised questions, explained and clarified their ideas and approaches, provided reasons for their assumptions, and drew upon experiences outside of mathematics class to support their ideas. I consider this an active evaluation and validation process, where students actively argued for and against various ideas based on their experiences and their understanding and interpretation of the problem. These episodes of active evaluation and validation are different from what literature describes during modeling tasks. Literature suggests that while expert modelers are likely to constantly validate their models, many modelers validate their mathematical model after they obtain mathematical results towards the end of the modeling cycle. Furthermore, novice modelers often do not bother to validate their mathematical models (Galbraith & Stillman, 2006; Blum & Leiβ, 2006). I believe the difference in validation and evaluation can be attributed to the embedded ambiguities in RCMTs. The ambiguities in RCMTs require students to consciously make assumptions and provide reasons for their assumptions, which is an active evaluation process. This happens mostly as students made assumptions about the situations and investigated the relevant factors to build the real model.

Real Model and Mathematization

Literature describes real model and mathematization as two distinctive stages in modeling cycles. In this study, however, results indicate that mathematization began as students were creating a real model for the task. In other words, students began the mathematization process prior to the completion of a real model. For example, students started to calculate the amount of gasoline required for a round trip based on the distance and fuel efficiency before they considered fuel price and other factors under fuel cost.

Difficulty and Focus

The second difference between these tasks is the difficulties students experience during the tasks. Similar to modeling tasks, a major difficulty students experienced in this study is understand the problem context (Galbraith & Stillman, 2006). Students in this study spent most of their time and energy on understanding the context and finding relevant information to solve the task. Another most challenging transition in the modeling process is mathematization. However, this is not the case in this study. Students did not seem to experience much difficultly in transitioning between reality and the mathematical world. This difference makes sense because while RCMTs are complicated, the mathematics required to solve RCMTs are rudimentary. The difficulty of RCMTs lies in understanding and making sense of the problem, determining the relevant variables involved in the solution, making suitable assumptions, and making appropriate evaluation and validation of approaches instead of mathematization.

Intergroup Communication

Other than the validation of ideas and difficulties students experienced, this study also notices interesting conversations between the groups. Since students worked in close proximity within each other, it was impossible to not overhear the conversations in other groups. Therefore, the ideas from one group easily spread to other groups like wildfire. However, the spreading of ideas was not the surprising element here. What was surprising was while students constantly provided reasons to support their own ideas and calculations, they did not validate the ideas and calculations from other groups. For example, when one group decided the cost of gas for a round trip was roughly $16, a few other groups blindly followed and decided that the cost of gas was also \$16. Students seem to put a lot of trust in their friends and rely on them to validate the information prior to sharing the information with them.

In summary, while data in this study shows that the RCMT process parallels modeling cycles in general, modeling cycles do not exemplify all that happened during RCMTs. Based on the results and the analysis, I propose the following RCMT process:

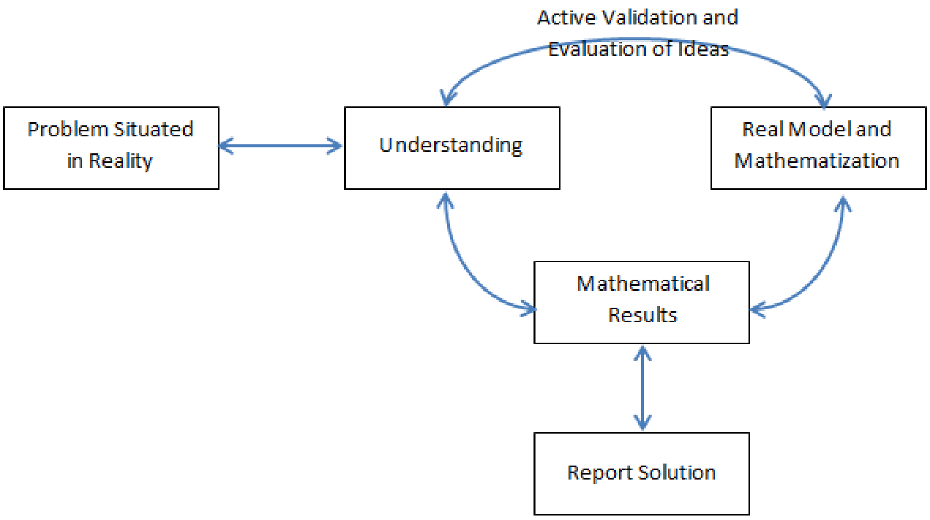


Figure 2: A proposed RCMT process

Compared to the modeling cycles found in the literature, the proposed RCMT process emphasizes understanding the context and active validation and evaluation of ideas. Also, the process of creating a real model and mathematization are not distinctive stages but overlap each other. Furthermore, since RCMTs focus on the flexible use of rudimentary mathematics, mathematization and the resulting mathematical models are downplayed in the process.

CONCLUSION

In this study, I investigated students' RCMT process using modeling cycles as a framework. While some similarities exist, there are also differences. Compared to what is suggested in the modeling literature, students spent more time and energy to understand the situation and to investigate factors involved in the solution, and less time in mathematization during RCMTs. Also, students validated their own ideas very often during RCMTs, except when the ideas come from their peers. Finally, this study proposes a RCMT process to describe students' behaviors during the task.

References:

Ferrucci, B., & Carter, J. (2003). Technology-active Mathematical Modelling. *International Journal of Mathematical Education in Science and Technology*, *34*(5), 663-670.

Blum, W., & Borromeo Ferri, R. (2009). Mathematical Modelling: Can it be Taught and Learnt? *Journal of Mathematical Modelling and Application*, *1*(1), 45-58.

Blum, W., & Leiβ, D. (2006). How do Students and Teachers deal with Modelling Problems? In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical Modeling (ICTMA 12): Education, Engineering and Economics* (pp. 222-231).

Borromeo Ferri, R. (2006). Theoretical and Empirical Differentiations of Phases in the Modelling Process. *ZDM, 38*(2), 86-95.

Galbraith, P., & Stillman, G. (2006). A Framework for Identifying Student Blockages during Transitions in the Modelling Process. *ZDM*, *38*(2), 143-162.

Howison, S. (2005). *Practical Applied Mathematics: Modelling, Analysis, Approximation.* Cambridge University Press.

Kaiser, G., & Sriraman, B. (2006). A global survey of international perspectives on modelling in mathematics education. *ZDM, 38*(3), 302-310.

Lesh, R., & Doerr, H. (2003). Foundations of a Models and Modeling Perspective on Mathematics Teaching, Learning, and Problem Solving. In R. Lesh, & H. Doerr, *Beyond Constructivism: Models and Modeling Perspectives on Mathematics Problem Solving, Learning, and Teaching* (pp. 3-33). Mahwah: Lawrence Erlbaum Associates.

Liljedahl, P. (2010, October 12). *Numeracy Task Details*. Retrieved November 22, 2012, from http://www.peterliljedahl.com/teachers/numeracy-tasks/numeracy-task-details

Maaβ, K. (2006). What are Modelling Competencies? *The International Journal on Mathematics Education*, *38*(2), 113-142.

SOFYA KOVALEVSKAYA: MATHEMATICS AS FANTASY

Veda Roodal Persad

Simon Fraser University

What do accounts by and about mathematicians of their involvement with mathematics tell us about the nature of the discipline and the attendant demands, costs, and rewards? Working from an autobiographical sketch and biographies of the first woman in the world to achieve a doctorate of mathematics, Sofya Kovalevskaya (1850-1891), and using the Lacanian notion of desire, I examine the forces that shape and influence engagement with mathematics. I contend that involvement with mathematics is impelled, fuelled, and sustained by desire.

Accounts by and about mathematicians regarding their journeys in mathematics have long been neglected as a source of knowledge about the discipline. My study, in general, focuses on finding out what can be learned from these accounts about the discipline of mathematics, and seeing what they tell us about the mathematical subject, the person who mathematics calls us to be in order to engage with it. In this paper, I examine the journey of Sofya Kovalevskaya (1850-191), the first woman in the world to achieve a doctorate in mathematics. I use the Lacanian notions of the subject, subjectivity, and desire in order to see what impels involvement and achievement in mathematics.

Lacanian theory has been used by researchers in mathematics education such as Baldino and Cabral (1999; 2005; 2006; 2008), Brown (2008; 2011), Cabral (2004), and Walshaw (2004), to describe the constitution of the mathematical subject and the pedagogic transference that takes place in teaching. This is a recognition of the view that while the usual considerations of students, teachers, tasks, technology, and classrooms are important in the endeavour of teaching and learning mathematics, the root of the engagement lies with the subject, the person or individual confronting the discipline.

In his theory of the subject, Lacan posits three psychic registers or orders of experience: the Imaginary, the Symbolic, and the Real. These registers are not to be understood as developmental stages as they obtain at every sphere of human activity. The Imaginary is the realm of "images, conscious or unconscious, perceived or imagined" (Lacan, 1973/1981, p. 279) of the people and objects in the world present to us. These idealized images are formed in childhood and persist even into adulthood. One's sense of 'self' starts from the mirror-stage, that is, from the child seeing its specular image in a mirror or in beholding another child. This marks the beginning of a *méconnaissance* or misrecognition of 'self' as the child imagines the image to be whole and coherent while perceiving itself as fragmented. The Symbolic, derived from the "laws" of the wider world in its structure and organization, disturbs the shaping or the interpellating (Latin: *inter*/between, within, *pellere*/push) of the subject. The Symbolic is enabled by language as it is language that gives us the structures for the signifiers for the "I" and the Other, for loss, lack, and absence, for the misidentification of the self with the Other, and for the formation and experience of desire. The Real is the unmarked backdrop against which the Imaginary (image-based) and the Symbolic (word-based) come into play, the screen on which images and words unfold and move.

For Lacan, desire is to be distinguished from need and demand. Examples of need are hunger and thirst in that they can be satisfied. Greater than need is the subject's demand in its dawning recognition and search of self in relation to others and the Symbolic order. In Lacanian arithmetic, desire is what remains when need is subtracted from demand. "[I]t is this irreducible 'beyond' of the demand that constitutes desire" (Homer, 2005, p. 77). Desire is a manifestation of a lack in the subject as the subject seeks to separate itself from the Other and to differentiate her desire from the Other's desire. Against this framework, I consider Kovalevskaya as a mathematical subject and demonstrate her desire in her mathematical journey.

FINDING KOVALEVSKAYA

I did not know of Kovalevskaya as a mathematician despite all my years of learning and teaching mathematics. I came upon her serendipitously from a collection of stories by Alice Munro (2009), *Too much happiness*. Munro had been looking for something else when she came upon Kovalevskaya, and was struck by the unusual combination of mathematician and novelist. I soon found that there was much more material on her; indeed there is a small industry on her life and work among historians of mathematics and science and a few mathematicians. I found her memoir, *A Russian childhood*, which includes an autobiographical sketch. There were several biographies which gave different perspectives on her life and the myths that have been created around her. One review of three biographies of her in *Physics today* is titled *A divergence of biographies: Kovalevskaya and her expositors* (Grabiner, 1984); this is a play on the title of one biography, *A convergence of lives; Sofia Kovalevskaia, scientist, writer, revolutionary* (Koblitz, 1983), a title which is, if anything, overblown. Besides the biographies, there are many other secondary sources of books, reviews, and journal articles. A second challenge was that of translation into English and a related matter, citation. Much of the primary source material about Kovalevskaya is in other languages (Russian, French, and German) and the English translations are not uniform. Different translations give different words with different meanings and nuances (one example is the word, imagination, in place of fantasy).

I begin by describing Kovalevskaya as a mathematical subject with the various subject positions (attitudes, ideals, values, beliefs) that are involved in taking up and doing mathematics. Then I study the refrains of desire as they manifest in her engagement with mathematics, while noting the various other factors in her life and times that interacted with and shaped her desire with respect to mathematics.

HER CONSTITUTION AS A MATHEMATICAL SUBJECT

In the autobiographical sketch that Kovalevskaya has provided, the first indication of a stirring of intellectual ideas comes from her father's brother, Pyotr Vasilievich Korvin-Krukovsky. She writes that "[her] love of mathematics first showed itself" (Kovalevskaya, 1889/1978, p.213) in the stories he told and in their conversations about the things that he had taught himself from reading widely:

> It was during such conversations that I first had occasion to hear about certain mathematics concepts which made a very powerful impression upon me. Uncle spoke about "squaring the circle," about the asymptote – that straight line which the curve constantly approaches without ever reaching it – and many other things which were quite unintelligible to me and yet seemed mysterious and at the same time deeply attractive. And to all this, reinforcing even more strongly the impact of these mathematical terms, fate added another and quite accidental event. (Kovalevskaya, 1889/1978, p. 214)

The accidental event occurred when she was eleven years old; the family moved to the country and the new wallpaper that had been ordered proved insufficient for all the rooms. The one room left over, the nursery, was papered with the pages of lecture notes of a course in differential and integral calculus. This was a course which her father had taken in his training as an Army officer and was given by the Academician Ostrogradsky, member of the Petersburg Academy of Sciences. Kovalevskaya spent hours with the wallpaper:

> As I looked at the nursery walls one day, I noticed that certain things were shown on them which I had already heard mentioned by Uncle … It amused me to examine these sheets, yellowed by time, all speckled over with some kind of hieroglyphics whose meaning escaped me completely but which, I felt, must signify something very wise and interesting. And I would stand by the wall for hours on end, reading and rereading what was written there. I have to admit that I could make no sense of any of it at all then, and yet something seemed to lure me on toward this occupation. As a result of my sustained scrutiny I learned many of the writings by heart, and some of the formulas (in their purely external form) stayed in my memory and left a deep trace there. I remember particularly that on the sheet of paper which happened to be on the most prominent place of the wall, there was an explanation of the concepts of infinitely small quantities and of limit. (Kovalevskaya, 1889/1978, pp. 215-216)

Later, when she was presented with the subject by her professor in Petersburg, it was all familiar to her: "You have understood them as though you knew them in advance"

(p. 216). That Kovalevskaya held the symbols and "hieroglyphics" in her mind without any understanding of what they represented shows that the Symbolic is deeply important to her. Later I will discuss the significance of the Symbolic to her and her attempt to reconcile the various aspects of herself with respect to the registers of the Imaginary, the Symbolic, and the Real.

Besides the fascination with mathematics and the world of mathematics, Kovalevskaya brought deep concentration to her study of mathematics. She was intensely absorbed when doing mathematics. She refused activities with her friends to work on mathematics: 'Now I am sitting at my writing desk in bathrobe and slippers, deeply absorbed in mathematical thoughts, without the slightest desire to take part in your excursion.' (Kennedy, 1983, p. 36) Mathematics required complete absorption and concentration in a world that Kovalevskaya kept to herself; she did not share her thoughts with her close friend, Julia Lermontova (the first woman to obtain a doctorate in chemistry) with whom she lived, presumably because Julia was not engaged in the mathematical endeavour. While Kovalevskaya could lose herself in the mathematics, she had to return to the struggles of the real world.

REFRAINS OF DESIRE

I now examine the refrains and reverberations relating to her desire. I have chosen the words, refrains and reverberations, deliberately for their acoustic connotation because, it seems to me that as I read the various sources on Kovalevskaya, I was listening for the resonances and themes relating to her desire. I discuss four refrains which contribute to what Freud calls the melody of the drive. I then show how the leitmotif of her life can be seen as asymptotic desire. The refrains relating to desire that stand out in Kovalevskaya's life are absorption, substitutes, fake, and fantasy.

Absorption refers to the process of taking in or being taken by, leading to both an inward and an outward captivation. An early instance of absorption can be seen in her childhood passion of staring at the wallpaper in the nursery. This was no passing attraction; she spent hours every day absorbed in and by the mathematical hieroglyphics. Kovalevskaya's time in front of the wall was well-spent in that it produced a subliminal, unconscious understanding. Later when she was introduced to the mathematics depicted, "the concept of limit appeared to me as an old friend".

Also, as a young woman studying mathematics, she spent long hours by herself engrossed in the mathematics; she willingly gave up social activities with her friends to spend time with the mathematics on which she was working. Further, there was an outward absorption in her strong identification with the style and spirit of her teacher, Weierstrass. By fashioning herself along his principles, she came close to losing her mathematical self in her relationship with him, to the extent that she left herself open to the Klein's charge of it being impossible to tell what was her work and what was that of Weierstrass.

The second refrain in Kovalevskaya's life is that of substitutes, in that the essential supports of her life from beginning to end were gratified by substitutes. To begin, she was not a boy. The eldest child in the family was a girl, Anyuta. Kovalevskaya was the second child and unwelcome as her parents were hoping for a boy. Cooke (1983, p. 7) describes her as having "a dark complexion with a very intense and serious personality." Her mother preferred her first and third, Anyuta with her blonde curls and pleasing manner and Fedor because he was a boy. Besides this, by engaging in a fictitious marriage, Kovalevskaya assumed a substitute husband, Vladimir Kovalevsky. Their relationship was fraught with tension as Kovalevsky was supposedly no more than a convenience to her. Another substitute is in her life is seen in her relation with Weierstrass; he was a substitute father to her.

Closely tied to substitutes is the third refrain of fake. In a letter to Anyuta dated 1868, Kovalevskaya writes: "In my present life, despite its seeming logic and completeness, there is a certain false note that I cannot determine, by which I feel nonetheless" (Kochina, 1981/1095, p. 51). It is to her credit that she noted the inauthenticity of her life but how could it have been otherwise? Both her experience of being parented and her marriage were fake, thereby leaving her with a desire to be desired. Her marriage, after many tensions and misunderstandings, was not consummated until after eight years or so. She had found it difficult to keep up the fiction to her parents and to deal with the inauthenticity of a pretend marriage and a pretend life.

On the other hand, her life was full of fantasy. In a letter from Weierstrass to her:

> "How fine it would be were we both here. You with your soul full of fantasy, and I, excited and refreshed by your enthusiasm. We could dream and think here about the many problems that we have to solve: of finite and infinite spaces, of the stability of the world systems, and about all the other great problems of mathematics and physics. But long ago I resigned myself to the fact that not every wonderful dream is realized" (Kochina, 1981/1985, p. 75).

Her fantasies were broad and wide-ranging as seen in her literary pursuits but with respect to mathematics, she was trying to find herself in it. This is one of the universal fantasies of mathematics, that we can find ourselves in it and that it will give us back ourselves but the sad truth is that it cannot and does not. The other realization is that to seek to possess mathematics is to undertake a journey towards it. It requires special effort and does not yield its secrets too easily. She learned of her mathematics from staring at the wall and from the stories the uncle told; her life was a journey in search of the fascination it portended. In some sense, mathematics is the place of the things that she wanted to be true and to come true. The fantasy of mathematics for

Kovalevskaya is that it is the place of truth. Indeed, the ultimate fantasy of mathematics is that it gives us a false sense of power. Ah yes, but that is what mathematics is about, power!

The leitmotif of her life: Asymptotic desire

These refrains underpin the central theme of asymptotic desire in Kovalevskaya's life. Lacan had used the notion of the asymptote to describe desire, always approaching but never attaining an object because there is no object of desire, only an object-cause of desire. Grosz (1990), in her feminist introduction to Lacan, elaborates the origin, nature, and path of desire:

> Lurking beneath the demands for recognition uttered by the cogito (this is Hegel's 'solution' to the problem of the solipsism of the cogito), by the subject (to the other) and by the masculine subject (to an unknowable femininity) is a *disavowed*, repressed or unspoken *desire*. Desire is a movement, a trajectory that asymptotically approaches its object but never attains it. Desire, as unconscious, belies and subverts the subject's conscious demands; it attests to the irruptive power of the 'other scene', the archaic unconscious discourse within all rational discourses, the open-endedness of all human goals, ideals, aspirations, and objects. (Grosz, 1990, p. 188, original emphasis)

This is a powerful evocation of desire as it points out the unconscious, unknown, and unacknowledged aspects of desire. Desire is not only asymptotic; it is ever-circling. As Žižek explains, it is not the goal but the aim (the path towards the goal) that gives enjoyment. For Lacan, the subject is constituted by its lack which gives rise to desire. Kovalevskaya's desire arose out of various sense of lack, the sense of not being male, of not being allowed to take her place as a mathematician, and of not being complete as a mathematician.

That she was not allowed to take her place as a mathematician is seen in her not being able to get a teaching position in Russia or Paris as she desired (Paris being the centre of cultural and political activity). In her desire to be desired, she was trying to find a place where she was wanted ("[t]ake away your savage, she is not wanted here" from her mother to her nurse); to some extent mathematics did not want her either. Keen (1986, p. ix) writes: "Sonya Kovalevskaya was a distinguished mathematician who was considered among the best of her generation by her contemporaries", but this was not enough to have given her a position that she wanted.

Finally, Kovalevskaya did not feel complete as a mathematician in that she had other aspirations pertaining to writing. Her writing included theatre reviews, poetry (for herself), plays, an autobiographical memoir, and a novel. Mostly she was influenced by the quotation attributed to Weierstrass, of not being a complete mathematician without having the soul of a poet. She yearned to be both. Kovalevskaya yearned for more out of life, more involvement in social, political and cultural dimensions, perhaps to her detriment. In responding to the *che vuoi* (what do You want?) of the Other, she seemed to be always looking for another mountain to climb. Her friend, Julia, writes that she set herself difficult goals but “I never saw her so dismal and depressed as when she reached her goal” (Kochina, 1981/1985, p. 88).

CONCLUSION

In a letter to a young Russian woman writer, Kovalevskaya writes:

> I understand your surprise that I can work at the same time with literature and mathematics. Many who have never had an opportunity of knowing any more about mathematics, confound it with arithmetic and consider it an arid science. In reality however, it is a science which requires a great amount of fantasy, and one of the leading mathematicians of our century states the case quite correctly when he says that it is impossible to be a mathematician without being a poet in soul … one must renounce the ancient prejudice that a poet must invent something that does not exist, that fantasy and invention are identical. It seems to me that the poet has only to perceive that which others do not perceive, to look deeper than others look. And the mathematician must do the same thing. (Kovalevsky, 1889/1978, p. 316)

One might ask, why does she say mathematics requires great fantasy and not intuition or imagination, say? She worked with partial differential equations, rotations of a rigid body, and elliptic integrals, all constructs and concepts which require more than a leap of intuition or imagination. There is little in the accounts that help with understanding of her use of the word, fantasy, but it may be an effect of the translation as, in another translation, the word imagination is used. Earlier I pointed to the quote from Julia who wrote that after long hours immersed in her work, Kovalevskaya appeared transported to another world (‘carried by fantasy’) that she could not or chose not to express in words but could only find release in rapid walking back and forth. Generally we see a fantasy as a product of caprice and fiction. For Lacan, a fantasy is how we stage our desire in pursuit of fulfillment of our desire. So in this sense mathematics was fantasy to Kovalevskaya.

Why does Kovalevskaya distinguish fantasy and invention and go on to say that we must abandon the notion that fantasy and invention are the same? Both fantasy and invention are creative acts in all fields, not just mathematics. In both, there is the bringing forth of thoughts and ideas that had not existed before. Perhaps, she means to indicate that they have different ends and that fantasy must not be associated with any use-value. The final thought in the quote is helpful in showing her position that mathematics requires a depth of looking, of looking deeper than others. She writes:

It is the philosophical aspect of mathematics which has attracted me all through life. Mathematics has always seemed to me a science which opens up completely new horizons/ (Kovalevsky, 1889/1978, p. 216).

At every turn, Kovalevskaya's life and work were dominated by the signifiers of 'woman', 'Russian' and 'mathematician'. None of these would have come up as an issue of struggle in a given society or community. Only when her desire was hemmed in by these that they became forces by which she was buffeted. There was little possibility that the society in which she found herself could acclimate or integrate her desire. Besides mathematics, she looked to other avenues such as literature and a second marriage in which she hoped for love. In another time and another place, her desire of taking a place in mathematics (in the positions and situations that she hope for, suitable to her talents and abilities) may have been possible and may have led to great fulfillment as a subject.

Looking back on Kovalevskaya's life, it seems to me that the distance from the place of mathematics as fantasy that she accessed through her mathematical work to the reality of her life in the circles in which she moved was too great. The metric needed to conceptualize that distance would take a century and more of social upheaval. The costs were too inordinate to bear and the cold realization is that mathematics is indeed, even with the gifts of genius and charm, not for the faint of heart. Kovalevskaya could do mathematics but she could not be a mathematician as she had hoped. She was capable in the doing of mathematics, in her research work and in her teaching of mathematics and science but she was constrained by the symbolic order of being a mathematician in that time and that society. She could not take her place with the other mathematicians in the positions and institutions of the time – the highest position to which she could aspire was to teach in schools for girls and women.

In the end, she was unable to realize her dreams to the extent that she desired. She had started with quadratures and asymptotes. Her life was an ode to her attempts of squaring the circle amid the trajectory of asymptotic desire in search of her old friend and lost object, the limit.

References

Baldino, R. & Cabral, T. (1999). Lacan's four discourses and mathematics education. In O. Zaslavsky, *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education, Vol. 2,* 57-64. Haifa, Israel: Technion University.

Baldino, R. & Cabral, T. (2005). Situations of psychological cognitive no-growth. In H. Chick & J. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education, Vol. 2,* 105-112. Melbourne: PME.

Baldino, R. & Cabral, T. (2006). Inclusion and diversity from Hegel-Lacan point of view: Do we desire our desire for change? *International Journal of Science and Mathematics Education 4*, 19-43.

Baldino, R. & Cabral, T. (2008). I love maths anxiety. In T. Brown (Ed.), *The psychology of mathematics education: A psychoanalytic displacement*, 61-92. The Netherlands: Sense Publishers.

Brown, T. (2008). *The psychology of mathematics education: The psychoanalytic displacement.* The Netherlands: Sense Publishers.

Brown, T. (2011). *Mathematics education and subjectivity: Cultures and cultural renewal*. The Netherlands: Sense Publishers.

Cabral, T. (2004). Affect and cognition in pedagogical transference a Lacanian perspective. In M. Walshaw (Ed.) *Mathematics education within the postmodern*, 136-179.

Grabiner, J. (1984). A divergence of biographies: Kovalevskaya and her expositors. *Physics Today*, April 1984, 79-81.

Homer, S. (2005). *Jacques Lacan*. Oxon, OX: Routledge.

Lacan, J. (1973/1981). *The four concepts of psychoanalysis*. New York, NY: W. W. Norton.

Koblitz, A. (1983). A convergence of lives: Sofia Kovalevskaia, Scientist, Writer, Revolutionary. Boston: Birkhäuser, Inc.

Kovalevsky, S. (1889/1978). A Russian Childhood, tr. by B. Stillman. New York: Springer Verlag.

Walshaw, M. (2004). The pedagogical relation in postmodern times: Learning with Lacan. In M. Walshaw (Ed.), *Mathematics education within the postmodern*, pp. 121-139. Greenwich, CT: Information Age.

DESIGNING STUDENT ASSESSMENT TASKS IN A DYNAMIC GEOMETRY ENVIRONMENT

Marta Venturini

Simon Fraser University & University of Bologna

This paper explores the design of assessment tasks involving the use of Dynamic Geometry Environments (DGEs). I adapt the work of Laborde and the results of Sinclair, which focus on the design of DGE tasks, to the context of formative assessment. I provide an initial framework, along with illustrative examples, for different types of DGE-based assessment tasks that can be used in the classroom but also to study technology-based teacher practices. This research develops new directions in finding how to design suitable tasks for student mathematical assessment in a DGE.

INTRODUCTION

While many institutions and teachers have introduced the use of digital technology into the teaching and learning of mathematics, nobody seems to know how to evaluate the mathematical learning developed through the use of technology: “The use of technology in assessment continues to be a complex issue across Canada” (Caron & Steinke, 2005). My goal in this paper is not to list and explain all the challenges about the integration of technology in the assessment of mathematics. Instead, I want to prove that it is possible to design suitable tasks in a DGE to evaluate students, although technology asks for rethinking assessment in mathematics. According to Heidenberg & Huber (2006): “what you test is what you get”, which means that if digital technology is never part of the mathematical tasks, students (and perhaps even teachers) will not see how to effectively use it in mathematics as a learning goal.

Why using digital technology to assess students?

I summarize the main issues on digital technology use in assessment as follows:

Teachers	Students
teachers can see student reasoning through the “actions” that they do in the technology environment	technology helps students in expressing and communicating their ideas
teachers have the opportunity to design tasks that enable certain mathematical thinking that is not accessible with paper-and-pencil tasks	technology offers students the possibility to show different kinds of abilities and knowledge

technology helps teachers in recognizing student misconceptions	technology allows students to continue learning while they are taking a test

Table 1: Digital technology in student assessment.

Technology allows new opportunities to capture and replay student work: "By studying a student's script, a teacher can infer ways that the student is thinking about the object or procedure" (Wilson, 2008). Moreover, Laborde et al. (2006) indicate that feedback through technology offers a great deal of opportunity for new ways of understanding mathematics. Feedback from student interactions with technology can have a strong impact on their mathematical understandings and practices:

> "Even without sophisticated constructions, a student's simple action of manipulating a dynamic figure can already be a meaningful mode to demonstrate their understanding of geometric concepts" (Sangwin et al., 2010, p. 235).

What roles does the technology play in the assessment of mathematics?

Technology offers new ways of doing mathematics, which means that it is possible to design new and different tasks for student assessment. However, the content of the assessment has to change, in order to include questions that students are not able to solve in a paper-and-pencil context, like Caron and Steinke (2005, p. 3) state:

> "We must also look at what mathematical problems could now be tackled by students with the use of technology, what concepts and techniques ("new" and "old") would be mobilized in solving these problems and how the solving of such problems could contribute to the development of a creative, powerful and rigorous mathematical practice."

The main affordances of digital technology use relevant to the design of innovative tasks for assessment are summarised below:

1. *Answer recording*: explaining the answer using tools like screenshot, script, and recording voice or video;
2. *Validation/verification*: interpreting an answer or a non-answer from a machine;
3. *Illustration/visualization*: observing objects and phenomena, and making deductions or inferences about them;
4. *Dragging objects*: exploring a domain in order to find relationships among objects, or the laws that drive a certain environment;
5. *Construction*: creating objects, and ideating examples and counter-examples;
6. *Simulation and Measure*: observing and modelling a real life phenomena;
7. *Solving Problems*: putting forward a conjecture;
8. *Motion*: moving objects to obtain a result, or to solve a situation;

9. *Symbolic computation and Numerical computation*: focusing on the concept, leaving the calculation to the machine;
10. *Collaboration*: communicating mathematical concepts among students.

Using these points it is possible to assess student understanding on content and process in mathematics, and, in particular, the reasoning process, because "technology can be designed in such a way as to enhance the implementation of didactical principles" (Mackrell, Maschietto & Soury-Lavergne, 2013, p. 79).

THEORETICAL FRAMEWORK

Mackrell, Maschietto and Soury-Lavergne (2013) state that "both the design of tasks and the design of technology have been identified as important factors in the effective use of technology-based tasks in the classroom".

Laborde (2001) describes a case study on teachers designing tasks for a DGE, it analyses every task considering the place in the mathematics curriculum, the role that teachers assigned to the technology, and the degree of change of the designed task for the DGE compared to the paper-and-pencil context. It came up with four different categories that were used to drive the teachers' tasks:

Category	Description	Characteristics
Tasks in which the DGE facilitates the material aspects of the task.	The task is not changing conceptually, it is only facilitated by some drawing tools of the DGE.	The solution strategies of both tasks do not differ deeply.
Tasks in which the DGE facilitates the mathematical task.	The DGE is supposed to facilitate the mathematical task that is considered as unchanged.	The DGE is used as a visual amplifier in the task of identifying properties.
Tasks modified when given in a DGE.	The DGE is supposed to modify the solving strategies of the task due to the use of some tools and to the chance that the task might be rendered more difficult.	The task in the DGE requires more mathematical knowledge, which students find difficult to put into action.
Tasks only existing in a DGE: - "black box" - prediction	The task itself takes its meaning or its "raison d'être" from the DGE. It requires reasoning and knowledge.	Such tasks require identifying geometrical properties as spatial invariants in the drag mode and possibly performing

experiments with the tools of the DGE on the diagram.

Table 2: Laborde's categories.

Sinclair (2003, p. 313) points out that:

> "Pre-constructed dynamic sketches are central elements of the learning activity, and therefore, decisions about their design have the potential to support or impede the development of exploration strategies and geometric thinking skills."

I decided to use a DGE to design my sketches, in order to show some of the advantages in using this environment for the assessment. In (Sinclair, 2003) I found some interesting suggestion for the characteristics that a task designed in a DGE should have depending on the aim of the task. She provided some hints to design the sketches in order to address the questions and the instructions:

Aim of the Task	Characteristics of the Sketch
focus student attention	visual stimulus (color, motion, markings…)
prompts action (drag, observe, deduce…)	provide affordances so that students can take the required steps
exploration	provide options and alternate paths
surprise	it must be flexible enough to help students examine cases for further exploration and experimentation, but constrained enough to prevent frustration
check understanding	the sketch can aid peer-interactions by providing a shared image for students to consider and discuss

Table 3: Characteristics for sketch design in (Sinclair, 2003).

METHODOLOGY

Sinclair (2003, p. 293) observes: "In the case of pre-constructed sketches, the task also includes the sketch itself, together with any special investigative software tools created by the designer". Teachers can include details and already assembled objects in order to check student understanding on specific properties. Moreover, designing the tasks for the iPad allows teachers to give students a "restricted environment" for the assessment, where students can explore certain objects/situations under set conditions, and use only the tools provided by the teacher.

I designed some tasks to assess student knowledge on Circle Geometry. Clearly, some mathematical topics are more suitable to be assessed in a DGE, in particular geometry, because students can drag objects, construct figures, and explore a domain. The *BC*

Curriculum on Mathematics grade 9 describes the competence on Circle Geometry that students are expected to have: they need to know some specific properties of the circle, and they have to achieve some learning outcomes, like providing examples, solving problems, measuring, and explaining relationships.

Some tasks come from the textbooks, the web, and game-competitions in mathematics. I adapted them to be designed in Sketchpad, and I invented other tasks. It is interesting to notice that some textbooks contain a technology based lesson plan in a DGE, but no examples of assessment with technology are provided. I used (Laborde, 2001) as framework to design the tasks, and (Sinclair, 2003) to implement the sketches in Sketchpad.

DESCRIPTION OF SKETCHES

I'm describing some of the sketches that I designed in Sketchpad for student assessment on Circle Geometry, one for each category of (Laborde, 2001).

First category

> If you think this statement is true, make a drawing to represent it; if you think it's not, create a counterexample: *The centre of any circle is the intersection of the perpendicular bisectors of any two chords in the circle.*

This task is particularly subtle, because there is only one case in which the statement is not true: when the two chords are parallel.

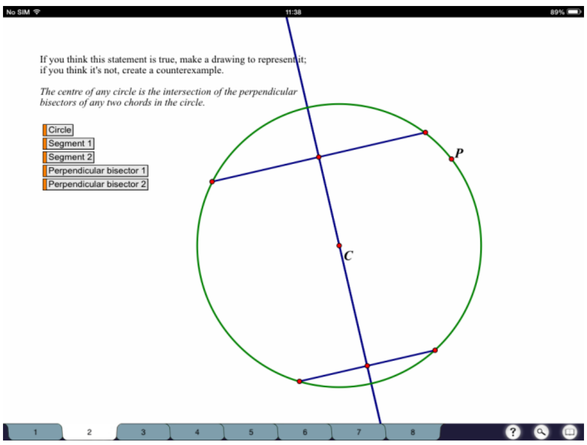


Figure 1: The Counter-example.

In this task, Sketchpad facilitates the material aspects of the task in making the drawing. The task isn't changed conceptually compared with a paper-and-pencil environment. The solution strategies of both tasks do not differ deeply: the only difference could be that students can move the chords all around the circle, and they can trip over the case in which the two chords are parallel.

Second category

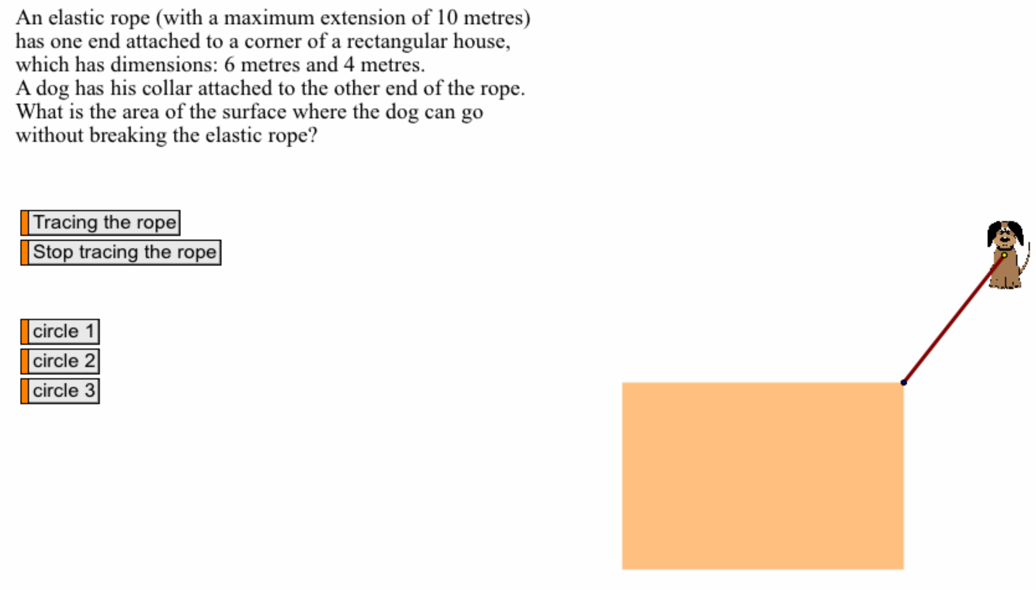


Figure 2: The Dog

In this sketch students can explore the situation by dragging the dog around the screen. Sketchpad allows them to see where the dog can go within the constraints set by the rope and offers visual cues into the problem's solution that would be difficult to obtain through paper and pencil alone.

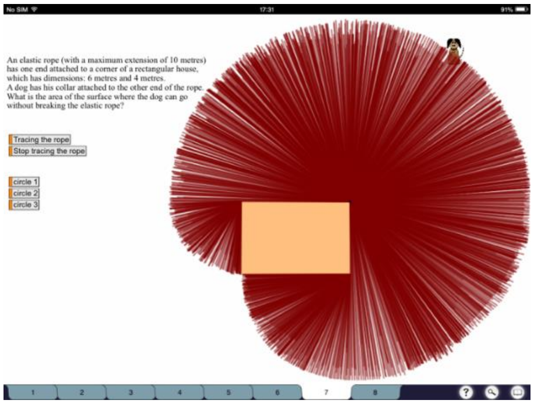


Figure 3: Tracing the rope.

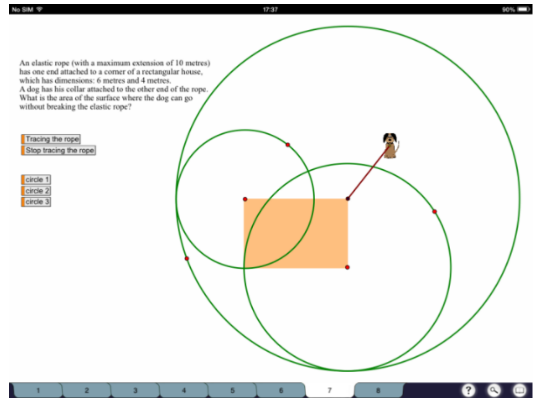


Figure 4: Using the circles.

To gain a better sense of the bounds imposed by the rope, students can press the button "tracing the rope", so that the rope leaves behind a trace of all its locations as they drag the dog. Pressing the buttons "circle1", "circle2", and "circle3" students view three circles, whose location and size they can change simply by dragging them. Thinking about the radii of the circles and their placement may help students determine the area where the dog can roam. The mathematical task is considered unchanged, but the DGE facilitates it: Sketchpad is used as a visual amplifier in the task of solving the problem.

Third category

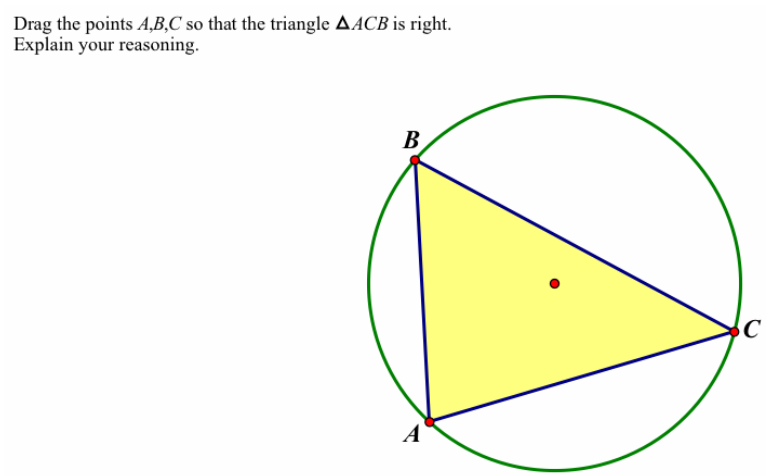


Figure 5: The Right Triangle.

In this sketch students don't have the measure tool for the angles, they have to explain how to obtain a right triangle inscribed in a circle depending on the position of the points. The task is modified in Sketchpad, compared to a paper-and-pencil context. The solving strategy is different, because students have to drag the points on the circle so that the triangle *ABC* is right. Students need to know that in a right triangle inscribed in a circle the hypotenuse is the diameter, but the task in Sketchpad actually requires more mathematical knowledge: students can move the third point around the circle, thus they have to know that wherever they decide to place it, the triangle will be right, and this is something that students usually find difficult to put into action.

Fourth category

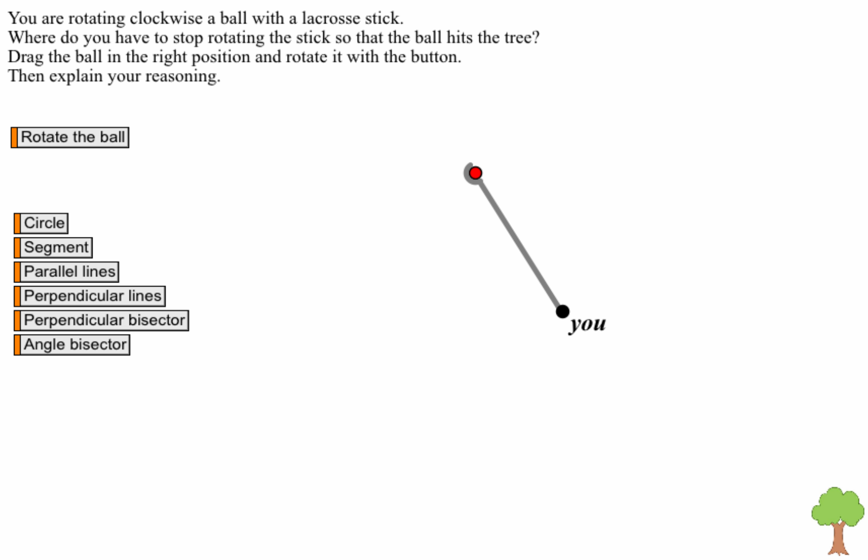


Figure 6: The Ball.

In this sketch students are supposed to explore the situation pressing the button “Rotate the ball”, and observing what happens. They have to find the right initial position for the stick, so that the ball hits the tree after one rotation of the stick. It is quite easy to predict that students will try to rotate the ball until they find the solution, but then they need to explain why that one is the right position. At this point students will use the tools, and they will find that the trajectory of the ball is the tangent to the circle whose centre is the point “you”, and radius is the length of the stick.

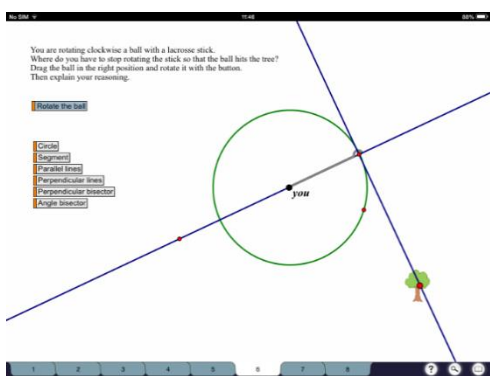


Figure 7: The trajectory of the ball.

This task can exist only in a DGE, and it takes its meaning from it. Students have to guess where the ball is going when the stick stops, and they can check their answer with the button. Then, they have to find the “hidden construction” of the situation. This task requires reasoning and knowledge: students need to know that a tangent to a circle is perpendicular to the radius at the point of tangency. They have to identify the geometrical properties as spatial invariants in the drag mode, because as long as they drag the stick, the trajectory of the ball is always tangent to the radius of the circle, and they can perform experiments with the provided tools on the diagram. The identification of the underlying property that the trajectory of the ball is the tangent isn't easy and constitutes the question.

An exploratory problem can be included in student assessment, because the goal is to see if students know how to use technology in mathematics to solve problems and make conjectures. Following (Heidenberg & Huber, 2006) taking a test can be an opportunity for learning, so that students continue to learn during the assessment.

CONCLUSIONS

I investigated the ways in which the opportunities that technology offers can be used to design assessment tasks for the evaluation of student mathematical learning in a DGE. In the sketches I exploited the affordances of digital technologies (DGEs in particular) in order to assess students in a different way, including questions that cannot be asked in a paper-and-pencil environment. Laborde (2001) suggests it is easier for teachers to adapt paper-and-pencil tasks for a DGE, but much more difficult to create novel technological tasks different in nature from what one might do with paper-and-pencil. My research is driven by the wish to identify which kinds of tasks teachers might be more willing or interested in using. In particular, I will investigate whether they confirm Laborde's conclusion or whether exposure to different types of tasks involving pre-made sketches might change their approach to technology-based assessment.

References

Caron, F., & Steinke, T. (2005). Learning in the presence of technology: Report of the CMEF working group. *Proceedings of the 2005 Canadian Mathematics Education Forum*.

Heidenberg, A., & Huber, M. (2006). Assessing the Use of Technology and Using Technology to Assess. *Supporting Assessment in Undergraduate Mathematics*.

Laborde, C. (2001). Integration of Technology in the Design of Geometry Tasks with Cabri-Geometry. *International Journal of Computers for Mathematical Learning, 6(3)*, 283-317.

Laborde, C., Kynigos, C., Hollebrands, K., & Sträßer, R. (2006). Teaching and learning geometry with technology. *Handbook of Research on the Psychology of Mathematics Education: Past, Present and Future*, 275–304.

Mackrell K., Maschietto M., & Soury-Lavergne, S. (2013). The interaction between task design and technology design in creating tasks with Cabri Elem. *Proc. of the 22nd ICMI Study: Task Design in Mathematics Education*, 79-87.

Sangwin C., Cazes C., Lee A., & Wong K.L. (2010). Micro-level Automatic Assessment Supported by Digital Technologies. *Proc. of the 17th ICMI Study: Mathematics Education and Technology-Rethinking the Terrain, 10*, 227-250.

Sinclair, M. P. (2003). Some implications of the results of a case study for the design of pre-constructed, dynamic geometry sketches and accompanying materials. *Educational Studies in Mathematics, 52*, 289-317.

Wilson, P. (2008). Teacher Education: Technology’s Conduit to the Classroom. *Research on technology in the learning and teaching of mathematics*, *2*.

STUDENTS' USE OF GESTURE AND POSTURE MIMICRY IN DEVELOPING MUTUAL UNDERSTANDING

Kevin J. Wells

Simon Fraser University

In this paper I focus on observations made regarding students mimicking of each other's gestures in face-to-face conversation while problem solving. The data supports the idea that the students may use such gestures to subconsciously signal acceptance. Through talk, gesture, prosody, and intonation, combined with context, the interlocutors may develop a better connection with each other, enabling a belief in having achieved a shared understanding of each other's contribution. In so doing, they are positioned to develop their understanding of the problem. In addition, recordings of students working together on problem solving show evidence of posture mimicking during times of effective collaborative. The results suggest that teachers' recognition of such mimicry may help in knowing when to successfully intervene.

INTRODUCTION

In this report I address the question of what clues a teacher can look for as indicators of when to intervene in student group work. My consideration of the use of mimicked gestures arose on reviewing recordings of students engaged in mathematical problem solving. While not initially looking for such gestures it stood out that the students demonstrated mimicry of both gesture and posture, prompting deeper analysis. My initial question, arising from recognition of this phenomenon, was whether or not there seemed to be any relation between such gesturing and the students' ability to progress with the problem. If so, could this be an indicator of the group's progress? The evidence presented here indicates that a teacher can look for gesture and posture mimicry as guides to appropriate intervention timing.

BACKGROUND

The reform-based shift towards a sociocultural approach in mathematics teaching, associated with the Vygotskian school of thought, takes a view of human thinking as being essentially social. There has been a push to replace the traditional classrooms featuring an outspoken teacher and silent students with small groups of learners talking to each other and expressing their opinions in whole class settings (Sfard, Forman, & Kieran, 2001). The need for a teacher to carefully facilitate the discourse in these situations has been noted by many researchers (e.g. Sfard et. al, 1998; Jaworski, 2004). While there is much research on how a teacher can successfully intervene (e.g. Ding et al. 2007), knowing when to intervene has been a less discussed but is an equally important aspect of such facilitation. The close presence of a teacher can stymy the flow of the group, while at other times the teacher needs to intervene in order to encourage and give critical feedback.

When students engage in mathematical problem solving in a group situation, there is a clear need for good communication to occur within the group if all participants are to gain from the collective experience. In everyday talk, gestures have been considered to be an integral part of communication (e.g. McNeil, 2005), and linked to speech in a semantic and temporal way. Radford (2009) notes that 'thinking does not occur solely in the head but also in and through a sophisticated semiotic coordination of speech, body, gestures, symbols and tools' (p.111). Sfard (2009) also considers gestures to be 'crucial to the effectiveness of mathematical communication (...) to ensure that all the interlocutors speak about the same mathematical object' (p. 197). Other researchers (e.g. Goodwin, 2000) have examined the role of gesture on the sequential organization of conversation. Clark and Wilkes-Gibbs (1986) argue that interlocutors in a conversation create meaning jointly, with the aim of creating mutual understanding. The process is considered to be in constant need of attention since, at best, the interlocutors can only believe that they have understood what each other meant. Such a belief, however, may be sufficient to allow the dialogue to continue based on the situation. The impression, then, of students working together on a problem, is one of a continuous need to repair meaning and make connections to each other. If we hold the view that learning mathematics is akin to developing a special type of discourse (Sfard, 2001) then observing students participating in such discourses is important. If, in addition, the important feature of group problem solving is in the activity rather than the end result, then being aware of that activity is a more important outcome than viewing the final answers. If we are interested in the unfolding understanding within the group then we 'must focus on the various forms of signs that speakers make available to others as well as themselves. These signs comprise words, gestures, body positions, prosody, and so on' (Roth & Radford, 2011, p. 55). With this in mind, students taking on, or mimicking, each other's words and gestures may be an important and visible part of the process.

There is evidence that people mimic a wide range of behaviours, including postures and mannerisms (Chartrand & Bargh, 1999). The occurrence of mimicry in physical behaviour during mathematics group work has been noted by Gordon-Calvert (2001). Holler and Wilkin (2011) found that mimicry in co-speech gestures does occur and concluded that 'mimicked gestures play an important role in creating mutually shared understanding' (p. 148). Holler and Wilkin also found that mimicked gestures were used to express acceptance of group members, suggesting that such gestures were an important part of the conversational structure, even when such acceptance was not expressed verbally. Gestures were also found to be important in signalling incremental understanding, something the authors paraphrased as 'I am following what you are saying in an effort to reach shared understanding with you' (p. 145). This view supports that of Roth (2000) who notes that 'the human body maintains an essential rationality and provides others with the interpretive resources they need for building common ground and mutual intelligibility' (p. 1685).

A limitation of many gesture studies, however, is that they are focussed on tangible objects that one party is attempting to describe to another (e.g. in Holler and Wilkin case it is abstract shapes with figure like qualities). A similar limitation can be seen in the work of McNeil (2005), wherein participants are asked to recall scenes from a cartoon they have watched. Students working in a classroom are generally describing or talking about mathematics that is not a recollection of an action but rather an ongoing action. Some of the actions involved may be hard for a student to put an image to in quite such a dynamic way as McNeil's subjects. As a result, it might be expected that the gestures can often be more subtle, especially in the early stages of working together. In the case of mathematical problem solving the participants in the dialogue are trying to create a solution without one member having a privileged informational position (such as would occur if a teacher was present). In addition, any power relations within the group may lead to a particular student being granted a dominant starting position. Mimicked gestures may be an attempt by a student to reflect the mannerisms of his/her interlocutor with the aim of acceptance.

METHODOLOGY

The video clips were taken from a larger study in a school in which two classes of grade 5 students (aged 10-11 years) were videoed over the course of an academic year. A camera was set up and left unattended with the intent that neither researcher nor the classroom teacher was a direct part of, or influence on, the conversation. The school is located just outside of a large city in Canada and reflects a very multicultural population, with several ESL students. Economic background is not considered to be an obvious factor in the school. Recordings were made weekly while the students were engaged in problem solving and transcribed using a framework of Conversation Analysis. A second viewing was made paying attention to gestures and body language. As part of the transcription process the occurrence of mimicked gestures became apparent, and led to this reported study. Going through a collection of clips looking for a particular but different event can bring out common features that were not seen as significant on initial observation. On becoming aware of this mimicry in more obvious cases, a random selection of 20 of the recordings was re-examined explicitly for mimicked gestures and posture. The clips discussed here were selected as exemplary of different forms of observed mimicked gesturing and posture. For the purposes of this report, only clear cases of mimicry were included, where a hand gesture or body position was mimicked either collectively or within two turns at talk. A deeper analysis of smaller gestures over the period of the discourse may prove interesting, but in this case I focussed on what might be seen by a teacher in a classroom setting observing several groups from a distance.

RESULTS

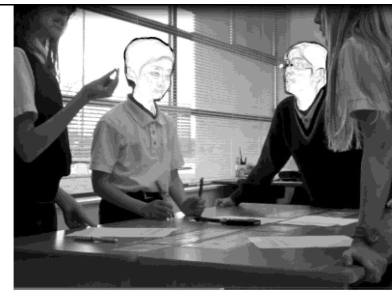
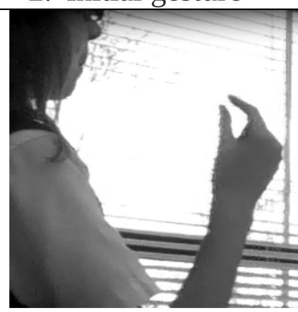
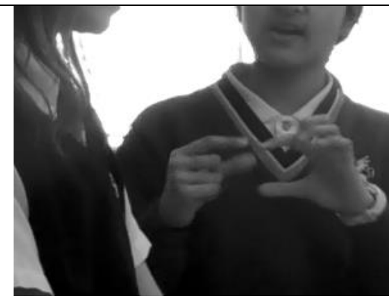
Table 1 illustrates a conversation between Gina and Susan. The problem concerns the change in area of a desk reduced to half its length but doubled in width. This example matches several recorded in this lesson and is of interest because, while gestures used differed between groups, there was evidence of gesture mimicry between interlocutors when the students were able to make progress. In examples where the students were unable to make progress, there was no clear evidence of gesture matching. In this example Gina initiated by describing the desk using large gestures. Susan, in her adjacent turn, mimicked the dynamic gesturing of Gina in describing the table.

		
1.You’re taking it in half	2. (..) and then ….	3. doubling one side, right?
		
4. You take some of it off	5. and you add it to	6. the other side (0.5)

Table 1: Gina and Susan describe the same process

Table 2 also shows another example of gesture mimicry between two girls working in a group on a problem where they were asked to estimate the size of a bag required to hold a million dollars in $100 dollar notes. Panel 2 shows one girl, Jasmine, making an initial gesture which is then mimicked by Gina (panel 3) as they engaged in conversation. As the conversation develops Jasmine moved gradually closer to Gina until their gesture space became shared. They continued to mimic each other’s gestures as they did so. During this time, the conversation was rich, and led to a clear progression in the problem’s solution.

Table 2 also shows the group engaging in posture mimicry. The three girls adopted an almost identical posture once they started to work on the problem together. The male member of the group, Jason, seemed to be shut out by this common posture and found it very difficult to gain attention (panel 1) until he adopted a similar posture (panel 3). A male-female dynamic or other social situation, may account for this early barrier to Jason’s inclusion, and he may not be aware of his own change in posture during the process, but in order to participate he appears to need to connect through posture first.

		
1. Common posture	2. Initial gesture	3. Gesture mimic
		
4. Repeated gesture	5. Adjacent gesturing	6. Closing gesture space
Table 2: An example of gesture mimicry within a group		

The group shown in Table 3 also showed signs of gestural mimicry, but in this case it was rare. Panel 5 illustrates the only clear mimicked gesture, a cutting motion used in conjunction with talk of division. A common deictic gesture, as shown in panel 3, seemed to serve the similar purpose of connecting the group while talking. While there were other gestures which were repeated by different members of the group, such as the spread fingers shown by the girl on the left side of panel 5, these may or may not be mimicked gestures since they occurred more than two turns after the initial gesture.

A second example of posture mimicry is illustrated in table 3. Panels 1 and 2 show three of the group have adopted a pose while the fourth student has become disengaged, initially standing while the others leaned, and then a different student sitting while the others stood. Throughout this problem session the group came together in this way, either in pairs, as a threesome, or all together whenever they were successfully sharing something about the problem (as indicated by the conversation transcript). The common posture varied, as shown between panel 1 and 2, but was generally shared by the members of the group. There were occasions when a student stepped back from this shared gesture space, as illustrated in panel 4. This was followed by a return to the group posture, perhaps when the student felt they had something to share, or had given up on an idea.

		
1. Posture mimicry	2. Participation involves mimicry	3. Deictic gestures
		
4. Independent thinking?	5. Mimicking a cutting gestures related to division	
Table 3:		

SUMMARY

Of twenty recordings analysed there were twenty-one clear incidents of gesture mimicry where students reproduced a given gesture exactly within two turns at talk. In four of the twenty recordings no clear gesture mimicry was observed. Only two recordings demonstrated no posture or gesture mimicry and in both of these recordings the students made little progress with the problem. In all cases gesture mimicry accompanied conversational adjacent pairs rather than an isolated utterance. Groups generally demonstrated several adoptions of posture mimicry and, in all but one case, this coincided with on-task work and resulted in progress with the problem. Gesture mimicry tended to be associated with actions, such as the description of shapes or objects, or mathematical operations such as divide, increase and counting. Very little mimicry was associated with student activities centred on calculating. In seven of the recordings the students were standing and in these recordings gesture mimicry was seen in six cases. These tended to involve a larger gesture space than when the students were seated. There was only one case involving three students mimicking gestures in succession. Generally, only pairs of students mimicked gestures whereas posture mimicking tended to involve more members of the group.

Overall, mimicked gestures clearly occurred but were not seen to be used extensively while students were working on the mathematical processes. Gesture mimicking was predominantly used, and seemed important, in establishing the situation in which the mathematics was framed. When gesture mimicking was observed as related to the actual mathematics, the gestures were seen to represent ‘cutting’ (as in division), ‘framing’ (as in framing a shape such as a circle), ‘counting’ (particularly the action of skip counting using a bouncing motion) and a ‘this-and-that’ gesture where the flat hand was rotated at the wrist in a back and forth motion (as in referring to two cases). The predominant gesture seen during discussion about mathematical processes was deictic, with students pointing to the pages being working on. While these gestures often looked similar, there is not enough evidence to suggest mimicking, given the limited variations of pointing. Table 3, panel 3, illustrates this type of gesture.

This study indicates that posture imitation is an important part of group work. When students were working productively on a problem, or exploring an idea together, they tended to imitate each other’s posture, whether standing or sitting. These common postures shifted throughout the working session and demonstrated enough variation to indicate that it was not merely coincidental. When a student opted out of the common posture they rarely added to the thinking of the group, or their attempted contribution was less well-received. In some cases it appeared that a student removed themselves from the group so that they could think through a situation independently as in these cases the student self-gestured (table 3 panel 4) before re-joining the group. In just over half of such cases the students made a positive contribution to the group. In other situations a student moved out of the group and showed no signs of thinking independently about the problem (i.e. using some kind of self-gesturing or facial expression); in none of these cases did the student return to offer anything new.

The study suggests that mimicked gestures can play a role in creating a mutually shared understanding of the situation within which the problem is set. The mimicked gestures may help to coordinate a mathematical process amongst the group so that mathematic actions are seen to be agreed upon. This communication of acceptance in a process has been seen as a core step in the process of reaching a shared understanding in dialogue (Clark and Wilkes-Gibb, 1986). While gesture-mimicking may not be significant in advancing the mathematical process itself, it may be seen by the interlocutors as an acceptance that the speaker is understood and seen to be making progress. Gesture mimicry is part of the collaborative process but relies on the belief of the interlocutors that they have interpreted each other’s’ intent in the same way. It must also be noted that such gesturing may be subject to interpersonal relationship issues. Students with a strong rapport with each other may be more likely to mimic gestures.

In conclusion, analysis of the recordings of student work provides evidence that students mimic each other’s posture when being collaborative, and also mimic each other’s gestures as a means to establish a common process. As such, mimicked gestures may play an important part in helping to establish a shared understanding amongst the interlocutors and assist in progression of the collaborative effort. Given this possibility, there is an opportunity for teachers’ observing from afar to recognise good opportunities to intervene in order to best facilitate the group’s progress. When a group is seen to mimic each other's posture or gestures then this may be an indication to stay away from the group and allow them to continue to develop their ideas. If there is no evidence of such mimicry then that may indicate a good time to offer support to the group. This result may also tie in with the findings of Gerofsky (2008), in being another observable feature that students who are more confident of their ideas tend to use larger gestures.

References

Clark, H. H., & Wilkes-Gibbs, D. (1986). Referring as a collaborative process. *Cognition, 22(1)*, 1-39.

Ding, M. L. (2007). Teacher interventions in cooperative learning mathematic classes. *Journal of Educational Research*, 163–175.

Gerofsky, S. (2008). *Gesture Diagnosis and Intervention in the Pedagogy of Graping: Pilot Studies and Next Steps*. Retrieved from Academia: http://www.academia.edu/999238/Gesture_as_diagnosis_and_intervention

Goodwin, C. (2000). Action and embodiment within situated human interaction. *Journal of Pragmatics,32(10)*, 1489–1522.

Gordon-Calvert, L. M. (2001). *Mathematical conversations within the practice of mathematics.* New York: Peter Lang.

Holler, J., & Wilkin, K. ((2011)). Co-Speech Gesture Mimicry in the Process of Collaborative Referring During Face-to-Face Dialogue. *Journal of Nonverbal Behavior, 35*, 133–153.

Jaworski, B. (2004). Grappling with complexity: Co-learning in inquiry communities in mathematics teaching development. *In M. Høines & A. Fuglestad (Eds.), Proceedings of the 28th conference of PME* (pp. 17–36). Bergen: Norway.

McNeil, D. (2005). *Gesture and Thought.* Chicargo: University of Chicargo Press.

Radford, L., Edwards, L., & Arzarello, F. (2009). Introduction: Beyond Words. *Educational Studies in Mathematics (70)*, 91-95.

Roth, W.-M. (2000). From Gesture to Scientific Language. *Journal of Pragmatics, 32*, 1683-1714.

Roth, W.-M., & Radford, L. (2011). *A Cultural-Historical Perspective on Mathematics Teaching and Learning*. Rotterdam: Sense.

Sfard, A. (2001). Learning mathematics as developing a discourse. *In R. Speiser, C. Maher, C. Walter (Eds), Proceedings of 21st Conference of PME-NA* (pp. 23-44). Columbus, Ohio:: Clearing House for Science, mathematics, and Environmental Education.

Sfard, A. (2009). What's all the fuss about gestures? *Educational Studies in Mathematicsl, 70*, 191–200.

Sfard, A., Forman, E., & Kieran, C. (2001). Cognition as communication: Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, and Activity 8(1)*, 42-76.

WHO SHOULD TEACH PROSPECTIVE MATHEMATICS TEACHERS: MATHEMATICIANS WITH EDUCATION BACKGROUND OR ONLY EDUCATION EXPERTS?

Jamshid Nejad, Masomeh

Simon Fraser University

Teachers' knowledge plays a vital role in developing students' academic achievements. Therefore, training prospective teachers for the purpose of developing their professional knowledge considers as one of the most important issues for teacher trainers. The question raises here is who should train prospective mathematics teachers, a mathematician who is an educator or an expert in education? In this research study, a mathematician who has experience of teaching prospective teachers had been interviewed to investigate how he tended to develop prospective teachers' professional knowledge.

INTRODUCTION

Teachers' knowledge has been the main focus of many educational researchers for more than twenty years. For some researchers, what is described as teachers' knowledge is dissimilar to what other educational researchers describe. In other words, researchers have looked at teachers' knowledge from different angles. For instance, Shulman (1986) (who is the main focus of this paper) divided teachers' knowledge into three groups: subject matter content knowledge, pedagogical content knowledge and curricular knowledge.

Since introduction of Shulman's model, many research studies focused on teachers' knowledge (especially content knowledge) and its influence on teaching and learning. For example, a number of researchers (such as Ball, 1991, Berenson et al., 1997, Hill, Rowan and Ball, 2005, Baumert and Kunter, 2010, Mullens, Murnane and Willett, 1996, Sherman, 1992 and Simon, 1993) investigated the ways of developing subject matter content knowledge of teachers and its impact on teaching and learning. On the other hand, other research studies (such as Hill, Rowan, and Ball, 2005, and Mullens et al. 1996) focused on the influence of teachers' content knowledge on students' learning and their mathematics achievements.

Besides the effect teachers' content knowledge had on pupils' math scores, Baumert, et al. (2010), Borgen (2010), Charalambous (2010), Hill, & Ball (2009) and Hill, Blunk, Lewis, Phelps, Sleep, and Ball (2008) investigated its influence on the quality of teachers' teaching methods and instructions. In other words, research has shown that those teachers who have more mathematical knowledge would have different and more effective teaching styles than those who do not have enough mathematical content knowledge. The question raised here is “how teachers’ knowledge could be developed and what role educators might play in developing teachers’ knowledge.”

There are some researches (e.g. Ball, 2008) indicated that the number of university courses teachers taking during university would directly influence teachers’ knowledge development. However, there is no research available to investigate how educators are often teaching prospective mathematics teachers and whether it is important that mathematics prospective teachers have been trained by educational educators, mathematicians or educational educators who are mathematicians too. To figure it out, the initial step is investigating the influence of rich mathematics knowledge of teacher trainers on developing prospective teachers’ professional knowledge. As such, in this research, a teacher trainer who is a mathematician has been interviewed to investigate the main objectives he often pursues while teaching prospective teachers.

Methodology

A mathematician, who has experience of teaching prospective teachers for more than a decade in a large Canadian university, is the only subject of this research. He has been selected since he often applies his rich mathematical ideas when teaching prospective mathematics teachers and also because he is the only mathematician educator at education department. In a two hours interview, he had been asked about his ultimate goals of using his mathematics background during teaching practice. The main focus of interview questions were around teaching trigonometry as some research (e.g. Orhun, 2011 and Weber, 2005) indicated that this topic is a challenging concept for teachers and students as well. To analyse the data, I listened carefully to the audio recordings and then I transcribed them. I noticed several themes the participant emphasized on and, then, I assign a section for each theme in this paper. My reflections on the themes and their relationship with literature review which is related to mathematics teachers' knowledge will be described in each part.

Theoretical framework

A theoretical framework for this paper is related to Shulman’s classification of teacher’s knowledge. Shulman (1986) divided teachers’ knowledge into three groups: subject matter content knowledge, pedagogical content knowledge and curricular knowledge. Shulman (1986) defined subject matter content knowledge as “knowledge of discipline, facts, concepts, principles, structures and ways in which they are developed.” In other words, he indicated that not only do teachers need to know the concepts but also that teaching requires going beyond knowledge of facts and concepts.

The second category of teachers’ knowledge is Pedagogical content knowledge. Shulman referred to pedagogical content knowledge as “the ways of representing and formulating the subject that make it comprehensible to others” (1986, p. 9) including …the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the most useful ways of representing and formulating the subject that make it comprehensible to others. . . . Pedagogical content knowledge also includes an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons" (p. 9). In other words, Shulman's definition of pedagogical content knowledge is a combination of both content knowledge and pedagogy knowledge.

The third type of knowledge that Shulman (1986) defined is curriculum knowledge. He indicated that curriculum knowledge is knowledge of available instructional material such as textbooks and curriculum as well as knowledge of the topics taught and how they were addressed in the previous years' books so as to incorporate this into the following years (Rowland & Ruthwen, 2011).

Designing appropriate mathematics tasks

Mathematics tasks play a big role in learning mathematics, so that teachers are responsible for designing appropriate tasks for students of different level of interest and need (Crespo, 2003). During the interview, the participant mentioned that his current prospective teachers need opportunities to improve their sense of being mathematics teachers since they are still students the transition from students to teachers is difficult. He indicated that, therefore, one of the important responsibilities of every educator is making them familiar with their teaching duties. The participant continued that every successful mathematics teacher needs to know how to adapt and generate mathematics tasks. He added that:

> Elizabeth: .... I like tasks that have a number of ways of being approached, so there are not automatic solutions. So in this course, I choose tasks for students to work on themselves and then I talk about what I see as mathematics in those tasks. ...for example, working on square grid dot papers. I might talk about the question of what is a circle on this grid, how many points can you get on a circle, which is a question that has no meaning.

It is noticed from the above excerpts that the participant tend to improve prospective teachers' sense of being a teachers through engaging them in open-ended mathematics tasks followed by a discussion about the mathematics facts embedded in the given tasks. It appears that the participant interested to develop prospective teachers' pedagogical content knowledge (designing appropriate mathematics tasks) when applying his rich content knowledge during the discussions (Shulman, 1986).

Furthermore, it is believed that teachers besides being teachers, they are always learners who need to update their knowledge. In other words, as Garmston and Wellman (1998) argued teachers are influenced by the extent of their repertoire of teaching methodologies and their ability to experiment with their own practice, by working through a learning cycle of: activity, reflection and evaluation, extracting meaning from this review, and planning how to use the learning in future. In particular, when teachers plan for students' learning, their "bag of tricks" includes tasks and processes to promote active learning, collaborative learning, learner responsibility and learning about learning, and skills related to handling relationships.

In this particular study and during the interview, the participant expressed that prospective teachers often have the lack of situating themselves as learners. He continued, therefore, prospective teachers need to participate in activities focused on posing mathematics tasks.

Interviewer: What do you think teachers do not know and you wish them to know?

Elizabeth: They come from mathematics departments and I think with a range of technical knowledge but often with no strong sense of mathematical learners. One of the things I do is work on mathematics together to give them a better sense of mathematics knowledge. I have them write up both a mathematical task to explore mathematics and also what they notice about themselves as a learner, where they are stuck, what they notice when working on problems.

From the above excerpt, it is noted here again that the participant's background knowledge as a mathematician and a mathematics educator helped him to design an educational course to benefit prospective teachers. He, in fact, was interested to invite his prospective teachers to "write up a mathematical task" and then as a whole group have a discussion about it in order to not only develop their content knowledge as still learners (to better sense of mathematics knowledge) but also to explore any difficulties they students may face. In other words, the participated mathematician employed his rich mathematics/educational background to develop his prospective teachers' content knowledge as well as pedagogical knowledge (exploring pupils' difficulties is a pedagogical matter according to Shulman' category).

Broadening prospective teachers' horizon about mathematics

It is indicated that prospective teachers require developing the skills of widening their view about the variety of ways through which mathematics facts could be proved rather than following those written in mathematics textbooks. This is the case since ability to re-think about mathematics concepts from different angles would benefit students because it would provide pupils with different abilities and interest with opportunities to learn through the ways are more understandable for them. In other words, teachers cannot teach students by focusing on the same strategy for all students.

In order to extend prospective teachers' learning and developing this skill, the participated mathematician added that:

Elizabeth: To show them how the mathematical results that are usually in the sellubes are tools for solving problems and not end-points themselves....., asking them to discuss about the formula of the area of triangle…......during the class discussion, one of the students, for example, came up with the idea of looking at a triangle with all three heights drawn in and then showing how each height divided the area of the triangle into three right angle triangles...

The participant was eager to provide prospective teachers with opportunities of being involved in a rich class discussion through which they could look at mathematics with different lenses. It is believed that leading a rich discussion often requires a leader with strong related knowledge. If leaders have lack of knowledge in the area, the classroom conversations would not end with learning. In other words, the participant’s higher mathematics knowledge enabled him to lead his prospective teachers properly and, as it is clear from the last excerpt, the conversation wrapped up with a great strategy to prove the formula of the area of triangle. He not only used his mathematics training to increase his prospective teachers' content knowledge, which agrees with the results of Watson and Harel (2013) (teachers who have advanced mathematical knowledge have more successful students), he also developed their knowledge of teaching that is under the category of pedagogical content knowledge too (Ball et al., 2008).

Furthermore, re-thinking about mathematics concepts and the order in which they should be taught could enhance prospective teachers’ capacity to take appropriate decision making. When the participant was asked about how he teaches trigonometry, which is described as one of the most challenging topics for teaching (Weber, 2005), he expressed:

Elizabeth: ...In my course, I was using trigonometry as an example to show them how you do not have to teach the school curriculum in the way it has been traditionally taught. I like them to be flexible about curricular order… I am interested to the quality of discussion of sin and cos..., before getting a numerically ac curate graph.... . You could just read it of this triangle and another place to start is with the unit circle especially if students know where tangent line is…One figure is always twice the area of the other.

It seems from the above excerpt that the participant was interested to develop prospective teachers’ curricular knowledge (one of the knowledge classification of Sulman) since he found teaching trigonometry based upon following curriculum order a challenging topic for undergraduate students based on his mathematics teaching experience at undergraduate level, he wanted to give the examples of the flexibility a teacher could have in teaching trigonometry.

Final marks

In analysis of the interview with the participant, who is a mathematician and has teaching experience of teaching educational courses in a large university in Canada, I noticed several themes in which he used his rich background knowledge. Analyze of the data illustrate that the mathematician participated in this study applied both his rich mathematics and educational knowledge to develop their pedagogical, curricular and content knowledge.

To enhance his prospective teachers’ skills of generating appropriate mathematics, he gave them a set of mathematical tasks that could be solved through various ways (he called them open-ended mathematical tasks). He expressed that it is a proper way of developing prospective teacher’s sense of being mathematics teachers since ability to design mathematics tasks is an important skill all teachers required to have. It seems that the participant implemented his mathematics knowledge to improve his prospective teachers' content knowledge as according to Shulman (1998) teaching knowledge model, asking for reasoning and posing open-ended mathematics tasks are the special skills for teachers.

He also stated that often mathematics teachers are graduate students from the mathematics department who do not have a strong sense of being mathematical learners. He indicated that having this sense is important because teachers need to experience how a mathematics task could be solved and the challenges they may face. To do so, he asked them to generate a mathematics task and asked them to think deeply about the mathematics embedded in the task and the way it could be solved. In fact, he applied his background knowledge to improve the pedagogical content knowledge (knowledge of teaching) of his students.

Furthermore, increasing prospective teachers' opportunities for rich mathematics discussion is another theme I noticed that the participant used his knowledge for. He expressed that how giving the prospective teachers a formula (e.g. area of triangle) and asking them to discuss and prove it through other ways than those mentioned in the textbooks would result in a great conclusion. Again, his knowledge let him to generate appropriate mathematics facts and to lead his prospective teachers' discussion, because a good discussion requires an expert leader (in mathematics). As it is clear, the participant was eager to develop prospective teachers’ pedagogical knowledge through this activity.

From the interview, I also found that the participant rich knowledge of mathematics and teaching provided him with occasions to enhance his prospective teachers' decision-making opportunities. In other words, he gave prospective teachers different examples of teaching a particular concept of trigonometry to show them how flexible they need to be while making decisions in their classrooms. He wanted to show them that they need to know different ways of proposing a particular topic to be ready for unanticipated situations.

In this study, the participant who was a mathematician and had experience of teaching prospective teachers for many years tended to develop prospective teachers' content knowledge, pedagogical content knowledge as well as curriculum knowledge by applying his background knowledge. Future research is needed to investigate how educators (in general) who are not mathematicians could enhance prospective teachers' knowledge. Also, it would be interesting to investigate how educators (in general) could develop prospective teachers' knowledge compared with mathematicians who have expertise in education.

References

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? Journal of Teacher Education, 59(5), 389–407.

Bordie, K.(2010). Teaching mathematical reasoning in secondary school classrooms. Springer, Science+ Business media, LLC.

Charalambous, C.Y. (2010). Mathematical Knowledge for Teaching and Task Unfolding: An

Exploratory Study. Elementary School Journal, 110(3), 247-278.

Garmston, R. & Wellman, B. (1998, April). Teacher talk that makes a difference. Educational Leadership, 55 (7), 30-34.

Hill, H.C., & Ball, D. L. (2004). Learning mathematics for teaching: Results from California's

mathematics professional development institutes. Journal for Research in Mathematics Education, 35 (5), 330-351.

Hill, H. C., Rowan, B.,&Ball, D. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. American Educational Research Journal, 42(2), 371–406.

Hill, H., Ball, D. L., & Schilling, S. (2008). Unpacking "pedagogical content knowledge": Conceptualizing and measuring teachers' topic-specific knowledge of students. Journal for Research in Mathematics Education, 39(4), 372-400.

Mullens, J. E., Murnane, R. J., Willett, J. B. (1996). The contribution of training and subject matter knowledge to teaching effectiveness: a multilevel analysis of longitudinal evidence from Belize. Comparative Education Review, 40, 139-57.

Orhun, N. (2001).Student's Mistakes and Misconceptions on Teaching of Trigonometry, Mathematics Education Into The 21st Century Project Proceedings of the International Conference New Ideas in Mathematics Education, 19-24 August 2001, Australia.

Sherman, J. G. (1992). Reflections on PSI: Good news and bad. Journal of Applied Behaviour Analysis, 25, 59 – 64.

Simon, M. (1993). Prospective elementary teachers' knowledge of division. Journal for Research in Mathematics Education, 24, 233-254.

MATHEMATICAL ABSTRACTION AND TEACHING CHALLENGES: TEACHING ACTIVITY THROUGH THE LENS OF REDUCING ABSTRACTION IN TEACHING (RAIT)

KRISHNA SUBEDI

Simon Fraser University, Burnaby, Canada

Mathematics is an abstract subject. One of the most important challenges for mathematics teachers therefore involves the task of dealing with mathematical abstraction and figure out ways of translating them into understandable ideas for their students. By analysing teaching episodes through the lens of reducing abstraction in teaching (RAiT), this paper explores the notion of mathematical abstraction and illustrates various strategies and tendencies of teachers dealing with mathematical abstraction.

1. INTRODUCTION

Abstraction has been an object of discussion across several discipline. Particularly in philosophy and in philosophy of mathematics, it has been the central topic of intense inquiry from the days of Plato and Aristotle. As Hershkowitz, Schwarz & Dreyfus (2001) put it, "not only did Plato and his followers see in abstraction a way to reach "eternal truths," but modern philosophers such as Russell (1926) characterized abstraction as one of the highest human achievements" (p.196). In this paper, rather than providing the detail review of research on abstraction in philosophy and other discipline, I focus on the notion of abstraction as used in mathematics education and its implication in teaching and learning. I then provide a brief overview of theoretical framework of Reducing Abstraction in Teaching (RAiT) followed by methodology. Finally, the result and discussion are presented.

1.1 Abstraction in mathematics education and the teaching challenges

Abstraction in the mathematics education has long been discussed by educators and researchers (e.g. Piaget 1970; Dienes, 1989). Piaget (1970) observed that abstraction is the skills required for learning elementary through advanced mathematical concept. Ferrari (2003) also noted the role of abstraction in mathematics learning and stated that "abstraction has been recognized as one of the most important features of mathematics from a cognitive viewpoint as well as one of the main reasons for failure in mathematics learning" (p. 1225). As such, the issue related mathematical abstraction in teaching and learning mathematics has long been the topic of discussion among researchers and mathematics educators.

Since mathematical concepts are abstract in nature as commonly understood, the debate on what is the effective way of teaching mathematics- whether proceed from abstract to concrete or the other way around has been an important topic of discussion in mathematics education. The abstract- concrete order of learning seems to rely on two assumptions. The first is a Platonic philosophy in which mathematics is viewed as an objective reality existing in the platonic realms, which is not accessible to our senses but can be revealed by good teaching (Ernest, 1991). The second is based on the assumption that "knowledge acquired in 'context-free' circumstances is supposed to be available for general application in all contexts" (Lave, 1988, p. 9). However this view of teaching as the transfer of abstract, decontextualized mathematical concept has proven to be ineffective and "much of what is taught turns out to be almost useless in practice" (Brown, Collins & Duguid 1989, p. 32).

What is the effective way of teaching mathematics then? Should we proceed from abstract to concrete or the other way around? Davydov's (1990) dialectical materialistic account of abstraction offers a way of resolving the debate revolving around the trajectory of how to teach abstract mathematical concept. Davydov (1990) thought that the concrete is correlated with the abstract and learning does not follow the trajectory from concrete to abstract but a dialectical, two way relationships between the concrete and the abstract. In his view, abstraction process results in the discovery of the essence which ultimately needs to ascend back to the concrete.

In Wilensky's (1991) account of abstraction, the debate on the teaching trajectory of abstract- concrete or concrete -abstract order as commonly understood collapsed and takes a different turn. Instead of locating the assessment of abstraction solely in the object, he redefined abstraction as the relationship between the person and object of thought thereby promoting a more subjective conception of abstraction. According to Wilensky (1991), mathematical concepts are neither more nor less abstracts in their own right; it depends on the internal connection of the learner's with the concepts. He further said, "concepts that were hopelessly abstract at one time can become concrete for us if we get into the 'right relationship' with them" (Wilensky, 1991).

2. LITERATURE AND THEORETICAL FRAMEWORK

Much has been written about abstraction in mathematics education and how mathematics can be taught well and poorly, particularly in school level. However the study that aimed to examine the mental process of learners while dealing with abstraction in learning new mathematical concepts is slim. Most notable study in this camp is carried out by researchers such as Hazzan (1999), Raychaudhuri, (2013), Hazzan & Zazkis (2005). These researchers examined learners' mental tendency with regard to dealing with mathematical abstraction and found that learners have a mental tendency of reducing level of abstraction of a task or concept while learning new a concept. There is however no study found in my literature review which aimed to look specifically at how teachers deal with abstraction in teaching.

Having reviewed the literature, I was convinced that both students and teachers reduce abstraction level in learning and teaching respectively, but for different reasons. Hazzan (1999) for example, found that since learners do not have sufficient mental resources *"to hang on"* to cope up with the same level of new mathematical concept as intended by authorities (such as teacher, textbook), they tend to reduce the abstraction level to make it mentally more accessible which often happens *unconsciously*. In contrast, reducing abstraction in teaching is a conscious and intended act. From teachers perspective, the choice of the words and phrases such as *'unconscious'*, '*lacks of the mental construct', 'to hang on to'* are problematic. The assumption here is that teachers are the experts and usually have sufficient mental resources to deal with the abstraction of a mathematical concept in the same or even higher level as given in the textbook. So, the assumption here is that teacher reduces abstraction appropriately in contrast to their students and therefore reducing abstraction in teaching has pedagogical value.

This shift in perspective necessitated a different interpretation of reducing abstraction in teaching. Hence, taking the ideas from literature, particularly the ideas from Hazzan's (1999), Wilensky (1991) and Sfard (1991) in to account; the notion of reducing abstraction was redefined and reinterpreted from teacher's perspective. In so doing, a modified framework of reducing abstraction emerged which I call Reducing Abstraction in Teaching (RAiT). Detail discussion of RAiT framework is out of the scope of this paper. I however provide a brief overview of the framework.

The thematic categories and subcategories of the framework of"Reducing Abstraction in Teaching" (RAiT) are given below:

Category 1: Abstraction level as the quality of the relationships between the mathematical concept and the learner

Teachers task implementation behaviour in which teacher makes an attempt to establish a right relationship (in the sense Wilensky, 1991) between the students and the abstract mathematical concepts.

Subcategories:

1a) Reducing abstraction by connecting unfamiliar mathematical concept to real- life situations

1 b) Reducing Abstraction by Experiment and Simulation

1 c) Reducing abstraction by Storytelling

1d) Reducing abstraction by using familiar but informal language rather than formal mathematical language

1e) Reducing abstraction by the use of pedagogical tools (such as model, manipulative, metaphor, analogy, gesture etc.)

Category 2: Abstraction level as reflection of the process-object duality

Teachers' task implementation behaviour in which teacher shifts the emphasis to a process or correctness of the answer rather than the concept itself is considered as reducing abstraction in this category.

Subcategory:

2a) Teacher reducing abstraction by shifting the focus on procedure

2b) Reducing abstraction by shifting the focus on end-product (answer).

Category 3: Abstraction level as degree of complexity of mathematical problem/concept

Here abstraction is determined by the degree of complexity. Hence, teacher's task implementation activity in which teacher attempt to reduce the complexity of the problem in various ways was considered as reducing abstraction in this category.

3a) Reducing abstraction by shifting focus on particular rather than general

3b) Reducing abstraction by stating the concepts rather than developing it.

3c) Reducing abstraction by giving away the answer in the question or provide more hints than necessary (Topaze effect- See Brousseau, 1987)

Although I considered the three interpretation of abstraction from Hazzan (1999), I would like to emphasise that these three categories of reducing abstraction in RAiT are defined from different perspectives than found in Hazzan's work. In Hazzan's work reducing abstraction is concerned with students' mental action and their coping strategies where as RAiT focuses on teachers' action with regard to dealing with abstraction in teaching.

3. METHODOLOGY

While selecting the research methodology appropriate to the aim of this study, I considered Bogdan and Biklen (1998) suggestion that if a study aims to "better understand human behaviour and experience" (p. 38) quantitative positivist approach is of little help. Since this study also involves human and social phenomena, I began to investigate qualitative research methodology which led me to modified analytical induction. Modified analytic inductions is a methodology that allows for the researcher to begin with her/his pre-held theoretical perspectives (often emerged from the literature) and test the theoretical frameworks with empirical cases and thus amend and improve the theory (Bogdan & Biklen 2007).

This paper reports a part of the preliminary result of the larger study for which the strategy for gathering data consisted of my observation of nine college preparatory classes (each lasted about an hour and half) taught by three different instructors , who were well experienced, and professionally trained mathematics educators. I attended the classes, audio taped all the classroom interaction and then transcribed. As much as possible, I also noted down all the phrases, statements or sentences the instructors used to explain the concepts including some observable behaviour such as ‘gestures’ and students’ responses that I found relevant for my study. Due to the risk of influencing the natural classroom situation, I avoided the video recording. I now present one representative example with analysis and interpretation using (RAiT) framework.

4. RESULT AND DISCUSSION

The Box Problem:

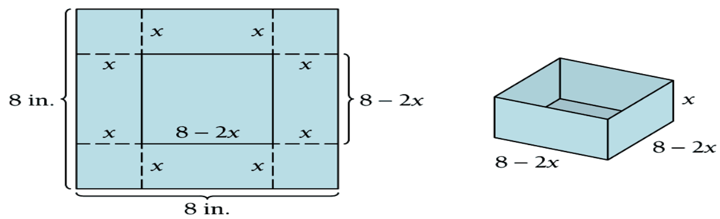
The objectives of the lesson was to introduce the concept of polynomial function of degree three or greater and sketch their graphs using leading coefficient, x- intercepts and the multiplicity of the zeros. The class began with a very brief review of quadratic function. The teacher then put up the following box problem in the board and instructed students to work in a group of three or four and try to find the equation to model the volume of the box. Because of the space limitation, I illustrate only few tendencies of teacher’s reducing abstraction while implementing the box problem.

Question: *Suppose that you are a manager of a packaging company that manufactures identical rectangular boxes from square sheets of card board each sheet having the dimension 8 inches by 8 inches. To save money, you want to manufacture boxes of maximum volume by cutting out a square of x inches by x inches from each corners of a sheet and then folding their sides up to make an open toped box. What length should you select for x in order for the maximum possible volume of the box?*

Polynomials in a box:

All the students were discussing with their neighbours. Each group seemed to agree on the fact that the height of the box would be x. The debate was however on the dimension of the bases of the box, particularly on whether each dimension of base of the box would be $8 - x$ or $8 - 2x$. After about two minutes or so, while students were still working on the problem, the teacher handed out a card board, a pair of scissors and a masking tape to each group. Each group cut the square corner and folded the side to make a box. At this point everyone seemed to be convinced that each dimension of the base of the box is actually $8 - 2x$ and hence the function representing the volume of the box is given by: $\text{Volume}(V) = \text{ length x width x height} = (8-2x).(8-2x).x = 4x^3 - 32x^2 + 64x$ (RAiT-1d).

The teacher then put the following figure on the board .



The teacher then moved to the idea of the polynomial function and its degree. She wrote the definition of the general form of polynomial functions in the board as follows:

Let n be a nonnegative integer and let $a_n, a_{n-1}, a_{n-2} \ldots a_2, a_1, a_0$ be real number with $a_n \neq 0$. The function defined by

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$$

is called a polynomial function of x of degree n.

She mentioned that the function V(x) is also a polynomial function of degree 3. At this moment, there was a realization among students that the polynomial functions are not an abstract entity but something that relates to the real world situation- Polynomial in a box. (RAiT-1a).

Leader, attitude and the shape of the graph of polynomial

After briefly discussion on theoretical and practical domains of the polynomial in this context, the teacher then shifted the focus to the concept of leading term, relationship between leading term and the *end behaviour* of the graph. Considering the function $f(x) = 4x^3 - 32x^2 + 64x$, she told that the term with highest degree is called the leading term and hence in this example the leading term is $4x^3$ and the leading coefficient is 4. She explained that *end behaviour* of the graph as the nature of the graph (goes up or down) of a function to the far right as $x \to \infty$ and to the far left as $x \to -\infty$. The dialogue continues:

T: OK guys, as you have seen in the last example (pointing to the function $V(x) = 4x^3 - 32x^2 + 64x$ that the graph of third degree polynomial is like this (left hand up and right hand down). I will tell you one thing right now. To find the end behaviour of the graph, what you need is just the leading term, ok, the term with highest degree. In this function (pointing to $V(x)$) what we need to look at is just the leading term $4x^3$. We don't care about the other terms. Ok.

S: Ms. I don't get it. Why you don't care the other terms?

T: I mean, when x goes to positive or negative infinity (*writes in the board,* $x \to \infty$ or as $x \to -\infty$) the function $V(x) = 4x^3 - 32x^2 + 64x$ and the function $f(x) = 4x^3$ behave the same way. There is a theorem that tells that (she writes in the board) if $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$ is an n degree polynomial function of x, then the "end behavior" of P(x) is the same as that of

$y = a_n x^n$. Ok.

T: Now I will tell you how to figure out the shape or the end behaviour of different type of polynomial functions. If the degree is odd, then think of "odd" as meaning "different". OK. That means the end part of the graph on the left and the right must go in the different directions like this or like this (with hand gesture) .OK.

T: Now if the degree is even, think of "even" as meaning "the same" . For example, if two players are "even", in a game, what would you think? They have the same score, right. This will help you to remember that if n is even, both ends of the graph must go in the same direction like this (both hands up) or this (both hands down).

S: But how to find which one is which?

In response to the students question, the instructor explained by using her hand gesture and the odd and even metaphor that the direction of the end part of the graph can be determined just by looking at the degree (odd or even) and the sign (+ or -) of leading coefficient. The class seemed to be really interesting as everyone was actively participating in modelling the end behaviour of the graph using their hands. However, the later discussion shows that some of the students still struggling in grasping the concept.

The teacher wrote the following two functions in the board: 1) $f(x) = 2x^3 + 2x - 3$ and 2) $g(x) = 3\,x^5 - 4\,x^2 + 6$ and goes near to one group:

T: Ben, can you tell me what is the end behaviour of the first function?

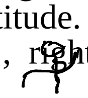
Ben: It's like this (gesture)  Oh, wait a minute, it's like this  Umh.., I forgot, I really don't get it.

When she realized that students were still struggling to remember the end behaviour, she used another strategy to explain it:

T: alright guys, think this way. Your leading term is the leader or manager of company, OK. If the leader of a company is positive or say has a positive attitude, the business finally goes up even though it was down before, right. And If the leader is negative or say, has a negative attitude, the business goes down even though it was up earlier, right.

SS: Oh, I see!

T: Alright. Ben, do you want to try the second function?

Ben: Ok. so the leading term is $3\,x^5$. So, 5 is odd and 3 is negative. so the guy (the guy, he meant the leader or manager) has negative attitude. So, the business finally goes up . The graph is like this (gesture)  , right?

T: Yes Ben, you are absolutely right.

Ben: Oh, I get it!

T: Remember guys, that the positive and negative leader metaphor works for even degree of function as well. OK. If the leading coefficient is positive, uhmm.., both ends go up like this (raised her both hands up) and if the leading coefficient is negative, both ends go down like this (raised down her both hands). Ok, now let's try few more functions.

The teacher writes few more functions with odd degree and even degree and with both positive and negative leading coefficient. Every student seemed to be able to answer end behaviour of the functions correctly. In fact, the teacher's use of leader of a company as a leading term of a polynomial metaphor, and her gesture reduced the abstraction of the unfamiliar and abstract concept of end behaviour of polynomial functions thereby making it a familiar hand motion idea ((RAiT – 1d). Then the teacher moved to the concept of zeros and multiplicity discussion of which is not presented in this paper because of the space limitation.

One of the points to note however is that in response to the students query regarding the end behaviour of polynomial and the end behaviour of leading term, she did not developed the concept but stated it (RAiT-3b) by saying that if $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ is an n degree polynomial function of x then the "end behaviour" of P(x) is the same as that of $y = a_n x^n$. As a result, some of the students developed rather vague and inappropriate rule as I observed in one of the students work who was sitting next to me . To my query to his work with regards to the function $f(x) = 3\,x^5 - 4\,x^2 + 6$, he mentioned his approach for this rule is the following: $as\ x \to \infty,\ g(\infty) = 3\,\infty^5 - 4\,\infty^2 + 6$ becomes positive infinity because any positive quantity to the power odd becomes positive and the first term has bigger infinitive. Therefore $x \to \infty, g(x)$ equals positive infinity. Similar reason was given for g(x) as $x \to -\infty,$ and he concluded that $x \to -\infty,, g(x)$ negative infinity. In fact, if x tends to some finite value, say, $x \to 2$, the functional value of g(2) can be found by plugging in 2 for x in the function such $g(2) = 3\,(2)^5 - 4\,(2)^2 + 6 = 86$. But for $x \to \infty,$ we know that $3x^5 \to \infty\ and\ 4\,x^2 \to \infty$. However, we cannot simply add these two limits together to find the limit of g(x), since limits of the form $(\infty - \infty)$ is indeterminate.

This misconception could have been possibly avoided if the concept was developed instead of stated. For example, the polynomial function $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$ could be shown to equal $a_n x^n$ easily for very large $|x|$ by using simple algebra as follows:

$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$ can be rewritten as

For very large x , $P(x) = a_n x^n \left(1 + \frac{a_{n-1}}{a_n}\frac{1}{x} + \frac{a_{n-2}}{a_n}\frac{1}{x^2} + \ldots + \frac{a_0}{a_n}\frac{1}{x^n} \right) = a_n x^n,$ (since $\frac{1}{x^k} \to 0\ when\ x$ is very large) . Similar results hold for $x \to -\infty$. Hence for large $|x|$, $P(x) \approx a_n x^n$

To understand the reason for stating the concept rather than developing, I decided to talk to the teacher after the lesson. This informal interview shed some lights on this issue:

> You know, the time allotted for this course is too short. I have to cover so many things and I always run out of time. So, I decided not to go in detail of that concept. Moreover, the theorem that you are referring, the proof involves the limit concept and they do not have that concept yet. The limit concept generally introduced in calculus courses. If I had imposed the concept on them, many of them would find it too difficult to understand and lose their interest. I did not want them to develop any anxiety for the rest of the class just because of that theorem. After all, I told them that they won't be tested on the proof of the theorem in the exam. So, it was not that important to dig deep in to that concept.

Her views regarding reducing abstraction in this subcategory was that the time allotted for the course was too short and could not work on the concept in detail. Further, the concept was a complex or difficult one for the student at this level. She also pointed out the fact that student would not be tested on the proof of the theory behind it. This result supports the finding of other studies. Doyle (1988), for example found that high level tasks were perceived by the teachers (and students) as risky and ambiguous and therefore there was a tendency of reducing the complexity of the task so as to manage the accompanying anxiety.

5. CONCLUSION

The findings of this study suggest that while dealing mathematical abstraction in teaching, teachers often reduce abstraction level of the problem or concept whose main goal is to make the concept mentally accessible to their students. As a result, it is expected that such tendency of task implementation would enhance learning. However, the findings also suggest that reducing abstraction in teaching may not necessarily promote learning. Thus, the results emphasize the importance of paying attention to how the abstraction is reduced during task implementation as it is closely related to the students learning opportunity and nature of understanding and possible misconceptions.

Reference

Bogdan, R. C., & Biklen, S. K. (1998). Qualitative research in education: An introduction to theory and methods (3rd ed.). Needham Heights, MA: Allyn & Bacon.

Bogdan, R.C., & Biklen, S.K. (2007). Qualitative research for education: An introduction to theories and methods (5th ed.). Boston: Pearson Education

Brousseau, G. (1987). Theory of Didactical situations in mathematics 1970-1990. Translation from French M. Cooper, N. Balacheff, R. Sutherland & V. Warfield. Kluwer Academic Publishers, 304 p

Danesi, M. (2007). A conceptual metaphor framework for the teaching of mathematics. Studies in Philosophy and Education, 26 (3), 225-236

Davidov, V. V. (1990). Types of generalisation in instruction: Logical and psychological problems in the structuring of school curricula (Soviet studies in mathematics education, Vol. 2; J. Kilpatrick, Ed., J. Teller, Trans.). Reston, VA: National Council of Teachers of Mathematics. (Original work published 1972)

Doyle, W. (1988). Work in mathematics classes: The context of students' thinking during instruction. Educational Psychologist, 23, 167-180.

Ferrari P.L. (2003). Abstraction in mathematics. Philosophical Transactions of the Royal Society London B 358, 1225-1230.

Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. Educational Studies in Mathematics 40 (1), 71–90.

Hazzan, O. & Zazkis, R. (2005). Reducing abstraction: the case of school mathematics. Educational Studies in Mathematics, 58, 101–119

Hershkowitz, R., Schwarz, B.B. and Dreyfus, T. (2001). Abstraction in context: Epistemic Actions. Journal for Research in Mathematics Education 32, 195–222.

Lave, J. (1988). Cognition in practice: Mind, mathematics and culture in everyday life. Cambridge: The University Press

Piaget, J. (1970). Genetic epistemology. New York: W. W. Norton and Company.

Raychaudhuri, D. (2013). Adaptation and extension of the framework of reducing abstraction in the case of differential equations. International Journal of Mathematical Education in Science and Technology, (ahead-of-print), 1-23..

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. Educational Studies in Mathematics, 22, 1–36.

Wilensky, U. (1991). Abstract meditations on the concrete and concrete implications for mathematical education. In I. Harel, and S. Papert (eds.), Constructionism, Ablex Publishing Corporation, Norwood, NJ, 193-203.

THE RELATIONSHIP BETWEEN MOTIVATION, ACHIEVEMENT GOALS, ACHIEVEMENT VARIABLES AND MATHEMATICS ACHIEVEMENT

Ike Udevi-Aruevoru

Simon Fraser University, Burnaby, BC, Canada

The purpose of this study is to investigate the relationship between motivation, achievement goals, and achievement variables on academic achievement. Using the six variables from the motivation questionnaire (Glynn, et. al, 2009) and five other variables associated with academic achievement namely, academic achievement emotions – hope and pride, academic interest, academic achievement goal, and the importance of mathematics to future career goal, this study will use multiple regression analysis and also, path analysis to determine which of these variables accounts for the most variance in student's academic achievement.

Introduction

The connection between a person's level of education, income, social status, and the overall health and wellbeing of his/her family is transparently evident across all cultures. Education has become a vehicle for upward social mobility and perhaps, the quickest way of lifting families out of poverty within just a few decades. While many scholars may see education as a social activity, for many students, education is an economic activity; an investment by them for themselves. It is for this investment that many students travel thousands of miles to distant lands to pursue a good education. It is also for this reason that one's program of study is usually, not an arbitrary decision, but one made in consultation with friends, family and often, with career counsellors.

Quantitative research on the variables that impact student's academic achievement is currently pursued through two research frameworks. One through the relationship between a student's motivation and his/her academic achievement. This research is generally conducted using motivation questionnaires (Bryan, Glynn & Kittleson, 2011). Other studies have focused on the impact of academic achievement goals, academic achievement emotions, and other achievement variables on academic achievement. The variables most frequently referenced in these studies are student's perceived self-efficacy belief (Pajares, & Miller, 1994), academic achievement emotions – hope and pride (Pekrun, et.al, 2011), student's academic interest and academic achievement goals (Harackiewicz, J., M., et. al, 2008), and the importance of the subject of study (here, mathematics) to one's future career goals.

The research on motivation and academic achievement

The research on motivation and academic achievement centers on the premise that we can understand, describe and even predict a student's academic achievement based on her academic motivation as determined usually, through self-report questionnaires. For example, Bryan, et. al (2011) investigated the motivation of 288 public high school students (14-16 years old) to learn science using a three component motivation questionnaire on self-determination, self-efficacy belief, and intrinsic motivation; student essays about their motivation and also, an interview of a sample of the students. The student's academic achievement was accessed through their final grades in the assigned courses. In their findings they report that the correlation between self-determination, intrinsic motivation, and self-efficacy belief, with the students' academic achievement were 0.31, 0.37, and 0.56 respectively. They concluded that since self-efficacy belief has the highest correlation, that it was the most influential factor in determining academic achievement (p. 1060).

In a similar study, Kim, et. al (2012) investigated the factors that might impact student's achievement in mathematics courses offered online. They used self-report questionnaires on student's motivation that consists of self-efficacy belief and intrinsic motivation (Pintrich, & DeGroot, 1990), and academic achievement emotions (Pekrun, Goetz, & Frenzel, 2007). Student's academic achievement was accessed using their final grades in the assigned course. They used a multistep regression analysis to determine how each of these factors impacts student achievement. From their results, they report that initially, motivation accounted for about 13% of the variance in academic achievement with self-efficacy belief as a significant contributor. However, when achievement emotions were added to the analysis, they accounted for 37% of the variance and, self-efficacy belief was no longer a significant contributor. They wrote:
"Self-efficacy was not a predictor any more once achievement emotions were taken into account. Achievement emotions were useful in explaining student motivation and performance in online learning environments" (p. 2).

The research on achievement goals, emotions, and achievement variables

At the heart of the control-value theory of achievement emotions is how one answers two fundamental questions associated with academic achievement. The first is an appraisal of control; one's ability to control activities that determine achievement outcomes. The second question is that of value and significance. It is the diversity of how different students might answer these questions that accounts for the diversity of the emotions and motivations that many students bring to the same academic activity (Lazarus, 1991).

The research on self-efficacy belief

At the core of self-efficacy theory is the agency perspective; that people know what they can or cannot do. That a belief in oneself is not only a requirement to engage in any serious endeavor, but also that it provides the necessary staying power that allows one to persevere in the face of difficulties. (Bandura, 1997). Pajares & Miller (1994) used the agency perspective to investigate the impact of self-efficacy belief on mathematics achievement. They used path analysis to explore the causal relationship between mathematics self-efficacy belief, self-concept belief, and mathematical achievement. From their findings, they wrote the following:

Of all path coefficients from the independent variables to performance, only those from math self-efficacy ($\beta = .545$, $t = 10.87$, $p < 0.0001$), math self-concept ($\beta = .163$, $t = 3.07$, $p < .005$), and high school level ($\beta = .099$, $t = 2.22$, $p < 0.05$) were significant. The magnitude of the self-efficacy/performance path coefficient is such, however, that the answer to the substantive question of study is readily apparent (p. 198).

In a related study, Beghetto & Baxter (2012) also used path analysis to investigate the relationship between mathematics self-efficacy beliefs, and science efficacy beliefs of 270 students from twelve elementary schools, and their academic achievement. Student's data were collected using self-report questionnaires while their academic achievement was based on their teacher's assessment of their performance. On the significance of self-efficacy belief to academic achievement, they wrote the following:
"ability alone is not sufficient. Students who otherwise have the ability to be successful in learning science and math, yet believe they are not capable of success, likely give up in the face of challenge, under perform, and ultimately, focus their effort and attention on other pursuits and endeavours. Put simply: 'student's beliefs matter' "(p. 942).

The research on academic interests/perceived usefulness of subject of study

Koller, et. al. (2001) described academic interest "as a person-object relation that is characterized by value commitment and positive emotional valences" (p. 449) and argued that student's academic interests and their educational outcomes are positively correlated. They tested their hypothesis using a longitudinal study of German high school students. Data was collected from the students at the ends of Grade 7, Grade 10, and in the middle of Grade 12. Each time, the students completed standardized tests for many subjects including mathematics and also completed intelligence tests. The students' interest in mathematics was measured using a questionnaire. They hypothesized that male students will express higher levels of interests in mathematics than female students by opting to take more advanced mathematics courses and, that in the end, they will outperform the female students in standardized tests.

The results of their study show that of the proportion of students who selected advanced mathematics courses, male students were twice as likely to do so as female students. On mathematics achievement, their results show that while all students gained in mathematical knowledge over time, the gain was more for those students who selected advanced courses in mathematics. They report that male students outperformed the female students at each measurement point with a gap that increased from 8 points in Grade 7 to more than 24 points in Grade 12. They observed that interest in mathematics decreased for those students who took basic mathematics courses as they progressed from Grades 7 to 10, while for those who opted for more advanced courses, their interests remain stable during the same period. Overall, the male students showed a higher level of interest in mathematics than female students and it increased over time. Using path analysis, they traced the causal paths from these variables to academic achievement. From their results, they wrote:

> Of particular importance here are the paths leading from interest to achievement. Whereas the path from interest in Grade 7 to achievement in Grade 10 fails to reach significance, the picture changes from Grade 10 to Grade 12, where the path (0.30) is significant. As expected, the transition to upper secondary school involves institutional structures in which the role of academic interest becomes important for learning mathematics. (p. 459).

They argued that all direct paths to achievement support their hypothesis that interest initiates the learning activities that ultimately leads to academic achievement, and that their findings are not consistent with claims of prior achievement as a driver for interest. (p. 459)

The research on the importance of subject of study to a student's career goals

Hull-Blanks, et. al (2005) from their review of research data observed that about 50% of first year students do not complete their degree programs, and that about 32% of them quit after or during their first year of studies. They argued that career goals are important retention factors for college students because attrition rates are directly associated with the absence of career goals. Also, that career goals are associated with academic interests, persistence, motivation, self-esteem, and self-efficacy belief and for female students, are predictors of their career aspirations (p. 17). They identified four types of career goals namely: job related, value related, school related and unknown. Using this classification, they hypothesized the following relationships between student's career goals, academic achievement, and school retention rates:

Hypothesis 1: That first year students with differences in their career goals would also differ in their academic persistence decisions and continued enrolment in school.

Hypothesis 2: That first year students with differences in their career goals would also differ in their academic performance, school and career commitment, and self-belief.
Hypothesis 3: There would be no difference amongst male and female first year students in their career aspirations. (p. 19)

They tested their hypothesis using a sample of 433 first semester first year students and using their career goals, gender, and continued enrollment as explanatory variables. The dependent variables were students' academic performance (GPA), academic persistence, self-esteem, educational self-efficacy, and commitment to school and career. From an analysis of their data, they observed that first year students with unknown career goals were less persistent in those decisions that might positively impact their achievement than other students. They argued that without an identified goal, students have very little incentive to persist in their decision-making. They also report that the differences they expected amongst students in their academic performance, school and career commitment, and self-efficacy belief based on differences in their career goals were not found in this study (hypothesis 2). They noted that previous studies had reported such differences and its absence in their study might be because such difference takes a longer time to emerge than they had in their study. Also, their hypothesis of no difference amongst the male and female students in their career choices (hypothesis 3) was not supported by their results. Their results show that the female students had more job related and fewer value related goals than they hypothesized, while the male students had more value related and fewer job related goals than they expected. (p. 25)

The research on achievement goals and academic achievement emotions

Pekrun, Maier, & Elliot (2006) used the control value theory of academic achievement to argue that a student's emotions can impede or facilitate how she regulates her learning activity. They defined a goal as a cognitive representation of a future outcome that a student commits to (p. 585). They argued that students through their academic goals regulate their achievement related thoughts and actions. That it is with their goal in mind that students appraise a learning activity with respect to its value and significance to their future goals. They argued that when a person has control over achievement activities associated with a valued outcome, that this fosters positive outcome emotions. However, a lack of control over achievement activities for a valued outcome might foster negative outcome emotions.

Why this study and positioning the research questions

The research on the relationship between motivation, achievement goals, and achievement variables on academic achievement usually centers on eleven explanatory variables. So far, there is no clear distinction in the literature as to the specific role of each of these variables in a student's academic achievement. However, a closer scrutiny of these variables shows that they have different properties. They are different conceptually and functionally and thus expected to play different roles in academic achievement. To advance our understanding of their impact on academic achievement, we need a better understanding of the role that each variable plays in a student's academic achievement.

Also, intrinsic and extrinsic motivations are frequent components of motivation questionnaires. There appears to be a suggestion in the literature albeit implicit, that one form of orientation is more preferable than the other and ought to be desired or pursued. For example, it is frequently stated in the literature that through intrinsic motivation a student learns more and retains more information overtime than a student who engaged a learning activity using extrinsic motivation (Vansteenkiste, et. al, 2005). The literature is however silent on how and why a student makes the determination to be intrinsically or extrinsically motivated with regards to a learning activity; e.g., is a learning orientation an attribute of the learner where learner A is intrinsically motivated where as learner B is extrinsically motivated? Or is it the case that some students are unaware of the benefits of the various learning orientations? Or is it the case that a learning orientation is a wilful decision made by each student with full knowledge of its pros and cons? The suggestion been advanced here is that one may not separate how knowledge is acquired from its intended use. Also, Glynn, et. al, (2009), Bryan, et. al, (2011), & Kim, et. al, (2012) all used different number and composition of motivation elements in their motivation questionnaires. Thus, the question of how and why one makes a determination of how to constitute the motivation elements remains an important research question. Finally, the inconsistencies in the results reported in the literature, e.g., Pajares & Miller, (1994), Kim, et. al, (2012), and the strong correlation of these variables with each other, suggests that a full and accurate understanding of their collective and separate impacts on academic achievement requires that all of them be investigated together.

Research goals

The purpose of this study is to: Using multiple regression analysis (model selection), determine an optimal set of these variables that accounts for the most variance in academic achievement, and the contribution of each variable.

Using PLS SEM, determine which of these variables has a causal path to academic achievement, the strength of each path.

Research hypothesis

Following Lazarus (1991, p. 94), we argue that academic emotions and motivations are derived through one's wants and needs. That through want and needs, a person may be compelled to take action to pursue a desired goal. The stronger they are, the more likely that one would take action to achieve the desired goal. With respect to this study, we hypothesize as follows:

Hypothesis 1: The causal variables of academic achievements are: personal relevance of learning mathematics, academic achievement emotions (hope, pride), anxiety about

mathematics learning, academic interest, academic achievement goals, and the importance of mathematics to one's career goals. These variables inherently, can induce both an outcome and also, differences in outcomes.
Hypothesis 2: The moderator variables of academic achievement are derived from one's academic abilities (potentials). With respect to this study, they are: self-determination and mathematics self-efficacy belief. These variables can predict differences in outcome however, they cannot by themselves, induce an outcome. They are static potentials and must be activated in order to generate the expected differences in academic outcomes.
Hypothesis 3: Another variable of academic achievement are those that describe how one engages a learning activity. With respect to this study, they are: intrinsic and extrinsic motivation. Also, while these variables can predict differences in outcome, they cannot by themselves induce an outcome, Also, they have nominal properties. They are called mediators of academic achievement in this study.

Data collection methods

The data for this study will be collected from university students enrolled in a mathematics course. The students would be required to complete a total of 51 survey questions online. The questions are from the Science Motivation Questionnaire (Glynn, et. al., 2009); Academic Achievement Emotions Questionnaire (Pekrun, et. al, (2011**), A**cademic Interest and **A**cademic Achievement Goals Questionnaires (Harackiewicz, et. al, 2008). One-item question on the importance of mathematics to a student's career goal. The students' academic achievement will be accessed through their final grades in their assigned mathematics course.

Data analysis

The data from this study would be analysed using exploratory data analysis, multiple regression methods and structural equation modelling.

Results/Discussion/Conclusion

The variables in this study only estimate the portion of the variance in academic achievement that the students could accomplish by themselves. The quality of the teacher in the classroom also impacts academic achievement. For a full picture, the impact of teachers has to be part of the model.

References

Bandura, A. (1997), Self-Efficacy, the exercise of control, W.H. Freeman & Company, New York.

Beghetto, R. A., & Baxter, J. S., (2012), Exploring student beliefs and understanding in elementary science and mathematics, Journal of Research in Science Teaching, 49 (7), pg. 942-960

Beghetto, R., A. (2006) Creative Self-Efficacy: Correlates in Middle and Secondary Students, Creativity Research Journal, 18(4), pp. 447-457

Bryan, R. R., Glynn, S.M., & Kittleson, J.M., (2011), Motivation, achievement, and advanced placement intent of high school students learning science, Science Education, pp. 1049 –1065

Cano, F., (2005), Epistemological beliefs and approaches to learning: Their change through secondary school and their influence on academic performance, British Journal of Educational Psychology, 75, pp. 203-221

Cano, F., & Berben, A., B., G., (2009), University student's achievement goals and approaches to learning mathematics, British Journal of Educational Psychology, 79, p.131-153

DeCatanzaro, D.A. (1999), Motivation and Emotion, evolutionary, physiological, developmental, and social perspectives, Prentice Hall, New Jersey

Glynn, S., M., Taasoobshirazi, G., & Brickman, P., (2009), Science Motivation Questionnaire: Construct Validation With Nonscience Majors, Journal of Research in Science Teaching, 46(2), pp. 127-146

Harackiewicz, J., M., Durik, A., M., Barron, K., E., Linnenbrink-Garcia, L., & Tauer, J., M. (2008), The role of achievement goals in the development of interest: Reciprocal relations between achievement goals, interest, and performance, Journal of Educational Psychology 100(1), PP. 105-122

Hull-Blanks, E., Kurpius, S. E., R., Befort, C., Sollenberger, S., Nicpon, M., F., & Huser., L., (2005), Career goals and retention-related factors among college freshmen, Journal of Career Development, 32 (1) pp.16-30

Kim, C., Park, S. W., Cozart, J. (2012), Affective and motivational factors of learning in online mathematics courses, British Journal of Educational Technology, p. 1-15

Koller, O, Baumert, J., & Schnabel, K., (2001), Journal for Research in Mathematics Education, 32 (5), pp.448-470

Lazarus, R. S. (1991), Emotion and Adaptation, Oxford University Press, New York

Pajares, F., & Miller, M., D. (1994), Role of self-efficacy and self-concept beliefs in mathematical problem solving: A path analysis, Journal of Educational Psychology, 86 (2), pp.193 – 203

Pekrun, R., Elliot, A., J., & Maier, M., A., (2006), Achievement goals and discrete achievement emotions: A theoretical model and prospective test, Journal of Educational Psychology, 98 (3), pp. 583 – 597

Pekrun, R., Goetz, T. & Frenzel, A. C. (2007). Perceived learning environment and students' emotional experiences: a multilevel analysis of mathematics classrooms. Learning and Instruction, 17, 5, 478–493.

Pekrun, R, Frenzel, A.C., Goetz, T., & Perry, R., P.(2007, The control-value theory of achievement emotions: An integrative approach to the emotions in education, In Emotion in Education, Schutz, P., A, (Ed.s) pp. 13-36

Pekrun, R. & Stephens, E. J. (2009), Goals, emotions, and emotion regulation: Perspectives of the control-value theory, Human Development, p. 357 – 365

Pekrun. R., Goetz, T., Frenzel, A.C., Barchfeld, P., & Perry, R., P. (2011), Measuring emotions in students' learning and performance: The Achievement Emotions Questionnaire (AEQ), Contemporary Educational Psychology 36, pp. 36-48.

Pintrich, R. R. & DeGroot, E. V. (1990). Motivational and self-regulated learning components of classroom academic performance. Journal of Educational Psychology, 82, 33–40.

Vansteenkiste, M., Simons, J., Lens, W., Soenens, B., Matos, L. (2005), Examining the motivational impact of intrinsic versus extrinsic goal framing and autonomy-supportive versus internally controlling communication style on early adolescents' academic achievement, Child Development 76 (2), pp483 – 501

Villavicencio, F., & Bernardo, A.,B., I., (2012), Positive academic emotions moderate the relationship between self-regulation and academic achievement, British Journal of Educational Psychology, www.wileyonlinelibrary.com, pp.1-12

SUBJECTIVE PROBABILITY AS PRESENTED IN BC CURRICULUM AND TEACHER PREPARATION TEXTBOOKS

Simin Chavoshi Jolfaee

Simon Fraser University

Abstract: Today, unlike the curriculum documents, among the practitioners of probability, subjective probability (Bayesian methods) is a known term. The Bayesian probabilistic models have been receiving considerable attention over the last few decades from the users of probability, i.e. scientists and engineers. In many fields including computer science, Biostatistics, cognitive science, medicine, and meteorology using Bayesian models are common practice. In this paper I have tried to identify and discuss various ways in which the subjective probability is implicated in K-12 mathematics education with respect to documents such as the BC mathematics education curriculum, and the mathematics education research literature.

METHOD:

I have looked into the BC curriculum documents (IRP's 1-12), two teacher-training textbooks, and into some research in mathematics education literature looking for definitions, examples, tasks, and learning outcomes addressing subjective probability.

BC mathematics curriculum: In BC mathematics IRP's chance is described as a communication tool which addresses the "predictability of the occurrence of an outcome" and mathematical probability is defined as a tool that describes "the degree of uncertainty more accurately" (IRP, p.14). Later the same curriculum document states probability as an aspect of constancy: "the theoretical probability of getting head with a coin is constantly 0.5" (IRP, p.14).

What follows is a brief overview of what K-12 IRP's prescribe as learning outcomes for probability and statistics:

Grade level	Probability and Statistics Prescribed learning outcome
Grade 2	Gather and record data about self and others. Construct and interpret concrete graphs and pictographs to solve problems.
Grade 3	Collect first-hand data and organize it. Construct, label and interpret bar graphs to solve problems.
Grade 4	Demonstrate an understanding of many-to-one correspondence. Construct and interpret pictographs and bar graphs involving many-to-one correspondence to draw conclusions.

Grade 5	Differentiate between first-hand & second-hand data. Construct and interpret double bar graphs.
Grade 6	Create, label, and interpret line Graphs. Select, justify, and use appropriate methods of data collection. Graph collected data and analyze the graph to solve problems. Demonstrate an understanding of probability by identifying all possible outcomes of a probability experiment, differentiating between experimental and theoretical probability, determining the theoretical probability of outcomes in a probability experiment, comparing experimental results with the theoretical probability for an experiment.
Grade 7	Demonstrate an understanding of central tendency, outliers & range. Determine the most appropriate measures of central tendency to report findings. Determine the effect on the mean, median, and mode when an outlier is included in a data set. Construct, label, and interpret circle graphs. Express probabilities as ratios, fractions, & percents. Identify the sample space (where the combined sample space has 36 or fewer elements) for a probability experiment involving two independent events. Conduct a probability experiment to compare the theoretical probability (determined using a tree diagram, table or another graphic organizer) and experimental probability of two independent events.
Grade 8	Critique ways in which data is presented. Solve problems involving the independent events.
Grade 9	Describe the effect of bias, language, ethics, cost, time, privacy, and cultural sensitivity on the collection of data. Select and defend the choice of using either a population or a sample of a population to answer a question. Develop and implement a project plan for the collection, display, and analysis of data. Demonstrate an understanding of the role of probability in society, explain, using examples, how decisions based on probability may be a combination of theoretical probability, experimental probability, and subjective judgment
Grade 11 Foundations	Demonstrate an understanding of normal distribution including standard deviation and z-scores, Interpret statistical data, using confidence intervals, confidence levels, and margin of error.
Grade 12	Interpret and assess the validity of odds and probability statements. Solve problems about probability of two dependent /independent, mutually exclusive and non-mutually exclusive events. Solve problems

	that involve fundamental counting principle, permutations and combinations.
Grade 12 app-work	Solve problems that involve measures of central tendency. Analyze and describe percentiles. Analyze and interpret problems that involve probability. Describe and explain the applications of probability. Calculate the probability of an event based on a data set. Express a given probability as a fraction, decimal and percent and in a statement. Determine the probability of an event, given the odds for or against. Explain, using examples, how decisions may be based on a combination of theoretical probability calculations, experimental results and subjective judgments.

DISCUSSION OF PROBABILITY PORTRAYED BY BC CURRICULUM DOCUMENTS:

Based on the above table, the curriculum suggests two avenues to teach and learn probability: classical (theoretical in IRP's words) and frequentist (experimental), and both in a very objective sense. It (the curriculum document) also believes in the idea of an underlying and knowable probability in a deterministic way: "the theoretical probability of getting head with a coin is constantly 0.5" (IRP, p.14).

However believing in the notion of a True and Knowable probability makes probability easy to teach and learn, but it cannot go unnoticed that even when calculating the most typical examples of classical probabilities related to objects such as a die, the physical probability cannot be measured free of subjective considerations including the fairness of die, fairness of flipping process, and the fairness of observing and reporting the outcomes. Moreover, additional subjectivity arises because of our ignorance of the precise initial conditions of the mechanical systems, the die in this example.

The curriculum does not very much attend to such subjective considerations that are inevitable from either of frequentist or classical standpoints. There are very few cases that the curriculum brings up the subjective aspect of decision making in uncertain situations, here is one example: in the suggested achievement indicator of grade 9 and grade 12 foundations, the students are expected to "explain, using examples, how decisions based on probability may be a combination of theoretical probability, experimental probability, and subjective judgments" (grade 9 math IRP p.86). No further discussion of these "subjective judgments" and the ways in which the students are supposed to develop reliable ideas about them is provided. One thing is apparent though; the curriculum is referring to subjective aspects of decision-making based on probabilities and not to the probability itself.

SUBJECTIVE PROBABILITY AS REFLECTED BY TEACHER TRAINING TEXTBOOKS

I have looked into probability chapters of two textbooks used for teacher training at SFU:

1) Elementary and Middle School Mathematics by Van de Walle, the textbook for Designs for Teaching Elementary Mathematics Course.

2) Reconceptualizing Mathematics for elementary school teachers by Judith and Larry Sowder and Susan Nickers, the textbook for Principles of Mathematics for Teachers course.

Prospective teachers take both of the courses mentioned above as a required part of their training program.

On page 473 of the Van de Walle (2011) book where the authors lay the big ideas of the whole probability chapter, they say: “The probability of an event occurring is a number between 0 and 1. It is a measure of the chance that the given event will occur ... The relative frequency of outcomes of an event (from experiments) can be used as an estimate of the exact probability of an event (my italics)... For some events, the exact probability (my italics) can be determined by an analysis of the event itself. A probability determined in this manner is called a theoretical probability”. In Sowder book, probability is defined as a feature of an event: “An event is an outcome or a set of outcomes of a designated type. The probability of an event is the fraction of the times the event will occur when some process is repeated a large number of times” (p614). This frequentist approach is later supplemented by a possibility of skipping the repeated experiments if some theory of the likelihoods arisen by situation is at hand:

“A probability that can be arrived at by knowledge based on a theory of what is likely to occur in a situation, such as when a fair coin is tossed, is a theoretical probability” (p. 617).

Both textbooks put forward examples and tasks that involve probability calculation via coin flipping, spinner spinning, and like. Similar to the curriculum documents the idea in these books is that there exists a True probability and we either determine it based on the physical features of the experiment (e.g.: a square divided by into four equal parts each colored differently) or we achieve a good estimate by repeating the experiment patiently.

In Sowder book there is a section on conditional probability and Bayes’ formula in which I hoped to come across some ideas related to subjective probability. The book instructs the students on how to use the Bayes’ formula to calculate certain probabilities given certain set of occurrences. What seems to be missing to me in this chapter is any awareness of the fact that the important nuance offered by subjective probability as captured by Bayes formula is that after each intake of new evidence, the decision maker (the person who is assigning probability to an event) decides whether this new observation or evidence changes the initial arrangement of the possibilities and the already assigned distributions and based on that modifies and changes the probabilities. With no mention of prior probabilities, evidence-based probability assignment, re-defining initial distributions after each occurrence, and posterior probabilities Bayes' formula seems very out of context and miss represented.

IN PROBABILITY EDUCATION RESEARCH LITERATURE (PRE):

Subjective probability in the mathematics education research literature is strongly associated with fallacious personal beliefs and incorrect reasoning about probability. Here is an example: statements such as: "the next ball drawn from the urn is going to be red because it is my favorite color" (Jones[4], 2005 p. 132)-my underlines-, and "the next role of the die will be a three because I just know it's going to happen" (p. 474 van de wall) –my underlines- are typical examples of subjective probabilistic ideas of students presented in research in probability education literature. In some instances the term subjective probability is associated with the probability assignment to the events without any formal calculations, sometimes referred to as probability estimation.

Chernoff in his PhD disquisition presents a detailed account of the lack of a unified definition for subjective probability and the polysemic nature of this term and later in his 2008 work he offers distinctive terminologies that enable the researchers bring new nuances into the vocabulary of the subjective probability as well as into the philosophical underpinnings of the probability.

Subjective randomness:

In looking for subjective probability in curriculum documents and PRE, another term I kept an eye for was subjective randomness. The notion of randomness is a basis for probabilistic modeling, and mathematicians have taken painstaking efforts to identify it through solid mathematical terms. Also, perceived randomness versus true randomness has been investigated by practitioners of probability (and not for educational purposes). I was hoping that by looking for randomness in curriculum and PRE documents to find more evidence of treatment of subjective probability.

My findings are that to IRP's, randomness is either assumed obvious or unimportant. No definition of either objective or subjective randomness is hinted at the textbooks and curriculum documents that I have looked at. I was very glad to find that some research is done on how students conceptualize randomness, some examples are included here:

Bognar & Nemetz (1977) long before the probability appears in the curriculum, express a need to get the children acquainted with the notion of randomness. They contend that it is very essential that the students must get some ideas about random phenomena and they suggest that this goal can be achieved by letting the students meet numerous practical situations having simple random structure such as games. Kahneman & Tversky

[4] In Jones (2005) Exploring Probability in Schools there are around 80-90 instances of the word subjective probability/judgment/assessment/etc, nearly almost all of them referring to the situations that correspond to wrong answers, assessments that are done without any reasonable explanation, fallacious ideas, and misconceptions.

(1972), in an attempt to explain the decision making behavior in uncertain situations suggested that two general properties namely irregularity and local representativeness, seem to capture the intuitive notion of randomness.

Pratt and Noss (2002) observed 10-11 year old children using four separable resources for articulating randomness: unsteerability, irregularity, unpredictability, and fairness. Burril (cited in Bennete 1990) finds that the idea of fairness is an important intuitive in children's notion of randomness. When asked by psychologists why they used counting-out games, such as one potato/two potato, ninety percent of the time children responded that counting out gave them an equal chance of being selected. Soon enough children will discover that chance is not as fair as it seems first. For example a random selection of a particular child for some privilege may seem fair at first but it is possible that one person might be selected more than once while another kid is never selected. Falk & Konold (1998) investigate the subjective aspects of randomness through tasks in which people are asked to simulate a series of outcomes of a typical random process such as tossing a coin (known as generating task) or to rate the degree of randomness of several sequences (known as perception tasks). Their findings say that perceived randomness has several subjective aspects that are more directly reflected in perception tasks. Batanero & Serrano (1999) identify students' thinking about randomness as initially deterministic in nature and propose that the intuitions about randomness develop through experience and instruction.

Fischbein & Schnarch (1997) point out that the learners' personal beliefs pertaining to real-life context accompanies them for a very long time and hence it is a significant challenge to lay out a gradual process in which students' subjective intuition builds up appropriately. I would propose that the task of overcoming this challenge would be aided by introducing the subjective randomness inherent to the sequences of outcomes generated by randomizers at the early stages of probability education.

Why subjective probability is as important as the other two?

Shafer (1992) discusses in detail that both belief type probability and frequency type of probability share the common grounds of mathematical probability. In other words they both satisfy Kolomogorov's axioms and there is a transition from one to another and each one of the three takes of frequency, belief, or support can be taken as a starting point for the mathematical theory of probability. He also draws links between the three frequency-classical and subjective probabilities by giving examples of the shortcomings of each approach that another approach can overcome. For instance he claims that the frequency interpretation is less widely applicable than the belief interpretation. For example a person can hold beliefs about any event, but the frequency interpretation applies only when a well-defined experiment can be repeated and the ratio always converges to the same number. Many events for which we would like to have probabilities clearly do not have probabilities in the frequency and classic sense. There is also a need to acknowledge the subjective aspects of the frequency story. A full account of a frequentist probability must go beyond the existence of frequencies since the randomness of the sequence of the outcomes is not an objective fact about the sequence in itself. It is a fact about the relation between the sequence and the knowledge of a person, contends Shaffer (1992).

In a frequency based setting of probability, events with probability zero can never occur and events with probability one have to occur all the times. Therefore the nature of experiment plays a more significant role than the outcome of the experiment. As for the classic probability one can easily see that such events do not entertain a meaningful classical probability, for example consider the most commonly believed sure event according to the participants of a study (described in response for the third question), in which several participants presented the "I will die" example for an event with 100% probability of happening. Let's try to assign a classical probability to this event: we first need to define a sample space consisting of equiprobable events, count the number of events in which "I will die" and divide it by the total number of the events in the sample space. The inherent difficulty in doing so may lie in the idea that equiprobability bears a connotation of speaking about more than one thing, and in this example there is no more than one outcome (unless we have pre-determined the probability) and therefore it is not very meaningful to assign classical probabilities to such events.

Shaffer adds: when teaching probability it is a subtle and crucial task to adopt a practice that appeals to all the aspects of probability. We need to appeal to frequency in order to explain why probabilities add and in order to attend to the concepts such as random variable and expected value. At the same time we need ties to belief type of probability in order to mediate conditional probabilities and statistical inference in data analysis.

Concluding remarks:

Looking into BC curriculum documents shows that subjective probability is not yet acknowledged and defined in the curriculum. Also most of the research in probability education that have taken to themselves to address subjectivity, don't seem to recognize subjective probability as a possible form of independent approach to probability suitable for K-12 students The present situation reflects that Subjective Probability framework through the works of Bayes, De Finetti, Ramsey, Keynes and others have not met the same acceptance and enthusiasm as the objective probabilities from the people in the mathematics education field to the extent to be found worth distilling in the curriculum. Instead, several researchers acknowledge the importance of subjective considerations about the whole continuum of probabilistic notions in the role they play in building a sound intuition that can account for objective probabilities both in mathematics and real-life context probabilities. It is true that subjective probability as an independent approach to probability is more of a mathematical construct with a set of certain axioms and assumptions (in its game-theoretic and betting behavior form) and is wrapped up with complex calculus (in its Bayesian form) that makes it difficult to enter the K-12 curriculum. Notwithstanding, the subjective aspects and considerations of frequentist and classical probabilities are both easy and essential to implement into the curriculum.

Finally I wish to add that in the same way that no one no longer believes that the exact measure of physical quantities is attainable (or even exist), when measuring uncertainty (a commonly used term to describe probability e.g.: NCTM standards), it is only fair to have the same awareness of if the uncertainty measure sought after exists and whether it is obtainable. At any rate if the idea of teaching and learning probability is to train human mind to measure and weight uncertainty and make probabilistic decisions that matter, then the personal and subjective issues will be inseparable parts of it.

I here end with a quote from Shaffer (1992, p.18):

"The ways in which probabilities are used, in statistical inference and elsewhere, are varied, and they are always open to criticism. We should guard, however, against the idea that a correct understanding of probability can tell us which of these applications are correct and which are misguided. It is easy to become a strict frequentist—or a strict Bayesian—and to denounce the stumbling practical efforts of statisticians of a different persuasion. But our students deserve a fair look at all the applications of probability".

References:

Bennett, D. (1998). Randomness. *Harvard University Press*. USA.

Bognar, K., & Nemetz, T. (1977). On the Teaching of Probability at Secondary Level, Educational Studies in Mathematics, 8(4), 399-404.

Batanero, C. & Serrano, A. (1999). The meaning of randomness for secondary school students. *Journal for Research in Mathematics Education*, 30(5), 558–67.

Chernoff, E. J. (2008). The state of probability measurement in mathematics education: *A first approximation. Philosophy of Mathematics Education Journal*, 23.

Falk, R., & Konold, C. (1998). Subjective Randomness. In S. Kotz, C.B, Read, & D.L. Banks (Eds.) *Encyclopedia of statistical sciences*, Wiley & sons, NY, 653-659.

Fischbein, E. & Schnarch, D. (1997). The evolution with age of probabilistic intuitively based misconceptions. Journal for Research in Mathematics Education, 28(1), 96–105.

Jones, G. A. (Ed.). (2005). Exploring Probability in School: Challenges for Teaching and Learning. New York: Springer.

Kahneman, D., & Tversky, A. (1972). Subjective probability: A judgment of representativeness. *Cognitive Psychology*, 3, 430-454.

Pratt, D., & Noss, R. (2002). The Microevolution of Mathematical Knowledge: The Case of Randomness. The Journal of the Learning Sciences, 11(4), 453-488.

Shafer, G. (1992). What is probability? *Perspectives on Contemporary Statistics*.

Sowder, J., and Sowder L., and Nickerson, S. (2010) *Reconceptualizing Mathematics for elementary school teachers,* New York, USA, W. H. Freeman and Company Press.

Van de Walle, H., & Folk, S., & Karp, K., & Bay-Williams, J. (2011). *Elementary and Middle School Mathematics*. Pearson. Toronto.

UNDERSTANDING MATHEMATICAL LEARNING DISABILITIES (MLD): DEFINITIONS AND COGNITIVE CHARACTERISTICS

Peter Lee

Simon Fraser University

While the amount of research on reading disabilities far exceeds that of mathematical learning disabilities (MLD), research on MLD over the past decade has shown significant growth. This paper examines some of this emerging research. More specifically, this paper reveals the challenges of defining MLD and examines the cognitive deficits that characterize children with MLD.

INTRODUCTION

This paper is a modest attempt to understand some of the emerging research on mathematical learning disabilities (MLD). While the amount of research on reading disabilities far exceeds that of MLD, research on MLD over the past decade has shown significant growth. This is a heartening trend as there is a growing awareness of the importance of quantitative literacy in everyday life. To keep the discussion focused, this paper critiques articles from the book *Why is Math So Hard for Some Children?: The Nature and Origins of Mathematical Learning Difficulties and Disabilities*. This book, edited by Daniel Berch and Michèle Mazzocco, gives a good introduction and overview of the field of MLD research. Of course, the scope of MLD research is too vast to be contained in any one book. But the articles in this book provide a beginning for us to understand some of the challenges and complexities in the field, and the reference list to each article gives the reader further opportunities to explore the topic.

The following questions will be addressed in this paper: How do we define and recognize MLD? How does MLD differ from dyscalculia? How do we make a distinction between a student who is just having difficulties with math and a student with an actual disability? What are the behavioral characteristics of MLD and how do they originate? What are the different pathways to understanding MLD? That is, what different methodological approaches are used to get at the construct of MLD? How can findings from research with children with MLD help us better understand the mathematical thinking of typically achieving students?

DEFINING MATHEMATICAL LEARNING DISABILITIES

Defining MLD is not easy, and neither is sorting through its variously related—and often inconsistent—terminology: acalculia, the aforementioned dyscalculia, arithmetic disorder, and specific learning disabilities in mathematics are some examples. Gersten, Clarke and Mazzocco (2007) suggest that these inconsistencies in terminology are due in part to the complex history of MLD research. While relatively short, this history draws upon the histories of medicine (neurology specifically), developmental psychology, cognitive science, mathematics education, special education, and even law—each of these fields contributing their own unique terms, definitions and perspectives to MLD research. The lack of communication between these disciplines creates just such inconsistencies in contemporary terminology. These authors argue, however, that it is through this type of multidisciplinary work major gains are made in MLD research.

Despite these challenges in terminology, Mazzocco (2007) argues correctly that, in order to make substantial progress in the field of MLD research—including efforts to develop and test appropriate interventions—a consensus definition using standardized criteria will be helpful. Towards this end, she makes an important distinction between the terms MLD and mathematical difficulties (MD). MLD suggests a *biologically* based disorder characterized by specific *cognitive* deficits. As such, Mazzocco uses the terms MLD and dyscalculia to refer to the same population, "as both imply an inherent disability rather than one caused predominantly by environmental factors" (p. 30). (By "environmental factors," she means things such as poor mathematics instruction or socioeconomic factors.) Similarly, MLD would subsume the terms arithmetic disorder, specific learning disabilities in mathematics, and mathematical disability.

Mazzocco (2007) uses the term MD to refer to children with low average performance in math (operationalized as a standardized math achievement test score that falls below a cutoff point of approximately the 35^{th} percentile). This is a much broader population than children with MLD, as many children with low average performance do not have MLD whereas children with MLD often do exhibit poor mathematics achievement (the prevalence rate for MLD is approximately 6% of the general population). An advantage of having the less strict criterion in MD is to enable researchers to study a larger group of children who struggle with mathematics.

Moreover, drawing a distinction between MLD and MD is useful in unpacking any studies that treat both populations as the same when in fact only one of the populations is the implied target of interest (often the MLD one). Mazzocco (2007) notes that the degree to which combining both MLD and MD is problematic depends on the extent of overlap—which can be small—between the two groups. For instance, students with MLD, may, despite their cognitive impairment, perform above the 35^{th} percentile on a standardized mathematics test through sheer effort and hard work. On the other end of the spectrum, the poor achievement of students with MD may be the result of low socio-economic status rather than any inherent disability in mathematics.

Thus, the danger in combining the two groups is that the target of interest (the MLD population) may be missed entirely and so will the degree to which they struggle with mathematics and the reasons for the struggle. The main distinction between MD and MLD is that MLD is *biologically* based, and attending to this distinction is crucial when attempting to study or establish any definition of MLD. Mazzocco's dissociation of MD from MLD can act as a small step towards reconciling the inconsistencies of some of the terminology used across disciplines. Later in the paper, I will discuss how the category MLD itself can be broken down into further subtypes.

It is curious that Mazzocco does not mention the oft-used term acalculia (a term which preceded dyscalculia by nearly 50 years) in her discussion of MLD and MD. Acalculia is often used in the cognitive psychological literature to indicate an acquired mathematical disability as a result of a brain injury. This is in contrast to a disability with genetic or developmental origins (sometimes referred to as *developmental* dyscalculia for this reason, with dyscalculia referring to a general difficulty in understanding arithmetic whether biological or environmental in nature). Using Mazzocco's classification, I assume that acalculia would fall under the category of MLD as it refers to a disability of the brain.

But whatever one thinks of Mazzocco's taxonomy, this section acts as a caution to all MLD researchers to be wary of the terminology used when reading a research study on mathematics learning disabilities and be clear on exactly what the study sample is and how inclusive it is. Moreover, inconsistencies in terminology do not only pose challenges when comparing study results across disciplines or developing and testing appropriate interventions, they have a significant effect on establishing accurate prevalence rates of MLD as they vary as a function of the definitions used to classify children with MLD (Shalev, 2007). For instance, is the threshold for MLD set at the 6th percentile on standardized achievement tests or 25th percentile? Or is a discrepancy model used to determine MLD? That is, how low must a child's mathematics ability be compared to his general intelligence (i.e. IQ) before a child is to be classified as having MLD? The answers to these questions may mean a high or low prevalence rate of MLD.

COGNITIVE DEFICITS ASSOCIATED WITH MATHEMATICAL LEARNING DISABILITIES

We move now from a discussion about the terminology and semantics used to describe MLD to a discussion of the actual cognitive deficits that characterize those with MLD. While MLD has biological origins, there is no consensus as to what those biological markers are (although cognitive neuroscience has recently made good inroads in this area (see Simon and Rivera (2007)). MLD, therefore, needs to be behaviorally defined, and much research has been focused on identifying those core deficits that characterize children with MLD (e.g. working memory functioning, phonological processing, visuospatial thinking, or processing speed).

As we will discover, however, researchers have been unable to isolate the cognitive variables that underlie MLD due to many confounding factors. For instance, difficulties in math and reading acquisition are often attributed to the same cognitive processes (phonological in nature). Some consensus exists, however, that children with MLD have difficulty efficiently accessing math facts (Swanson, 2007). As a result of these confounding factors, current definitions of MLD do not share a common single core deficit, although children with MLD may share many cognitive and behavioral characteristics. We now turn our attention further to some of these characteristics.

Butterworth and Reigosa (2007) explore what are the basic information processing impairments of children with developmental dyscalculia (DD) compared to typically achieving children. They first ask themselves a fundamental question: Are these impairments domain general or domain specific? In other words, are the cognitive tools that children use to learn arithmetic specific to arithmetic, or are these tools the same ones children use to learn most other school subjects? This is an important methodological question and suggests that if impairments are domain specific, it is possible for a child to excel at all school subjects except for math. If impairments are domain general, a child's poor arithmetic performance may be the result of poor language skills (including dyslexia), low IQ, or anxiety as arithmetic often requires the use of a diverse skill set: good memory, strategy use, focus, or conceptual understanding to name a few.

Butterworth and Reigosa's (2007) review of the evidence favors a domain specific interpretation of DD where the cognitive deficit is at a very basic level:

> The individuals with DD described here are not only poor in school arithmetic and on standardized tests of arithmetic, they are slower and less efficient at recognizing the numerosities of displays of objects (typically dots) and/or at comparing numerosities in a variety of number comparison tasks. (p. 78)

These authors note, also, that is it unclear whether the impairment in detecting and comparing numerosities is an isolated impairment or whether the impairment interacts with other cognitive deficits to create a possible subtype of dyscalculia. Moreover, their analysis does not preclude the possibility that there may exist other domain specific information processing impairments yet to be identified. Nevertheless, this important finding regarding numerosities suggests that strengthening their recognition in children with MLD may serve as part of an intervention before any subsequent understanding of numbers and arithmetic can be made.

Geary, Hoard, Nugent, and Byrd-Craven (2007) conclude instead that deficiencies in *working memory* of children with MLD interfere with their ability to learn more complex mathematics. By examining the strategies and procedures that children use to solve simple and complex addition problems, typically achieving students in the elementary years shift from using counting procedures (such as finger counting) to the recalling of arithmetic facts. A defining feature of children with MLD, on the other hand, is a continued reliance on finger counting and a difficulty in accessing arithmetic facts from memory.

Using a different approach to getting at the cognitive deficits of MLD, Jordan (2007) examines the connections between mathematics and reading difficulties. Jordan reviews the literature of past authors that have suggested that MLD is related to language and that there is a commonality between math and reading. She takes particular issue, however, with the claim that problems with number and fact retrieval may be related to a *phonological* processing difficulty. When asked to retrieve arithmetic facts quickly, her own research shows that children with just reading difficulties (RD) perform better than students with math difficulties (MD) and those with both math and reading difficulties (RD+MD). In general, she argues that both MD and RD+MD groups show similar functional profiles in number processing: they show difficulty in counting procedures, mathematical operations, and computational fluency. She also concludes that children with MD perform better than both the RD and RD+MD groups at complex word problems because their verbal strengths compensate for their weaknesses in number processing.

Jordan's research illustrates a common theme in much of the MLD literature: there is little consensus as to the underlying cognitive mechanisms of MLD other than a general difficulty in accessing math facts accurately and efficiently. Indeed, further questions are raised: Are difficulties in recognizing numerosities a primary or secondary deficit of children with MLD? How important a role is working memory in comparison to basic information processing? Can we truly separate language ability from math fact retrieval? Compounding these problems in isolating the key cognitive mechanisms in MLD is the aforementioned diversity that researchers use in defining their study sample across disciplines.

These studies also illustrate how researchers differ in their choice of behaviors to examine when looking for key cognitive deficits in MLD. Should researchers of MLD place the emphasis on conceptual or procedural understanding? Should they focus their attention on the speed and accuracy of child performance or the quality of strategy use? The answers to these methodological questions will strongly impact the study results. Thus, establishing non-arbitrary criteria for defining and examining MLD remains a challenge.

CONCLUSION

MLD research is clearly an emerging field. This is exemplified by the lack of consistency in terminology across disciplines, the lack of consensus as to the core cognitive deficits of MLD, and the sometimes contradictory findings of MLD research. Although it is a challenge to state exactly what MLD is, researchers are beginning to define some of its boundaries. Mazzocco has made a small but important step in this direction by making a distinction between MLD and MD.

Future research may bring in more theory from math education and its theories of mathematical cognition. What does Anna Sfard and her notion of commognition have to say about MLD? What are the implications of embodied mathematics for MLD research? Since definitions of MLD must reflect stability as a trait, more longitudinal studies of both children and adults with MLD need to be done. What are the differences between children with MLD and adults with MLD? Is it easier for still developing children to recover from MLD? What are the implications of neuroplasticity for adults recovering from brain injury? Finally, while most studies on MLD focus on arithmetic performance, more studies are needed on the influence of MLD on higher levels of mathematics such as algebra or even calculus. There have been studies of students who overcome their MLD to perform mathematics at high levels. Indeed, this is the goal that educators and MLD researchers should have for all their students.

References

Butterworth, B. & Reigosa, V. (2007). Information processing deficits in dyscalculia. In D.B. Berch & M.M.M. Mazzocco (Eds.), *Why is math so hard for some children?: The nature and origins of mathematical learning difficulties and disabilities* (pp. 65-82). Baltimore: Brookes Publishing.

Geary, D.C, Hoard, M.K., Nugent, L., & Byrd-Craven, J. (2007). Strategy use, long-term memory, and working memory capacity. In D.B. Berch & M.M.M. Mazzocco (Eds.), *Why is math so hard for some children?: The nature and origins of mathematical learning difficulties and disabilities* (pp. 83-106). Baltimore: Brookes Publishing.

Gersten, R., Clarke, B., & Mazzocco, M.M.M. (2007). Historical and contemporary perspectives on mathematical learning disabilities. In D.B. Berch & M.M.M. Mazzocco (Eds.), *Why is math so hard for some children?: The nature and origins of mathematical learning difficulties and disabilities* (pp. 7-28). Baltimore: Brookes Publishing.

Jordan, N.C. (2007). Do words count? Connections between mathematics and reading difficulties. In D.B. Berch & M.M.M. Mazzocco (Eds.), *Why is math so hard for some children?: The nature and origins of mathematical learning difficulties and disabilities* (pp. 107-120). Baltimore: Brookes Publishing.

Mazzocco, M.M.M. (2007). Defining and differentiating mathematical learning disabilities and difficulties. In D.B. Berch & M.M.M. Mazzocco (Eds.), *Why is math so hard for some children?: The nature and origins of mathematical learning difficulties and disabilities* (pp. 29-48). Baltimore: Brookes Publishing.

Shalev, R.S. (2007). Prevalence of developmental dyscalculia. In D.B. Berch & M.M.M.Mazzocco (Eds.), *Why is math so hard for some children?: The nature and origins of mathematical learning difficulties and disabilities* (pp. 49-60). Baltimore: Brookes Publishing.

Simon, T.J. & Rivera, S.M. (2007). Neuroanatomical approaches to the study of mathematical ability and disability. In D.B. Berch & M.M.M. Mazzocco (Eds.), *Why is math so hard for some children?: The nature and origins of mathematical* *learning difficulties and disabilities* (pp. 283-306). Baltimore: Brookes Publishing.

Swanson, H.L. (2007). Cognitive aspects of math disabilities. In D.B. Berch & M.M.M.Mazzocco (Eds.), *Why is math so hard for some children?: The nature and origins of mathematical learning difficulties and disabilities* (pp. 133-145). Baltimore: Brookes Publishing.