Faculty of Education

Simon Fraser University

MEDS-C 2008 Proceedings

PROCEEDINGS OF THE 3RD ANNUAL MEDS-C:

MATHEMATICS EDUCATION DOCTORAL STUDENTS' CONFERENCE

NOVEMBER 22, 2008

MATHEMATICS EDUCATION DOCTORAL STUDENT CONFERENCE 2008 - PROGRAM

9:00 – 9:25	Welcome and coffee	
9:30 – 9:55	Pedagogical content knowledge in mathematics for elementary teachers courses: Two preliminary cases	Susan Oesterle
10:00 – 10:25	Connecting patterns and that mumbo jumbo stuff we have to teach: A collaborative lesson design	Paulino Preciado
10:30 – 10:55	An account of a lesson study on the parabola: Insights into building the effective practitioner	Natasa Sirotic
11:00 – 11:25	Access to education for aboriginal students: Measurement of effectiveness of outreach programs	Melania Alvarez
11:25 – 11:45	Break	
11:50 – 12:15	Inequalities in the history of mathematics: From peculiarities to a hard discipline	Elena Halmaghi
12:20 – 12:45	Independent component analysis and its application to mathematics education research	Olga Shipulina
1:00 – 2:00	Lunch: Himalayan Peak Restaurant	
2:15 – 3:00	Plenary: Mathematics education as an interdisciplinary endeavour: over thirty years of looking elsewhere	David Pimm
3:00 – 3:30	Break / Discussion Period	
3:30 – 4:00	Plenary Q&A	
4:05 – 4:30	A functional role for the cerebellum: Implications for mathematics education	Kerry Handscomb
4:35 – 5:00	Beyond static imagery: How mathematicians think about concepts dynamically	Shiva Gol Tabaghi
5:05 – 5:30	How to act? A question about encapsulating infinity	Ami Mamolo

Plenary Session:

MATHEMATICS EDUCATION AS AN INTERDISCIPLINARY ENDEAVOUR: OVER THIRTY YEARS OF LOOKING ELSEWHERE

Dr. David Pimm

University of Alberta

David.Pimm@ualberta.ca

Description:

I have been working in mathematics education since switching out of a pure mathematics doctoral degree in algebraic topology in 1975. In my main work, examining mathematics and its classroom teaching at various levels, I have drawn on the fields of mathematics (especially its history and philosophy), linguistics and quite recently poetry and poetics. In this talk, I will describe how and why I see myself having worked interdisciplinarily, then illustrate my abiding interest in mathematics classroom language by means of thinking aloud about a short transcript before ending with some short comments about poetry.

Speaker info:

My main research is in exploring the inter-relationships between language and mathematics. My work has focused both on analyses of mathematics classroom language and on producing theoretical accounts of linguistic aspects of mathematics itself. I am particularly interested in the roles of metaphor and metonymy in creative mathematical endeavour. My secondary research interest is in the potential influence of studies of the history and philosophy of mathematics on the teaching of mathematics.

http://www.quasar.ualberta.ca/cpin/edstaffweb/davidpimm/

Research Reports Abstracts:

ACCESS TO EDUCATION FOR ABORIGINAL STUDENTS: MEASUREMENT OF EFFECTIVENESS OF OUTREACH PROGRAMS

Melania Alvarez Adem

Simon Fraser University

melania@pims.math.ca

My research problem consist in developing, implementing and assessing outreach activities that could help aboriginal students to improve their mathematical knowledge and access to a higher education and a better job. We started our outreach activities with a six-week summer camp for students entering high school, the key part of the camp was to provide intensive instruction in Mathematics and English: the initial goal of this program was to build a strong foundation for success in grade 8. It is our hypothesis that if we help students to feel more confident academically when they start high school then they will do better academically throughout their high school years, and they will have a better chance to graduate.

BEYOND STATIC IMAGERY: HOW MATHEMATICIANS THINK ABOUT CONCEPTS DYNAMICALLY

Shiva Gol Tabaghi and Nathalie Sinclair

Simon Fraser University

shivagol@gmail.com

Researchers have emphasized the role of visualization, and visual thinking, in mathematics, both for mathematicians and for learners, especially in the context of problem solving (see Presmeg, 1992). In this paper, we examine the role that motion and time play in mathematicians' conceptions of mathematical ideas, focusing on undergraduate concepts. In order to expand the traditional focus on (and distinction between) visual and analytic thinking (see Zazkis, Dubinsky, and Dautermann, 1996), we employ gesture studies, which have arisen from the more recent theories of embodied cognition. Expanding on Núñez's (2006) work, we show how mathematicians' gestures express dynamic modes of thinking that have been hitherto underrepresented.

INEQUALITIES IN THE HISTORY OF MATHEMATICS: FROM PECULIARITIES TO A HARD DISCIPLINE

Elena Halmaghi

Simon Fraser University

ehalmagh@sfu.ca

In this theoretical contribution history of inequalities is looked into in a search for an answer to the question: Why are inequalities hard to meaningfully manipulate and understand? Memorable dates in the development of inequalities and the symbols for representing inequalities are highlighted. Well known inequalities are presented and some novel proofs will be shown. Implications for the teaching of mathematics are identified.

A FUNCTIONAL ROLE FOR THE CEREBELLUM: IMPLICATIONS FOR MATHEMATICS EDUCATION

Kerry Handscomb

Simon Fraser University

khandsco@sfu.ca

A characteristic of mathematical reasoning is a focus on the essential aspects of any given situation. Mason and Pimm (1984), in their seminal paper, refer to this as "seeing the general in the particular." I will argue that activity of the cerebellum with respect to the cerebral cortex is the neural correlate for "seeing the general in the particular." In other words, a functional role of the cerebellum is to facilitate the precise, focused reasoning that is necessary for mathematics. There are implications for mathematics education because of the structure of the cerebellum and its connections with the cerebral cortex. These are that repetition, decontextualization, and decomposition of concepts can play an important role in mathematical learning.

HOW TO ACT? A QUESTION ABOUT ENCAPSULATING INFINITY

Ami Mamolo

Simon Fraser University

amamolo@sfu.ca

This report is part of a broader study that investigates the specific features involved in accommodating the idea of actual infinity. It focuses on the conceptions of two participants – a mathematics university student and graduate – as manifested in their engagement with a well-known paradox: the ping-pong ball conundrum. The APOS

Theory was used as a framework to interpret their efforts to resolve the paradox and one of its variants. These two cases suggest there is more to encapsulating infinity than just the ability to 'act' on a completed object – rather, it is the manner in which objects are acted upon that is also significant.

MULTI-LAYERS OF NUMERACY

Paulino Preciado

Simon Fraser University

apreciad@sfu.ca

Predicting students' struggles and possible approaches in problem solving is part of Lesson Study strategy. In this paper a team of teachers, including the author, made use of previous experience, knowledge of current students, and some theoretical background from the literature in order to prepare suitable responses in advance to students' questions and thoughts in the designed tasks. While making such predictions, beliefs of mathematics and mathematics learning were discussed and negotiated, and we developed theoretical statements about students' learning process. In conclusion, I argue that predicting such possible students' struggles and approaches not only provides an arena to analyze and negotiate teachers' mathematical and pedagogical knowledge, but also is a critical factor contributing to the improvement to educational systems.

PEDAGOGICAL CONTENT KNOWLEDGE IN MATHEMATICS FOR ELEMENTARY TEACHERS COURSES: TWO PRELIMINARY CASES

Susan Oesterle

Simon Fraser University

oesterles@douglas.bc.ca

This paper offers some preliminary results of a qualitative research project whose aim is to study pedagogical content knowledge in the context of Mathematics for Elementary Teachers courses. Grounded theory methodology is applied to interviews with two teachers of this course. Themes that emerge from analysis of the transcripts are identified and discussed, and implications for future directions of the project are considered.

INDEPENDENT COMPONENT ANALYSIS AND ITS APPLICATION TO MATHEMATICS EDUCATION RESEARCH

Olga Shipulina

Simon Fraser University

oshipuli@sfu.ca

Educational neuroscience in mathematics education research can provide better empirical ground for developing more accurate theories of mental processes during mathematical thinking and learning. Electroencephalography (EEG) is a technique for noninvasive measurement of electrical characteristics of brain function. Scalp measurements, nevertheless, include activities generated within a large brain area. This paper reports on the roles of independent component analysis (ICA) for analysis EEG data. ICA provides separation of different signals related to different brain activators. It also calculates relative projection strengths of the respective components at all scalp sensors. As such, ICA is shown to be a useful tool for imaging brain activity and isolating artifacts from EEG data. An overview of these application areas is provided in the study on the example of data set capturing an 'AHA moment'.

AN ACCOUNT OF A LESSON STUDY ON THE PARABOLA: INSIGHTS INTO BUILDING THE EFFECTIVE PRACTITIONER

Natasa Sirotic

Simon Fraser University

nsirotic@telus.net

We report on how a group of practicing teachers experiences a school based professional development initiative by way of implementing lesson study, and how this process facilitates the development of teachers' knowledge for teaching mathematics. The report presented here is taken from an ongoing study situated in a school based community of practice on how teachers' key competences for teaching mathematics develop, refine, and even transform as a result of their participation in a collaborative enterprise of lesson study. Our focus is on the outcomes, taken here as teachers' gains in their expertise in mathematics teaching, and which have the capacity to transform teachers' everyday practice in the classroom. They are embodied in the knowledge, skill, attitude, and capability, both in the individuals and in the community as a whole. In this report we present these outcomes and the conditions in which they were achieved, through a particular account of the lesson study on the topic of parabolas. This lesson study has been the sixth lesson study implemented in the school since the inception of the initiative 18 months ago.

Research Reports:

ACCESS TO EDUCATION FOR ABORIGINAL STUDENTS: MEASUREMENT OF EFFECTIVENESS OF OUTREACH PROGRAMS

Melania Alvarez
Simon Fraser University

My research problem consist in developing, implementing and assessing outreach activities that could help aboriginal students to improve their mathematical knowledge and access to a higher education and a better job. We started our outreach activities with a six-week summer camp for students entering high school, the key part of the camp was to provide intensive instruction in Mathematics and English: the initial goal of this program was to build a strong foundation for success in grade 8. It is our hypothesis that if we help students to feel more confident academically when they start high school then they will do better academically throughout their high school years, and they will have a better chance to graduate.

Introduction

Canada's Aboriginal population is growing faster than the general population, increasing by 20.1% from 2001 to 2006. This is due to a higher fertility rate among Aboriginal women than among other Canadian women. Population projections estimate that Aboriginal people could account for 4.1% of Canada's population by 2017, but this proportion would be significantly larger in Saskatchewan (20.8%) and Manitoba (18.4%) (Statistics Canada).

Due to the fact that Aboriginals make up a growing portion of the student population, in the future they will become a substantial proportion of the population that should be participating in the workforce. If we do not address the great disparity in educational achievement of aboriginals compared with the rest of the population, the repercussions will be disastrous.

The mathematical skills of aboriginal students recorded in ***How are we doing?*** (http://www.bced.gov.bc.ca/abed/performance.htm), show that even thought there has been a slight improvement in the last years, aboriginal students in British Columbia still face great challenges. Most aboriginal students do not take Mathematics 12, and this usually means that they do not have access to scientific and technological careers; as a result there are very few aboriginal representatives in the technological sector.

It is generally known that skills obtained throughout K-12 play an essential role in determining access to college or university. Basic numeracy is essential to be able to obtain a job and function as a citizen in our modern society; as an adult one has to be able to balance budgets and know about percentages when dealing with taxes, shopping, etc, which are habitual activities in our modern daily life. In addition, mathematics is the key that opens the door to careers in science and technology, and these disciplines are the keys to our industrialized society. Advanced mathematical skills are essential for higher-level jobs in research and development that are the key to economic progress.

The topic of my research is the study of outreach activities that could help address the transition from K-12 to post-secondary education in the Aboriginal community. The goal of these activities is to tackle key pipeline issues arising for students before entering high school, and throughout their high school years.

Development and Importance of Outreach Programs:

One of the first major studies that looked at the effectiveness of outreach programs was Gandara and Bial (2001). They found that in general the assessment of most programs was faulty, five years later Schultz and Mueller (2006) made a review of new programs and still found that most of them have problems in assessing their effectiveness.

Some of the most common limitations of evidence are:

- Most assessments do not use control groups (Gullatt and Jan, 2003)
- Most programs do not mentioned the participants selection criteria (Gullatt and Jan, 2003; Swail and Perna, 2001)
- Students who leave the program are usually not included in the evaluations (Gullatt and Jan, 2003; Swail and Perna, 2001)
- Few programs have long term follow up assessment of effectiveness (Gandara and Bial 2001; Perna and Swail, 2002)
- Program implementation varies from site to site and this variation is usually not address adequately in the evaluations.
- Description of evaluation methodology is often incomplete
- Statistical reports are usually incomplete.
- Evidence is limited because programs are usually not widely spread.
- Many times negative results are not reported.
- In the survey done by Schultz and Mueller (2006) the analysis of the programs with the strongest evidence for effectiveness, was mainly statistical analysis (quantitative analysis), the researchers seems to avoid making any conclusions from qualitative data.

These are issues that we should be careful to address to make any evidence meaningful in this project, and in any project related to outreach assessment.

According to Gandara and Bial (2001) there are five main impediments that low-income minorities of underrepresented youth face which hinder their access to higher education:

- Lack of access to information and resource networks
- Lack of peer support
- Segregation
- Ineffective high school counselling
- Low expectations and aspirations

In addition researchers show that social support (Perna 2000a, 2000b, 2000c; Hossler, Schmit and Vesper, 1999), strong social networks (Cabrera and La Nasa, 2001), mentorships (Levine and Nidiffer, 1996), parental involvement (Corwin et al 2005), and in general a comprehensive long term support (Cabrera and La Nasa 2001; Perna and Swail, 2002; Gandara and Bial 2001, Hossler et al., 1989; Hossler et al., 1999; Perna, 2000a) are key predictors of a successful access to college education.

One of the main consequences of low expectations is the low academic preparation of students. Researchers agree that it is essential to pay greater attention to outreach programs that would focus on improving academic knowledge and university readiness (Fenske, Geranios, Keller, & Moore, 1997), and it is particularly important that students take rigorous math courses in high school, given that this was one of the greatest predictors of successful college completion (Adelman 1999; Horn, 1997; Perna, 2000c). Cabrera and La Nasa (2001) indicate that in order for the students to even think about the possibility of having a choice they need to feel that they have an opportunity to become academically qualified to attend college and graduate from high school.

Early intervention programs are key to raise a better predisposition for college, most researchers recommend interventions before 9 grade (Corwin et al., 2005, Perna 2002, Levine and Nidiffer, 1996, Cabrera & La Nasa, 2000b, 2001, Cabrera and La Nasa, 2000c). Cabrera and La Nasa (2000a) strongly recommend that precollege outreach programs should make sure that students and parents know during the sixth, seventh, and eighth grades what is required to go to college.

Studies made by Hossler et al. (1999) and Choy et al. (2000) point out that it is important for a project to focus on a particular school and work on connections between and among students given that students are more likely to plan to attend the university if their friends plan to attend as well.

Theoretical Framework and Methodology:

The diverse realities in which children are situated, as well as the various influences in their lives, not just from home but school, the neighborhood, television, etc.; make it necessary for us to use the combination of quantitative and qualitative analysis (James and Prout 1990; Brooks-Gunn et al., 1993). There is a dynamic between children being ‘participant agents’ in social relations and being able to change their circumstances and social structure, while at the same time there is a dependency on the adults, the community and in general the social structures that surround them (White, S. 2002:1103; Holland et al, 2006; Mayall, 2002). Jones and

Summer (2007:3) summarized it well: children's reaction are "grounded in local cultural contexts and specificity of experience; emphasis in particular 'new' areas including autonomy, enjoyment/fun, relatedness and status."

Using a mixed methods approach means using both the quantitative method and qualitative method in our research. A key reason for using the mixed methods approach would be "to enrich or explain, or even contradict, rather than confirm or refute. It may even tell 'different stories' on the same subject because quantitative methods are good for specifying relationships (i.e. describing) and qualitative for explaining and understanding relationships" (Thomas and Johnson, 2002:1).

According to Brannen (2005:12) four good reasons for combining methods are:

- Elaboration or expansion: One can get a better understanding of data collected by one method, by using the other method.
- Initiation: the use of one method can initiate a new hypotheses or research question from which we can get a better insight by using the other method.
- Complementarity: by combining both methods one can arrive at a better understanding than if only one method were used.
- Contradictions: if there seems to be a conflict between the qualitative data and quantitative findings, by exploring these contradictions one can simply juxtapose the contradictions for others to explore in further research.

I will use an embedded mixed method design where quantitative and qualitative data will be collected whenever possible, and both sets will support each other in helping to develop new outreach programs and in explaining results.
We will compare both data sets, hoping to find some correlation between attitudes and marks, however qualitative data will be mainly use to help us design better programs. In the end the data that will have greatest priority will be the quantitative data which will include marks in tests, drop-out rates etc. This is what in the politics of life people use to determine success.

Being able to replicate results is another of the main arguments against the qualitative approach, especially for those who strongly believe in the scientific method; however for researchers working within a social context it is clear that social circumstances change, and we cannot always replicate them. On the other hand we can look at how some social variables affect quantitative results and how consistent these results are. One of the ways we can observe this consistency is by analyzing how transferable our findings are in a different setting. In our research we implemented summer camps at two locations, the Sk'elep School of Excellence, a First Nations School in Kamloops British Columbia, and at Britannia Secondary an urban school where 30% of its students are aboriginal. We will compare results in looking for consistencies and differences in academic achievement and in qualitative data.

How data sets are being collected:

- Students are evaluated at the beginning and at the end of the summer camp to determine their level of mathematical knowledge at these two points in time and to determine their level of improvement (quantitative data)
- Teachers write diaries of their daily impressions while teaching in the summer camp, including change in children's attitudes and goals (qualitative data).
- Once students start high school we will try to find if students are taking mainstream courses in other areas besides mathematics. We will compare if there are any differences in choices made between students who took the summer camp and those who didn't. (Qualitative data)
- Students will be interviewed during their high school years about their impressions about the camp, and their career expectations and goals. (Qualitative data)
- Reduction in the dropout rate (quantitative data).
- We will monitor results of standardized Provincial tests and such (quantitative data).
- We will follow the academic progress until they drop out or complete high school (quantitative data)

We will constantly compare both data sets, hoping to find some correlation between attitudes and marks, however qualitative data will be mainly use to help us design better programs. In the end the data that will have greatest priority will be the quantitative data which will include marks in tests, drop-out rates etc.

Setting of our Research:

The Pacific Institute for the Mathematical Sciences (PIMS) has been implementing various outreach activities in several First Nation schools and schools with a significant aboriginal population in British Columbia. Two years ago it started working with Britannia Secondary in Vancouver, looking for ways to improve the high school graduation rate of aboriginal students, as well as to increase the level of math preparation among these students. I have been working with PIMS in the development of these programs.

More than 30% of the students attending Britannia Secondary are aboriginal, and no one can recall at any time in the history of the school when an aboriginal student graduated having taken principles of math 12 or its equivalent.

Several outreach activities focusing mainly on acquisition of mathematical knowledge and understanding have been implemented by PIMS in order to improve aboriginal students' access to the university and in particular to a science career. It is especially important that students take rigorous math courses in high school, given that this is one of the greatest predictors of successful college completion (Adelman, 1999). By leaving behind the philosophy of reduced expectations, in introducing new interesting and challenging programs and exciting ways to learn mathematics, researchers at PIMS hope to be able to provide aboriginal students with the tools they need to be able to make a career decision of their choice, including a career in science.

In general researchers recommend that the type of outreach programs, which PIMS is implementing, should begin by eighth grade or earlier and not later than ninth grade (Corwin et al., 2005, Perna 2002, Nidiffer, 1996). Conversations we had while working with aboriginal students at Britannia Secondary, the First Nations Secondary School at Lytton, and the Sk'elep School of Excellence in Kamloops transitioning into secondary school, seemed to confirm these findings. We found out that many of the "delinquent behaviors" in class or skipping class altogether started in 8th grade due to feelings of not being able to cope with the courses from the beginning, and not being able to foresee any possibilities of going to the university and getting an education which could provide them with a better future.

In general the transition from seventh to eighth grade is a difficult one for many children, however for aboriginal students it seems to be particularly harsh. For the first time children are streamed and in the case of most aboriginal students, they are placed in courses with the lowest academic expectations.

Aboriginal children are aware of the schools' low expectations towards their group, and they also become aware that with the course load that has been assigned to them, after graduation they would not be able to enroll at the university or in trade programs that could interest them. As a result, they do not see why they should continue at school and by 10th grade most aboriginal students have stopped attending school on a regular basis.

After we became aware of these facts, we realized that one of the best uses of our resources was to support summer camps for students transitioning from elementary school to high school. We were able to identify methods and materials that seem to be successful, and a six weeks summer camp for aboriginal children transitioning from elementary to high school was implemented during the summer of 2008 at Britannia Secondary in Vancouver, and the Sk'elep School of Excellence in Kamloops.

Students took an intensive math course, with a 'master' teacher who fully understands the subject and enjoys teaching it, and complemented this camp with a an English and reading comprehension class, since many of these kids seem to need some extra instruction in this area as well.

In Vancouver the mathematics materials that were used were developed by Dr Rahael Jalan and Vicki Vidas, the head of the mathematics department at Britannia. Dr Jalan is a mathematician who for 2 years worked with teachers at Britannia Secondary in their classrooms observing the level of students' knowledge and what was needed for students to better understand and achieve real success in mathematics. Dr Jalan and Vicki Vidas' goal was to be able to prepare 8th graders with the necessary mathematical understanding to be able to take principles of math 10 in two years and succeed. In the program they developed, mathematics is taught as a universal language using numbers, the history and evolution of numbers, arithmetic and algebra with emphasis placed on the elements of the language of mathematics, rules of operations and relations. Foundation courses at the grade 8 and 9 level were designed to help students make a smooth transition back into the regular mathematics stream by grade 10.

In addition to the summer camps, mentorship programs are being implemented to help these students with their math courses throughout their high school years.

Two more initiatives are in the works: a) to provide scholarships throughout the year to students who attend school regularly and have good work ethics; b) work with teachers to improve their math knowledge as well as to develop better pedagogies, and to work with them to attain a better understanding of the needs of aboriginal students.

At Sk'elep a mathematician taught the math class supported by teachers, and one of the teachers at the school, a language specialist taught the English class. Rahael Jalan's materials were used, but supplemented with the Math Power 8 book, which is still one of the standard textbooks in grade 8.

The format of the summer camps was a follows:

The weekly schedule:□**Monday – Friday**□Morning:□ 9:00am - 9:30am Breakfast□ 9:30 am - 11:00 pm Mathematics/English□11:00am - 11:15 pm Snack□11:15am - 12:45 pm Mathematics/English□12:45pm - 1:30pm Lunch□Afternoon: □Four days a week from 1:30 to 4:00, students will participate in a variety of athletic activities and one day a week meeting with an elder.

We are following the development of these children through their high school years, and during this time we will develop and implement new programs to improve their chances of graduation and success.

A key component of my research is to be able to being constant contact with students, teachers and administrators in order to become aware of where a problem exists and to develop possible solutions that will bring all these parts together towards their common goal, the academic success of these children.

Preliminary Results and Conclusions:

20 students participated in the Summer Camp at Britannia Secondary and 19 finished the six weeks, at Sk'elep 10 children started in the camp and only 5 finished. We found that the long day was not a good setting for Sk'elep whereas for Britannia parents were happy to have their kids taken care of for the whole day. At Sk'elep parents complained that the evening activities interfered with traditional summer ceremonies, and the six weeks commitment interfered with family travelling. We paid $50 a week to each child for perfect attendance and hard work, however this was not enough enticement for the most of the kids at Sk'elep. We realised that different settings need to be looked at more carefully, activities need to be tailored for each

location. If we do this activity again at Sk'elep we will seek more support on behalf of the Shuswap, and Chuachua Band and possibly just run it for half a day.

At Britannia the preliminary results have been encouraging. We gave students a test on the first day of the camp and a test on the last day, testing similar mathematical operations: addition, subtraction, multiplication, division, knowledge about integer, primes, fractions, problem solving, etc. From comparing the results on these tests we can see a significant improvement in most kids. There was also a second test on the first day of class based on the MCAT test, unfortunately the teacher did not give this test to all the students again at the end of the camp and we cannot compare. The tests from which we have consistent results are close to what teachers in Britannia Secondary use to evaluate students in mathematics when they enter grade 8. We give a detailed list of the marks that children got in the first and final test as wells as the overall marks students got, two children did not take the final exam because of a death in the family. Overall we saw significant improvement in most students.

Child	***Overall Marks***	***Final Test***	***Preliminary Test***	***Child***	***Overall Marks***	***Final Test***	***Preliminary Test***
1	93.00%	95.00%	57.00%	11	66.00%	50.00%	20.00%
2	92.00%	92.00%	47.00%	12	63.00%	60.00%	23.00%
3	92.00%	88.00%	50.00%	13	60.00%	30.00%	20.00%
4	90.00%	87.00%	33.00%	14	40.00%	11.00%	0.00%
5	89.00%	89.00%	50.00%	15	35.00%	18.00%	23.00%
6	82.00%	82.00%	37.00%	16	27.00%	17.00%	0.00%
7	82.00%	68.00%	53.00%	17	17.00%	12.00%	0.00%
8	81.00%	80.00%	37.00%	18	90.00%		47.00%
9	69.00%	71.00%	40.00%	19	70.00%		43.00%
10	68.00%	51.00%	37.00%				

Out of these 19 students, 10 are currently going to Britannia and they are doing well. Three of these students are now in the Venture program in Britannia Secondary, taking grade 9 math instead of grade 8, and they are in line to go to the IB program. This is the first time in the history of Britannia that these many aboriginal children are immediately and so far successfully inducted in this program.

So far we have anecdotal data from teachers and some parents stating that the students are doing well, that they work hard and do not get easily distracted in class.

They are enjoying the school and the friendships they started in the summer have continued.

We also underestimated the English component of the camp. The teacher who took care of the English component was extraordinary. She introduced the book Touching Spirit Bear by Ben Mikaelsen and in the beginning most students were not interested in reading it at all. This change within a few weeks, and by the end of the camp the students were anxiously waiting for the sequel, which they intended to buy with their own money. Some students commented that finally they understood why reading could be fun.

We will have to wait for a real proof of success, as we follow their progress throughout high school.

References

Adelman, C. (1999). Answers in the toolbox: Academic intensity, attendance patterns, and bachelor's degree attainment. Washington, DC: U.S. Department of Education, Office of Educational Research and Improvement.

Brannen, J. (2005), 'Mixed Methods Research: a discussion paper', Retrieved June 23, 2006 from the ESRC National Centre for Research Methods website: http://www.ncrm.ac.uk/publications/methodsreview/MethodsReviewPaperNCRM-005.pdf

Brooks-Gunn, J., Duncan, G., Klebanov P., and Sealand, N. (1993), 'Do Neighborhoods Influence Child and Adolescent Development?' *American Journal of Sociology* (2), 353-95.

Cabrera, A. F., & La Nasa, S. M. (2000a). Overcoming the tasks on the path to college for America's disadvantaged. In A. F. Cabrera & S. M. La Nasa (Eds.), Understanding the college choice of disadvantaged students (pp. 31-44). San Francisco: Jossey-Bass.

Cabrera, A. F., & La Nasa, S. M. (2000b). Three critical tasks America's disadvantaged face on their path to college. In A. F. Cabrera & S. M. La Nasa (Eds.), Understanding the college choice of disadvantaged students (pp. 23-30). San Francisco: Jossey-Bass.

Cabrera, A. F., & La Nasa, S. M. (2000c). Understanding the college-choice process. In A. F. Cabrera & S. M. La Nasa (Eds.), Understanding the college choice of disadvantaged students (pp. 5-22). San Francisco: Jossey-Bass.

Cabrera, A. F., & La Nasa, S. M. (2001). On the path to college: Three critical tasks facing America's disadvantaged students. Research in Higher Education, 42(2), 119-149.

Creswell, John W. (2007) Educational Research: Planning, Conducting, and Evaluating Quantitative and Qualitative Research. Prentice Hall; 3 edition (Jul 26 2007)

Choy, S. P., Horn, L. J., Nunez, A. M., & Chen, X. (2000). Transition to college: What helps at-risk students and students whose parents did not attend college. In A. F. Cabrera

& S. M. La Nasa (Eds.), Understanding the college choice of disadvantaged students (pp. 31-44). San Francisco: Jossey-Bass.

Corwin, Z., Colyar, J.E. and Tierney, W.G. (2005) Introduction: Engaging research and practice -- Extracurricular and curricular influences on college access. In W.G. Tierney, Z.B. Corwin and J.E. Colyar (eds), *Preparing for College: Nine elements of effective outreach* (*pp 1-9)*. Albany NY: State University of New York Press.

Fenske, R. H., Geranios, C. A., Keller, J. E., & Moore, D. E. (1997). Early intervention programs: Opening the door to higher education (ASHE-ERIC Higher Education Report Volume 25, No. 6). Washington, DC: The George Washington University, Graduate School of Education and Human Development.

Freire, P. (1998). *Pedagogy of the Oppressed*. (New Revised 20th-Anniversay ed.). New York: Continuum Publishing Co.

Gandara, P. and Bial, D. (2001) *Paving the way to postsecondary education: K-12 interventions for underrepresented youth.* Washinton, D.C.: National Center for Education Statistics. Retrieved August 10, 2008 from: http://nces.ed.gov/pubs2001/2001205.pdf

Gullat, Y., and Jan, W. (2003) *How do pre-collegiate academic outreach programs impact college-going among underrepresented students?* Washington, D.C.: Pathways to College Network Clearinghouse. Retrieved August 25, 2008 from http://www.pathwaystocollege.net/webarticles/pdf/gullat_precollegiate.pdf

Holland, J., Thomson, R. and Henderson, S. (2006), 'Qualitative Longitudinal Research: A Discussion Paper', *Families & Social Capital ESRC Research Group Working Paper No 21,* London, South Bank University.

Horn, L. J. (1997). Confronting the odds: Students at risk and the pipeline to higher education. Postsecondary Education Descriptive Analysis Report (PEDAR). Washington, DC: U.S. Department of Education, Office of Educational Research and Improvement. (NCES 98-094)

Hossler, D., Braxton, J., & Coopersmith, G. (1989). Understanding Student College Choice. In John C. Smart (Ed.), Higher education: Handbook of theory and research (Vol. 5, pp. 231-288). New York: Agathon Press.

Hossler, D., & Gallagher, K. S. (1987). Studying student college choice: A three-phase model and the implications for policymakers. College and University 2(3), 207-221.

Hossler, D., Schmit, J., & Vesper, N. (1999). Going to college: How social, economic, and educational factors influence the decisions students make. Baltimore: Johns Hopkins University Press.

James, A. and A. Prout. (1990), *Constructing and Reconstructing Childhood: Contemporary Issues in the Sociological Study of Childhood*, Routledge: London.

Jones, N. and Summer, A. (2007) WeD 40 'Does Mixed Methods Research Matter to Understanding Childhood Well-being?'
http://www.welldev.org.uk/research/workingpaperpdf/wed40.pdf

Levine, A. and Nidiffer,J. (1996) *Beating the odds: How the poor get to college.* San Francisco, CA: Josey Bass Publishers.

Mayall, B. (2002), *Towards a sociology for childhood: Thinking from children's lives*, Milton Keynes, UK: Open University Press.

Perna, L. W. (2000a). Differences in the decision to attend college among African Americans, Hispanics, and Whites. Journal of Higher Education, 71(2), 117-141.

Perna, L. W. (2000b). Racial and ethnic group differences in college enrollment. In A. F. Cabrera & S. M. La Nasa (Eds.), Understanding the college choice of disadvantaged students (pp. 65-84). San Francisco: Jossey-Bass.

Perna, L. W. (2000c, April). Racial/ethnic group differences in the realization of educational plans. Paper presented at the annual meeting of the American Educational Research Association, New Orleans, LA.

Perna, L.W. (2002) Pre-College outreach programs: Characteristics of porgrams serving historically underrepresented groups of students. *Journal of College Students Development, 43, 1, 64-83.*

Perna, L.W. and Swail, W.S. (2002). Pre-college outreach and early intervention programs. In D.E. Heller (Ed.), *Condition of access: Higher education for lower income students* (pp 97-112). Westport, CT: American Council on Education and Praeger Publishers.

Schultz, J.L., Mueller, D. (2006) *Effectiveness of programs to improve postsecondary education enrollment and success of underrepresented youth:*A literature review of the effectiveness of outreach programs to improve postsecondary education enrollment and success of underrepresented youth. Wilder Research. Retrieved August 11, 2008 from: http://www.wilder.org/download.0.html?report=1948

Stage, F. K., & Hossler, D. (1989). Differences in family influences on college attendance plans for male and female ninth graders. Research in Higher Education, 30(3), 301-315.

Swail, W.S. and Perna, L.W. (2001) A view of the landscape:Results of the national survey of outreach programs. *College Board Outreach Porgram Handbook: Washington, D.C.: Educational Policy Institute.* Retrieved August 1, 2008 from hhttp://www.educationalpolicy/pdf/OutreachHandbookEssays.pdf

Statistics Canada. *Projections of the Aboriginal Populations, Canada, Provinces and Territories 2001 to 2017.* Ottawa, Statistics Canada, 2005 (Cat. No. 91-547-XIE).

Thomas, A. and Johnson, H. (2002), 'Not Only Reinforcing But Also Different Stories: Combining Case Studies and Survey to Investigate How Postgraduate Programmes Can Build Capacity For Development Policy and Management', *Paper for 'Combined Methods' conference,* Swansea: Centre for Development Studies, 1-2 July, 2002.

Thompson, P. (2004), 'Researching family and social mobility with two eyes: some experiences of the interaction between qualitative and quantitative data', *International Journal of Social Research Methodology, 7(3),* July 2004, 237 – 257.

White, S. (2002), 'Being, becoming and relationship: conceptual challenges of a child rights approach in development', *Journal of International Development,14 (8),* pp 1095 – 1104.

White, H. (2002), 'Combining Quantitative and Qualitative Approaches in Poverty Analysis', *World Development, 30(12)*, 511-522.

BEYOND STATIC IMAGERY: HOW MATHEMATICIANS THINK ABOUT CONCEPTS DYNAMICALLY

Shiva Gol Tabaghi

Nathalie Sinclair

Simon Fraser University

Abstract: Researchers have emphasized the role of visualization, and visual thinking, in mathematics, both for mathematicians and for learners, especially in the context of problem solving (see Presmeg, 1992). In this paper, we examine the role that motion and time play in mathematicians' conceptions of mathematical ideas. In order to expand the traditional focus on (and distinction between) visual and analytic thinking (see Zazkis, Dubinsky, and Dautermann, 1996), we employ gesture studies, which have arisen from the more recent theories of embodied cognition. Expanding on Núñez's (2006) work, we show how mathematicians' gestures express dynamic modes of thinking that have been hitherto underrepresented.

INTRODUCTION

There has been a growing body of research on students' ways of thinking in problem solving situations since the 1970s. While researchers initially identified two different modes of thinking (visual and analytic), they later became more concerned with the interrelationships between these modes. Zazkis, Dubinsky, and Dautermann's (1996) study, in particular, argues the two modes of thinking are not dichotomous. They draw specific attention to the visualization of dynamic objects and processes, and argue that perceiving dynamic processes and objects creates more complex mental images than perceiving static objects. Their study motivated us to further probe the interaction between analytic and visual thinking. We also draw on theories of embodied cognition to suggest that dynamic thinking is potentially a bridge between visual and analytic thinking. In this paper, we first present a brief overview of research on the different modes of thinking, and connect this research to emerging theories and methodologies from embodied cognition. We then present the analysis of mathematicians' verbal and non-verbal expressions in describing two concepts: quadratic functions and eigenvectors. Finally, we present a discussion about our findings, and offer suggestions regarding the use of gestures in teaching mathematics.

BACKGROUND

In mathematics education literature on students' mathematical thinking, researchers have proposed distinct modes of thinking. Krutetskii (1976) distinguishes verbal/logical thinking from visual/pictorial thinking. The former is an indicator of level of mathematical abilities whereas the latter indicates a type of mathematical giftedness. Krutetskii's work led mathematics educators to inquire further about the visual/pictorial mode of thinking, and to emphasise its importance in mathematical thinking (see Bishop, 1989, Eisenberg and Dreyfus, 1986 and 1991, Presmeg, 1992; Zimmermann and Cunningham, 1991). In her study of mathematicians' ways of coming to know mathematics, Burton (2004) interviews seventy mathematicians working in different fields of mathematics. She identifies three primary modes of thinking: visual/pictorial, analytic/symbolic and conceptual.

While many researchers have distinguished the visual from the analytic, as two modes of thinking (see Clement, 1982), Zazkis et al. (1996) focus on the relationships between the visual and the analytic modes of thinking. They argue that these two modes of thinking are not dichotomous, and propose the Visualization/Analysis (VA) model to describe students' ways of thinking in problem solving. In defining the visual category, Zazkis et al. draw attention to the sometimes dynamic nature of visual imagery, something that Burton (2004) also does, in describing her visual/pictorial category as "often dynamic." According to Zazkis et al., perceiving dynamic processes and objects creates more complex mental images than perceiving static objects. Further, the very act of perceiving static objects involves dynamic actions, as the eye moves across the visual field to build the static objet (Piaget, 1969). Although both Burton and Zazkis et al. recognise the presence of dynamic visual imagery, they do not offer many examples of such imagery, nor do their models of mathematical thinking accord it a primordial role.

More recent research, drawing on theories of embodied cognition (see Lakoff and Núñez, 2000), suggests that dynamic thinking (and not just image-based dynamic thinking) plays an important role in conceptual development. For example, Núñez (2006) argues that mathematical ideas and concepts are ultimately embodied in the nature of human bodies, language and cognition. He has shown that static objects can be unconsciously conceived in dynamic terms through a fundamental embodied cognitive mechanism called 'fictive motion;' he illustrates this mechanism using the concepts of limits, curves and continuity.

In addition to studying mathematicians' linguistic expressions, Núñez broadens the methodological scope by including analyses of mathematicians' metaphors and gestures, which are key to revealing more dynamic thinking processes. As such, Núñez's approach differs from that taken by the researchers cited above, who focus mainly on linguistic expression, and who minimize the role of dynamic thinking in their models of mathematical thinking. We note that the inattention to (and sometimes ignorance of) the role of time and motion in mathematical thinking has strong historic roots: not only does mathematics tend to detemporalise mathematical processes (Balacheff, 1988; Pimm, 2006), but several mathematicians have expressed discomfort at the idea of moving objects (see Frege, 1970).

Núñez writes that gestures have been "a forgotten dimension of thought and language" (p. 174). Recent research, however, has shown that speech and gesture are two facets of the same cognitive linguistic reality. In particular, research claims that gestures provide complementary content to speech content (Kendon, 2000) and that gestures are co-produced with abstract metaphorical thinking (McNeill, 1992). This research supports our methodological approach in this paper, which is to analyse both speech and gesture in describing mathematical thinking. In particular, given the motion aspect of gesturing, we hypothesis that analysing gestures will provide more insight into the dynamical thinking process of mathematicians.

RESEARCH CONTEXT AND PARTICIPANTS

In our larger study, we extend Núñez's work to explore concepts other than limits and continuity. This paper focuses on concepts relating to functions, matrices and eigenvectors. While Núñez studied mathematicians as they gave lectures, we chose to adopt the approach of Burton (2004), who used interviews to examine the nature of mathematical thinking. We designed our interviews using a set of questions aimed at eliciting mathematicians' concept imagery around a variety of mathematical concepts, spanning K-12 and undergraduate mathematics. We interviewed four mathematicians whose interests were in both pure and applied mathematics, and who were all members of a medim-sized mathematics department in Canada. Each interview lasted between 1 and 1.5 hours. Interviews were videotaped and transcribed. We reviewed the video clips and selected to analyse their speech, gestures, analytic and visual thinking about quadratic function and eigenvector.

ANALYSIS OF STUDY

We refer to Núñez's framework, conceptual metaphor and fictive motion to analyse mathematicians' linguistic and non-linguistic expressions. We also use McNeill's gesture classification and transcription to analyse the movements of the mathematicians' hand and arm as they described mathematical concepts. Verbal and gestural excerpts from interviews follow.

ANALYSIS OF SPEECH AND GESTURES: QUADRATIC FUNCTION

In our first analysis, we illustrate the way in which linguistic expression by itself can include evidence of dynamic thinking. In response to our prompt about quadratic function, LG first says: "Well I guess I see, I picture a parabola, right, a parabola which is, um, or a conic section if it's a quadratic function of two variables." In addition to the visual image of a graph of a parabola, he also talks about variables, which point to more analytic/symbolic thinking. It seems that his thinking moves flexibly between visual and analytic thinking which supports the VA model.

LG then continues to say: "I don't think I picture just one. […] I know that there's only one parabola up to scaling. If you took any two parabolas, you can always rotate it, put them side by side, zoom in on one and it will look just like the other." LG not only visualizes the graph of a parabola, but also thinks about the graph in motion, as he translates it, rotates it, and zooms in on it. He conceives a static entity (the parabola, the equation of the parabola) in dynamic terms, as illustrated by the verbs *translate*, *rotate*, *zoom*. In other words, his concept of quadratic function doesn't include just the graph, or the equation, but the parabola in motion: in the language of

Sfard (2008), he uses the dynamic aspect of the parabola as a “saming” technique, to make all the parabolas, whatever their shape, size, orientation, be one single object; as he later says “there’s only one quadratic function really.”

In our next example, we show how the linguistic expression and the non-linguistic expression can illustrate different aspects of mathematical thinking. Once again, in response to our prompt of quadratic function, NN begins by referring to a real object: “Something like a goblet, yeah so both a parabola and a goblet.” She uses the goblet metaphorically to describe the shape of a parabola. While both ‘parabola’ and ‘goblet’ evoke visual images, instead of dynamic ones, her speech coincidences with a set of gestures. In Figure 1 below, her right hand is cupped under, with fingers pointed upward, as if holding the goblet. Then, she uses her index finger to trace out a parabola starting from left to right and then returning from right to left (see Figure 2). In MacNeill’s scheme, this is a *metaphoric gesture*, which ‘points’ to an abstract object. Note that in this gesture, the finger is moving, as if tracing a curve, or drawing a parabola; it is not a static gesture, as the one used to accompany the word “goblet.”

	
Figure 1. shows NN’s right hand which depicts a parabola	Figure 2. shows NN’s index figure while tracing out a parabolic curve

In our third case, instead of producing the gesture along with the speech, the mathematician replaces speech by gesture. Again, in response to our prompt, JJ says: “initially I thought of algebraically, then I thought of [index figure depicts a concave down parabola] one of these [index figure depicts a concave up parabola] one of these.” His gesture resembles that of NN, but differs also in several ways: he draws two different parabola, one concave and one convex, and also, draws them right in front of his body, at chest level. In contrast, NN goes back and forth along one parabola, and draws her in a region above, and to the right or her head. For both NN and JJ, the gestures are metaphorical, referring as they do to abstract objects. However, whereas NN evoked the metaphor of the goblet and the visual imagery of the parabola, JJ speaks first about the algebraic interpretation of the quadratic function, signalling an initial analytic—and very static—conception.

Our fourth and final case PT, combines various aspects of the first three, but in slightly different ways. His thinking is analytic/symbolic, while he says “this would be a function that is ay ex squared plus bee ex plus cee, and then, you could represent that by a parabola.” But, he uses a set of gestures (see Figure 3) to actually write out the symbols ax^2+bx+c. He then draws out a very big parabola (see Figure 4), in his upper left spatial field, with his index finger, and says "going like that." His gesture points to an abstract object. He then says "of course, in that you can include a line, you can imagine a line in there, […] though a line is technically a quadratic function." In his accompanying gesture, his whole hand moves from left to right, fingers extended, as if cutting out a plane (see Figure 5). That he sees the parabola becoming a line (as the parabola flattens out), it also appears that he sees the parabola moving continuously from a curved line to a straight one—whereas LG saw the parabola move continuously across transformations.

		
Figure 3. PT gestures the quadratic equation.	Figure 4. PT draws a parabola.	Figure 5. PT's gestures line as parabola.

ANALYSIS OF SPEECH AND GESTURES: EIGENVECTORS

In response to our prompt about eigenvectors, LG first says: "I guess eigenvector might be a resonance so if you are in a big tunnel and you start singing and hit the right note it starts to go really loud resonating your ears." He uses resonance metaphorically to describe abstract objects, eigenvectors. He evokes a visual image and conceives it in dynamic terms, as he uses the verb *to go*. LG's linguistic expression alone reveals the presence of dynamic thinking. Unlike the examples above, LG's dynamic thinking is not necessarily image-based; rather, the dynamism is in the echo, which starts to "go really loud."

In our next example, we analyse NN's linguistic and non-linguistic expressions to illustrate dynamic aspects of her mathematical thinking. In response to our prompt about eigenvectors, she says: "stresses, so if I am thinking about a plate being pulled out so it's gonna move along principles." She uses 'stresses' as a metaphor that refers to eigenvectors. She evokes a visual image of a plate and uses motion, as illustrates by the verb *pull out*, to describe her concept image of eigenvectors. Her speech coincidences with a set of gestures: Figure 6 shows how she embodies a dynamic imagine of eigenvector in the context of a real world example, "a plate being pulled out." She clenches her hands and moves her arms back and forth, as if holding a horizontal steering wheel, to accompany her verbal expression. This is another *metaphoric gesture*.


Figure 6. Shows NN’s hand and arm movements in describing eigenvector.

Our third case, JJ, first says “one idea is, you have the idea of matrix as a linear transformation and, um, so you take a vector and you map it to something else.” This seems to describe a visual image of mapping on vector to another, though his description “matrix as a linear transformation” also indicates a more analytic conception of eigenvector. He then continues to say that “you set the matrix up by some inputs, they’re gonna come inside and then obviously you say what is the important direction when the two line up of course. So, that is one idea that I use to say that there is something special about that direction.” Here, he uses his hands to demonstrate a vector and its transformations: with his index fingers (on both hands) he rotates one finger toward the other (see Figures 7 and 8). His hand movements, which coincidence with the verbal description quoted above, show how he conceives the process of transformation dynamically, as something coming together in the same “direction.”

	
Figure 7. Shows JJ’s use of hands to depict a vector and its linear transformation.	Figure 8. Shows how JJ’s line up a vector and its transformation to illustrate the concept of eigenvector.

Our fourth and final case PT, in response to our prompt, says “I think of a matrix, I think of applying the matrix to the vector, and then what you get out is another vector that’s in the same direction but either stretched or shrunk.” Again, his linguistic expression reveals the presence of fictive motion in his conception of an eigenvector, which he describes as something you “get out,” that is “stretched or shrunk.” His speech coincidences with arm and hand movements that are similar to NN’s gesture: starting with his hands and arms extended (as in Figure 9), he brings them toward each other as he says “same direction” and moves them away again when he says “stretched or shrunk.” Unlike NN, who is referring to plates and stresses, PT seems to be thinking about the vectors themselves, and also using metaphorical gestures in describing them.

Figure 9. Shows PT's arm movements in illustrating result of transformation, stretched or shrunk in eigenvector.

DISCUSSION AND REFLECTIONS

The results of our analysis indicate that: first, the mathematicians use gestures and metaphors to express their thinking about concepts. Second, their linguistic and non-linguistic expressions comprise a dynamic component. However, while sometimes this dynamic component is visual in nature, other times it is no. This would suggest that some forms of dynamic thinking are non-visual, and more time-based. In fact, in Thurston's (1994) categorisation of the different "facilities of mind," he includes both a "vision, spatial, kinaesthetic (motion) sense" category and a "Process and time" category, where the latter refers to a facility for thinking about processes or sequences of actions.

As Núñez (2006) points out, the dynamic component of gestures and metaphors promote understanding mathematical concepts (Núñez, 2006). Following Zazkis et al.'s (1996) work, which draws attention to the important interaction between the visual and the analytic, we hypothesise that dynamic thinking is potentially a bridge between visual and analytic thinking: further research on this hypothesis seems warranted.

On a final note, turning now to the teaching and learning of mathematics: we suggest that the instructional use of gestures warrants further study. Cook, Mitchell and Goldin-Meadow (2008) reported that requiring students to gesture while learning a new concept helped to retain the knowledge they had gained during instruction. It seems reasonable to assume that not all gestures will work in this way; however, drawing on the gestures that mathematicians use to think about concepts may well provide guidance to educators looking to identify productive gestures for instruction.

REFERENCES

Balacheff, N. (1988). Aspects of proof in pupils' practice of school mathematics (Pimm, D., trans.). In Pimm, D. (Ed.), *Mathematics, Teachers and Children.* London, Hodder and Stoughton, pp. 216-235.

Bishop, A.J. (1989). Review of research on visualization in mathematics. *Focus on Learning Problems in Mathematics*, 11(1-2), pp. 7-16.

Burton, L. (2004). *Mathematicians as Enquirers: learning about learning mathematics*. Dordrecht: Kluwer.

Clements, K. (1982). Visual imagery and school mathematics. *For the Learning of Mathematics*, 2 (2), pp. 33-39

Cook, S. W., Mitchell, Z., Goldin-Meadow, S. (2008). Gesturing makes learning last. *Cognition*, 106, 1047-1058.

Eisenberg, T., and Dreyfus, T. (1986). On visual versus analytical thinking in mathematics. *In proceedings of the tenth international conference for psychology of mathematics education* (pp. 153-158). London: University of London Institute of Education.

Eisenberg, T., and Dreyfus, T. (1991). On the reluctance to visualize in mathematics. In W. Zimmermann and S. Cunningham (Eds.), *Visualization in teaching and learning mathematics*. (pp. 153-158). Washington D.C.: Mathematical Association of America.

Frege, G. (1970). What is function. In Geach, P. and Black, M. (Eds.), *Translations for the Philosophical Writings of Gottlob Frege*. Oxford: Blackwell.

Kendon, A. (2000). Language and gesture: unity or duality? In D. McNeill(Ed.), *Language and gesture* (pp. 47-63). Cambridge: Cambridge University Press.

Krutetskii, V.A. (1976). *The psychology of mathematical abilities in schoolchildren*. Chicago: University of Chicago press.

Lakoff, G., and Núñez, R. (2000). *Where Mathematics Come From: How The Embodied Mind Brings Mathematics Into Being*. New York: Basic Books.

McNeill, D. (1992). *Hand and Mind: What Gestures Reveal about Thought*. Chicago: University of Chicago Press.

Núñez, R. (2006). Do Real Numbers Really Move? Language, Thought, and Gesture: The Embodied Cognitive Foundations of Mathematics. Reprinted in R. Hersh (Ed.), 18 *Unconventional Essays on the Nature of Mathematics* (pp. 160-181). New York: Springer.

Pimm, D. (2006). Drawing on the image in mathematics and art. In N. Sinclair, D. Pimm, & W. Higginson (Eds.), *Mathematics and the aesthetic: New approaches to an ancient affinity* (pp. 160–189). New York: Springer.

Presmeg, N.C. (1992). Prototypes, metaphors, metonymies and imaginative rationality in high school mathematics. *Educational Studies in Mathematics*, 23, 595-610.

Sfard (2008). *Thinking as communicating: human development, the growth of discourses, and mathematizing*. New York: Cambridge University Press.

Thurston, W. P. (1994). On proof and progress in mathematics. *The American Mathematical Society*, 30 (2), pp. 161-177.

Zazkis, R., Dubinsky, E., and Dautermann, J. (1996). Coordinating visual and analytic strategies: a study of students' understanding of the group D_4. *Journal for Research in. Mathematics Education*, 27(4), pp. 435-457.

Zimmermann, W., & Cunningham, S. (1991). What is mathematical visualization? In W. Zimmermann & S. Cunningham (Eds.), *Visualization in teaching and learning mathematics* (pp. 1-8). Washington D.C.: Mathematical Association of America.

INEQUALITIES IN THE HISTORY OF MATHEMATICS: FROM PECULIARITIES TO A DIFFICULT DISCIPLINE

Elena Halmaghi

Simon Fraser University

In this theoretical contribution history of inequalities is looked into in a search for an answer to the question: Why are inequalities hard to meaningfully manipulate and understand? Memorable dates in the development of inequalities and the symbols for representing inequalities are highlighted. Well known inequalities are presented and some novel proofs are shown. Implications for the teaching of mathematics are identified.

WHY ARE MATHEMATICS EDUCATORS LOOKING AT THE HISTORY OF A CONCEPT?

A valid question that someone could ask is why do researchers in mathematics education involve in the study of the history of mathematics? We are not historians and the lens that we are using when checking a history book are not those of qualified historians. Moreover, we are looking not into historical documents, but into secondary sources of history. Radford (1997) argues that mathematics educators first look into history of mathematics in a naïve way; to find anecdotes to make class more interesting and to motivate study. Naïve is considered as well the search for old problems to be solved in class when introducing a concept. What approach to the history of mathematics can be considered less naïve?

The history of mathematics could be viewed as an "epistemological laboratory in which to explore the development of mathematical knowledge" (Radford, 1997, p.1). In this type of research lab, informing about the evolution of a concept, the history of mathematics can inform about the epistemological obstacles. A parallel could be drawn between the obstacles encountered in the historical development of a concept and the problems nowadays students have in understanding that concept (Sfard, 1995). Historical studies of the origins of a mathematical concept can inform curriculum designers, teachers and instructors as well as the epistemological theorists (Dennis, 2000).

The special case of checking the history of inequalities: were to look

"Researchers witnessed students' and teachers' frustration with the difficulties encountered when dealing with inequalities" (Tsamir & Bazzini, 2002, p.2). Research on inequalities reports mostly on students' misconceptions on inequalities or on obstacles in understanding inequalities (Linchevski & Sfard, 1991; Bazzini & Tsamir, 2002, 2003; Tsamir, Tirosh & Tiano, 2004; Boero & Bazzini, 2004; Sackur, 2004; Vaiyavutjamai & Clements, 2006). With a concept prone to misconceptions and misunderstanding, between many lenses one could use to find a cause, or possibly a solution for problems, history of inequalities seems a promising one.

Checking the history of inequalities for periods of hardship to inform why the concept is difficult as a school subject is not an easy task. Why? The answer resides first in the difficulty to find references for this task. Even though a web search brings almost fifty thousand results on history of mathematical inequalities, there are only a few that could help the journey. What about searching for books in the library? In school mathematics inequalities are placed under Algebra. In undergrad mathematics as well, there is a section on inequalities in the algebra preview of Precalculus. In Calculus, inequalities have no special treatment; they are tools for proving limits or analysing functions. So, a first attempt to find references was looking for inequalities in history of algebra books.

Is Algebra a good place to search for the history of inequalities?

The material about inequalities in the history of algebra is scarce. What is Algebra? Using a simple definition, Algebra is the science of generalized computation (Garcia and Piaget, as in Sfard, 1995).

In the history of algebra three developmental stages are identified: rhetorical algebra, syncopated algebra, and symbolic algebra. This division is due to Nesselmann, based on the notion of mathematical abstraction (Radford, 1997). Rhetorical algebra is the algebra of words. Syncopated algebra uses a mixture of words and symbols to express generalities. This is the algebra of Pacioli, Cardan, and Diophantus. It is Francois Viete who introduced the *species* and made the distinction between a given quantity, which is constant but represented by a letter in equation and the variables; he was the first one who could solve parametric equations (Bagni, 2005; Sfard, 1995). Before Viete, algebra was at an operational level. After that the equations became objects of higher order processes. Viete purified algebra from all the noise of words and presented it in abstract form, the encapsulation of a pure mathematics idea (Radford, 1997). From Viete on, it was time to talk about structural algebra. The structure in algebra influenced geometry. The works of Descartes and Fermat, on the shoulders of Viete, helped geometry capture generality and express operational ideas. In early years algebra needed geometry for reification and verification, now geometry will be using algebra for new reifications and development (Sfard, 1995).

Before the invention of symbols, algebra was a verbal interpretation of computational processes. Could inequalities emerge from rhetoric or syncopated algebra? What if inequalities are originally of a different essence than algebra? It is possible that the invention of a symbol for inequalities to help the manipulation of the known inequalities, but it took more than the symbol to help the rise of a discipline of inequalities. It took the initiative of a great mathematician, Hardy, in the 20th century, to carefully look into the subject, collect, prove and publish inequalities. The volume *Inequalities* published in 1934 was the first monograph of inequalities. The apparition of the *Journal of the London Mathematics Society* marks the most important date in the history of inequalities. The dates marked by Hardy, Polya and Littlewood in the history of inequalities are very recent, compared to the history of mathematics. Can we trace somehow inequalities in older mathematics text? Were inequalities forewings to Ancient mathematicians? (Fink, 2000). Let's have a closer look into the old history of mathematics.

Inequalities in Antiquity

The ancient mathematicians knew "the triangle inequality as a geometric fact" (Fink, 2000). They also knew the arithmetic-geometric mean inequality as well as the "isoperimetric inequality in the plane" (Fink, 2000). Euclid used words like 'alike exceed', 'alike fall short' or 'alike in excess of' to compare magnitudes (Kline, 1972, p.69). The contemporary translation of Euclid's words uses the inequality symbols to help the reader understand the old text, but those symbols were foreign to Euclid. In the Pickering version of Euclid's *Elements* the symbols introduced by Oughtred are used to write geometric inequalities. Working on π and on calculations for approximating square roots of numbers, Archimedes was in fact manipulating inequalities arithmetically (Fink, 2000).

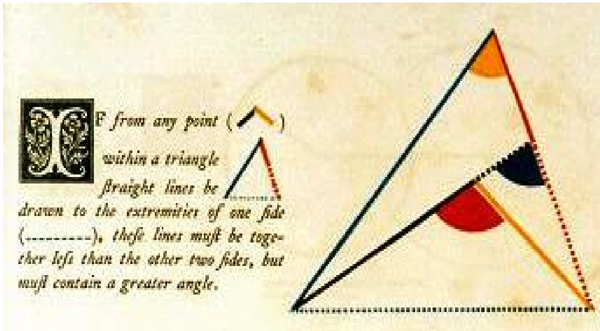
> Using inequalities to measure awkward quantities dates back to Euclid and beyond. Archimedes in particular was skilled in using inequalities to deduce equalities, and after translating his method into algebra, such proofs were used by Fermat (1636) and are accessible to undergraduates today (Burn, 2005, p.271).

Inequalities in Geometry

Thus, for noticing inequalities in a history of old mathematics book, one needs the awareness of what should be looking for: Inequalities could not emerge from rhetorical algebra, but are to be found embodied in Geometry. How do inequalities look like in old geometry texts? The following figures represent inequalities well known in antiquity. Figure 1 is a picture of one page form Byrne's (1847) *The First Six Books of the Elements of Euclid.* In this edition of Euclid's works Byrne used colours to make the book attractive and appealing to students. The proofs were presented as pictures. Figure 1a) represents Proposition XXI from Book one. In plain language, the proposition reads:

> If from the ends of one of the sides of a triangle two straight lines are constructed meeting within the triangle, then the sum of the straight lines so constructed is less than the sum of the remaining two sides of the triangle, but the constructed straight lines contain a greater angle than the angle contained by the remaining two sides (Joyce, 1996-1998).

Figure 1b) is the pictorial proof of proposition 21 from Euclid's Book 1. For inequalities, Oughtred's symbols were used to supply the pictures with inequality meaning without using too many words.



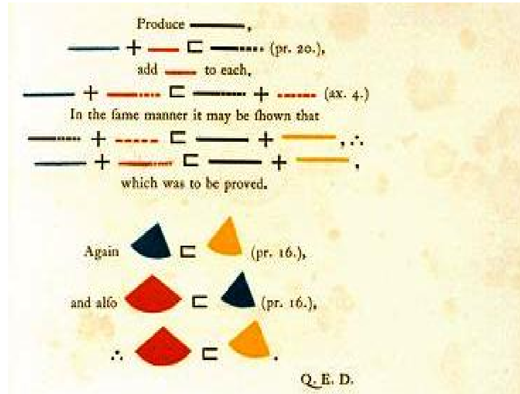
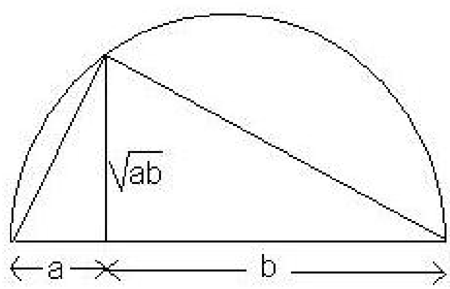


Fig 1a) Fig 1b) (Byrne, 1847, p.21)

Another proof without words or a geometrical proof of the inequality of the means, $\sqrt{ab} \le \frac{a+b}{2}$, can be seen in Fig2, which represents a right triangle inscribed in a circle. The proof of the inequality is based on the result that the hight of a right triangle is the geometric mean of the segments that it divides the hypotenuse into. This proof of the inequality of the means as looks as Euclid could have imagined (Steele, 2004).



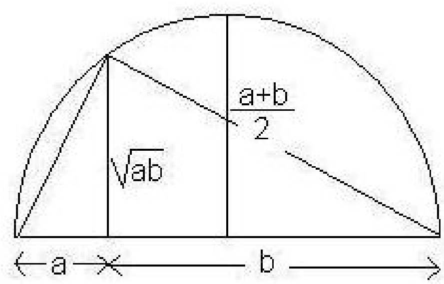


Fig 2a)

The height of the right triangle is the geometric mean of the projections of the legs over the hypotenuse: $h = \sqrt{ab}$

Fig 2b)

The radius of the circle is the arithmetic mean of the projections of the legs over the hypotenuse: $\sqrt{ab} \le \frac{a+b}{2}$ (Steele, 2004)

From the second picture can be seen the radius $\left(\frac{a+b}{2}\right)$ as being the highest of the all projections from points on the circle over the diameter, which proves the inequality $\sqrt{ab} \le \frac{a+b}{2}$.

Inequalities in Artefacts

Nelson claims that he saw the proof of the famous Cauchy-Swartz inequality on the tiling found in *The Courtyard of a House in Delft,* a painting by Pieter de Hooch. When painting, was the painter aware of this inequality as well? When arranging the tiles, was the tilling artist aware of the mathematics we can see in his work? Changing the size of the tiles, Nelsen created a new tiling where he shows the proof of the famous inequality of the means. Also, using the fact that a parallelogram has an area smaller than the area of a rectangle whose sides are equal with the sides of the parallelogram, Nelsen proves without words the Cauchy-Swartz inequality (Nelsen, 1997). Figure 3a) represents Nelsen's tiling. Figure 3b) shows the proof of the second one of the two simultaneous inequalities comprising the Cauchy-Swartz inequality: $|ax+by| \le |a||x|+|b||y| \le \left(\sqrt{a^2+b^2}\right)\left(\sqrt{x^2+y^2}\right)$. The first inequality can be proved using triangle's inequality and the properties of absolute value: $|ax+by| \le |ax|+|by| \le |a||x|+|b||y|$.

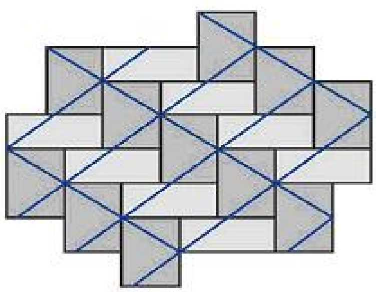


Fig 3a)

The tilling as seen in the painting.

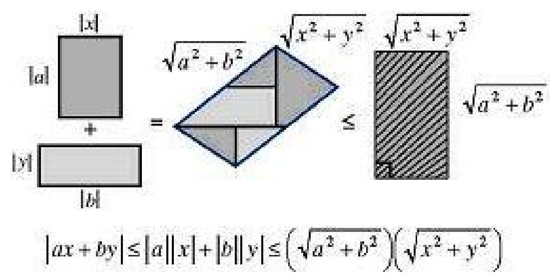


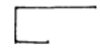
Fig 3b) (Nelsen, 1997, p.8)

The decomposition of the tiling and the composition of the Cauchy-Swartz inequality.

The history of the Inequality symbol(s)

> "It may be hard to believe, but for two millennia, up to the 16th century, mathematicians got by without a symbol for equality" (Lakoff & Núñez, 2000, p.376).

The symbols < and > were first introduced in mathematics related texts by Thomas Harriot (1631). He was a mathematician who worked for Sir Walter Raleigh as the cartographer of Virginia, North Carolina now. Harriot is considered to be the founder of "the English School of Algebraists" (Howard, 1983, p. 249). An anecdote says that Harriot got inspired from the symbol ⪤ seen on the arm of a Native American to get the symbols for inequalities (Johnson, p.144).

The mathematics community did not adopt Harriot's symbols immediately, because exactly at the same time, in 1631, Oughtred suggested  for *greater than* and  for *less than*. Oughtred's *Clavis Mathematicae* was more popular than *Artis Analyticae Praxis ad Aequationes Algebraicas Resolvendas* (The Analytical Arts Applied to Solving Algebraic Equations), Harriot's posthumously published work (Howard, 1983). In 1734, the French geodesist Pierre Bouguer invented the symbols ≤ and ≥. These new symbols were used to represent inequalities, on the continent (Smith, 1976, p.413).

I wonder how the 17th-18th centuries symbols for inequalities would had managed the differences acknowledged nowadays when using for example < or ≤. More precisely, the < symbol is used to represent quantities that are different, the first one being less than the second one. The ≤ symbol incorporates the equality as well; it allows the first magnitude to be equal with the second one. The ≤ symbol recognizes that the absolute extrema one quantity could touch in the solution as well.

In conclusion, the inequality symbol allowed for compression and aesthetic presentation of many old inequalities and permeated the development of a concept from a peculiarity.

Why this section dedicated to the history of the inequality symbol? It is well known that way before the aparition of the symbolic algebra people were writing all arguments in longhand. There were no symbols to represent the unknowns and there were no symbols to represent the relationship between unknowns as well. That was before Diophantus, during the 'Rhetorical algebra' stage (Harper, 1987). There is noting wrong in writing mathematical statements in plain language, but it is well known that it may take several pages to describe a statement when in mathematical symbols the same job could be done, possibly, in one line. The use of symbols allow for more work to be performed in a shorter time. To the best of my knowledge, I am not aware of any research that would argue that even when symbols are well known and the best way to represent some piece of mathematics, one would use, written or verbal plain language to describe the same idea. Is getting meaning of a formal mathematical statement associated with symbolic notation, or to be able to reason about a mathematical statement, someone should 'read it' in plain language as well?

It is my perception that the symbol < is more easily assimilable than  to represent *less than*. Comparatively looking at the two symbols, one can associate a metaphor with <. Or is this only a cultural perception or habituation with the symbol that is not foreign to us? Is the symbol  counterintuitive? I wonder how Oughtred explained the choice for this symbol. What about < as well? Does this symbol allude to action or contemplation? Is the symbol getting in the way of creating good metaphors that could help understanding inequalities?

Is there a discipline of Inequalities?

Geometry, Arithmetic, or Number Theory were well established disciplines from Antiquity. With the stage of symbolic algebra, new mathematical discipline, as

Algebraic Geometry, evolved. Sfard (1995) argues that geometry helped reification of heavy computations in algebra, and then algebra helped geometry evolve and answer many of the problems that were posed and unsolved from Antiquity. Initially, inequalities did not have a special status in mathematics; they were considered either mathematics peculiarities or tools for developing other theories. Two millennia and personal action changed the status of inequalities form support for some mathematics to inequalities as a discipline of study. Fink (2000) acknowledged that the history of inequalities had been written when Hardy et al. wrote the 300 pages of inequalities and their proofs. Moreover, today there are two journals of inequalities – JIA[1] and JIPAM[2] – as well as many other mathematics publications that print papers "whose sole purpose is to prove an inequality" (Fink, 2000).

Implications for mathematics education

When teaching, learning, or understanding a concept encounters problems, there is a tradition in research in mathematics education to turn the search for the solution of the problem toward the history of the concept (Cornu, 1991). In the development of the concept one may find information about periods of slow development. There could be an indication somewhere that the concept had been created problems to mathematicians first. As is well known, Hippasus died for discovering the irrational numbers. Even if mathematicians of his time experienced incommensurability, they had problems accepting it. Such an incident informs about epistemological obstacle associated with that concept. Teaching a concept linked to epistemological obstacles and being aware of that, the educator could plan when and how would be more appropriate to introduce it to the students to avoid, if possible students' cognitive conflicts.

At a shallow search into the waters of history of inequalities, no apparent epistemological obstacles were encountered. But it is recorded and documented that inequalities are not easy concepts to manipulate. Even Hardy, the man who can be called the father of inequalities, confessed:

> There are, however, plenty of inequalities which are hard to prove; Littlewood and I have had any amount of practice during the last few years, and we have found quite a number of which there seems to be no really easy proof. It has been our unvarying experience that the real crux, the real difficulty of idea, is encountered at the very beginning (Hardy, 1928).

Research on inequalities reports mostly on students' misconceptions on inequalities. Students encounter problems in their process manipulation, as well as at the level of interpretation of what an inequality is and what does a solution of an inequality represent. Are inequalities hard to manipulate? The answer to this question resides in

[1]Journal of Inequalities and Applications issued the first volume in 1997. JIA is a multi-disciplinary forum of discussion in mathematics and its applications in which inequalities are highlighted.

[2] Journal of Inequalities in Pure and Applied Mathematics, founded in 1999 by the Victoria University members of the Research Group in Mathematical Inequalities and Applications (RGMIA).

the history of inequalities. However, there is still at least another important question whose answer is nor in the history of inequalities neither in the research on inequalities: Why are inequalities hard to meaningfully manipulate? Inequalities are the back bones of many concepts and mathematical areas, therefore is worth the effort of doing more research for clarifying what makes them hard to process.

References

Bagni, T. G. (2005). Inequalities and Equations: History and Didactics. *Proceedings CERME 4*. Working Group 6. Algebraic Thinking. (652-663)

Byrne, O. (1847). *The First Six Books of the Elements of Euclid.* William Pickering. London. http://www.math.ubc.ca/~cass/Euclid/byrne.html

Dennis, D. (2000). The Role of Historical Studies in Mathematics and Science Educational Research. Kelly and Lesh (Eds.), *Research Design in Mathematics and Science Education*. Mahwah, NJ: Lawrence Erlbaum.

Fink, A. M. (2000). An essay on the history of inequalities. *Journal of Mathematical Analysis and Applications.* 249 (2000), no. 1, 118–134.49.

Hardy, G. H., Littlewood, J. E. and Polya, G. (1934). *Inequalities*. Cambridge Univ. Press, Cambridge, UK, 1934.

Hardy, G. H. (1929). Prolegomena to a chapter on inequalities *Journal of London Mathematical Society.* 4, 61–78.

Howard, E. (1983). *An Introduction to the History of Mathematics.* Fourth Edition, Holt Rinehart Winston.

Joyce, D. E. (1996-1998). Euclid's Elements. *Mathematics web pages.* http://aleph0.clarku.edu/~djoyce/java/elements/elements.html

Kline, M. (1972). Mathematical Thought from Ancient to Modern Times**.** New York: Oxford University Press.

Lakoff, G. & Núñez, R., (2000). Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being. New York: Basic Books.

Nelsen, R. B., (2003). Paintings, plane tilings, and proofs, *Math Horizons,* Nov., p. 5-8.

Radford, L. (1997). On Psychology, Historical Epistemology and the Teaching of Mathematics: Towards a Socio-Cultural History of Mathematics. *For the Learning of Mathematics*, 17(1), 26-33.

Smith, D. E. (1958). *History of Mathematics*. New York: Dover Publications.

Steele, J. M. (2004). *The Cauchy-Schwarz Master Class: An Introduction to the Art of Mathematical Inequalities***.** Cambridge University Press and the Mathematical Association of America. Cambridge UK and Washington DC.

A FUNCTIONAL ROLE FOR THE CEREBELLUM: IMPLICATIONS FOR MATHEMATICS EDUCATION

Kerry Handscomb

Simon Fraser University

A characteristic of mathematical reasoning is a focus on the essential aspects of any given situation. Mason and Pimm (1984), in their seminal paper, refer to this as "seeing the general in the particular." I will argue that activity of the cerebellum with respect to the cerebral cortex is the neural correlate for "seeing the general in the particular." In other words, a functional role of the cerebellum is to facilitate the precise, focused reasoning that is necessary for mathematics. There are implications for mathematics education because of the structure of the cerebellum and its connections with the cerebral cortex. These are that repetition, decontextualization, and decomposition of concepts can play an important role in mathematical learning.

OBJECTIVE

A peculiar characteristic of mathematical reasoning is perception of the general in the particular. A particular triangle, in other words, is viewed *schematically*, so that any of its properties that are perceived in this light are true of *any* triangle. The question for mathematics educators is how to foster this kind of reasoning in children. The technique I have used to answer this question is to identify the brain structures and activities that correspond, or correlate, to schematic reasoning. I argue that the cerebellum and its connections with the parietal lobe of the cerebral cortex play a major role in this respect. If this is the case, then there are significant implications for mathematics education. Curiously, these implications recall methods of mathematics education in which mathematical situations are decontextualized and decomposed, and learning is accomplished by repeated attention to similar problems. This paper is the précis of a doctoral dissertation. Of necessity, the full argument has been curtailed for reasons of space.

THEORETICAL FRAMEWORK

The investigation needs a theoretical framework that permits correlation of subjective cognitive functioning with objective neurophysiological activity. The classical solution is the neutral monism of Spinoza's *Ethics*, in which the substance of the world has two attributes by which it may be known, thought and extension. These two epistemological categories are subjective knowledge of mind and objective knowledge of materiality.

The notion that first-person cognition is embodied in physical activity was investigated in some depth in Varela, Thompson, and Rosch (1991). Campbell (2001) extended the ideas of Varela et al. in the formulation of a *radical* embodied cognition. The philosophical father of embodied cognition is Merleau-Ponty, and Campbell suggests that Merleau-Ponty's neutral monism of *flesh* is an appropriate metaphysical foundation for embodied cognition. This strategy produces a theory that is remarkably close to the neutral monism of Spinoza (Handscomb, 2007). According to Spinoza, the "order and connection of ideas is the same as the order and connection of things" (E2, P7). In other words, within a radical understanding of embodied cognition, the *structure* of subjective cognition mirrors the *structure* of neurophysiological activity.

Neurophysiological activity in the cerebral cortex, particularly electrical activity, is especially relevant for examining higher cognitive functions (e.g., Varela, Lachaux, Rodriguez, & Martinerie, 2001). Fuster's (2006) cognitive network theory correlates certain aspects of the structure of electrical activity in the cerebral cortex with certain aspects of subjective cognition.

According to Fuster's (2006) analysis, the Rolandic and Sylvan fissures separate the cerebral cortex into an *anterior hemisphere* and *posterior hemisphere*. Brain activity in the anterior hemisphere is correlative to action; brain activity in the posterior hemisphere is correlative to perception. Consider the posterior hemisphere. For example, activity of a specific cognitive network is the neural correlate of a triangle *percept*. Activity of another cognitive network, at a higher level, a "network of networks" would be correlative to the *concept* of triangle. Each level in the hierarchy of cognitive networks is localized to the specific region of the posterior cortex that contains the highest-level connections. The top level in the hierarchy of the posterior hemisphere is localized to the association cortex at the confluence of parietal and temporal lobes. At the lowest level of a cognitive network are patterns of activity in the primary sensory cortex (Hubel, 1988), and these are referred to as *properties*.

An analogous hierarchical structure pertains in the anterior hemisphere, in which the cognitive networks for acts are networked together to form the cognitive networks for *procedures*. The lowest level of the cognitive network system in the anterior hemisphere consists of patterns of activity in the primary motor cortex. The highest level of the anterior hierarchy is localized to the prefrontal cortex. Most of my argument refers to the posterior hemisphere, although the anterior hemisphere is important for a broader discussion of mathematical reasoning.

Of the many possible kinds of cognitive networks, those correlative to concepts, percepts, and properties in the posterior hemisphere, and to a lesser extent their analogues in the anterior hemisphere, are the only types of cognitive networks of relevance for this dissertation.

Reciprocal neural connections join analogous hierarchical levels in the anterior hemisphere and posterior hemisphere. At the lowest level, percepts and acts are linked through movement and sensation in the *perception-action loop* (Fuster, 2006). Higher-level links between the two hemispheres are also perception-action loops, although they do not circle through the external world.

Cognitive network theory correlates certain aspects of the structure of neural activity to certain aspects of the structure of cognition. An alternative approach is Brown's (1988, 2002) microgenetic theory. Microgenesis utilizes the knowledge of large-scale brain structure, brain phylogenesis, and data concerning the cognitive effect of brain lesions to develop a theory of cognition. Microgenetic theory and cognitive network theory are synergistic. Microgenesis refers to the linear evolution of a cognitive state from the potential of a concept to the actuality of a percept.

Bergson's (1889/1960) duration concerns the subjective aspect of cognition, without reference to neural correlates. The philosophy of duration inspired microgenetic theory. Bergson's (1896/1991) metaphor of the cone describes the emergence of concepts from deep duration and their surge toward actuality as percepts.

An alternative approach to duration, inspired by cognitive network theory, converts the cone to a cylinder. Concepts and percepts share the same *extent*, in that properties of percepts correspond on a one-to one basis with *potential properties* of concepts. Concepts at one end of the cylinder emerge as percepts at the other end, with properties running like fibers the length of the cylinder. This metaphor is valuable in that a particular cognitive ability facilitates a narrowing of the cylinder, focusing cognition, as it were. Identification of the neural correlate of this cognitive ability is one of my main goals.

If an object is *recognized*, then a specific cognitive network has been activated. This is the meaning of recognition. It is not somehow a separate function. Recognition may invoke language or some other symbolic designation, but not necessarily. However, different cognitive networks can be activated for the same recognition. What is recognized is the object's *identity*. It should be noted that the object, as in *external* object, is no doubt real, but for the cognizing subject all that matters is the activation of cognitive networks.

A percept will have *essential* properties and *incidental* properties with respect to its identity. For example, the property of having three sides is essential for a triangle, but the property of being green is incidental. Those properties that are essential will have the same value in every percept of the object; those properties that are incidental will vary over different percepts. This is what essential and incidental properties are—their designation as essential or incidental does not arise from some metacognitive analysis. The potential of an essential property in the concept is limited, whereas the potential of an incidental property is correspondingly broad.

Within the cognitive network correlative to a concept or percept, various components can be differentially excited or inhibited. If the components of the cognitive network correlative to essential properties are excited, and those correlative to incidental properties are inhibited, then the concept or percept is *schematic*. Because of the equality of extent of concepts and percepts, a schematic concept will always correspond to a schematic percept, and vice versa.

The meaning of the narrowing of Bergson's cone (or cylinder) of duration is the schematizing of concepts and percepts. When a schematic concept corresponds to a schematic percept, it can be said that the subject "*sees the general n the particular*" (Mason & Pimm, 1984). This, I believe, is the real meaning of the phrase.

METHOD

The method of this paper lies broadly within the research program of neurophenomenology (Varela, 1996, 1999). According to Varela (1996), "Phenomenological accounts of the structure of experience and their counterparts in cognitive science relate to each other through reciprocal restraints" (p. 343). I utilize a psychological analysis appropriate to mathematics education, rather than rigorous phenomenological investigation, and research on the cerebellum in cognitive science. The conclusions arise from mutual constraints offered by the two disciplines.

EMPIRICAL STUDIES IN COGNITIVE NEUROSCIENCE

Many regions of the cerebral cortex are linked with the cerebellum in closed, distinct cerebrocerebellar loops: cerebral cortex → pons → cerebellar cortex → deep cerebellar nuclei → thalamus → cerebral cortex (Allen et al., 2005; Leiner, Leiner, & Dow 1986; Schmahmann & Pandya, 1997). In particular, the top level in the posterior hierarchy of cognitive networks is linked through the lateral cerebellar cortex and neodentate nucleus, all of which are phylogenetically recent neural accretions.

The cerebellar cortex has a highly uniform structure (Ramnani, 2006). It contains around 5000 microcomplexes, which are indivisible modules of cerebellar activity (Ito, 2006). The microcomplexes receive cerebral input through the cerebrocerebellar loops, and respond combinatorially to this input (Imamizu et al., 2000).

For each cerebrocerebellar loop, an adjunct loop passes from the cerebral cortex to the cerebellum through the inferior olive and red nucleus (Habas & Cabanis, 2007; Leiner et al, 1986). This adjunct loop selectively suppresses microcomplex response, enabling the cerebellum to "learn" (Kawato, 1999). Cerebellar learning is a form of supervised learning; supervision is accomplished by means of recognition at the cerebral source of the loop (Doya, 1999).

RESULTS AND CONCLUSIONS

The functional role of the cerebellum is to schematize concepts. Analogously, the functional role of the cerebellum is to schematize procedures. The hypothesis is falsifiable and therefore scientific—it is "true" only to the extent of its explanatory power. The following are arguments in favour of the hypothesis:

- *Phylogenesis*. Some cerebrocerebellar loops are composed of neural structures that are of very recent phylogenetic origin (Leiner, Leiner, & Dow, 1993). The functional correlates of these loops will also be of very recent phylogenetic origin. Sophisticated conceptual thought, presumably, arose only with humans. It is consistent with the hypothesis that recently evolved cerebellar structures are implicated in conceptual thought.
- *Structural connections*. Cerebrocerebellar loops connect cerebellum and top level in the posterior cognitive network hierarchy (Allen et al., 2005). Conceptual thought is associated with this region of the cerebral cortex (e.g.,

Dehaene, 1997) It is consistent with the hypothesis that cerebellar structures that connect to this region are implicated in conceptual thought.

- *Uniformity of cerebellar microstructure*. Orthodox theory postulates that the effect of the cerebellum with respect to the anterior hemisphere is to make motor behaviour smooth and efficient (Schmahmann, 1997). This can be interpreted as schematization of procedures. There is a remarkably uniform microstructure across the entire cerebellar cortex—uniformity of structure implies uniformity of function (Ramnani, 2006). In the posterior hemisphere, this implies schematization of concepts.
- *Analogy to the anterior hemisphere*. This argument presupposes the notions of concept, percept, and property, and their anterior hemisphere analogues. A functional role of the cerebellum with respect to the anterior hemisphere is to overcome inefficiencies in the perception-action loop by means of forward models (e.g., Ito, 2006) The cerebellum accomplishes this by emphasizing those action properties in the cognitive network of the action plan that will belong to the successor act. Because of the temporal relationship between the action plan and its individual acts, the effect will be to select few action properties from many. If the role of the cerebellum with respect to the posterior hemisphere is the same, then the effect in the posterior hemisphere will also be to select few properties from many—in other words, the concept is schematized. Moreover, schematization of concepts is the posterior analogue of smooth, efficient motor behaviour.
- *Modularization*. Relatively few cerebellar microcomplexes respond combinatorially to a vast number of possible cognitive networks (Imamizu et al., 2000). Cognitive networks can be modularized by means of their properties. Modularization of cognitive networks by properties, such that a collection of microcomplexes corresponds to a given property, is consistent with the argument by analogy to the anterior hemisphere.
- *Supervised learning*. In its input and output characteristics the cerebellum is a supervised learning system (Doya, 1999). An alternative name for supervised learning is concept learning, in which a diverse input is classified by the learning system. In this case cognitive networks are input and the method of classification is by concept recognition. It is consistent with the hypothesis that the cerebellum is a concept learning system.
- *Cerebellar lesions*. Lesions of phylogenetically recent cerebellar structures produce cognitive dysfunctions that are consistent with an impaired ability to schematize concepts; these same lesions do not result in motor dysfunction (Schmahmann, 2004).
- *"Seeing the general in the particular."* It was argued that schematization of concepts may be regarded as seeing the general in the particular. There must be a neural correlate for this cognitive facility. It is proposed that activity of the cerebellum with respect to the cerebral cortex is this neural correlate.

IMPLICATIONS FOR MATHEMATICS EDUCATION

An essential characteristic of mathematical reasoning is its focus and precision. In other words, mathematical reasoning is the epitome of reasoning with schematic concepts and percepts. In subjective, functional terms, the role of the cerebellum is to schematize concepts and percepts. In other words, the cerebellum sharpens and focuses cognition. The manner in which this is accomplished in objective, physiological terms has significant implications for mathematical reasoning and learning. Three of these implications are *repetition*, *decontextualization*, and *decomposition*. Although mathematical reasoning consists of the balanced application of concepts and procedures, the presentation below will focus solely on the posterior hemisphere and the conceptual aspect of mathematical reasoning. The summary argument below does not do justice to the full argument.

- *Repetition* refers to presentation of a mathematical concept multiple times. On a single presentation the concept may not be schematized accurately. However, subsequent presentations of the same concept will correct the schematization (Imamizu et al., 2000). Cortical recognition of the concept supervises the accuracy of cerebellar learning (Doya, 1999).
- *Decontextualization* of refers to presentation of the object that represents a concept in such a way that the percept, and therefore also the concept, will involve few incidental properties. Cerebellar schematization will be less efficient if a concept has associated with it a large number of incidental and essential properties which are emphasized equally.
- *Decomposition* of a mathematical concept refers to breaking the concept into simpler constituents, each of which is learned separately. Decomposition may be regarded as serial decontextualization, and efficacious for that reason. Synthesis of the learned components is a separate issue.

These three implications for mathematical learning of the cerebellar schematization hypothesis recall traditional forms of mathematics education, with rows of students bent over exercises consisting of large numbers of largely identical symbolically presented problems. I do not claim that this is the way for all mathematics education. The theoretical efficacy of repetition, decontextualization, and decomposition can *inform* mathematics education. On the other hand, Anderson, Reder, and Simon (2000) argue that constructivist and situated learning theories of mathematics education have misapplied the results and ideas of cognitive psychology. It is too simplistic to suppose that mathematical learning can *only* take place with complex, varied, contextualized presentations of mathematical ideas. On the other hand, I am not arguing that mathematical learning can take place *without* constructivist elements. Mighton (2007) writes,

> People are seldom more destructive than when they invent simple theories to solve complex problems. In education there are too many variables to control for a researcher to prove that one method of teaching or one philosophy is necessarily, under all conditions, superior to another. (p. 226)

A good metaphor is the driving range. Few people would claim that the decontextualized driving-range can replace real golf. On the other hand, golf can be decomposed into various elements, one of which is hitting the ball a long way. It is beneficial, surely, for golfers to improve this element of their game by practicing it in isolation. Moreover, repetitive activity, swing after swing, allows golfers to perfect their technique. The driving range is a good way to improve certain aspects of one's golf. And it can be fun . . . .

References

Allen, G., McColl, R., Barnard, H., Ringe, W. K., Fleckstein, J., & Cullum, C. M. (2005). Magnetic resonance imaging of cerebellar-prefrontal and cerebellar-parietal functional connectivity. *NeuroImage*, *28*, 39-48.

Anderson, J. R., Reder, L. M., & Simon, H. A. (2000). Applications and misapplications of cognitive psychology to mathematics education. Texas Educational Review.

Bergson, H. L. (1960). *Time and free will* (F. L. Pogson, Trans.). New York: Harper and Brothers. (Original work published 1889)

Bergson, H. L. (1991). *Matter and memory* (N. M. Paul & W. S. Palmer, Trans.). New York: Zone Books. (Original work published 1896)

Brown, J. W. (1988). *Life of the mind*. Hillsdale, NJ: Erlbaum Associates.

Brown, J. W. (2002). *The self-embodying mind: Process, brain dynamics and the conscious present*. Barrytown, NY: Station Hill.

Campbell, S. R. (2001). Enacting possible worlds: Making sense of (human) nature. In J. F. Matos, W. Blum, S. K. Houston, & S. K. Carreira (Eds.), *Modelling and Mathematical Education* (pp. 3-14). Chichester, UK: Howard Publishing.

Doya, K. (1999). What are the computations of the cerebellum, the basal ganglia and the cerebral cortex? *Neural Networks*, *12*, 961-974.

Fuster, J. M. (2006). The cognit: A network model of cortical representation. *International Journal of Psychophysiology*, *60*, 125-132.

Habas, C., & Cabanis, E. A. (2007). Cortical projections to the red nucleus: Complementary results with probabilistic tractography at 3 T. *Neuroradiology*, *49*, 777-784.

Handscomb, K. (2007). Enactive cognition and Spinoza's theory of mind. Proceedings of the North American Chapter of the International Group for the Psychology of Mathematics Education, 2007, Lake Tahoe, USA.

Hubel, D. (1988). *Eye, brain, and mind*. New York: Freeman.

Imamizu, H., Miyauchi, S., Tamada, T., Sasaki, Y., Takino, R., Pütz, B., Yoshioka, T., & Kawato, M. (2000). Human cerebellar activity reflecting an acquired internal model of a new tool. *Nature*, *403*, 192-195.

Ito, M. (2006). Cerebellar circuitry as a neuronal machine. *Progress in Neurobiology*, *78*, 272-303.

Kawato, M. (1999). Internal models for motor control and trajectory planning. *Current Opinion in Neurobiology*, *9*, 718-727.

Leiner, H. C., Leiner, A. L., & Dow, R S. (1986). Does the cerebellum contribute to mental skills? *Behavioral Neuroscience*, *100*(4), 443-454.

Leiner, H. C., Leiner, A. L., & Dow, R. S. (1993). Cognitive and language functions of the human cerebellum. *TINS,* 16(11). 444-447.

Mason, J., & Pimm, D. (1984). Generic examples: Seeing the general in the particular. *Educational Studies in Mathematics*, *15*, 277-289.

Mighton, J. (2007). *The end of ignorance: Multiplying our human potential*. Toronto, ON: Alfred A. Knopf Canada.

Ramnani, N. (2006). The primate cortico-cerebellar system: Anatomy and function. *Nature Reviews: Neuroscience*, *7*, 511-522.

Schmahmann, J. D. (1997). Rediscovery of an early concept. In J. D. Schmahmann (Ed.), *The cerebellum and cognition* (pp. 3-27). San Diego, CA: Academic Press.

Schmahmann, J. D. (2004). Disorders of the cerebellum: Ataxia, dysmetria of thought, and the cerebellar cognitive affective syndrome. *Journal of Neuropsychiatry and Clinical Neuroscience, 16*(3), 367-378.

Schmahmann, J. D., & Pandya, D. N. (1997). The cerebrocerebellar system. In J. D. Schmahmann (Ed.), *The cerebellum and cognition* (pp. 31-60). San Diego, CA: Academic Press.

Spinoza, B. (1677/1996). *Ethics* (E. Curley, Ed. & Trans.). London: Penguin Books.

Varela, F. J. (1996). Neurophenomenology: A methodological remedy for the hard problem. *Journal of Consciousness Studies*, *3*(4), 330-349.

Varela, F. J. (1999). The specious present: A neurophenomenology of time consciousness. In J. Petitot, F. J. Varela, B. Pachoud, & J-M. Roy (Eds.), *Naturalizing phenomenology: Issues in contemporary phenomenology and cognitive science* (pp. 266-329). Stanford, CA: Stanford University Press.

Varela, F., Lachaux, J-P., Rodriguez, E., & Martinerie, J. (2001). The brainweb: Phase synchronization and large-scale integration. *Nature Reviews: Neuroscience*, *2*, 229-239.

Varela, F. J., Thompson, E., & Rosch, E. (1991). *The embodied mind*. Cambridge, MA: The MIT Press.

ENCAPSULATING INFINITY: RECONSIDERING ACTIONS

Ami Mamolo

York University

This report is part of a broader study that investigates the specific features involved in accommodating the idea of actual infinity. It focuses on the conceptions of two participants – a mathematics university student, and a graduate – as manifested in their engagement with a well-known paradox: the Ping-Pong Ball Conundrum. The APOS Theory was used as a framework to interpret their efforts to resolve the paradox and one of its variants. These two cases suggest there is more to encapsulating infinity than just the ability to 'act' on a completed object – rather, it is the manner in which objects are acted upon that is also significant.

The focus of this study is shaped by the conceptions of infinity of university students and graduates of mathematics as manifested in their engagement with variations of a well-known paradox: the Ping-Pong Ball Conundrum (presented and discussed below). The APOS (Action, Process, Object, Schema) Theory (Dubinsky & McDonald, 2001) postulates a framework for interpreting learners' understanding of mathematics. Dubinsky, Weller, McDonald, and Brown (2005a) proposed an APOS analysis of two distinct ideas of mathematical infinity: *potential infinity* and *actual infinity*. According to Fischbein (2001), *potential infinity* can be thought of as a process which at every moment in time is finite, but which goes on forever. In contrast, *actual infinity* can be described as a completed entity that envelops what was previously potential. Through the mechanisms of internalisation and encapsulation, Dubinsky et al. (2005a,b) suggest that learners construct meaning for the concept of mathematical infinity as a process and infinity as an object. Further, they relate these two conceptualisations to the ideas of potential infinity and actual infinity, respectively. This study takes a closer look at the specific features connected to the encapsulation of infinity as an object.

THE PING-PONG BALL CONUNDRUM

The Ping-Pong Ball Conundrum can be presented in the following way:

> Imagine you have an infinite set of ping-pong balls numbered 1, 2, 3, …, and a very large barrel; you are about to embark on an experiment. The experiment will last for exactly 1 minute, no more, no less. Your task is to place the first 10 balls into the barrel and then remove ball number 1 in 30 seconds. In half of the remaining time, you place balls 11 – 20 into the barrel, and remove ball number 2. Next, in half of the remaining time (and working more and more quickly), place balls 21 – 30 into the barrel, and remove ball number 3. Continue this task ad infinitum. After 60 seconds, at the end of the experiment, how many ping-pong balls remain in the barrel?

The normative resolution to the Ping-Pong Ball Conundrum (PP) involves coordinating three infinite sets: the in-going ping-pong balls, the out-going ping-pong balls, and the intervals of time. In order to make sense of the resolution to this paradox, a normative understanding of actual infinity is necessary. Although there are more in-going ping-pong balls than out-going ping-pong balls at each time interval, at the end of the experiment the barrel will be empty. An important aspect in the resolution of this paradox is the one-to-one correspondence between any two of the three infinite sets in question (see Mamolo & Zazkis, 2008, for a more detailed discussion). Given these equivalences, at the end of the experiment, the same amount of ping-pong balls went into the barrel as came out. Moreover, since the balls were removed in order, there is a specific time for which each of the in-going balls was removed. Thus at the end of the 60 seconds, the barrel is empty.

A variation of the Ping-Pong Ball Conundrum can easily be imagined. Consider the following:

> Rather than removing the balls in order, at the first time interval remove ball 1; at the second time interval, remove ball 11; at the third time interval, remove ball 21; and so on… At the end of the experiment, how many balls remain in the barrel?

This Ping-Pong Ball Variation (PV) begins in much the same was as the original Ping-Pong Ball Conundrum (PP). In one minute, an experiment involving inserting and removing infinitely many ping-pong balls from a barrel is carried out. However, the distinction lies in the fact that the Ping-Pong Ball Variation calls for the removal of balls numbered 1 at time one, ball number 11 at time two, ball number 21 at time three, and so on. Thus, despite the one-to-one correspondences between all of the infinite sets in question, at the end of the 60 seconds there will remain infinitely many balls in the barrel. In this experiment there will never be a time interval wherein balls 2 to 10, 12 to 20, 22 to 30, and so on, are removed. The seemingly minor distinction between removing balls consecutively, as in PP, versus removing them in a different ordering, as in PV, has a profound impact on the resolution of the paradoxes: while in one instance subtracting infinitely many balls from infinitely many balls yielded zero, in the other it yielded infinitely many.

BACKGROUND

The Ping-Pong Ball Conundrum, PP, and the Ping-Pong Ball Variation, PV, illustrate an essential feature of the conventional understanding of actual infinity: that the ‘sizes’ of infinite sets, or their ‘cardinalities’, are compared by establishing one-to-one correspondences. This understanding forms the basis of Cantor’s Theory of Transfinite Numbers, a theory in which Cantor developed several aspects of actual infinity, and which is accepted today as one of the cornerstones of mathematics.

A prominent trend in current research in mathematics education has been to analyse learners’ conceptions of actual infinity through their strategies of comparing infinite sets (e.g. Fischbein et al., 1979; Tirosh & Tsamir, 1996; Tsamir, 2003). In particular, when analysing infinite sets, students were observed to apply methods for comparing sets that are acceptable in the case of finite sets, such as the ‘inclusion’ (or ‘part-whole’) method, but which result in contradictions in the infinite case (Dreyfus &

Tsamir, 2004; Fischbein et al., 1979; Tall, 2001). Further, Tsamir and Tirosh (1999) observed that different presentations of sets, such as the set N = {1, 2, 3, …} of natural numbers and the set E = {2, 4, 6, …} of even numbers, elicited different strategies of comparison from high school students. For example, if the sets N and E were presented side-by-side, students tended to respond that N was the larger set since the set E was contained within it (the 'inclusion' method of comparison). Whereas if the sets N and E were presented one above the other, the tendency was to draw a one-to-one correspondence between each number and its double and thus conclude that the two sets were equinumerous. In a related study which examined the conceptions of prospective secondary school teachers, Tsamir (2003) observed, "Even after studying set theory, participants still failed to grasp one of its key aspects, that is, that the use of more than one … criteria for comparing infinite sets will eventually lead to contradiction" (p.90).

Complementing investigations into learners' conceptions of infinite set comparison includes research which examines learners' understanding of paradoxes of infinity. Mamolo and Zazkis (2008) explored the conceptions of undergraduate and graduate university students as they emerged in participants' engagement with Hilbert's Grand Hotel paradox and the Ping-Pong Ball Conundrum. They observed: (1) participants attended to the practical impossibility of the paradoxes and dismissed the normative solutions, and (2) participants who distinguished between their intuitive inclination and formal knowledge had the most success resolving the paradoxes. Mamolo and Zazkis (2008) concluded, in resonance with recommendations made by Dubinsky and Yiparaki (2000), that separating 'realistic' and intuitive considerations from conventional mathematical ones is important in helping learners appreciate properties of actual infinity.

This study extends on prior research by using paradoxes as a lens to investigate learners' understanding of 'acting' on infinite sets, in the terminology of the APOS Theory. Using paradoxes as a lens to investigate learners' understanding of different aspects of actual infinity has been acknowledged as an effective means to help identify specific difficulties inherent in conceptualising actual infinity (Dubinsky et al. 2005a; Mamolo & Zazkis, 2008). In particular, through an analysis of learners' engagement with the ping-pong ball paradoxes, this study offers a first look at the specific challenges associated with subtracting infinite quantities.

This report focuses on the responses of a mathematics university student and a graduate of mathematics, as they addressed the Ping-Pong Ball Conundrum (PP) and the Ping-Pong Ball Variation (PV), and analyses their responses through the perspective of the APOS Theory. Of particular interest are the specific attributes related to encapsulating infinity.

THEORETICAL PERSPECTIVES

The APOS Theory postulates a framework for interpreting learners' understanding of tertiary mathematics. Through the mechanisms of *internalisation* and *encapsulation* the learner is said to construct meaning for mathematical entities that are conceptualised with the 'structures' of the APOS Theory: *actions, processes, objects,* and *schemas* (Dubinsky & McDonald, 2001). In the terminology of the APOS Theory, an understanding of a mathematical entity begins with an *action* conception of that entity. Action conceptions are recognised by an individual's need for an explicit expression to manipulate or evaluate. Eventually, an action may be *interiorised* as a mental *process*. That is, once an action has been interiorised, the individual can imagine performing an action without having to directly execute each and every step. A process conception is recognised by qualitative descriptions which may describe actions though not execute them. If the individual realises the process as a completed totality, then *encapsulation* of that process to an *object* is said to have occurred. Encapsulation of a process is considered a sophisticated step in an individual's conceptualisation. It requires appreciating the mathematical entity as a completed object that can be acted upon. In other words, the entity is conceived of as an object upon which transformations or operations may be applied. These three structures of the APOS Theory – the action, process, and object – describe how the idea of a single mathematical entity may develop. However, it is possible that a mathematical concept may be composed of more than one entity, involving several actions, processes, and objects that must be coordinated into a mental *schema*.

Relating this discussion to the concept of mathematical infinity, Dubinsky et al. (2005a,b) suggested that interiorising infinity to a process corresponds to the idea of potential infinity, that is, infinity is imagined as performing an endless action, though without imagining carrying out each step. Encapsulating this endless process to a completed object is said to correspond to a conception of actual infinity. Connecting Dubinsky et al.'s (2005a,b) classification to this study, in the case of the Ping-Pong Ball Conundrum, the action of cutting the remaining time in half can be imagined to continue indefinitely, and would thus describe potential infinity. Whereas actual infinity would entail the completed infinite process of halving time intervals, and would describe the set of time intervals as a completed entity, where each interval exists within the 60 seconds.

As in the general case, encapsulation of infinity is considered to have occurred once the learner is able to think of infinite quantities "as objects to which actions and processes (e.g., arithmetic operations, comparison of sets) could be applied" (Dubinsky et al., 2005a, p.346). Dubinsky et al. (2005a) also observed that "in the case of an infinite process, the object that results from encapsulation transcends the process, in the sense that it is not associated with nor is it produced by any step of the process" (p.354). Brown, McDonald, and Weller (in press) introduced this possibility, and termed the encapsulated object of infinity a transcendent object.

Two questions arise: (1) *How* does a learner act on infinity (i.e. how are arithmetic operations applied)? and (2) What can the 'how' tell us about an individual's understanding of infinity? This study is a first attempt at addressing these questions.

SETTING AND METHODOLOGY

Data for this study were collected from two participants: Jan and Dion. Jan was mathematics major in a southeastern state university in the USA. She was in her final year of the program and was very interested in the concept of infinity both from a mathematical and philosophical point of view. Jan had prior experience with Cantor's Theory of Transfinite Numbers through formal instruction during her undergraduate studies. In particular, she was familiar with comparing sets via one-to-one correspondences. Dion was an instructor in mathematics education at a university in eastern Canada. He held a master's degree in mathematics education and a bachelor's degree in mathematics. Dion taught prospective secondary school teachers in mathematics and didactics, the curriculum for which included aspects of Cantor's theory, such as establishing a one-to-one correspondence between the sets of natural and even numbers.

Data was collected from an interview with the participants, who were asked the Ping-Pong Ball Conundrum (PP) as stated above. Following their responses and a discussion of the normative resolution to PP, participants were asked to address the variant (PV) which calls for the removal of balls numbered 1, 11, 21, … A discussion of the normative resolution to PV, akin to the explanation above, ensued. After this discussion (at the end of the interview), participants were encouraged to reflect on the two thought experiments and their outcomes.

RESULTS

Contrary to prior research (Mamolo & Zazkis, 2008), both Dion and Jan were easily able to resolve the Ping-Pong Conundrum (PP) by establishing the appropriate one-to-one correspondences. They both realised that the question of 'how many' balls referred to the cardinality of the sets of balls, and as Jan stated:

> "equal cardinality of two sets is entirely determined by the existence or nonexistence of a bijection [one-to-one correspondence] between the two sets in question."

When explicitly addressing the comparison between sets of in-going and out-going balls, both participants realised that "every ball that is put into the barrel is removed." In Jan's words:

> "if a ball is placed in the barrel in the nth step, then it is removed in one of the steps 10n-9, 10n-8, ... ,10n-1, 10n. So if a ball is placed in the barrel during the minute, it will be taken out. Conversely, if a ball was taken out of the barrel, it must have been put in at some point during the minute. This establishes a bijection between the balls put in the barrel and those taken out."

Similarly, both Jan and Dion also recognised a one-to-one correspondence between the in-going and out-going balls in the Ping-Pong Variation (PV). Dion commented on the similarities between PP and PV as well as the relevance of Cantor's Theory of Transfinite Numbers to his solutions. When addressing PV, Dion reasoned that, as in PP, there existed one-to-one correspondences between the pairs of sets of in-going and out-going ping-pong balls and time intervals. He concluded that the variant and the "ordered case" should yield the same result: an empty barrel. Dion argued that the barrel would be empty because "after you go [remove] 1, 11, 21, 31, …, 91, etc, you go back to 2". He described a "strong leaning to Cantor's theorem" (Cantor's Theory of Transfinite Numbers), and although he insisted that "at some point we'll get back to 2", he could not justify the claim.

During the interview, Dion grappled with the possibility of a nonempty barrel. He stated:

> "If ball number 2 is there, so is ball 2 to 10, etc… so, infinite balls there? I have trouble with that."

Eventually, Dion conceded he was "convinced" of the normative solution to PV. Dion went on to observe that while "on one hand infinite minus infinite equals zero, on the other it's infinite" – a property of transfinite arithmetic that was absent in his prior knowledge of Cantor's Theory of Transfinite Numbers. Engaging with the two paradoxes contributed to Dion's discovery of the indeterminacy of subtracting infinite quantities. Further, Dion's revelation that "on one hand infinite minus infinite equals zero, on the other it's infinite" suggests that accommodating the idea of actual infinity goes beyond the ability to act on an object, and includes an understanding of *how* to act on that object.

In contrast to Dion's struggle, Jan was able to resolve PV, recognising that "transfinite cardinal arithmetic doesn't work exactly like finite cardinal arithmetic". Jan connected her understanding of correspondences between infinite sets to explain the indeterminacy of transfinite subtraction. She remarked:

> "Even though there is a bijection [one-to-one correspondence] between the set of balls put into the barrel and the set of balls removed, there are still an infinite number of balls left in the barrel after the minute is up! … we can easily create an infinite sequence of balls that are not removed".

Jan realized that although the quantity of balls taken out of the barrel was the same as the quantity put in, this was not sufficient to conclude that *all* of the balls had been removed. Jan observed that remaining in the barrel was the set of balls numbered

> "$\{10n+2 \mid n=0,1,2,...\}$. This set is clearly infinite, and represents a subset of the balls left after the minute. Since the set of all balls left after the minute contains an infinite subset, it too must be infinite."

Further, Jan recognised the significance of the ordering of out-going balls as it determined which balls were removed from or remained in the barrel. Jan reflected on this issue as well as the relationship between her intuition and properties of actual infinity:

> "So, if we think about both the original question [PP] and its variation [PV], we seem to have done the exact same thing (physically) in both cases, but due to some arbitrary numbering system that we have imposed upon the set of balls removed, we have changed the remaining number from zero to infinity! But why should numbering matter? We seem to have done the same thing in both cases. This is [a] case where the intuition we've learned from the physical world fails us when it comes to the infinite".

DISCUSSION

(1) How does a learner act on infinity?

Dubinsky et al. (2005a) suggest two ways an individual may act upon the object of infinity – by applying arithmetic operations and by comparing cardinalities of sets. Focusing on the former, this study identifies two different ways learners 'acted' on infinity. Dion, who revealed a normative understanding of infinite set comparison in his resolution of PP, suggested that 'anything' subtracted by itself should be zero, and expressed 'trouble' with the idea that the barrel in PV would not be empty. When Dion was faced with a non-routine problem regarding transfinite subtraction, he 'acted' by generalizing his intuition of subtracting real numbers, and had difficulty with the indeterminacy of subtracting infinite quantities. Dion's struggle was surprising in light of his comfort and understanding of the normative approach to infinite set comparison. In contrast, Jan's ability to deduce consequences of a set being equinumerous with one of its proper subsets contributed to her understanding of the indeterminacy of transfinite subtraction. It allowed her to 'act' – both by comparing sets and by applying arithmetic operations – in a way that was consistent with normative standards.

(2) What can the 'how' tell us about an individual's understanding of infinity?

Dion's difficulty acting on actual infinity via applying arithmetic operations in the normative way, and his resistance toward the indeterminacy of transfinite subtraction, suggest that acknowledging the distinction between *how* actions, such as arithmetic operations or set comparisons, behave differently when applied to transfinite versus finite entities is an integral part of accommodating the idea of actual infinity. Further, it suggests that how actions are applied may be relevant to the encapsulation of an object. Dion's struggle exemplifies the intricacies involved in the mechanism of encapsulation – although Dion seemed able to consider the infinite sets of ping-pong balls as 'completed' entities which could be compared, he nevertheless was 'troubled' with transfinite subtraction.

This study opens the door to further investigation into how learners act on infinity and what, if anything, can be inferred about an individual's conceptualisation based on how that individual applies actions to a mathematical entity and which actions are applied.

References

Brown, A., McDonald, M., & Weller, K. (in press). Students' conceptions of infinite iterative processes.

Dreyfus, T., & Tsamir, P. (2004). Ben's consolidation of knowledge structures about infinite sets. *Journal of Mathematical Behavior, 23*, 271–300

Dubinsky, E., & McDonald, M. A. (2001). 'APOS: A constructivist theory of learning in undergraduate mathematics education research.' In Derek Holton, et al. (Eds.), *The Teaching and Learning of Mathematics at University Level: An ICMI Study*, (pp. 273–280). Dordrechet: Kluwer Academic Publishers.

Dubinsky, E., Weller, K., McDonald, M.A., & Brown, A. (2005a). Some historical issues and paradoxes regarding the concept of infinity: an APOS-based analysis: Part 1. *Educational Studies in Mathematics*, 58, 335–359.

Dubinsky, E., Weller, K., McDonald, M.A., & Brown, A. (2005b). Some historical issues and paradoxes regarding the concept of infinity: an APOS-based analysis: Part 2. *Educational Studies in Mathematics*, 60, 253 – 266.

Dubinsky, E., & Yiparaki, O. (2000). On student understanding of AE and EA Quantification. In E. Dubinsky, A.H. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education IV*, (pp. 239-289). Providence: American Mathematical Society.

Fischbein, E. (2001). Tacit models of infinity. *Educational Studies in Mathematics*, 48, 309–329.

Fischbein, E., Tirosh, D., & Hess, P. (1979). The intuition of infinity. *Educational Studies in Mathematics, 10,* 3–40.

Mamolo, A., & Zazkis, R. (2008). Paradoxes as a window to infinity. *Research in Mathematics Education, 10,* 167 – 182.

Tall, D. (2001). Natural and formal infinities. *Educational Studies in Mathematics, 48*, 199–238.

Tirosh, D. & Tsamir, P. (1996). The role of representations in students' intuitive thinking about infinity. *International journal of mathematical education in science and technology, 27(1)*, 33-40.

Tsamir, P. (2003). Primary intuitions and instruction: The case of actual infinity. In A. Selden, E. Dubinsky, G. Harel, & F. Hitt (Eds.), *Research in collegiate mathematics education V* (pp.79-96). Providence: American Mathematical Society.

Tsamir, P., & Tirosh, D. (1999). Consistency and representations: The case of actual infinity. *Journal for Research in Mathematics Education, 30*, 213–219.

CONNECTING PATTERNS AND THAT MUMBO JUMBO STUFF WE HAVE TO TEACH: A COLLABORATIVE LESSON DESIGN

Paulino Preciado

Simon Fraser University

Predicting students' struggles and possible approaches in problem solving is part of Lesson Study strategy. In this paper a team of teachers—including the author—made use of previous experience, knowledge of current students, and some theoretical background from the literature in order to prepare suitable responses in advance to students' questions and thoughts in the designed tasks. While making such predictions, beliefs of mathematics and mathematics learning were discussed and negotiated, and we developed theoretical statements about students' learning process. In conclusion, I argue that predicting such possible students' struggles and approaches not only provides an arena to analyze and negotiate teachers' mathematical and pedagogical knowledge, but also is a critical factor contributing to the improvement to educational systems.

INTRODUCTION

This paper is a preliminary result of a wider research project which attempts to analyze teachers' interactions when designing mathematical lessons collaboratively—in particular by conducting lesson study—as described by James W. Stigler and James Hiebert in their book *The Teaching Gap* (1999). The research focuses on the potential teaching improvement by giving an account of teachers' changes in their beliefs and practices.

Teachers and educators have been using collective lesson design and analysis as part of their professional development. Watanabe (2007) explains that lesson study contributes to improving the curriculum and textbooks design; Marton and Tsui (2004) use learning study, based on variation theory, as means of improving learning. Both learning study and lesson study involve collective lesson design by teachers and educators in a reflexive way. Communities of teachers and educators pursuing learning improvement in a critical way are sustained on social practices of teaching—for instance, Jaworski (2006) argues in favor forming communities of inquiry, and Servage (2008) describes the critical and transformative practices of professional learning communities. This perspective situates the teachers in a more socially-engaged practice, going beyond just the classroom: critical collaborative work among peers is a part of teachers' practice.

It is easy to predict the kind of interactions that members of a lesson design team could have, as well as the knowledge they might share. Subject matter, pedagogical and curricular knowledge—as described by Shulman (1986)—is shared through the discussions teachers have while planning lessons. The reflexive process of teachers and the changes they undergo have been analyzed using narratives (e. g. Brown and

Jones, 2001). However, a good understanding of the factors which contribute to professional growth in a community and its members when working in a collaborative way is needed in order to implement and sustain such community' development. Which factors contribute or constrain the collaborative work? What kinds of interactions trigger the teachers' learning in a community? What are the individual and community's learning processes? The aims of this research is to take a look inside a small community in order to observe its members' interactions and learning while participating in a cycle of lesson design, implementation, and refinement.

Among the topics teachers use to discuss in lesson study are: the selected problem for the lesson, including its wording and numbers; anticipated solutions, thoughts, and responses which students might develop; and the kind of guidance, or questions, that could be given to support students showing some misconceptions in their thinking (Stigler & Hiebert 1999, p. 117). In this paper I show a case where teachers involved in lesson study engage in discussions using theoretical statements,both derived from the literature and developed by their own, while approaching those aforementioned topics.

THEORETICAL BACKGROUND

"Communities of practice" (Wenger, 1999) is a useful theory to describe social interactions of people having a common enterprise. The notions of negotiation of meaning and identity serve to describe how knowledge is generated in such a community. In order to describe the process of designing the lesson, and in particular how a team developing theoretical statements about student learning, it is important to consider the collective characteristics of the task. As a community, each participant has a way of engaging with the team. Each individual contributes different ideas and resources according to his or her own perspective. Each participant has an identity, and a role in this particular community. The meaning of the theoretical statements obtained in the discussion of the lesson is negotiated by the team.

I will consider professional development not only as an individual enterprise, but also as a matter of community learning. For instance, if some teachers leave the school, the community of the school keeps a legacy of the former members. From this point of view, improving teachers' practices also improve s the school in time.

Brown (2001) describes learning not just as adding knowledge, but as a transformative process: *"...knowledge, or at least our state of knowing, can be transformed in many ways; one subtracts from it as well as adds to it . . . one reorganizes so that new things get new meaning"* p. 84. I use this description of learning for both communities and individuals; the case of learning in a community is described by Wenger (1998). From a phenomenological perspective, Brown (argues) argues that *"It is the individual's experience of the world, of mathematics and social interactions which governs his actions rather than externally defined notion of mathematics itself"* p. 138. I consider not only mathematical notions, but also ideas we may have about any other subject; in particular teachers' notions of students' learning process. Teachers and researchers build their knowledge from personal experience, peers discussion, and literature reviews—which in any case are subordinated to personal interpretation. From this point of view, the individual—teacher or researcher—as well as the community, makes meaning of, describes, and predicts certain phenomena—for example, students' performance in a classroom. I will call these individual or collective interpretations *theoretical statements*.

METHODOLOGY

In order to give an account of the evolving process of teachers designing a lesson through several meetings, ethnography is a suitable means of describing social interactions and micro-cultural aspects of the community. A team of teachers was video recorded while designing and discussing a mathematical lesson. I split the video records in small segments and write a description according to what was discussed in each moment, generating pre-codes. After focusing on teachers' use of theoretical statements to predict student approaches to the mathematical tasks of the lesson, I selected some segments to transcribe and analyze. My participation in the research is both as a member of the team and as a researcher. The use of video allowed me to focus fully in the lesson design discussion, and observe the meetings later for research purposes.

As I mentioned above, this is a preliminary report of a design experiment (Cobb, Confry, diSessa, Lehrer, & Schuable, 2003) and some steps which will complement and give stronger validation to the study are still missing. Further interviews with participants will be conducted in order to discuss my conclusions, or detect new issues. New interventions in the future will be conducted with a possible shift in the research.

THE STUDY

The team was composed of three secondary mathematics teachers from the same school and me—I have experience teaching at this level, though in another country. I will refer to myself as Armando when describing and analyzing the video recorded meetings.

The lesson has been designed for a grade nine class in a secondary school in British Columbia. We held five meetings, one a week, before the implementation of the lesson. Teachers selected the goal of the lesson study: for students to write algebraic expressions from word sentences. We decided to use patterns for this purpose in part because it relates to the curricular prescribed learning outcomes students have to achieve.

From the video it is possible to distinguish some differences among the team members. For instance, Arnold (pseudonym) always brings some book, article, or other resource to the discussion; Brad (pseudonym) engages in a critical way in the discussion by questioning whether we will reach the desired goals of the lesson, Denzel (pseudonym) used to redirect the discussion of the meeting when we lost focus, and Armando used to refer to his previous experiences to explain ideas. These differences are instances of each member engaging in the community.

Theorizing in order to predict

Since the first meeting, when we decided to use patterns, Armando has hypothesized that students will make meaning of algebraical expression easier if they can verbalize mathematical procedures derived from finding the required number—for example, perimeter, amount of squares—in a sequence of shapes with some linear pattern. In this hypothesis, Armando was theorizing the way they can make sense of algebraic expressions.

When we started to predict possible student approaches and difficulties, theoretical statements were used in the discussion; some of them came from books or other material and some of them were generated by us. I will show first an example of the use of theoretical statements from the literature.

As a way of describing the students level of understanding, Arnold referred to an assessing scale which appears in Marzano (2007) and consists in four major levels—with some additional sub-levels in between.

> The lowest score value on the scale is a 0.0, which represents no knowledge of the topic. Even with help, the student demonstrates no understanding or skill relative to the topic . . . A score of 1.0 indicates that with help the student shows partial knowledge of the simpler details and processes as well as the more complex ideas. . . .[with] a score of 2.0, the student independently demonstrates understanding of and skill at the simpler details and processes, but not of the more complex ideas and processes. A score of 3.0 indicates that the student demonstrates skill and understanding of all the content—simple and complex—that was taught in class. A score of 4.0 indicates that the student demonstrates inferences and applications that go beyond what was taught at class (p. 104).

This scale was used by Arnold in the second meeting as both a reference to classifying students as well as a description of how we would like students to move forward.

> Arnold: More moving on the scale so that if we model here for students with help, then hopefully the students will be able to move into this area [pointing to the score above 1.0 in the assessing scale].

Further in the same session, Arnold kept using the same scale to describe students' possible paths when we were selecting the problem to be posed in the lesson.

> Arnold: If it was up to me, I would introduce that and really try to move them on this continuum to one and then set them independently in something more challenging, and then see where the students can get to [pointing out to the Marzano's scale].

A second case of the use of theoretical statements is the way it helps to make meaning of students' learning process. In the end of the second meeting Brad was questioning whether the use of patterns in the way we were discussing would be effective in making students translate words into algebraic expressions, while Armando argued it was necessary that students come up with the algebraic expressions from their own explanation of how to find the general term in the patterns.

Brad: After doing all those puzzle-solving [problems] and getting their own solutions and writing them down and talking about it, how is that help with specifically this task of translating? [words into algebraic expressions].

....

Armando: We must conclude this lesson.... with some algebraic expressions . But the idea is that these algebraic expression come from the wording of students.

However, Brad is still concerned with covering the topics in the curriculum, which are the same as in the book, and how to relate that with the use of patters in the lesson we are designing.

Brad: I'm just trying to find the connection, the link . . . . So, this is the class, we now did all this problem solving, and now lets see if they can do this. We still have to teach this, no matter what.... After all these funny games we still have to teach this [pointing to the page in the textbook related to writing algebraic expression from English sentences].

Next meeting Brad made meaning of the students' process of writing their ideas in order to write algebraic expressions. It seems that Brad is thinking and talking, questioning and answering at the same time.

Brad: What I am trying to do is [this], I'm fitting in what I have to teach in that section, where they translate words into algebra, with the activities here. So, I was thinking: lets say they come up with "there is two more than three times the stage". Well, that is good because then we now can express algebraically two more than three times the stage; "is this like this?" and you can write it like that.

In my mind, I am trying to blend in what we are doing here with what we have to teach, or what they have to know how to do in the textbook. I am trying to find that connection between this and those mumbo jumbo stuff they have to do. They say, "Oh this is just two more than the stage." How do we write two more than something?

Arnold: I think that will be amazing if they can verbalize that, because.[interrupted by Brad]

Brad: They can verbalize it down here: you know for the tenth stage you just have to add two more to the previous stage. But, what does it have meant? How do you write that in math instead of just writing it in words? I try to blend, I'm trying to fit in that section I want to teach with what we were doing here. I try to find the way to do that in my head.

So if I would do this in my first class, I would say O.K. lets take a look at all your answers down here. Is there a way to simplify that? Is there a way to make it easier to write instead of all the words? You know, it took two lines or three lines to explain your answers. Is there a simpler way, an easier way to do that? And they will say "okay. . . . lets write that algebraically, or may not call it algebraically, lets write that in an easier way, and see what it looks like."

And then Brad agrees with Armando about providing a hypothetical case of two different students' approaches to find the general term in one of the possible sequences we use in the lesson.

Brad: But one group may say "two more than twist the number" or one group may say "add one to the stage and double it." Well, let's say they are the same. Let's work it out algebraically and see they are the same... Why not use algebra instead of all these words?

After this discussion, Brad started participating in a more enthusiastic way with the lesson design, possibly because Brad has made meaning of the use of the patterns in getting students to translate word sentences to algebraic sentences.

A third case was the use of theory from other sources in order to design our questions for students' task. In the fourth session, Arnold brought a rubric to evaluate communication for students from a binder with many resources which have been used before. The rubric consists in three criteria and four levels for each criterion. Although we did not use this rubric to describe or assess students, it was useful in phrasing one of the questions for the students' task.

Armando: We are missing here the fourth question which is: could you explain how do you get it [the number of the n stage]?

Arnold: How do you phrase that? Explain your ... like it was consistent with this rubric: "my explanations are clear and complete, and easily understood." [showing the communication evaluating rubric].

We were discussing the point that the textbook presents sentences which students must translate into algebraic expression. These expressions have no context and Denzel, as a theoretical statement, though that it would be problematic for students. Brad agreed with that statement.

Denzel: I'm kind of thinking that when they describe something that is physical, then it's easier to translate it [to an algebraic expression] than just some sentence.

Finally, at the end of the fifth meeting, we came up with a chart of students' possible thought-pathways, struggles, and teachers' responses. This reflects the theoretical statements we came up after meaning discussion and negotiation. These statements provided a framework which was also proposed as a frame to observe students achievements in the lesson while it was being implemented (Table 1).

Student steps and potential gaps	Possible teacher responses
Student draws each pattern and counts number of squares/perimeter.	
Student uses recursive thinking (adding to previous stage) to determine number of squares/perimeter at each subsequent	"Where are you adding lines at each stage?"

stage.	
Student can predict number of squares/perimeter at any stage (non-recursively). Student can use words to describe how to determine number of squares/perimeter at any stage (non-recursively).	"Is there a more efficient way of adding [the same number] many times?" "For [a particular stage], how many times did you add [the same number]?" "Think aloud."
Student can write a mathematical formula to describe number of squares/perimeter at any stage.	For pattern A, work through this step with the class, using several different examples of student-generated words to come up with (hopefully) a few formulas which can be compared. Use n, as well as $n-1$. Use the word "previous" in relation to $n-1$. Use the word "formula". Refer to this example later when students try to find formula for other patterns.

Table 1. Students' steps and potential gaps, and teachers' responses.

After the implementation of the lesson, we could observe that almost all student teams, when presenting their answers to the group, came up with a general formula to describe the patterns in the tasks. This was beyond the expectations of the lesson. We also realized that students didn't have troubles in moving from the drawn figures to the recursive formula. However, they needed some guidance to explain their process. Some other students wrote algebraic general expressions as explanation for their procedures. Although they were correct, explanation with words were missing.

Discussion

The process of anticipating solutions, thought, and responses which students might develop, as well as planning the kind of guidance teachers will give to students, entails teachers' use of theoretical statements—which either come from a known cognitive theory or are developed by teachers. This process challenges teachers' beliefs and assumptions, and triggers an adjustment of both the individual teacher's conceptions and collective meaning.

The idea of generating and refining theories useful for teaching in a collaborative lesson design involving teachers has been applied in design experiments (Cobb, Confry, diSessa, Lehrer, & Schuable, 2003). For instance, learning study uses variation theory as a grounded theoretical framework (Marton & Tsui, 2004). The capability of describing and anticipating phenomena is characteristic of any theory. Therefore, describing and anticipating students' solutions or approaches to posed problems must entail the use or development of theoretical frameworks.

In addition to corroborating and refining teachers' theoretical statements, written reports of implemented lessons contribute to an improvement in the community itself, not only in the teacher. In this way, the educational system can be in a permanent process of enhancement, as Watanabe (2007) mentions with regard to Japanese elementary educational system. Implementing communities of teachers working collaboratively and contributing to next teachers generations in local settings—school or district—will contribute to the learning of such community, not only as a mentor-apprenticeship relation with novice and experienced teachers, but also as generating new knowledge from teacher's practice.

References

Brown, T. (2001). *Mathematics education and language: Interpreting hermeneutics and post-structuralism* (2nd ed.). Dordrecht, The Netherlands: Kluwer Academic Publishers.

Brown, T., & Jones, L. (2001). *Action Research and postmodernism: Congruence and critique.* Buckingham: PA: Open University Press.

Cobb, P., Confry, J., diSessa, A., Lehrer, R., & Schuable, L. (2003). Design experiments in educational research. *Educational researcher*, 32(1), 9-13.

Jaworski, B. (2006). Theory and practice in mathematics teaching development: critical inquiry as a mode of learning in teaching. *Journal of Mathematics Teacher Education*, 9, 187-211.

Marton, F., & Tsui, A. B. (2004). *Classroom discourse and the space of learning*. New Jersey: Erlbaum.

Marzano, R. J. (2007). Designing a comprehensive approach to classroom assignment. In D. Reeves, (Ed.), *Ahead of the curve: The power of assessment to transform teaching and learning.* Bloomington, Ind.: Solution Tree.

Servage, L. (2008). Critical and transformative practices in professional learning communities. *The Teacher Quarterly,* 35(1), 63-77.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.

Stigler, J. & Hiebert, J. (1999). *The teaching gap: Best ideas for the world's teachers for improving education in the classroom*. New York: The Free Press.

Watanabe, T. (2007). In pursuit of a focused and coherent school mathematics curriculum. *The mathematics educator*, 17(1), 2-6.

Wenger, E. (1998). *Communities of practice: learning, meaning, and identity.* New York: Cambridge University Press.

PEDAGOGICAL CONTENT KNOWLEDGE IN MATHEMATICS FOR ELEMENTARY TEACHERS COURSES: TWO PRELIMINARY CASES

Susan Oesterle

Simon Fraser University

This paper offers some preliminary results of a qualitative research project whose aim is to study pedagogical content knowledge in the context of Mathematics for Elementary Teachers courses. Grounded theory methodology is applied to interviews with two teachers of this course. Themes that emerge from analysis of the transcripts are identified and discussed, and implications for future directions of the project are considered.

BACKGROUND

Over the last twenty years mathematics education researchers have worked towards more clearly articulating the knowledge that teachers need to teach mathematics effectively. In the late 1980's Lee Shulman (1986) coined the phrase "pedagogical content knowledge", which captured the essential interplay between subject knowledge and knowledge of teaching and learning in the practice of teaching. The subsequent research on "mathematics for teaching", though far from attaining a definitive description of all it entails, suggests that beyond subject matter competence teachers need a profound understanding of mathematics (Ma, 1999) which allows them to plan for and respond to their students' needs (e.g. Simon, 2004). (See Ball, Lubienski, & Mewborn, 2004, for a review.)

In an effort to more appropriately respond to the needs of prospective elementary school teachers, university mathematics departments have developed Math for Teachers courses. These courses are described as mathematics <u>content</u> courses and are distinguished from pedagogically focussed mathematics <u>methods</u> courses that are offered by education departments. This apparent separation of content from pedagogy seems to go against Shulman's (1986) call for a more integrated approach. However the Math for Teachers courses do aim to develop a deeper conceptual understanding of elementary school mathematics than students would have encountered previously. The curriculum typically includes a wide variety of representations and models for arithmetic operations and procedures, which forms a part of Shulman's (1986) conception of pedagogical content knowledge. Opportunity exists for teachers of these courses to foster the development of pedagogical content knowledge in their students. But do they?

Math for Teachers courses are typically taught by instructors who have advanced mathematics degrees and little or no formal training in pedagogy, let alone experience in elementary school classrooms. Given the importance of pedagogical content knowledge in the development of good mathematics teachers, it would be useful to know to what extent this type of knowledge arises in the context of these courses.

This paper offers some very preliminary results of a larger research project that seeks to describe the types of pedagogical content knowledge that are addressed by teachers of Math for Teachers courses, and the methods that these teachers employ. Such an investigation has the potential to contribute to our conception of the mathematical knowledge needed for teaching, and provide a theoretical basis for future reform of these courses.

METHODOLOGY

This study applies grounded theory techniques as described in Cresswell (2008) and Corbin & Strauss (2008). This research methodology is especially suitable for developing rich descriptions of complex phenomena.

Theoretical sampling is employed to select interview subjects from the pool of instructors of Math for Teachers courses at various post-secondary institutions in British Columbia. These instructors are invited to participate in a semi-structured interview that takes approximately one hour. The interview begins with questions about the backgrounds of the instructors: education, number of years of teaching, and number of years of teaching Math for Teachers. These questions are followed with questions about their initial orientation (preparation) for teaching the course, about what they do differently with this group of students compared to their other mathematics students, about their goals in teaching the course, and the outcomes they believe they achieve. These interviews are then coded for emergent themes through a process of constant comparative analysis.

It is important that I acknowledge from the outset that though I am researcher and interviewer in this study, I am also an instructor in a mathematics department at a post-secondary institution, and have taught the Math for Teachers course over a dozen times in the last 18 years. My experience with the issues surrounding this course may help provide insight into the perspectives of my interviewees, but also introduces the possibility of bias. To compensate for this, interview coding is corroborated by a more neutral colleague, and narrative descriptions are read and verified by the interview subjects.

The preliminary results reported here are drawn from only two interviews, and as such only provide a starting point for the study. From these interviews I have been able to write brief narratives describing the interview subjects. Some initial themes have emerged offering insight into potentially fruitful avenues for further investigation.

THEORETICAL FRAMEWORK

Shulman's (1986) description of pedagogical content knowledge provides a valuable initial framework for recognising instances of it in the interviews. Under this type of teacher knowledge he includes

> the most useful forms of representation of those [mathematical] ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others (p.9)

He goes on to add

> an understanding of what makes the learning of specific topics easy or difficult: the conceptions and preconceptions that students of different ages and backgrounds bring with them to the learning of those most frequently taught topics and lessons (p.9)

This understanding of pedagogical content knowledge has influenced the formulation of the initial interview questions. In particular, the instructors are asked to describe how they approach teaching a unit on fractions, a topic in elementary school mathematics that is particularly rich in models and interpretations, and one that is traditionally seen as problematic.

Ball and Bass (2003) have also contributed to our understanding of the scope of pedagogical content knowledge through their analysis of the practice of teaching mathematics. They identify an ability to unpack (or break-down) mathematical ideas, to understand the connectedness of mathematics concepts both at a particular level and across levels, and how students conceptions of mathematical concepts will evolve over time, as examples of mathematical knowledge required for teaching. Furthermore, they include knowledge of conventional mathematical practices, such as the role of definitions, and what constitutes an adequate explanation.

This interpretation of pedagogical content knowledge as "mathematics for teaching" prompted me to include questions in the interview that ask the subjects to reflect on what might be different about teaching mathematics content to prospective teachers versus teaching mathematics to other groups.

Although the theories of Shulman (1986) and Ball and Bass (2003) have shaped the initial approach to this study and provide a lens for viewing these preliminary results, the grounded theory research design demands the researcher keep an open mind in order to allow the theory to emerge naturally from the data. In consequence it is likely that additional related theoretical perspectives will be incorporated as the study proceeds.

THE CASES

I will begin with a few general comments about the Math for Teachers course and narrative descriptions of the two interview subjects, Harriet and Bob, in order to provide some context for their interview responses. This is followed by a discussion of a selection of their comments which specifically mention pedagogical content knowledge items, and a brief mention of some of the other emergent themes.

General Comments

Harriet and Bob both teach the Math for Teachers course, but predominantly teach traditional post-secondary level mathematics courses. The curriculum for the Math for Teachers course is set by the mathematics departments at each institution; however the course is provincially articulated to allow transfer of credit from one institution to another. As a result the courses and their target audience are very similar for the two instructors.

Harriet

Harriet is an experienced Mathematics instructor who has been teaching for 22 years. She is relatively new to teaching Math for Teachers, but she has taught the course six times over the last three years. She has not taken any Mathematics Education courses, nor does she have a formal teaching designation. She has a Masters Degree in Mathematics, and has a special interest in the history of mathematics. Harriet was initiated into the teaching of this course by a colleague who has a Masters Degree in Mathematics Education, has taught Math for Teachers for many years, and has a particular passion for the course. This colleague provided information about course materials and the nature of the students and their difficulties. She also provided teaching resources, including suggestions for activities.

Harriet feels strongly about the need for good teachers of mathematics in the elementary schools, and has put a great deal of thought into what can be done in a Math for Teachers course. Her priority when teaching the course is to change students' attitudes towards mathematics and their own mathematical abilities. She hopes students will come to see mathematics as enjoyable, even when it is challenging, and will develop confidence, based on a solid conceptual understanding of elementary mathematics.

Bob

Bob has been teaching mathematics for 13 years and has taught the Math for Teachers course nine times over the last nine years. He has a Masters Degree in Mathematics, and has not taken any Mathematics Education courses, nor has he had any formal teacher training. Bob's first forays into teaching the course were guided by the established curriculum, the textbook that had been selected by colleagues who had taught the course before, and through informal discussions with those colleagues.

Bob is passionate about mathematics. He enjoys its logic, its structure, and the challenges presented by a good problem. He cares about producing students who will be successful elementary teachers in the future, and to that end he hopes to equip them with a solid understanding of fundamental mathematics concepts, good communications skills, and a capacity to enjoy mathematics.

Pedagogical content knowledge: Harriet

Instances of comments coded under the heading "pedagogical content knowledge" permeate Harriet's description of her goals and strategies for teaching the Math for Teachers course. When describing the content of her course she mentions varieties of algorithms for arithmetic operations, along with models for their representation. Although these topics are part of the prescribed curriculum, her comments indicate that she goes beyond merely delivering this as content. She explicitly considers its relevance for teaching mathematics:

> H: We spend some time on the basic algorithms and different approaches to them, and how those can lead into different understandings of what you're doing when you're multiplying, or adding…

When asked if there is anything that she teaches her Math for Teachers students about fractions that she wouldn't teach someone who just wanted to learn how to use fractions, she replies:

> H: The fact that there are different models, there are different ways of picturing what's going on, and that they are appropriate for...what may work well for some situation, or for some student, may not work for some other one

She specifically addresses issues of appropriateness at various grade levels.

> H: …what you can do with a grade three student, and what you can do with a grade six student are quite different and I want them to see that it's all interconnected…

And in particular the theme of connections between the mathematical ideas plays a central role in her conception of the course.

> H: I emphasize it [connections between topics] all the way through. I don't try to plan the course to start from the beginning and go through to the end with an obvious thread, because mathematics is way too big for that. […]But at all times I connect it, as far as I can, to what goes on at different levels. What you might do with a grade 1 class, how that connects to what they're going to see in, you know grade 4 or 5 or something like that, how that connects to what they might do in high school and how that connects to what I'm doing in Calculus. Because they've got to see how it's connected, and how we build bigger and bigger, you know, understandings of sets of numbers, or calculations, or whatever.

Harriet does not just pay lip-service to these ideas. She describes assignments and activities for her classes that provide them with opportunities to exercise their pedagogical content knowledge: her students engage in analyses of pupil errors, as well as activities that allow them to compare alternative methods for solving math problems. The ability to communicate mathematical understandings clearly is also promoted through group work and formal evaluations.

Pedagogical content knowledge: Bob

Bob's interview contained far fewer statements that could be coded as instances of pedagogical content knowledge. Recurrent in Bob's responses are the notions of strong understanding of fundamental mathematics and communication skills. When comparing the Math for Teachers course with his other mathematics courses he notes

> B: ...this one focuses on their ability to communicate and convey the ideas that they should, hopefully, be already familiar with and capable of, you know, doing.

Near the end of the interview when pressed by the interviewer to consider how what the students do in the course may be more specifically related to what they will do one day as teachers, Bob responds that he discusses their future role as teachers with them.

> B: …what kinds of questions will you encounter? And why is it important that you to be able to communicate your ideas effectively, […], why should you understand this material to the most, sort of, fundamental and basic level, and understand all of the structure?

He goes on to note that the understanding needs to be deep enough to not only answer questions that arise but to make pedagogical decisions about them.

> B: … when you get some of these obtuse questions, that are seemingly, you know, that are seemingly obtuse, (laughs), you have to be able to appreciate it and be able to differentiate whether that's something that can lead you into a teachable moment...

Bob describes teaching various algorithms and models as part of the course content, but does not specifically address any comments to consideration of how this information can be used differently at different grade levels. His main concern seems to be to help students improve their personal conceptual understandings of the mathematics and their mathematics skills, what Shulman(1986) would describe as subject matter content knowledge. When asked what his students leave the course with, he replies:

> B: …I think that they […] leave having had some sense of the structure of mathematics, because there's a sufficient amount (chuckle) of that in the course, and I think that they also leave the course feeling that they can solve problems, on their own. […] probably it's the technical skills that they have, probably, you know, solidified the most

This comment indicates a predominant focus on subject knowledge, though it would be premature to conclude that Bob undervalues or fails to address pedagogical content knowledge in his course. One possibility is that Bob, in fact, may discuss pedagogical content issues with his students regularly, but was unable to recall instances of this during his interview. Another is that although Bob does not appear to address mathematics knowledge for teaching explicitly, it may arise in more subtle ways. At one point Bob relates a class discussion of a variety of solutions that emerged from a problem. He was excited about this opportunity to discuss the equivalence of the different solution methods, though this occurrence was not a deliberate part of the days' lesson.

Other themes for Harriet and Bob

The two interviews were also coded for additional themes that may impact on pedagogical content knowledge. These categories include affective elements such as confidence, attitudes, motivations and self-awareness of the students in these courses; references to problem solving and other types of knowledge; the teaching strategies employed by the instructors; and the challenges of teaching this course.

Especially striking in Harriet's interview is her concern with student attitudes towards mathematics and their own abilities to do mathematics, as well as her efforts to encourage her students to become aware of their own learning. Ball (1990) indicates that prospective teachers' feelings about mathematics and their ability to do math can have an influence on their understanding of mathematics concepts and their significance.

Bob's interview triggered coding of "other types of knowledge" based on his use of phrases like "theoretical appreciation" and "mathematical thinking". These are evocative of sociomathematical norms (Yackel and Cobb, 1996), and suggest consideration of the relationship between these norms and the mathematics knowledge needed for teaching.

Though a deeper discussion of these themes is beyond the scope of this paper, their continued occurrence in future interviews would suggest that an expanded theoretical framework may be required to capture a more complete picture.

DISCUSSION

Harriet and Bob exemplify two very different approaches to teaching the Math for Teachers course. While Harriet's course is more grounded in the context of becoming a future teacher, Bob's course seems to centre on mathematics for its own sake. Though Bob acknowledges the importance for teachers of being able to provide clear explanations and deal with questions, it appears that from his perspective, these skills will both be addressed adequately by having a sound conceptual understanding of the mathematics. This is quite consistent with the traditional view that strong content knowledge is sufficient for teaching (Hill et al, 2007). As the data set grows, the prevalence of this view among instructors of Math for Teachers will be more evident.

While it is too soon for a description of an emergent classification of the types of pedagogical content knowledge that occur, we see that in Harriet's interview there is often an emphasis on the "pedagogical", with students explicitly given opportunities to practice application of mathematics to teaching situations. In Bob's case the explicit pedagogical content knowledge seems to lean more towards the "content".

But even this broad description must be considered with due caution. At this point the analysis is based only on the instructors' reported recollections, beliefs, and interpretations of the interview questions. Analysis of the interviews will need to be supported by other research instruments. Observations of classes, follow-up interviews, and examination of artefacts (course outlines and tests) will be useful in providing validation.

Though the data set at this stage is very small, the preliminary results show promise that this approach will help elucidate the process of fostering the development of pedagogical content knowledge in these mathematics content courses. Initial coding suggests the need for an expanded theoretical framework which may include attitudes and beliefs, as well as address the relationship of sociomathematical norms to the knowledge of mathematics for teaching. I am hopeful that a broader and deeper analysis will provide much needed insight into this important stage in the development of future teachers.

References

Ball, D. L., Lubienski, S. T., & Mewborn, D. S. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.), Handbook of research on teaching (4th ed., pp. 433-456). New York: Macmillan.

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. Proceedings of the Annual Meeting of the Canadian Mathematics Education Study Group Edmonton, Canada. 3-14.

Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. The Elementary School Journal, 90(4), 449-466.

Corbin, J. M., & Strauss, A. L. (2008). Basics of qualitative research : Techniques and procedures for developing grounded theory (3rd ed.). Los Angeles, Calif.: Sage Publications, Inc.

Cresswell, J. W. (2008). Educational research: Planning, conducting, and evaluating quantitative and qualitative research (3rd ed.). New Jersey: Pearson Education, Inc.

Hill, H. C., Sleep, L., Lewis, J. M., & Ball, D. L. (2007). Assessing teachers' mathematical knowledge: What knowledge matters and what evidence counts? In Lester Jr., Frank K. (Ed.), Second handbook of research on mathematics teaching and learning (pp. 111-155). Charlotte, NC: National Council of Teachers of Mathematics.

Ma, L. (1999). Knowing and teaching elementary mathematics : Teachers' understanding of fundamental mathematics in china and the united states. Mahwah, N.J.: Lawrence Erlbaum Associates.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. Educational Researcher, 15(2), 4-14.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. Journal for Research in Mathematics Education, 26(2), 114-145.

Yackel, E., & Cobb, P. (1996). Sociomathematical norms, argumentation, and autonomy in mathematics. Journal for Research in Mathematics Education, 27(4), 458-477. Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education*, *21*(2), 132-144.

INDEPENDENT COMPONENT ANALYSIS AND ITS APPLICATION TO MATHEMATICS EDUCATION RESEARCH

Olga Shipulina

Simon Fraser University

oshipuli@sfu.ca

Educational neuroscience in mathematics education research can provide better empirical ground for developing more accurate theories of mental processes during mathematical thinking and learning. Electroencephalography (EEG) is a technique for noninvasive measurement of electrical characteristics of brain function. Scalp measurements, nevertheless, include activities generated within a large brain area. This paper reports on the roles of independent component analysis (ICA) for analysis EEG data. ICA provides separation of different signals related to different brain activators. It also calculates relative projection strengths of the respective components at all scalp sensors. As such, ICA is shown to be a useful tool for imaging brain activity and isolating artifacts from EEG data. An overview of these application areas is provided in the study on the example of data set capturing an 'AHA moment'.

INTRODUCTION

A major reason for growing interest in research in educational neuroscience in mathematics education research is that there is a need for better empirical ground for developing more accurate theories of mental processes during mathematical thinking and learning (Campbell, 2006 a, b). According to Byrnes (2001), brain research is relevant to the field of psychology and education to the extent that it fosters better understanding of the mind, development and learning. The validity, reliability, and relevance of psychological theories and traditional psychological experiments may be corroborated, refined, or refuted through neuroscientific studies or the use of neuroscientific tools and methods to test hypothesizes of any particular theoretical account (cf. Byrnes, 2001; Kosslyn & Koening, 1992). Among the neuroscientific research methods the electroencephalogram (EEG) is one of the most beneficial as it is noninvasive, and provides the best temporal resolution. Since EEG is the measurement of brain electrical activity recorded from electrodes placed on the scalp its major problem is that every electrode records a composite signal from many different electrical generators of the brain. Generally speaking, it is not possible to derive from the scalp potential distribution the activity of each single neuron. In the microscopic level the 'inverse' problem of deriving the source configuration from scalp potentials cannot be solved. Nevertheless, the separation of compound activity of some distinct brain area can be realized mathematically as well as the unique solution to the inverse problem can be found in mathematical sense (Scherg, 1990).

Here we will discuss and illustrate ways in which one independent component analysis (ICA) algorithm is used for analyzing EEG data.

WHAT IS ICA

ICA is a method for extracting individual signals from mixtures of signals. It is based on physically realistic assumption that different physical processes can generate independent signals. Intuitively ICA can be understood in terms of the classic 'cocktail party problem' (Stone, 2002). Imagine that you are at a cocktail party where many people are talking at the same time. If there is a microphone, then its output is a mixture of voices. More precisely this problem is described in (Stone, 2005) by the following way. Consider two people speaking at the same time in a room containing two microphones (Fig.1)

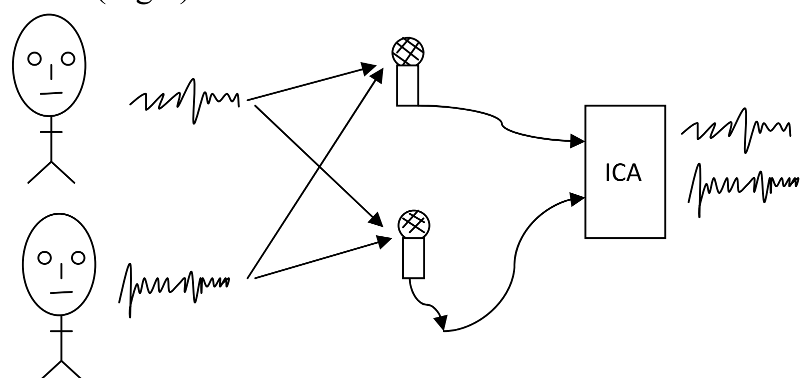


Figure 1. ICA decomposition if two people speak at the same time in a room with two microphones (after Stone, 2005).

Obviously, the voices of two different people are independent physical processes. This property has a fundamental importance for ICA and can be captured as statistical independence. If signals are statistically independent, then the value one signal provides no information regarding the values of the other signals. Another important assumption for ICA is that there must usually be as many mixtures (microphones) as there are source signals (speaking people). It is also assumed that original signals are mixed linearly. The source signals must have non-Gaussian distribution, in contrast, signal mixtures should have normal Gaussian histogram.

If we denote the speech signals by $s_1(t)$, $s_2(t)$, and microphone signals by $x_1(t)$, $x_2(t)$, the amplitude of both signals can be expressed as a linear equation:

$$x_1(t)= a_{11}\, s_1(t) + a_{12}\, s_2(t)$$

$$x_2(t)= a_{21}\, s_1(t) + a_{22}\, s_2(t)$$

where a_{ij} , i, j=1,2 are elements of a *mixing matrix* **A** that depend on the distances of the microphones. The matrix **A** defines a linear transformation on the signals $s_1(t)$, $s_2(t)$. Let us denote by **x** the random vector whose elements are the mixtures $x_1,\ldots,x_n$, and likewise by **s** the random vector with elements $s_1, \ldots, s_n$. Using this vector-matrix notation, the mixing model is written as

$$\mathbf{x}=\mathbf{A}\mathbf{s}.$$

Such linear transformations can usually be reversed to recover an estimate **u** of source signals **s** from signal mixtures **x.**

$$\mathbf{s} \approx \mathbf{u} = \mathbf{W}\mathbf{x}$$

where the separating matrix $\mathbf{W}=\mathbf{A}^{-1}$ is the inverse of **A**. A square matrix **W** specifies spatial filters that linearly invert the mixing process (Campbell, 2004; Jung, Makeig, McKeown, Bell, Lee, Sejnowski, 2001). However, the mixing matrix **A** is not known and cannot be used to find **W**. So, the numerical method find a separating matrix **W** which maps mixtures **x** to a close approximation to source signals $\mathbf{u} \approx \mathbf{s}$. Given that the method recovers an estimate $\mathbf{u}=\mathbf{W}\mathbf{x}$ of the source signals **s** which are assumed to be mutually independent this means that **W** should be adjusted (iteratively) so that to make the estimated source signals **u** mutually independent. According to Stone (2005), one common interpretation of ICA is as a maximum likelihood method for estimating the optimal unmixing matrix **W**. Maximum likelihood estimation (MLE) is a standard statistical tool for finding parameter values that provides the best fit to a given model. The detailed description of how MLE is applied in ICA can be found in Stone (2005).

Although independent component analysis was originally developed for problems related to separation of different speech signals, it is widely applied to EEG data analysis. The EEG data consists of recordings of electrical potentials in many different locations on the scalp. These potentials are presumably generated by mixing some underlying components of brain activity. This situation is quite similar to the cocktail-party problem: We would like to find the original components (signals) of brain activity, but we can only observe mixtures of the components.

ICA was first applied to EEG by Makeig, Bell, Jung, Sejnowski (1996) and now is widely used in EEG research community (Delorme, & Makeig, 2004). The special toolbox and graphic user interface, EEGLAB, designed and developed at Swartz Center for Computational Neuroscience (Delorme, Makeig, 2004), significantly simplifies ICA processing for single trial and/or averaged EEG data of different number of channels. Currently ICA is receiving increasing attention for its main advantage that it does not need specific information about the signals or about the medium they propagate through. The disadvantage of ICA is that there is no guarantee that this method can capture the individual signals in its components (Joyce, Gorodnitsky, Kutas, 2004). Another weakness is the key assumption that the sources are independent and this is not always strictly realistic in application to EEG analysis (Delorme, Sejnowski & Makeig, 2007). Conceptually, ICA in application to EEG could be used only for source identification without specifying directly where in the brain the activity arises (Makeig et al., 1996; Jung et al., 2001). ICA is also providing to be an efficient preprocessing step to source localization (Zhukov, Weinstein, & Johnson, 2000).

ICA FOR IMAGING BRAIN ACTIVITY

As discussed above, ICA finds an unmixing matrix, **W**, that linearly unmixes the multichannel scalp data into the number of temporary independent components, **u**=**Wx.** The rows of the matrix **u** are time –amplitude courses of activations of ICA components. According to Jung et al. (2001), the columns of the inverse matrix, $\mathbf{W}^{-1}$, give the relative projection strengths of the respective components at all scalp sensors. These scalp weights give the scalp topography for each component separately, and provide some evidence for the possible components' physiological origin . The projection of the *i*-th independent component onto channels is obtained by the outer product of the *i*-th row of the component activation matrix, **u**, with the *i*-th column of the inverse unmixing matrix $\mathbf{W}^{-1}$, and is measured in the corresponding units and located in original sensor locations. Thus, brain activities of interest can be obtained by projecting selected ICA components back onto the scalp. Fig. 2 illustrates component scalp projections for the data set capturing an "AHA! moment" in the research in the area of mathematical problem solving. The data set was processed by software EEGLAB which allows to visualize component scalp projections and time-amplitude courses for original channel recordings and component decompositions. Ten second single trial EEG data set capturing an AHA! moment reveals that during that time the majority of brain areas were active.

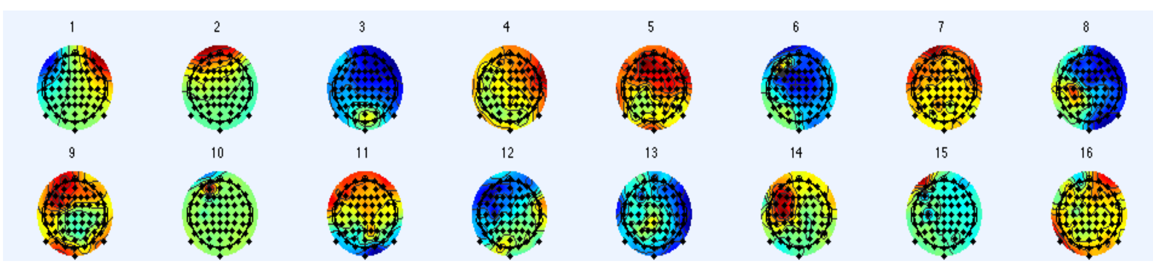


Figure 2. 2D- scalp projections of some components after ICA processing of EEG data set of "AHA! moment". The total number of components is 64, the same as the number of scalp electrodes.

Visual inspection of scalp projections together with corresponding component time - amplitude courses allows to preselect the activation areas of interest. For example, let us consider components 38 (Fig. 3).

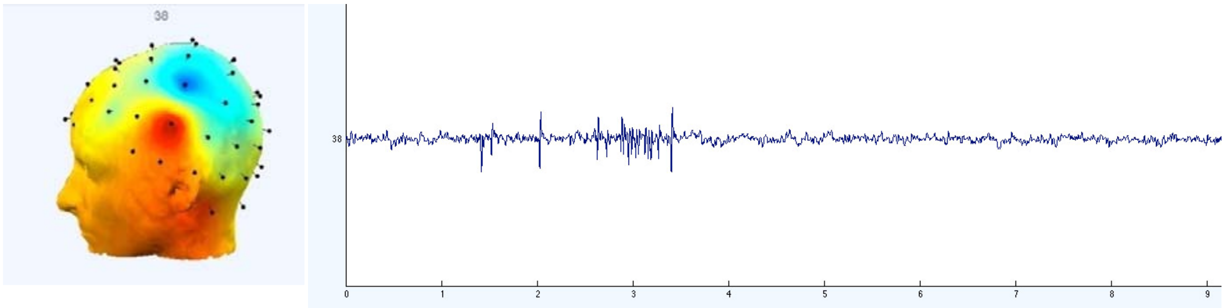


Figure 3. 3D- scalp projection (left part) of component 38 and its time - amplitude course (right part).

The activating source of component 38 can be located in brain's Broca area, which is involved in speech production. The burst of amplitude and frequency corresponds to the time segment of subject's speaking. Nevertheless, we cannot make any conclusions about the signal without having enough evidences that this component is not artifactual. Before making any conclusions about the nature of signals coming from any region of interest the first and the most important preprocess of data set is detection and removal artifacts which can not only mask and/or distort brain signals sufficiently. but to be treated as brain related activity instead of artifactual muscle activity.

ICA FOR DETECTION AND ELIMINATION ARTIFACTS IN EEG DATA

Because ICA algorithms have proven capable of isolating both artifactual and neutrally generated electrical brain activity sources ICA is widely accepted now as a useful tool for isolating artifacts EEG data (Delorme et al., 2004; Delorme et al., 2007; Iriarte, Urrestarazu, Valencia, Alegre, Malanda, Viteri, & Artieda, 2003; Jung et al., 2001; Makeig et al., 1996; Tran, Craig, Boord, Craig, 2004).

In the presented study of AHA! moment EEG data set, scalp topography and signals of components 1 and 2 (fig. 4) are typical for eye related behavior artifacts as they are of a low frequency and their topographical projection shows maximum in the frontal region (Delorme et al., 2007; Iriarte et al., 2003; Tran et al., 2004). Component 1 relates to saccadic horizontal eye movements; component 2 relates to vertical eye movements and/or blinks.

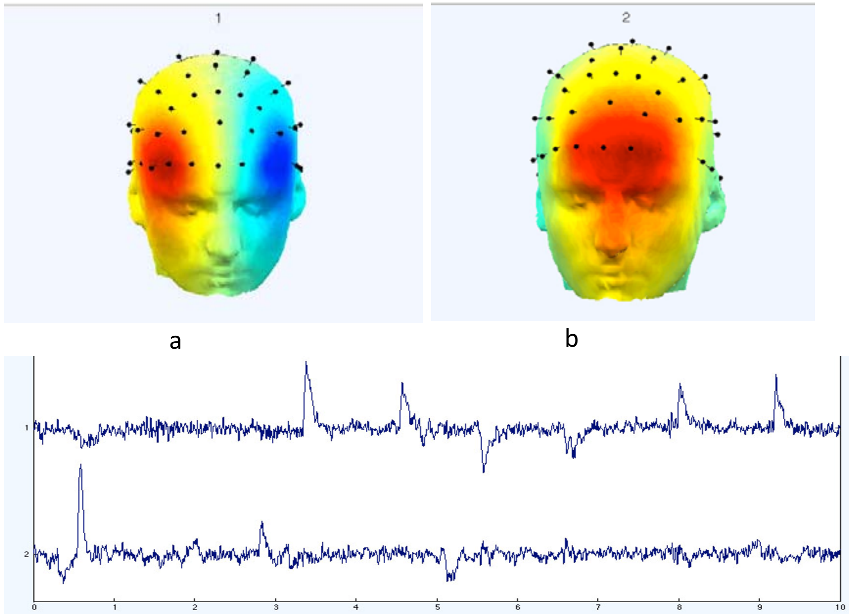


Figure 4. Eye movement artifactual components: a) component 1 is of horizontal eye movements; b) component 2 is of eye blinks and/or vertical eye movements. The time- amplitude courses correspond to the horizontal (1) and vertical (2) eye movements.

These time- amplitude courses were then synchronized with EOG and video recordings. The results of this synchronization showed that all picks in EEG time-amplitude courses coincided with the moments of corresponding eye movements. Analysis of time- frequency transformations also showed that bursts of low frequency energy related to the moments of eye movements. Hence, components 1 and 2 were identified as artifactual. The contribution of these artifact components to original EEG data records was removed by subtracting the components' projections onto the scalp electrodes from the electrodes' original EEG records (Jung et al., 2001).

CONCLUSION

Mathematical problem solving implies a high level of concentration when small body movements are often uncontrolled. Quite often people articulate their thoughts; the process also implies eye movements. Hence, sufficient amount of artifacts can be expected in the EEG data recordings during such kind of experiments. Sufficiently that AHA! moment experiments are principally of 'single trial' type and cannot be repeated fully or somehow be stimulated for getting expected time locked responses. This is why the procedure of artifact removal is so important for such kind of studies.

ICA algorithm is an effective tool for detection and removal artifacts from the EEG data recordings. This method is especially valuable for the 'single trial' type experiments as it removes only parts of EEG recorded signals related to artifact and saves information related directly to brain activity. ICA is also proved to be useful for imaging brain activity and identification the signals coming from the brain regions of interest connected, for example, with certain kind of cognitive process. As such, it has important methodological roles to play in educational neuroscience, and mathematics educational neuroscience in particular.

Acknowledgements

This research is a result of joint work with Prof. S. Campbell in his Educational Neuroscience Laboratory in the Faculty of Education at Simon Fraser University.

References

Byrnes, J. (2001). Minds, brains, and learning: Understanding the psychological and Educational relevance of neuroscientific research. New York, London: The Guilford Press.

Campbell, S. R. (2004) Forward and inverse modelling: Mathematics at the nexus between mind and world. In W. Blum, et al. (Eds.) Proceedings of the 14th International Conference in Mathematical Instruction Study Group on Applications and Modelling in Mathematics Education (pp. 59-64). Dortmund, Germany: University of Dortmund.

Campbell, S. R. (2006a). Defining mathematics educational neuroscience. In Alatorre, S., Cortina, J. L., Sáiz, M., & Méndez, A. (Eds.). Proceedings of the 28th annual meeting of the North American Chapter of the International Group for Psychology in Mathematics Education, (vol. 2; pp. 442-449). Mérida, México: Universidad Pedagógica Nacional.

Campbell, S. R. (2006b). Educational neuroscience: New horizons for research in mathematics education. In J. Novotná, H. Moraová, M. Krátká, & N. Steliková (Eds.) Proceedings of the 30th Conference of the International Group for the Psychology of Mathematics Education (vol. 2; pp. 257-264). Prague, Czech Republic: Charles University.

Dehaene, S., Izard, V., Pica, P., & Spelke, E. (2006). Core knowledge of geometry in an Amazonian indigene group [Electronic version]. Science, 311, 381-384.

Delorme, A., Makeig, S. (2004). EEGLAB: an open source toolbox for analysis of single-trial EEG dynamics including independent component analysis. Journal of Neuroscience Methods, 134, pp. 9-21.

Delorme, A., Sejnowski, T., & Makeig, S. (2007). Enhanced detection of artifacts in EEG data using higher-order statistics and independent component analysis. NeuroImage, 34, pp. 1443-1449.

Iriarte, J., Urrestarazu, E., Valencia, M., Alegre, M., Malanda, A., Viteri, C., & Artieda, J. (2003). Independent component analysis as a tool to eliminate artifacts in EEG: A quantitative study. Journal of Clinical Neurophysiology, 20(4), 249-259.

Joyce, C.A., Gorodnitsky, I.F., Kutas, M. (2004). Automatic removal of eye movement and blink artifacts from EEG data using blind component separation. Psychophysiology, 41, 313-325.

Jung, T.-P., Makeig, S., McKeown, M. J., Bell, A. J., Lee, T.W., & Sejnowski, T.J. (2001). Imaging brain dynamics using Independent Component Analysis. Proc. IEEE 89 (7), 1107-1122.

Kosslyn, S., & Koening, O. (1992). Wet mind: The new cognitive neuroscience. New York: Free Press.

Makeig, S., Bell, A. J., Jung, T.-P., & Sejnowski, T. J. (1996). Independent component analysis of electroencephalographic data. Advances in Neural Information Processing Systems, 8, 145-151.

Scherg, M. (1990). Fundamentals of dipole source potential analysis. In: Auditory Evoked Magnetic Fields and Electric Potentials. (eds. F. Grandory, M. Hoke and G.L. Romani). Advanced In Audiology, l6, 40-69.

Stone, J. V. (2002). Independent component analysis: an introduction. TRENDS in Cognitive Sciences, 6(2), 59-64.

Stone, J. V. (2005). Independent component analysis. In: Encyclopedia of Statistics in Behavioral Science (Ed. Brian S. Everitt & David C. Howell) , John Wiley & Sons, Ltd., Chichester, Vol.2, pp. 907-912.

Tran, Y., Craig, A., Boord, P., Craig, D. (2004). Using independent component analysis to remove artifact from electroencephalographic measured during stuttered speech. Medical & Biological Engineering & Computing, 42, 627-633.

Zhukov, L., Weinstein, D., & Johnson, C. (2000). Independent component analysis for EEG source localization. IEEE Engineering in Medicine and Biology, 19, 87-96.

AN ACCOUNT OF A LESSON STUDY ON THE PARABOLA: INSIGHTS INTO BUILDING THE EFFECTIVE PRACTITIONER

Natasa Sirotic

Simon Fraser University

This report is about how a group of practicing teachers experiences a school based professional development initiative by way of implementing lesson study, and how this process facilitates the development of teachers' knowledge for teaching mathematics. The report presented here is taken from an ongoing study situated in a school based community of practice on how teachers' key competences for teaching mathematics develop, refine, and even transform as a result of their participation in a collaborative enterprise of lesson study. Our focus is on the outcomes, taken here as teachers' gains in their expertise in mathematics teaching, and which have the capacity to transform teachers' everyday practice in the classroom. They are embodied in the knowledge, skill, attitude, and capability, both in the individuals and in the community as a whole. In this report we present these outcomes and the conditions in which they were achieved, through a particular account of the lesson study on the topic of parabolas. This lesson study has been the sixth lesson study implemented in the school since the inception of the initiative 18 months ago.

BACKGROUND

There is an increased attention to professional development initiatives that rely on building communities of teachers. While researchers may not yet fully agree on what mathematical knowledge, skills, and habits of mind teachers need to have to teach mathematics effectively, there seems to be a growing consensus that embedding teachers' learning in their everyday work, through a careful examination of their practice and classroom artifacts, increases the likelihood that this learning will be meaningful (Lampert & Ball, 1998; Lieberman, 1996; Stein, Smith and Silver, 1999).

It is believed that the Japanese succeeded in transforming on the national level something as complex and "culturally embedded as teaching" (Stiegler & Hiebert, 1999) by systemically implementing the well-defined and established professional development practice of lesson study (Chokshi & Fernandez, 2005; Lewis, 2000; Watanabe, 2002). Lesson study is a process where teachers and scholars engage as researchers by developing and testing lessons and studying their impact on students. The process itself builds up a repository of knowledge on teaching and learning of various content, which is then used to inform curriculum changes, textbook creation, and university programs for preservice teachers, thus impacting change on multiple levels, from individual teacher to the entire profession of teaching. Most importantly, with its being centred in the classroom and focused on students' learning, this practice provides a high-fidelity context, usually carried out in the form of school based lesson study, in which teachers can build their content knowledge and pedagogical skills and a shared understanding of what constitutes effective practice.

HOW LESSON STUDY WORKS

Activities in which teachers engage include detailed examination of the mathematical content they are supposed to teach, and how this content is approached in other curricula. Teachers confer and exchange ideas on how students learn, what common misunderstandings are and how to best address them, what teaching methods and pedagogical techniques will be used, how the mathematical content will be developed through the lesson, what instructional materials will be designed or used and for what purpose, how the teacher will know what the students did and did not understand. While the goal of the teachers is to create an effective lesson that will reach all students and produce lasting learning gains, through this process of collaboration and shaping of common goals teachers in fact access insights into teaching and learning that they may have never otherwise.

The centrepiece of lesson study is the *research lesson*, developed collaboratively, taught by one team member while observed by others, and finally discussed and reflected upon by the whole team. It should be noted that the term “research” in this context means teacher-initiated, practice-based inquiry. So what exactly is it about this context that allows for teachers to develop professionally?

While it is not the aim of this paper to explicate in any great detail the mechanisms by which lesson study results in instructional improvement, it is worth mentioning that there is a commonly held view that lesson study improves instruction through the refinement of lesson plans. To be sure, our team went through five draft revisions before the lesson plan was finalized; however, this is not a plausible conjecture given that teachers on average conduct about two lessons study cycles per year only. The aim of lesson study is not to produce perfect lesson plans; rather it is to engage teachers in reflective practice, which in turn builds teachers’ knowledge (of subject matter, instruction, curriculum, and of how students learn and think), teachers’ commitment and sense of community, well thought out learning resources (such as good problems and other learning tasks), and a clarity of goals as well as of expected standards for instruction.

For any community of teachers implementing lesson study, it is important to be aware of the pitfalls that have been identified and to avoid them at all cost (Lewis at al., 2006). Replicating the observable features of lesson study, which are: setting goals for student learning, study of existing curricula, planning and implementing a research lesson, collecting data during the research lesson, presenting and discussing data from research lesson, is not in itself a guarantee that improvement of instruction will result from these activities. This has been termed as “rote implementation of surface features”. Models are being developed, which specify the connections between lesson study’s observable features and instructional improvement, specifically to avoid this kind of surface level implementations.

BUILDING KEY COMPETENCIES AND CRAFT KNOWLEDGE

This paper does not aim to give a full description of the activities of the team, nor a comprehensive account of teacher learning. We focus here on certain specific gains in the teachers’ competencies for teaching mathematics that were documented and that could impact the effectiveness of instruction beyond the specific content of the particular research lesson. We define the term *teachers’ key competences for teaching mathematics* as the knowledge, skills, attitude and capability, which contribute to the effective instruction. We provide a preliminary analysis of how these gains are achieved by individual teachers or the group as a result of active participation in the lesson study process, and as evidenced from a concrete instance of lesson study. It has been the experience of this group of teachers that every lesson study cycle generated the kind of teacher knowledge that matters in the classroom. We invite the reader to consider the potential of lesson study as a possible way for all teachers to become experts in what they do. In this paper, we examine and discuss what kind of knowledge for teaching mathematics is being generated through one particular instance of lesson study, and how.

Let us first consider the ways in which lesson study acts as an incubator for building effective practitioners. To be sure, there are other ways to build effective practitioners, such as using various mentorship programs whereby an expert teacher initiates a novice practitioner into the profession of teaching, known also as instructional coaching. It seems that such settings are especially conducive to building effective practitioners because they provide a unified setting, with the totality of what teaching entails and an authentic environment for it to be experienced and learned from. As such, a single lesson study cycle touches upon all that a teacher needs to be able to do in a daily practice. It is a form of learning how to teach through teaching, a place where the divide between theory and practice is nonexistent. Teachers learn by carefully observing, analyzing, critiquing, and systematically reflecting upon one’s own or a colleague’s practice. In a sense, they engage in the discipline of noticing through which it is possible to research one’s own practice (Mason, 2002). This exposes the participants to the craft knowledge of the teaching profession, and it also deepens the knowledge of the discipline they are supposed to teach, also known as pedagogical content knowledge and subject matter knowledge respectively (Shulman, 1986). Craft knowledge is a phrase used to describe the particular knowledge generated through being a practitioner. It is a mixture of expertise, propositions and theories, and tacit knowledge applied in the daily conduct of teachers' practice, and it is different from the knowledge produced by formal research.

According to Burney (2004), craft knowledge is the road to transforming schools, if it were properly cultivated and disseminated. Craft knowledge is precisely the kind of knowledge that is being cultivated and shared through lesson study. Learning those kinds of skills is not a solitary endeavor but a highly social one. Furthermore, it depends on continual discussion and demonstration. People learn by watching one another, seeing various ways of solving a single problem, sharing their different "takes" on a concept or struggle, and developing a common language with which to talk about their goals, their work, and their ways of monitoring their progress or diagnosing their difficulties. When teachers publicly display what they are thinking, they learn from one another, but they also learn through articulating their ideas, justifying their views, and making valid arguments.

Trying to understand and account for what makes an effective practitioner in terms of one or more isolated constructs seems to create more and more unanswerable questions. For example, what teachers need to know to teach effectively has been investigated by a number of prominent researchers from the field of mathematics education. Various frameworks have been offered for investigation, interpretation, categorization, evaluation, and theoretical discussion of teachers' mathematical knowledge for teaching (e.g., Davis and Simmt, 2005; Ball and Cohen, 1999; Shulman, 1986, Leikin, 2006). While there is a consensus that teachers' mathematics-for-teaching is a complex, dynamic, and tacit body of knowledge, which is very difficult to assess reliably, there seems to be little agreement on what exactly this knowledge is.

Interestingly, while *what* should be known to teach well is elusive, *how* such knowledge should be held has been shown quite explicitly on several specific domains of mathematical knowledge for teaching (Ma, 1999). From Ma's research we learn that mathematical knowledge for teaching rests firmly on what has become known as the "profound understanding of fundamental mathematics". With such understanding teachers are seen to be able to move in their subject easily, naturally, and in a way that allows them to effectively plan for instruction avoiding the typical student misconceptions, and to respond efficiently to a great variety of possible student errors.

We could say that craft knowledge implies profound understanding of fundamental mathematics. But it implies more than that. As noted earlier, we refer to this multifunctional collection of a teacher's knowledge, skills, attitudes, and capabilities as key competences for teaching. Those competences enable their barer to act adequately in a multitude of situations and in various fields of activity within the teaching profession. Lesson study is a context in which these key competencies are cultivated on the basis of personal experience and activity as applied in practice.

In the remainder of this paper, we present and exemplify some instances and ways in which the above mentioned craft knowledge of mathematics teachers is being forged through the process of lesson study.

SUBJECTS AND CONTEXT

The team consisted of six practicing teachers specializing in mathematics teaching at secondary school level. Four of the teachers are based in Southpointe Academy, an independent school in British Columbia, where the implementation of the research lesson took place, while two of the teachers participated off site by providing feedback and suggestions for lesson refinement. This was the sixth lesson study cycle implemented in the school since the initiative started 18 months ago.

The chosen research lesson was on the topic of parabola. It was implemented in a class of 22 Grade 11 students. The lesson was developed over a period of six weeks during which teachers met weekly for about an hour. The first two meetings were used to establish the instructional goals of the lesson and to consider how it will support the previously established long term goals for student development, which are, "To nurture students' inquiry into mathematical ideas, to assist them in developing reasoning abilities, mathematical communication skills, and a willingness to persevere with difficult problems." Here teachers also decided who would be teaching the research lesson. Teachers take turns in teaching the research lessons from one lesson study cycle to another. It is seen as a challenge, honour, and a great opportunity for professional development, all at the same time. Kelly had volunteered to do it this time (pseudonym used).

The goal for the lesson was chosen based on what teachers agreed is the most difficult part of this instructional segment, which is for students to understand the effect of the coefficient a of the quadratic term and to be able to find its value based on a graph of a parabola. The effect of this coefficient is commonly referred to in school mathematics as "vertical expansion or compression" because of its effect on the graph, and it is often used interchangeably with "horizontal compression or expansion" respectively, which creates much confusion for the students. The lesson and the post lesson discussion were videotaped, and an interview was held with the teacher teaching the lesson two weeks after the implementation.

HOW KNOWLEDGE OF INSTRUCTION IS ADVANCED

In our ongoing effort to answer the central question of, "How do we develop effective teaching", we now turn to examine the gains in teacher competencies that were documented in this particular lesson study. A number of domains of teachers' knowledge for teaching mathematics were impacted, such as increased knowledge of instruction, of subject matter and curriculum, and of how students learn; however, in this paper we limit our discussion to the first, the knowledge of instruction. Within its domain we consider both the knowledge of how students think and learn, and what instructional support should be given to produce the desired learning outcomes, as it was developed in this community of practitioners. While these teacher gains were seen as true and real at the time when they were recorded and discussed, it is left for further research to determine which of these gains transfer into teachers' further practice and under what conditions.

To be sure, having a crystal clear instructional goal and a very well prepared lesson, together with a "research question" or even several of them of what is being investigated is inspiring enough to allow teachers to become especially sensitive and aware about how exactly students learn as a result of instruction that is being implemented. But first a great deal of thought must go into defining such clear instructional goals and then carefully designing such lesson that would allow to find out how students can learn most effectively the designated mathematical contents and processes. Here we examine the effects of lesson study in the domain of planning for instruction, and what exactly changed for teachers regarding this activity.

In this part of lesson study process teachers decided upon the instructional goals of the lesson, based on students' prior knowledge, and where they were heading in terms of their mathematical content learning. In this case, students have not yet been exposed to quadratic equation, and they have barely started the unit on quadratic function. In the lesson prior to this one, students learned to graph the basic parabola $y = x^2$ by hand using the table of values, and they investigated what happened to the graph if all y values are made negative, and what happened to the graph if a given constant is added to each of the y values. In effect they came to understand the functions and their related graphs of $y = -x^2$ and $y = x^2 + q$ as well as $y = -x^2 + q$. In school mathematics this is referred to as the reflection of the parabola in the x-axis and the vertical shift. This instructional segment is intended to build towards the standard equation of the parabola $y = a(x - p)^2 + q$ from which it is easy to determine how the corresponding graph should look like. Likewise, the standard equation is used to find the equation from a given graph most directly. While the values for constants p and q are easily determined from the vertex of the graph, determining the value of the coefficient of the square term a has been an ongoing struggle for the students, as reported by the teachers. Naturally they wanted to come of with a way to help students construct this knowledge in a way that would produce a robust understanding of the effects of this coefficient on the graph of the function, and how that coefficient can be found from the graph. This was the focus of the lesson.

Up to this phase there was nothing new in the ways teachers usually plan for instruction, until three options for instructional practice emerged from the planning sessions. First was an instructional approach using an *empirical learning process*. Second was a type of *discovery learning* with a kind of open investigation to be undertaken by the students – the learning task associated with this approach became known to the community as "Putting on the Fritz Face", and it had been used by one of the team members in the past. The third approach which ended up being the chosen one after a careful consideration and argued discussion was *structured problem solving* which is incidentally also the approach commonly used in Japan to develop mathematical concepts. It is also known as cognitively guided instruction, or teaching through problem solving, which incidentally while it was invented in North America it only really become well understood and developed for effective use in practice as a result of lesson study (Fenema, Carpenter, & Franke, 1992).

The empirical learning process was the instructional approach suggested in the standard textbook that the class had been using, and it is also the approach that Kelly had been using for years when teaching this particular concept. This instructional approach is commonly used by teaches who hold a *perception-based perspective* (Simon, 2007) on how students learn mathematics. Much of recent fascination with manipulatives can be attributed to the push for adopting this perspective, despite the fact that there is not much evidence about whether and how it works, and under which conditions. According to this perspective students develop mathematical understandings through their engagement with representations that make the concept under study clearly perceivable. Mathematical relationships exist as an external reality. Here we refer to this approach as the "show and tell" approach to teaching mathematical concepts. In our case of parabola the students are supposed to observe how varying the coefficient of the square term affects the graph. They commonly use the graphing calculator to do this; that is, by using a number of different values for a in the $y = ax^2$, usually building from large to small positive values, and then observing the ever wider parabolas this process produces. What they are learning from this approach is *that* the smaller this coefficient, the wider the parabola and not the logical necessity of that relationship (a concept). It is contended that empirical learning process does not result in conceptual learning, because mathematical concepts are the result of *reflective abstraction* and not of empirical learning (Simon, 2007).

The team members constructed the following problem with which students then engaged and through which they attained the concept in a more meaningful way, based on reflective abstraction.

> Problem: A bridge over a river is supported by a parabolic arch.
>
> At its centre it is 16 m above the water. Its supports are 56 m apart.
>
> a) Find the equation of the arch in standard form.
>
> b) How high above the river is the arch 8 m from one support?

Student engagement with the problem was very high. After some time spent on posing the problem and motivating the student buy-in, they pursued the construction of the equation actively. Indeed they explored and tested their hypotheses, and they learned along the way. In an interview with Kelly two weeks after the research lesson implementation, she indicated with much delight that this time around the concept had been learned much more robustly and that she "got a lot of mileage" in that class from that one lesson. In addition, she remarked that it would be the way she will teach this lesson in the future. Of course, there were some things she would change, but not what concerns the instructional approach and the learning task itself.

From a researcher perspective it is important to note that such research knowledge that is informed by practice should be more widely collected, tested, codified, and shared in the efforts to build a new foundation for the teaching profession. In addition, there also exists a rich body of knowledge produced by educational researchers, but which trickles to practitioners in very limited and haphazard ways. Take for example the research that had been done to understand how students attain proportional reasoning (Sowder, Armstrong, & Lamon, 1998). Despite the findings that clearly show the limitations of using the part whole interpretation of fractions as a primary mode for instruction on fractions, this approach is still widely used in practice, and sometimes it is even the only one used despite its ineffectiveness. Another such problem is the persisting problem of students' (non)understanding of the equals sign, which is well known amongst researchers, but rarely amongst teachers. Something like this would never happen in professions such as medical or accounting, where it would be rightly considered as malpractice. The responsibility lies in part with the way textbooks are being produced, without much reference to scientifically based findings but also with teachers, "who have come to regard autonomy and creativity – not rigorous, shared knowledge – as the badge of professionalism" (Burney, 2004). It is our hypothesis that teachers, after having the kind of experience just described, will be far more likely to critically examine and evaluate the instructional materials for their value in how well they align with their goals of instruction. Moving from teacher as consumer of tasks presented in the textbook to teacher as creator of learning situations requires a certain degree of professional judgment that is more likely to manifest after such experience.

There is a great deal more that teachers learned from this particular lesson study, but which cannot be fairly presented in this report given the space constraints. However, we have observed teachers generating knowledge from which powerful instruction can be understood and supported, and which has the capacity to transform what teachers do in the classroom to achieve greater learning outcomes for their students.

References

Ball, D. L., & Cohen, D. K. (1999). Developing practice, developing practitioners: Toward a practice-based theory of professional education. In L. Darling-Hammond, & G. Sykes (Eds.), Teaching as the learning profession: Handbook of policy and practice. San Francisco: Jossey-Bass, Inc.

Burney, D. (2004). Craft Knowledge: The Road to Transforming Schools. Phi Delta Kappan. v85, n7, p526.

Chokshi, S., & Fernandez, C. (2005). Reaping the systemic benefits of lesson study. Phi Delta Kappan, v86(n9)

Davis, B., & Simmt, E. (2006). Mathematics for teaching: An ongoing investigation of the mathematics that teachers (need to) know. Educational Studies in Mathematics, v61 n3 p293-319.

Fennema, E., Carpenter, T. P., & Franke, M. L. (Spring, 1992). Cognitively guided instruction. (Ncrmse Research Review: the Teaching & Learning of Mathematics, 1(2), 5-9) Madison, WI: National Center for Research in Mathematical Sciences Education (ERIC Document Reproduction Service No. ED 372929).

Fernandez, C., Cannon, J., & Chokshi, S. (2003). A U.S.-Japan lesson study collaboration reveals critical lenses for examining practice. Teaching and Teacher Education, v19(n2), p171.

Lampert, M., & Ball, D. L. (1998). Teaching, multimedia, and mathematics: investigations of real practice. New York, NY: Teachers College Press.

Leikin, R., & Rota, S. (2006). Learning through Teaching: A Case Study on the Development of a Mathematics Teacher's Proficiency in Managing an Inquiry-Based Classroom. Mathematics Education Research Journal, v18(n3).

Lewis, C. (2000). Lesson study: The core of Japanese professional development

Lewis, C., Perry, R., & Murata, A. (2006a). How should research contribute to instructional improvement? The case of lesson study. Educational Researcher, v35(n3), p3.

Lieberman, A. (1996). Practices that support teacher development: transforming conceptions of professional learning. In M. W. McLaughlin, & I. Oberman (Eds.), Teacher learning: New policies, new practices (pp. 185–201). New York, NY: Teachers College Press.

Ma, L. (1999). Knowing and teaching elementary mathematics: teachers' understanding of fundamental mathematics in China and the United States. Lawrence Erlbaum Associates Publishers, Mahwah, N.J.

Mason, John (2002) Researching your own practice: the discipline of noticing. Routledge, UK.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. Educational Researcher, 15(2), 4-14.

Shulman, L. (1987). Knowledge and teaching: Foundations of new reform. Harvard

Educational Review, 57, 1 – 22.

Simon, M.A. (1995). Reconstructing Mathematics Pedagogy from a Constructivist Perspective. Journal for Research in Mathematics Education, v26(n2), 114-145.

Simon, M. & Tzur, R. (1999). Explicating the teacher's perspective from the researchers' perspective: Generating accounts of mathematics teachers' practice. Journal for Research in Mathematics Education, 30, 252-264.

Sowder, J., Armstrong, B., Lamon, S. (1998). Educating teachers to teach multiplicative structures in the middle grades. Journal of Mathematics Teacher Education, v1(n2) p127-55.

Stein, M. K., Smith, M. S., & Silver, E. A. (1999). The development of professional developers: Learning to assist teachers in new settings in new ways. Harvard Educational Review, 69(3), 237–269.

Stiegler, J.W., & Hiebert, J. (1999). The teaching gap: Best ideas from the world's teachers for improving education in the classroom. New York: Summit Books.

Watanabe, T. (2002). Learning from Japanese lesson study. Educational Leadership, v59(n6), p36.

Wenger, E. (1998). Communities of Practice: Learning, Meaning, and Identity. NY: Cambridge University