MEDS-C 2007

Mathematics Education Doctoral Student Conference
2007 Program

9:00 – 9:25	Multi-layers of Numeracy	Kanwal Neel
9:30 – 9:50	Mediated successive refinement: A pedagogical tool for understanding mathematical structures	Shabnam Kavousian
9:55 – 10:20	Break / MEDS-C Reviews Session	Chairs: S. Kavousian and C. Newell
10:25 – 10:45	Electrooculography: Connecting brain and behaviour in mathematics education research	Olga Shipulina
10:50 – 11:10	Enactive cognition and image-based reasoning in geometry	Kerry Handscomb
11:15 – 11:35	The use of mathematical imagery in complexivist accounts of education and educational research	Craig Newell
11:45 – 12:45	Plenary: Perspectives on Dynamic Geometry Software	Nicholas Jackiw
12:45 – 2:30	Lunch: Eagle Creek Golf Course	
2:35 – 2:55	Conversations about connections: A beginning teacher struggles to teach for understanding	Aldona Businkas
3:00 – 3:20	Lesson study in Japan and Western Canada: Learning a collaborative Approach to improve classroom instruction	Melania Alvarez-Adem
3:30 – 3:50	A tale of teacher initiated inquiry into instructional practice: The case of division of fractions	Natasa Sirotic
3:55 – 4:15	Student generated worked-examples: A tool for enhancing the learning of mathematics	Elena Halmaghi
4:20 – 4:40	Break	
4:45 – 5:05	Constrained by knowledge: The case of infinite ping-pong balls	Ami Mamolo
5:10 – 5:30	Sample space rearrangement (SSR): The example of switches and runs	Egan Chernoff

ABSTRACTS

Plenary Session:

PERSPECTIVES ON DYNAMIC GEOMETRY SOFTWARE

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In considering the geometric figure, Kant distinguishes between image---the traditional visual diagram---and schemata, the generalized concept of that diagram that "can never exist anywhere except in thought." Dynamic Geometry figures produced by recent software such as The Geometer's Sketchpad have bridged this conceptual divide, through the introduction of flexible, rubbery diagrams that (under manipulation) can transform into all valid realizations of their defining geometric constraints, while at every instant retaining the immediacy and tangibility of specific images. In turn, they have spawned a (slow) revolution in schools: over the past decade, Sketchpad has become the most widely-used school mathematics software in Canada, the United States, and possibly the world. But the series of transitions involved in that movement---from propositional to image-based forms of mathematical argument, from paper-and-pencil to digital definitions of image, and most importantly from static to dynamic conceptions of mathematics---have hardly occurred without hiccup. Some of these transitions are only weakly understood theoretically; and some have encountered strong pedagogic and political resistance. In this talk, Sketchpad's author explores the implications of Dynamic Geometry visualization on mathematical inquiry and pedagogic practice in the context of school (6-12) mathematics, and surveys a variety of responses to the Dynamic Geometry phenomenon from mathematical, epistemological and historical perspectives.

Speaker bio:

Nick is the Chief Technology Officer of KCP Technologies, and is also the software designer responsible for The Geometer’s Sketchpad®. As one of the founding members of the Visual Geometry Project at Swarthmore College in 1987, he was responsible for the design and development of all of the VGP interactive software. In this capacity, he formulated the Dynamic Geometry® approach that defines the Sketchpad experience. Moving with the software from the academic environment to the publishing industry, Nick directed Sketchpad product development at Key Curriculum Press from 1990 through 1998, when with others he developed Key Curriculum Press’ software department into a separate company, KCP Technologies, where he presently works. In addition to designing software, he is the chief programmer of several incarnations of Sketchpad, and he directs programming staff in other areas. Nick also acts as a software design consultant to KCP Technologies data analysis software group. He works with schools in conducting field-testing and software evaluation, and also represents Sketchpad professionally in research, curriculum development, and professional development contexts. He has been PI and senior scientist on Small Business Innovative Research projects investigating Dynamic Geometry’s impact and potential, and has written numerous articles on the subject

Research Reports:

LESSON STUDY IN JAPAN AND WESTERN CANADA: LEARNING A COLLABORATIVE APPROACH TO IMPROVE CLASSROM INSTRUCTION

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Our purpose is to help increase collaboration between teachers, school administrators, math educators and mathematicians, with the goal of improving classroom instruction in the area of mathematics through the use of lesson studies. Lesson studies provide a process for teachers to collaborate in the design of lessons with the goal of identifying successful teaching strategies and to increase students' learning. Teachers work together in the development of a lesson, and when the lesson is implemented, observers take notes on students' questions and understanding. These observers are some of the teachers who helped to plan the lesson, other teachers or administrators in the school or school district, and invited math educators and mathematicians. The collaborative effort of all these parts allows for a better understanding of students' learning and the development of better teaching practices.

CONVERSATIONS ABOUT CONNECTIONS: A BEGINNING TEACHER STRUGGLES TO TEACH FOR UNDERSTANDING

Aldona Businskas
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This paper reports a single case (Josie) from a larger study in which secondary mathematics teachers were interviewed about their understanding of mathematical connections. Teachers participated in three interviews, each progressively more structured and focussed on their explicit connections related to a particular mathematics topic. Josie rarely explicitly uses the metaphor of "connections". When she does, she mostly refers to alternate representations. However, she is deeply committed to teaching concepts rather than procedures and struggles against what she perceives as obstacles to doing so.

SAMPLE SPACE REARRANGEMENT (SSR): THE EXAMPLE OF SWITCHES AND RUNS

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This study continues research in probability education by altering a "classical" problem, referred to as the sequence task. In this task, students are presented with sequences of heads and tails, derived from flipping a fair coin, and asked to consider their chances of occurrence. A new iteration of the task –that maintains the ratio of heads to tails in all of the sequences– provides insight into students' perceptions of randomness. Students' responses indicate their reliance on the representativeness heuristic, in that they attend to sequences by their representative features, rather than by considering independent events. The study presents an unconventional view of the sample space that helps situate students' ideas within conventional probability.

STUDENT GENERATED WORKED-EXAMPLES - A TOOL FOR ENHANCING THE LEARNING OF MATHEMATICS

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This study investigates learning from worked-examples as well as from generating worked-examples by first-year university students enrolled in a mathematics foundation course. Along with investigating the type of learning from examples, this inquiry also probes the role of generating examples as a research tool that helps the investigator understand the nature of learner's understanding of pre-algebra. The study tentatively uses the four-stage model of initial skills acquisition developed by Anderson, Fincham, and Douglass (1997) for a preliminary analysis and report on the collected data. It also shows the need for more search and work on developing a tool for analysing students' examples in order to see if the learning promoted by examples is learning with understanding as opposed to rote learning.

ENACTIVE COGNITION AND IMAGE-BASED REASONING IN GEOMETRY

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The purpose of this paper is (1) to introduce the lemniscus model for enactive cognition, (2) to apply the lemniscus model to image-based reasoning in geometry, and (3) to suggest ways in which these insights may improve the teaching of geometry in high schools. The discussion will clariy the relationship in enactive cognition between action and perception: external reality understood in terms of scientific materialism is incommensurable with subjective experience. Image-based reasoning in geometry may be regarded as a structured autonomous subsystem of the lemniscus. There may be metacognitive implications for the teaching of geometry. Ontogenetic implications of the structure may also suggest ways to increase the effectiveness of geometry teaching. It is hypothesized that fluency in the process of geometric reasoning is a more important goal than accuracy.

MEDIATED SUCCESSIVE REFINEMENT: A PEDAGOGICAL TOOL FOR UNDERSTANDING MATHEMATICAL STRUCTURES

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In this study, I will first introduce mediated successive refinement, a method that can be used as a pedagogical tool as well as a research tool. Subsequently, I will illustrate how I used this method to examine students' understanding of a combinatorial structure. I will also describe some of the pedagogical advantages of using this method.

In mediated successive refinement, learners are encouraged to generate an example of a situation or a problem whose answer is a particular mathematical structure. Afterward, the problems are gathered as a problem set and redistributed to them. Students are then asked to solve the problems, and reflect on them. In the final step, their responses are gathered and discussed again in the class.

CONSTRAINED BY KNOWLEDGE: THE CASE OF INFINITE PING-PONG BALLS

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This report is part of a broader study that investigates university students' resolutions to paradoxes regarding infinity. It examines two mathematics educators' conceptions of infinity by means of their engagement with a well-known paradox: the ping-pong ball conundrum. Their efforts to resolve the paradox, as well as a variant of it, invoked instances of cognitive conflict. In one instance, it was the naïve conception of infinity as inexhaustible that conflicted with the formal resolution. However, in another case, expert knowledge resulted in confusion.

MULTI-LAYERS OF NUMERACY

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Numeracy means different things to different people. In this paper I will argue that numeracy is multi-layered that requires the ability to integrate mathematics, situated and contextual problem solving, and communication skills. Numeracy is multi-layered: it is an entity, an embodied disposition, a language, a practice, and a cultural activity. Current school reform initiatives acknowledge the importance of connecting school mathematics with students' own experiences in social and cultural contexts. Familiar contexts can make mathematics more accessible to those who have been alienated from it.

THE USE OF MATHEMATICAL IMAGERY IN COMPLEXIVIST ACCOUNTS OF EDUCATION AND EDUCATIONAL RESEARCH

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Complexity thinking has been applied to educational contexts for about the past fifteen years. Of the many disciplines that have contributed to the development of complexity thinking, mathematics has played a key role – especially through recent advances in mathematics in the fields of nonlinear dynamics and fractal geometry. Some complexity thinkers have taken to using this "new" mathematics as a source of metaphors for talking about education. This paper documents and reviews a number of these efforts. The paper also discusses the value of such metaphors to educational discourses.

ELECTROOCULOGRAPHY: CONNECTING BRAIN AND BEHAVIOR IN MATHEMATICS EDUCATION RESEARCH

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Connecting brain and behavior is gaining increasing interest in mathematics education research, in areas as far ranging as mathematical problem solving and mathematics anxiety. Electrooculography (EOG) enables accurate measurements of eye-related behavior (e.g., blinks & movements) by detecting and recording changes in voltage potentials generated by contractions of extraocular and ocular muscles encasing and enclosing the eye. This paper reports on the potential roles and importance of EOG for mathematics educational neuroscience research. Here we identify and discuss three main uses of EOG. First, EOG provides insights into cognitive function and performance. Secondly, it can be used to attenuate eye-related artifacts in electroencephalography (EEG). Thirdly, when used in tandem with EEG, eye-tracking (ET), and audiovisual (AV) data sets, EOG serves as a helpful means for calibrating covert brain activity with overt behavior. Here, we provide an overview of the first two application areas, and we illustrate the third application using an example data set capturing an "aha moment" in our research in the area of mathematical problem solving. We conclude this paper with a summary discussion on these various uses of EOG in connecting brain and behavior in educational research.

A TALE OF TEACHER INITIATED INQUIRY INTO INSTRUCTIONAL PRACTICE: THE CASE OF DIVISION OF FRACTIONS

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This paper reports on the lesson study experience of a group of five mathematics teachers at Southpointe Academy. Teachers participated in collaborative six-week lesson study cycle on the topic of division of fractions. Two lessons with two different approaches to the topic were implemented in two Grade 8 classrooms in the form of “Lesson Study Open House”. Outside participants, who took part in the lesson observation and post lesson discussion phases included a mathematics educator, a mathematician, teachers from other schools, and preservice teachers. Here we examine, using the notion of critical lenses, some of the aspects of embedding teacher’s learning in their everyday work, or that of their colleagues, and the potential of this work to provide the conditions for increasing the effectiveness of mathematics instruction in schools.

PROCEEDINGS

LESSON STUDY IN JAPAN AND WESTERN CANADA: LEARNING A COLLABORATIVE APPROACH TO IMPROVE CLASSROM INSTRUCTION

Melania Alvarez Adem

Smon Fraser University

Our purpose is to help increase collaboration between teachers, school administrators, math educators and mathematicians, with the goal of improving classroom instruction in the area of mathematics through the use of lesson studies. Lesson studies provide a process for teachers to collaborate in the design of lessons with the goal of identifying successful teaching strategies and to increase students' learning. Teachers work together in the development of a lesson, and when the lesson is implemented, observers take notes on students' questions and understanding. These observers are some of the teachers who helped to plan the lesson, other teachers or administrators in the school or school district, and invited math educators and mathematicians. The collaborative effort of all these parts allows for a better understanding of students' learning and the development of better teaching practices..

LESSON STUDY: A COLLABORATIVE RESEARCH PRACTICE

What lesson study has shown is that through consistent examination of teaching practices and their effects on students' learning, teachers are able to change and improve their instructional practices (Watanabe, T. 2002, and Fernandez &Yoshida, 2004, Becker et al 1990, Takahashi 2000). Currently, if there is change, in most cases it is left to an individual teacher or school site who comes across "effective" teaching strategies. However, most times experienced teachers continue applying the same method they have used since the start of their careers (Stigler & Heibert, pp. 12-13).

Lesson study is a professional development practice where teachers are able to systematically examine the effectiveness of their teaching practices, and how to improve them. Working on these lesson studies involves planning, teaching, observing, and critiquing the lessons. To provide focus and direction to this work, the teachers select an overarching goal and a related research question that they want to explore. This research question serves to help them with their work throughout the planning of the lesson study. In fact, lesson study teachers become collaborative researchers, they not only plan a lesson but also observe and critically reflect on the lesson together. They carry out an in-depth study of the topic on which the lesson will focus. Those who planned the study lesson and those who are observing the lesson work together gathering data that the study group will analyse and discuss to find out how effective the lesson was in dealing with the theme, and finally they will write a report and in many cases they will publish their results.

In Japan, administrators also play an important role by providing behind-the-scenes support in the form of necessary time and resources for teachers. In Japan, faculty members at universities collaborate with elementary and high school teachers in this endeavour. University professors can be involved in several stages of the process. They can help in the teachers' planning either as an active member or as a consultant, and also after the lesson, during the debriefing, they can act as the “knowledgeable other” bringing together all the relevant points in the discussion and summarizing the process.

Wilms (2003) and Watanabe (2002) suggested that lesson study is not only a professional development activity but a culture. It requires teachers to plan, observe and reflect on a lesson through a collaborative effort. It also opens the doors to administrators and outside experts like university professors. This type of collaboration is almost non-existent in our society, and therefore, in order to implement this model, we need a shift in our culture of learning.

In the United States and Canada, mathematicians and math educators have been at odds for a long time about how mathematics should be taught, and teachers are usually caught in the middle of this discussion with very little say. Differences in theoretical perspectives about how students should be taught are at the core of the conflict. Traditional vs. constructivist, rote learning vs. cognitively guided instruction, etc. have been opposing points of view about teaching practice.

What was interesting to see in Japan was that all these practices were used, not in opposition but complementing each other; what brought them together were lesson study practices. Different ideas and perspectives were brought together for a lesson, and in the end what was added to the practice was what worked in the classroom. What lesson study requires is a collaborative effort from teachers, mathematicians and math educators, in an atmosphere of respect and where all parts are equal in importance and in the development of this practice. This is the cultural change that we need, and which lesson studies can facilitate.

How a Lesson Study Happens.

Following Watanabe (2003) description and notes from my experience in Japan, this is a description of how a lesson is developed. Initially a group of participating teachers identify a study theme, which comes about after a critical reflection on their goals for students and the realisation that there is a gap between the goal and the expected knowledge students should have. Sometimes these themes will focus on a particular concept, and other times they may focus more on student interaction and behaviour while learning. Many times the themes could combine the behavioural theme with a more subject-focused theme. Once the study theme is identified, teachers turn their attention to looking for ways to tackle the problem through a lesson. One of the first steps for the group is to find out how other lesson study groups may have dealt with similar themes, and how this topic relates to others topics throughout the curriculum. This research will allow them to develop a deeper understanding of subject-specific goals. As the teachers continue working on the lesson, they start focusing on the question they will ask the students to introduce the lesson and how they will pose this question. Teachers are also very thorough in selecting the instructional support materials to be used by carefully studying how they will facilitate class discussion. Included in the plan are ways of assessing students' acquisition of knowledge due to the lesson, and how successful they were in achieving the goals they set forth.

The lesson study research group will produce an instructional plan that will provide a clear idea as to how the lesson might develop. The lesson plan will include an explanation of the goals, how the goals might be achieved by the lesson, how this topic is related to other topics in the curriculum, and how the lesson will be assessed (see Lesson Study Group at Mills College, n.d. for sample plans)

Once the lesson study is developed, teachers will teach the lesson in public, and other teachers are invited to observe. Lesson studies can be school-based, district-wide or university-affiliated. At the former, only teachers from the same school participate. At the district-wide or a university-affiliated school level, teachers from many different schools attend the lesson. Sometimes there is a large lesson study open house, where several lesson studies happen at the same time within a school.

Lesson plans are distributed in advance to teachers participating in the lesson as observers. This will allow them beforehand to have an idea of how the lesson might develop, to look at the focus questions set by the lesson research group that develop the plan, and to come up with questions of their own to focus on while observing the lesson. Observing teachers are also provided with a student seating plan.

Throughout the lesson, teachers take notes, often times on the lesson plan section of the instructional plan or on the student seating plan they received. This depends on whether they are focusing on the class as a whole or on the understanding of particular students. During the lesson, while the instructor is teaching or giving instructions, observers stay at the back or the periphery of the classroom. Once the students start working either in groups or individually, observers can move around the classroom, taking notes of what students are doing. They are to minimize interference with the flow of the class.

After the lesson, there is a debriefing. This meeting is usually facilitated by the principal or the chairperson of the lesson study committee. The first comments come from the teacher who taught the lesson. Then, members of the study group provide their reflections on the planning and the lesson. Afterwards the discussion is opened to all participants. The facilitator makes sure that the discussion will not drift away from the topic of the lesson study. The debriefing usually concludes with comments from an invited outside observer. This outside observer is often a college or university faculty member. After the debriefing, the study group meets again to reflect on the comments and observations made. They may revise the study lesson, they may even teach the revised lesson in another classroom, where the only observers may be the members of the study group. The study group then prepares its final report. This report could be presented at a professional meeting or published online. Whatever form it takes, the creation of the report is a key component of lesson study, as members critically reflect on how well they were able to meet the goal(s) they had set for themselves.

Lesson Studies in Western Canada:

In Canada, one of the leaders in implementing professional development through lesson studies is Sharon Friesen of the Galileo Educational Network in Alberta (http://www.galileo.org/math/lessonstudy.html). She has been working with teachers using lesson study practices for over six years. Following her lead, the Pacific Institute for the Mathematical Sciences (PIMS) organized a series of five Saturday morning meetings at UBC in 2006 and 2007 on the topic of lesson studies. It is not surprising that their first guest speaker was Sharon Friesen. Workshops were developed based on the format used in Alberta. These are five key components of the workshops based on the Galileo workshops and modified by Klaus Hoechkman who is the PIMS education facilitator:

1. Workshop Leaders - Each workshop is led by a duo consisting of an experienced mathematician, and math educator. The role of the former is to ensure a sound mathematical basis, point out interesting connections, and fill in details where needed. The role of the math educator is to interpret the subject from a teacher's perspective and dovetail it with typical class room situations.

2. Mathematical Content – It has both structure and focus, but these are found interactively instead of being fixed in advance. The focus is agreed upon after a brief discussion of what the participants find most pressing and difficult for their classes. In this way, the discussions, explanations, and examples will concentrate on what is most important to participants.

3. Pedagogical Content – It is truly collaborative and supportive, counteracting the personal isolation and curricular pressure, which in many cases prevent teachers from examining in depth what and how their students are learning, and how they could be helped to overcome hurdles. In the process, the teacher's own understanding and enjoyment are nurtured.

4. Research Lesson Design – It provides opportunities for teachers to work together on creating curriculum-based research lessons, and to allow time for professional

dialog and support to develop, as well as practice and critique of new teaching methods.

5. Publish and Share – The collective insight of the group will be made available to the widespread teaching community via the Internet. It is imperative that local teachers have access to local examples of promising practices for their own use in discussion groups and exhibitions of student work.

Last year two math lessons studies were developed under the sponsorship of PIMS. Fred Harwood, a teacher and math Coordinator at Hugh McRoberts Secondary in Richmond, and his staff, together with Paulino Preciado, a Math Education Ph.D. student at Simon Fraser University, developed a lesson on “Confident Addition”. Natasa Sirotic, a teacher and a Ph.D. Student at Simon Fraser University, together with Paulino Preciado and Sachiko Noguchi, a student teacher, developed a lesson in development and understanding of algebraic equations.

To better illustrate what was done, in the case of the Fred Harwood lesson, it can be said that his main goal for the students was to be confident and able to add numbers without a calculator. Using the idea that numbers are not scratches on paper but a description of a compilation of “things”, numbers can be represented in many ways. “Numbers can be handled easier when they are “complete” (10 + 10 is easier than 8 + 12). Completeness may be factors of 10 or 5 or 25, wholes or ‘nicer’ fractions. One of the exercises in the lesson was posed as follows: “In your groups we want you to add up the following five numbers 84 + 72 + 87 + 95 + 47 . . . without calculators (cell phone calculators et al) . . .and we want you to do it five different ways. In about 10 minutes, we will want a representative of your group to write your group’s favourite way up on the board and some other(s) in your group will present to the class how you solved it and why you liked this method. Please name your favourite method so we can refer to it later. Go!” [from Fred Hardwood lesson plan one can find at http://public.sd38.bc.ca/~fharwood/LessonStudyLesson]. After each group posted their solutions on the board, the class discussed and decided which method was the most efficient and they were asked to look for common features of the methods. Fred assessed his students afterwards and they were able to make calculations more accurately and confidently.

A video of both lessons was created and they were presented at the workshops. Teachers attending the workshops provided interesting critical points to both lesson groups and found these workshops an enriching experience. Many of them saw the positive aspect of getting feedback from other professionals about their teaching practices.

As a follow up, Fred Hardwood presented his results at a province wide pro-D day this fall; more than 75 teachers attended his workshop. After the workshop, many teachers showed an interest in developing this practice at their schools or in workshops like the ones PIMS has supported.

Meanwhile, a lesson study group lead by Natasa Sirotic has been formed at Southpointe Academy in Tsawwassen, British Columbia. They hosted their first lesson study this fall, also during the province-wide pro-D day. They invited teachers from other schools to attend, as well as teachers within the school. This lesson study very closely followed the format of the Japanese lesson study process. This exercise showed that the lesson study experience can be replicated at least at a local level. The principal at the school was impressed with this exercise and given the teachers' positive feedback, the school is planning another lesson for next spring, not only in mathematics, but in other fields as well.

Conclusions:

Mathematicians have been active at our workshop meetings and have enjoyed working with teachers in an atmosphere of collegiality. Many teachers do not have much contact with mathematicians after they graduate, and the contact has proven beneficial for both parts. This is part of changing the culture: to initiate contact on a regular basis among mathematicians, teachers and math educators.

As James W. Stigler has pointed out, "[perhaps] we cannot implement lesson study in the United States [Canada] the same way it is implemented in Japan" (Yoshida 1998 p. xi). However the modified lesson study workshops organized by PIMS, together with the Fred Hardwood and Natasa Sirotic lesson studies have shown some success in attracting teachers to this process. And let us not forget that Sharon Friesen's lesson study group has been steadily growing in the last six years.

At least at a local level we have been able to reproduce the model of lesson study in Japan here in Canada. In the United States this practice is now being used by some schools (Chokshi & Fernandez 2003). District-wide administrators, and more faculty members from universities need to get involved. What we need to show them are the benefits in this change of venue, and that will take time.

Given the popularity of the two lessons study events at the provincial pro-D day, the PIMS education department was asked to continue the lesson study workshops at UBC. Five more workshops will be scheduled for this academic year and support has been secured for implementation of several lesson studies at schools that requested it.

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CONVERSATIONS ABOUT CONNECTIONS: A BEGINNING TEACHER STRUGGLES TO TEACH FOR UNDERSTANDING

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This paper reports a single case (Josie) from a larger study in which secondary mathematics teachers were interviewed about their understanding of mathematical connections. Teachers participated in three interviews, each progressively more structured and focussed on their explicit connections related to a particular mathematics topic. Josie rarely explicitly uses the metaphor of "connections". When she does, she mostly refers to alternate representations. However, she is deeply committed to teaching concepts rather than procedures and struggles against what she perceives as obstacles to doing so.

The National Council of Mathematics Teachers' (NCTM) document, *Principles and Standards for School Mathematics* (2000) and its earlier versions in 1989 and 1991 establish a framework to guide improvement in the teaching and learning of mathematics in schools. The documents identify "mathematical connections" as one of the curriculum standards for all grades K to 12. In this framework, "… mathematics is not a set of isolated topics but rather a web of closely connected ideas" (NCTM, 2000, p. 200). Making the connections is taken to promote students' understanding of new mathematical ideas (NCTM). However, in both the research and pedagogical literature the notion of what a mathematical connection is, often remains implicit.

In a previous paper (Businskas, 2007), I defined a mathematical connection as a true relationship between two mathematical ideas and drew on Skemp's notion of a person's mathematical knowledge as a set of hierarchical schemata (Skemp, 1987), composed of mathematical concepts and the connections among them. Also, I proposed a list (summarized below), of possible specific types of mathematical connections where A and B represent two mathematical ideas:

1. **A is an alternative representation of B.**
2. **A is equivalent to B.**
3. **A is similar to B (A intersects B).**
4. **A is included in (is a component of) B, B includes (contains) A.**
5. **A is a generalization of B; B is a specific instance (example) of A.**
6. **A implies B (and other logical relationships)**.
7. **A > B; B < A (and other order relationships)**.
8. **A is a procedure used to work with B.**

Problem Statement and Research Question

In an earlier studies (Businskas, 2005, 2007), secondary mathematics teachers, interviewed about their views of mathematical connections, spoke favorably of the importance and value of attending to making connections, but found it very difficult to present examples from their teaching or their own thinking of specific instances of mathematical connections.

In this study, I try to delve deeper into the question - what kinds of mathematical connections can teachers describe in their knowledge of a mathematical topic, by probing teachers' understanding of specific topics in mathematics to make their notions of mathematical connections explicit.

Research Setting

Participants in the study from which this case is derived were ten secondary mathematics teachers who volunteered for the study. Teachers were interviewed individually three times over a three-month period in their schools.

Interviews

The first interview was an acclimatizing interview and drew out information about teachers' background and general views about teaching mathematics and about mathematical connections.

The second interview was a semi-structured interview about a mathematical topic chosen ahead of time by the teacher that (s)he deemed conceptually rich. Prepared questions addressed teachers' own subject knowledge and their knowledge of the topic for teaching (Shulman, 1986), for example,

- Please tell me about your own understanding of this topic… What are the important concepts/ideas and procedures that make up your understanding of this topic?
- Please tell me how the ideas and procedures that you've identified are related to each other or related to other topics in mathematics.
- From your point of view as a teacher, what are the most important concepts and procedures that you want your students to learn?
- What are the important concepts that your students must already know before study this topic? How is each of these "prerequisite" concepts related to the ideas/procedures that you will teach?

Responses that teachers made that included some reference to connections were followed up with questions asking them to elaborate. If teachers did not naturally make any references to connections, they were asked further probing questions, and sometimes even leading questions in an attempt to get them to voice an opinion.

The third interview was a task-based interview in which all teachers dealt with the same topic – quadratic functions and equations. Teachers were given a set of 82 cards (see Appendix) containing mathematical terms, formulae and graphs related to this topic, gleaned from a selection of high school mathematics textbooks. They were asked to organize the cards in some way that showed the relationships among them. They were instructed, "Please group them in a way that will show how they are connected". After they completed the task (see photo in Appendix), they were asked to explain their organization, and were constantly pushed to elaborate their statements about connections.

Analysis

All interviews were transcribed. From the transcripts, I extracted particular statements the teacher made about connections either explicitly or implicitly. I then coded these statements according to the framework described earlier.

Data and Interpretations

Josie is in her first year of teaching. Teaching is her first career and she is in her first job. The year before, she completed her teaching practicum at another school in the same district. Josie has a Bachelor's degree with a major in Mathematics and sees herself as being very well prepared for teaching high school mathematics both in terms of her knowledge of the content and her knowledge of the BC curriculum and the NCTM standards. Josie is an immigrant, having come to Canada while she was in elementary school.

Background interview

Although Josie rarely used words like connection, relationship, link, she talked a lot about the importance of understanding and described a metaphor that exemplifies mathematics as "a web of closely connected ideas".

> I know the map of the city. I've not just memorized the map, but… I've been there and I lived in this neighbourhood, so… I know every building and I know everything around in the city, right? So that if I want to go from A to B, I can take any routes I want, but if there's someone who doesn't know this city very well, they will memorize oh, you turn left here, you turn right here.... If they get lost in one place, they wouldn't be able to turn around and go. But I know the whole thing, it's like I'm standing on a mountain, and I can see the whole thing. I know how each road will go. I think that's my way of saying deeper understanding… Math is about understanding the whole thing, and then you can go from one place to another.

Josie repeatedly distinguished between "understanding" and "just the steps". "I try to teach the core more…not the steps, but the reason behind it and the concepts." As an example, Josie offered "solving equations with variables on both sides… one way to do it, is to collect x first… or you can bring the numbers to the other side first, also two ways to write it… So this I think is different ways to go to the answer". When challenged to differentiate between concepts and procedures, she said

> They're all concepts right? First of all, $y = mx + b$ is a concept. It's a concept because it's a relationship between x and y… every line has an equation, and it can be changed, it will be in the form of $y = mx + b$. So it's not just a memorized thing… I just use the formula. But I have to teach why it works first… Even though they [students who memorize the formula] get the same answer, their understanding is very different.

Josie's model of understanding is based on reasoning from some basic principles. While she could not articulate how thinking like this differed from applying rotely-learned procedures, she was convinced of the difference. She is deeply committed to teaching for understanding, but sees herself as fighting an uphill battle:

> I kind of struggle, I don't want to teach just the steps… the reality is that some students… don't really want to connect them… I still have…my will to teach for understanding… we have common tests, so it's really restricting your teaching… the order of the textbook is really bad… I'm already in trouble because I have to teach that order and also don't have freedom, don't have as much freedom to go over something that I find more interesting that what the text has to say.

"Chosen topic" interview

Topic. Josie chose solving right triangles as an example of a topic that is conceptually rich. While she strained to express herself clearly, it was evident that she was using a framework reminiscent of Skemp for describing her knowledge –

> There's lots of different small concepts, and also bigger concepts, like, I would say, the main concept is similar triangles and they have the ratios of the sides are all the same … we're starting to learn trig ratios, tangents, sine and cosine. And these three ratios,… they stem from, like the concept that similar triangles have the same ratio of, for example, opposite to adjacent… So if you

> draw two different right triangles that are similar, if you divide, if you find the ratio between the opposite and adjacent, they will be the same … That's a big concept there… I think that's what I want to do is teach them the concept first instead of just teaching what is SOHCAHTOA.

Explicit connections. Even when pushed, Josie said very little explicitly about relationships of ideas, but what she did say is itemized below:

Statement	Type of Connection
the biggest angle is opposite the biggest side	A implies B
... relationships between legs and hypotenuse… the legs make up the right angle	A is included in (is a component of) B
the hypotenuse is opposite to the right angle	A implies B
the smallest angle is opposite the shortest side	A implies B
Pythagorean theorem is… just a method for us to find… one of the sides	A is a procedure used to work with B
angles' sum add up to 180 is just a tool to get the answer	A is a procedure used to work with B
You always do sine of the angle is opposite over hypotenuse… What does that number mean is that it's the opposite divided by the hypotenuse	A is equivalent to B
this notation [SOHCAHTOA] is… just a shortcut	A is a procedure used to work with B

What Josie re-iterated over and over again was her wish to teach "bigger" concepts, like similarity, ratio, which students could then apply in a variety of ways. Subsumed within these broader concepts, she saw smaller concepts, like the Pythagorean theorem, the sum of angles in a triangle is 180^{o}, which she thought of as procedures – "shortcuts… some ways for them to get the answer".

Task-based interview

Josie laid out the cards in ten groups, spending over half an hour on the task. She used 62 of the 82 cards and did not add any additional ones.

Josie's organizing principle for the task was to arrange her groups moving from "thinking about functions" to "thinking about equations". The connections that she described explicitly were the result of probing questions. Most of the "within-group" mathematical connections that she described were alternate representations – all algebraic/graphical, for example:

- the zeros of the function is the roots of the equation
- if a point is on the graph, then the x and y will satisfy the equation
- the graph is just a graphical representation of a function… parabola is the shape of the quadratic function

Although Josie was repeatedly asked to talk about how the objects were mathematically related, she several times linked objects from a teaching point of view. I called these "pedagogical connections", for example:

- perfect square, simplify, FOIL… they are all prerequisites kind of, skills and background knowledge

- the terminology for a quadratic function, there are maximum, minimum, vertex, symmetry, domain and range

When asked to extend the connections beyond this topic, she offered one generalization –

> cause they've seen how to solve a function by plugging in y=0, they solve the function… they get not only the zeros of the function but also the roots to the equation, so they will now be able to do any form of an equation, any form of a cubic, or maybe a … radical function, maybe rational functions.

Cards left out. Finally, I asked Josie about the cards that she left out. Josie's explanations for the cards she left out fall into two categories. Some she left out because she did not recognize them – "I don't remember what they are". These included associative, commutative, focus.

She left out a larger group, including co-ordinates, derive, exponent, expression, formula, operations, radical, square, value, variable, because she saw them as being "too simple" – "you should really have known that too long ago". Again, she was taking a pedagogical point of view, arguing that some ideas should have been learned long before and were therefore not relevant to her teaching of this topic.

Summary

Josie was able to explicitly describe some types of mathematical connections when pressed, but talking about connections was not natural for her. While she agreed with the statement that "students' mathematics concepts must be woven into a connected set of relationships", and incorporated the web metaphor in her own way of looking at mathematical knowledge, what she spontaneously talked about was understanding, not connections. Her constant theme was her desire to teach concepts rather than algorithmic procedures without understanding, but she saw herself struggling against obstacles to do so. She felt constrained by advice from colleagues, by textbooks, by department requirements for common testing.

With respect to her notions of mathematical connections, the following themes emerged from my conversations with Josie:

- the metaphor of "a web of closely connected ideas" seems to represent Josie's thinking about various topics and she was able to identify explicit connections
- talking about connections did not come naturally except for linking, in general terms, to prior knowledge and future applications – pedagogical connections
- assumptions are made that certain practices are obstacles to teaching for understanding, but these assumptions are not questioned.

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Appendix

List of Cards used in the Task-based Interview

algebra	domain	guess and check	quadratic
associative property	equation	inequality	quadratic formula
coefficient	expansion	intercept	radical
commutative property	exponent	inverse	range
complete the square	expression	isolate the variable	relation
compression	factor	maximum	remainder theorem
cone	factor theorem	methods of solution	root
co-ordinates	focus	minimum	simplify
curve	FOIL	operations	solve
derive	formula	parabola	square
directrix	function	perfect square	square root
discriminant	geometry	point	substitute
distributive property	graph	properties of roots	symmetry

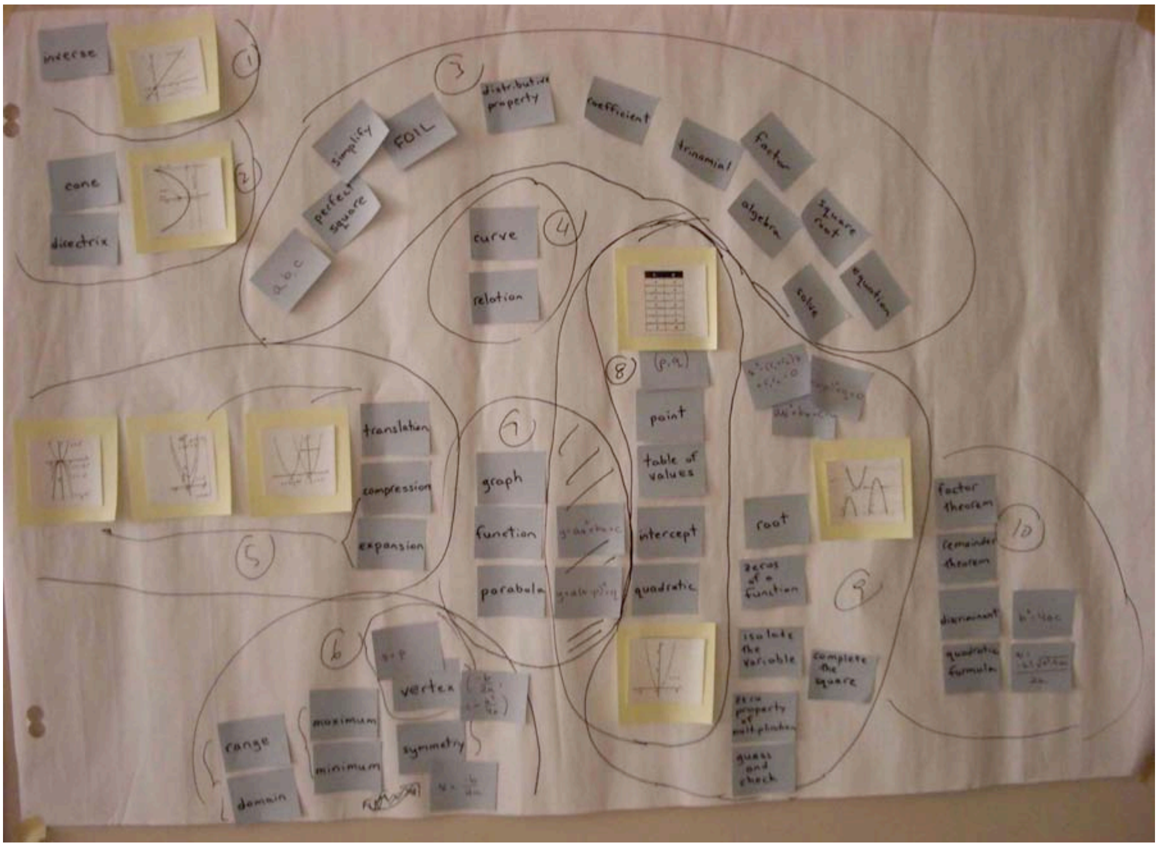
table of values
translation
trinomial
value
variable
vertex
zeros of a function
zero property of multiplication
GR: standard parabola

table of values example
GR: 3 example parabolas
GR: focus/directrix
GR: horizontal translation
GR: vertical translation
GR: compression examples

GR: inverse
a, b, c
(p, q)
$b^2 - 4ac$
$p \pm \sqrt{(-q/a)}$
$(-b/2a, (c - b^2/4a))$
$x = p$
$x = -b/2a$
$x = (-b \pm \sqrt{(b^2 - 4ac)})/2a$

$x = 2c/(-b \pm \sqrt{(b^2 - 4ac)}$
$ax^2 + bx + c = 0$
$x^2 - (r_1 + r_{2)}x + r_1 r_2 = 0$
$a(x-p)^2 + q = o$
$y = ax^2 + bx + c$
$y = a(x-p)^2 + q$

Photo of Josie's Arrangement



SAMPLE SPACE REARRANGEMENT (SSR): THE EXAMPLE OF SWITCHES AND RUNS

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This study continues research in probability education by altering a "classical" problem, referred to as the sequence task. In this task, students are presented with sequences of heads and tails, derived from flipping a fair coin, and asked to consider their chances of occurrence. A new iteration of the task –that maintains the ratio of heads to tails in all of the sequences– provides insight into students' perceptions of randomness. Students' responses indicate their reliance on the representativeness heuristic, in that they attend to sequences by their representative features, rather than by considering independent events. The study presents an unconventional view of the sample space that helps situate students' ideas within conventional probability.

The Sequence Task

Psychologists Tversky and Kahneman (1974) found that students, when considering tosses of a coin, determined the sequence HTHTTH to be more likely than HHHTTT, because the latter sequence did not appear random. Furthermore, they found HTHTTH more likely than HHHHTH, because the latter sequence was not representative of the unbiased nature of a fair coin (i.e., the ratio of the number of heads to the number of tails was not close enough to one). The caveat is that, normatively, all three sequences have the same chances of occurring. A number of researchers in mathematics education (e.g., Shaughnessy, 1977; Cox & Mouw, 1992; Konold, 1989) have continued Tversky and Kahneman's work and as such have furthered research involving the sequence task.

As Shaughnessy (1992) reflects: "There was no attempt made [by Tversky and Kahneman] to probe the thinking of any of these subjects" (p. 473). Shaughnessy's (1977) work brought two new elements to the sequence task. First, "[t]he subjects were asked to *supply a reason* for each of their responses. In this way it was possible to gain some insight into the thinking process of the subjects as they answered the questions" (p. 308). Second his tasks gave students the option of choosing "about the same chance" (p. 309) as one of the forced response items. The results echoed the conclusions of earlier work by Tversky and Kahneman. The sequence BGGBGB was considered more likely to occur than the sequences BBBGGG and BBBBGB. However, with the new "supply a reason" element to the task, Shaughnessy was able to confirm that subjects did find the sequence BBBGGG not representative of randomness, nor was the sequence BBBBGB representative of randomness, because it did not have a representative ratio of boys to girls: all of which had previously been inferred from the number of subjects choosing particular forced response options.

Konold's (1993) research built upon the two elements introduced to the sequence task by Shaughnessy. Konold found that when students were asked to determine which of the sequences were most likely to occur from flipping a fair coin five times, they often chose that all of the sequences were equally likely to occur. However, when students' were asked which of the sequences were the least likely to occur, they picked out a particular sequence. In fact, of the sequence options presented, HTHTH was deemed to be the least likely, because it was too representative. In other words, it did not reflect the random nature of the task; its archetypical appearance demoted its likelihood.

Researchers have found that humans are poor judges of randomness and, moreover, have determined the evaluation of randomness to be an important component of the sequence task. Batanero and Serrano (1999) found that students' ability to determine what was considered random, in results from flipping a fair coin, was derived from two variables. The proportion of the number of heads to the number of tails in the sequence and "the lengths of the runs and, consequently, the proportion of alterations" (p. 560). Reflecting these findings, Cox and Mouw (1992) state: "The sampling form [of representativeness] is seen both when a small sample is assumed to adequately represent some phenomenon in a population [e.g., 1:1 ratio for coin flips], and also when a sample is expected to appear random" (p. 164). Falk (1981) gave students two long sequences and asked which of the sequences appeared more random. The results show that sequences in which more switches occurred appeared more random. Furthermore, sequences in which a long run occurred appeared to be less random. In other words, randomness was perceived via frequent switches and thus short runs.

Sample Space Rearrangement

As presented, a number of issues have been addressed with subsequent iterations of the sequence task; however, one element remains the same. In each instantiation of previous research (by mathematics educators) presented, subjects were offered sequences with a different population ratio in each of their choices. For example, one option would be HTHTTH (3H:3T), while the second option would be HHHHHT (5H:1T). Cox and Mouw (1992) found disruption of one aspect of the representativeness heuristic, such as the appearance of randomness, did not exclude the population ratio being used a clue. Also, Shaughnessy (1977) found that "many students picked 'the same chance', but gave as a reason 'because each outcome has 3 boys and 3 girls'" (p. 310). While previous research has addressed sequences with a disparate number of heads and tails, this report examines responses from students when all of the choices presented contain the same ratio of heads to tails. As such, the task was constructed based upon an alternative view of the sequences: the view of switches and runs. A "traditional examination" of the sample space, when a fair coin is tossed five times, takes into account the numbers of heads and number of tails. However, rearranging the sample space (see Table 1) shows that the sequences can be organized by the number of switches, denoted S (a change from head to tail or tail to head), and, consequently, the longest run, denoted LR (the number of times the same result occurs).

0S & 5LR	1S & 3LR	1S & 4LR	2S & 2LR	2S & 3LR	3S & 2LR	4S & 1LR
HHHHH	**HHHTT**	HHHHT	HHTHH	HHHTH	HHTHT	**HTHTH**
TTTTT	TTTHH	TTTTH	TTHTT	TTTHT	TTHTH	THTHT
	TTHHH	THHHH	**HHTTH**	HTHHH	HTHHT	
	HHTTT	HTTTT	TTHHT	THTTT	THTTH	
			HTTHH	**THHHT**	**THHTH**	
			THHTT	HTTTH	HTTHT	
					THTHH	
					HTHTT	
P(0S5LR) =2/32	P(1S3LR) =4/32	P(1S4LR) =4/32	P(2S2LR) =6/32	P(2S3LR) =6/32	P(3S2LR) =8/32	P(4S1LR) 2/32

Table 1: Sample Space organized according to Switches and Longest Runs

Results and Analysis

Students were presented with the following:

Which of the sequences is least likely to result from flipping a fair coin five times:(A) HHHTT (B)HHTTH (C)THHHT (D)THHTH (E)HTHTH (F) All sequences are equally likely to occur. Provide reasoning for your response…

Although approximately sixty percent (30/49) of the students –14 to 15 year old high school students, from Vancouver, BC, Canada– incorrectly chose one of the sequences to be the least likely to occur, there was approximately an even split between those who chose *(A): HHHTT* (14 students) and those who chose *(E): HTHTH* (16 students). Justifications for why sequences *(A)* and *(E)* were not representative (of randomness) fell into two categories. Students who chose *(A)* said that the perfect alternation of heads and tails was not reflective of a random process, while students who chose *(E)* said that a run of length three was not indicative of a random process because it was too long. Given the ratio of heads to tails was maintained, determining randomness relied upon switches and, subsequently, longest runs.

This report suggests that heuristic reasoning (more specifically representativeness) is not incongruent with probabilistic reasoning. Illustration of this point draws upon the probabilities associated with the rearrangement of the sample space. For *(A)* there is a 4/32 chance that a sequence of five flips of a coin will have one switch with a longest run of three. For choice *(E)* there is a 2/32 chance that a sequence will have four switches and a longest run of one. From this alternative perspective of the sample space, organized according to switches and runs, all other sequences presented in the task have higher probabilities than sequences *(A)* and *(E)* (e.g., a sequence with three switches and a longest run of two has an 8/32 chance of occurring). As such, it can be inferred from the data that the students are unconventionally, but naturally, looking for features in the sequences which are least likely to occur. From this perspective of a rearranged sample space, it can be argued that the students were correctly choosing which of the sequences are least likely to occur.

Conclusion

An alternative view of the traditional sample space for five flips of a coin via switches and runs, not the ratio of heads to tails, suggests a probabilistic innateness associated with use of the representativeness when determining, heuristically, the randomness of a sequence. Given that "we need to know more about how students <u>do</u> learn to reason probabilistically" (Maher et. al., 1998, p.82), sample space rearrangement may become one such approach.

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STUDENT GENERATED WORKED-EXAMPLES - A TOOL FOR ENHANCING THE LEARNING OF MATHEMATICS

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This study investigates learning from worked-examples as well as from generating worked-examples by first-year university students enrolled in a mathematics foundation course. Along with investigating the type of learning from examples, this inquiry also probes the role of generating examples as a research tool that helps the investigator understand the nature of learner's understanding of pre-algebra. The study uses the four-stage model of initial skills acquisition developed by Anderson, Fincham, and Douglass (1997) for a preliminary analysis and report on the collected data. The data suggests a need to develop a framework that can identify if learning through examples promotes understanding.

Background and objectives

Examples play a key role in both the evolution of mathematics as a discipline and in the teaching and learning of mathematics. There is an abundance of research that acknowledges the pedagogical importance of examples in learning mathematics (Atkinson, Derry, & Renkl, 2000, Watson and Mason, 2005; Zhu and Simon, 1987; Kirschner, Sweller and Clark, 2006, Kalyuga, 2001, Bills, Dreyfus, Mason, Tsamir, Watson, & Zaslavsky, 2006, Zaskis & Leikin, 2007). Atkinson (2000) regards worked examples as "instructional devices that provide an expert's problem solution for a learner to study." Kalyuga (2001) reports that for students experienced in a concept, worked examples became redundant and problem solving proved superior, but for novice students, the study of worked examples turned out to be more beneficial than problem solving. Zhu and Simon (1987) bring substantial evidence for effective teaching by giving students worked examples to learn from. Worked-examples promoted learning with real understanding of the meaning of the rules and procedures they had to follow. They have also captured another aspect of teaching with worked-examples – freeing teacher's time from direct lecturing to focus on students having problems understanding the material. Zazkis (2007) used examples as a research tool that enabled the researcher to gain insight about students' understanding of mathematics. Students and instructors alike seemed to benefit from learning from examples. My study extends on prior research by transferring the responsibility of generating examples and worked-examples to students. I will examine the ways in which undergraduate students attempt to learn from worked examples.

Conceptual framework

The framework for my study is the four-stage model developed by Anderson, Fincham, and Douglass (1997). The four-stage model was developed within a cognitive theoretical framework called ACT-R. In the first stage, learners refer to known examples and try to relate them to the problem to be solved by analogy. In the second stage, the problem solving process will be guided by learners' newly developed abstract declarative rules. After extensive practice the performance becomes fluent and rapid. At this stage of production rules learners can deal with familiar problems quickly and automatically, without using many attention and memory resources. In the fourth stage, learners can often retrieve a solution quickly and directly from memory (Atkinson, 2000). Using this conceptual framework, I will examine students' understanding of order of operations, fractions and linear inequalities. My study examines students' emerging understanding as they learn from worked-example, and as they generate their own examples.

Modes of enquiry and data source

Participants in this research are first-year university students enrolled in a mathematics foundations course. Data collection comprises a series of written tests, in-class activities or homework assignments given over a period of 8 weeks. Students were familiar with learning from worked-examples; carefully chosen worked-examples were presented regarding concepts such as order of operations (BEDMAS), properties of fractions, and linear inequalities. Tasks for this study were designed to help students' acquisition of concepts and skills, and to promote a solid understanding of inequalities, and to avoid rote learning of "rules without reason. Specific questions and tasks for the data collection instruments include the items listed below. Item 4 was given as a home assignment, whereas all the other four items were either quizzes' items or tasks performed in class activities.

Tasks:

1. a) Create a worked-example that will help another student from this class learn how to solve order of operations exercises.
   b) Link the numbers 2, 5, and 3 to generate an example where only correctly applying the order of operations will help one calculate the product of your thought.

2. Give an example of a number that lies in between 1/5 and 1/7. How many numbers are in between 1/5 and 1/7?
3. Write a system of two linear equations with a solution (3, –2).
4. a) Study the **worked-examples** from the two sections:
   Section 4.1: 2, 5, 6, 7, 8, 9
   Section 4.2: 1, 2, 3, 4, 5, 6

   b) Create **worked-examples** that will help another student from this class learn how to solve linear inequalities.
5. a) Create a worked-example that will show how to solve linear inequalities.

   b) Is the one example provided at a) sufficient for someone to learn how to solve inequalities by following your work? Do you think you need to create more examples to demonstrate the full breadth of linear inequalities? If so, how many more examples you think you need?

Results and discussion

Invited to generate their own worked-examples, in ways which encourage explanation and reasoning, our students passed different stages: from being out of task when giving an order of operations example to working on identifying the essence of linear inequalities to be presented in a single worked example.

An analysis of the data based on four-stage model of initial skills acquisition developed by Anderson, Fincham, and Douglass (1997) suggests how students first solve problems by analogy. When asked to generate an example that shows how to solve order of operations, many students were out of task, providing an example similar to the first or the second task of the test, which were on different concepts – solving word problems involving fractions. The analogy worked, but not the expected analogy – the analogy with order of operations.

Reflecting on their work, students acknowledge the difficulty of a construct-an-example task. The majority of students recognized that they were less successful in a giving example than in a solving task. While some of them made comments about not understanding what to do or not being able to explain their thinking, there were a few students who noticed the essential difference between the two tasks – the novel nature of the second one. Here is Jane’s reflection on her performance:

> For me question numbers 1 and 2 [the first two items in the test] were more clear in the sense that a question is there & we have to answer [it]. I got kind of lost with question 3 because we’re not used to coming up with questions [examples] so question 3’s instruction wasn’t

something we've seen in the past. I was more successful in answering questions 1&2 because I was used to seeing that kind of question…

Jane acknowledged the fact that generating examples require more than solving a problem by analogy with a previous work. She admitted that it is difficult to construct examples when the knowledge of a subject is still immature. Seeing something in the past, retrieval from a studied example makes a task more approachable than constructing ones own example.

The examples provided in the second test also suggest that, from the first task on producing an example to the second one, students have developed some declarative knowledge. On the majority of papers BEDMAS was the key word and the explanations of what it stands for were present, but not everybody was successful in creating an example where respecting the order of operations to be vital to the process of calculating the value of the expression or even if the example captured the essence of order of operations, they failed to use the declarative rule properly and gave an erroneous result. For example, given the numbers 2, 5 and 3 and the task of generating an order of operations example, there were students linking the numbers in this way $2\cdot 5+3$, declaring the use BEDMAS and solving without any problem from the left to the right. Some students that connected the numbers in this form $2+5\cdot 3$ declared using BEDMAS, but then solved $2+5\cdot 3=10\cdot 3=30$.

A major proportion of students that that were able to solve task 1, constructed the example without attending to the level of complexity. Consequently, the example contained big numbers and heavy calculations and they ended up not being successful in solving their own production. For example, the students were not aware of the fact that if you want to incorporate a division and continue with the quotient embedded in some other calculations someone should check the divisibility rules and carefully choose the numbers. It is clear from the sample examples that the students were still at the declarative rule stage of skill acquisition. Their example space was limited to examples similar to the studied worked-examples were brackets as well as a multitude of operations were present. When restricted to use the 3 given numbers only and to capture the essence of order of operations they revealed the lack of such skills.

Task 2 was given first in a class activity and then in a midterm. How many numbers are in between 1/7 and 1/5? From 50 respondents, only 16 acknowledged that there are infinitely many numbers in between 1/5 and 1/7. 13 students answered that there are either 2, or 4, or 10 numbers. The majority of students said that there is only one number in between 1/5 and 1/7, and that is 1/6. Two students answered 6/35. At the reflective stage, students were asked to write the two given fractions 1/7 and 1/5 using common denominators. They were solicited to write them again and again with different denominators until the con of the many numbers appeared for them. A consecutive similar task given in the midterm, revealed that this soliciting of another and then another example helps increasing students' repertoire of examples.

Writing a system of equations with solution (3, -2) produces an ample range of examples and way of generating them. From graphing the point and taking two lines passing through it to writing the right side of the equation and from the top of ones head and calculating the free coefficient, from complicated equations to the simplest ones: $x = 3$ and $y = -2$, the horizontal and the vertical lines passing through that point. This examples shows that some students were up to the fourth stage of skill acquisition; they were able to retrieve from memory the most elegant and simple example for that situation. Unfortunately, not many students reached the fourth level. For the majority of them the rules have no reason attached to them and the examples are merely shabby reproductions of class work or provided worked-examples.

Conclusions

It is very well known that not only novices' but also expert mathematicians' self explanations and understanding of a concept depend on examples. While the experienced mathematician may need one example to promote understanding of a general method, most novices ask for numerous examples, in order to get some intuition about the situation and promote some generalization and reasoning from them. To understand something in general even experts report that they have to carry along in their mind specific examples and see how they work (Bills et al, 2006).

This paper gives an account of literature on exemplification as a learning, a teaching as well as a research tool, preliminary data on using worked-examples as a 'window' into student's mind and directions for future research. Examples could be a research tool for looking inside learners' mind and learning about their understanding of mathematics (Zazkis, 2007). Transferring the responsibility of generating examples and worked-examples to learners and responsibility of learning from their range, sample space, and explanations how to better use them in the teaching of mathematics to researcher becomes the focus of my further study.

Future questions and tasks for the researcher: Can the size of the intersection of a personal example space with the conventional example space explain the understanding of a mathematical concept? How can the personal example space be interpreted? Will examples, with the work involved in producing them, the reflection incorporated in the final production, and the reasons given for the steps in their work support the cognitive processing necessary for inducing change in the long-term memory? Will knowing more about how people learn help better decisions on whether or not using worked-examples and learner generated examples in teaching? How to sequence the worked-examples to learn from? Where and when will be better to incorporate examples in teaching? How to educate learner's perception about what makes a good example?

There is a good body of research on the role of examples on new concepts and skills acquisition. Further research is necessary to unveil the role of worked-examples in conceptual understanding.

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ENACTIVE COGNITION AND IMAGE-BASED REASONING IN GEOMETRY

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The purpose of this paper is (1) to introduce the lemniscus model for enactive cognition, (2) to apply the lemniscus model to image-based reasoning in geometry, and (3) to suggest ways in which these insights may improve the teaching of geometry in high schools. The discussion will clarify the relationship in enactive cognition between action and perception: external reality understood in terms of scientific materialism is incommensurable with subjective experience. Image-based reasoning in geometry may be regarded as a structured autonomous subsystem of the lemniscus. There may be metacognitive implications for the teaching of geometry. Ontogenetic implications of the structure may also suggest ways to increase the effectiveness of geometry teaching. It is hypothesized that fluency in the process of geometric reasoning is a more important goal than accuracy.

Any academic discipline requires a theoretical framework, a firm foundation to ground its empirical investigations. I have attempted in this paper to take some steps in that direction with regard to mathematics education. In the development of theory, however, we should not lose sight of the ultimate goal of the field of mathematics education, which is to understand how mathematics is learned and to improve the teaching of mathematics. With this in mind, the theoretical framework is made specific to geometrical reasoning, and concrete ways are suggested in which it can be applied to improve the teaching of geometry. The resulting interdisciplinary approach, involving philosophy, educational neuroscience, and mathematics education is consistent with the goals of PME-NA.

The last two sections of this paper build on previous work by the author in devising a model for image-based reasoning in geometry and in suggesting ways in which this model can inform the teaching of geometry. The final section, on the teaching of geometry, indicates some hypotheses that may be tested by empirical research. The bulk of the present paper, and its major contribution, is an elucidation of the lemniscus model for enactive cognition. The discussion attempts to clarify the roles of the subjective and objective perspectives and suggests a direction for epistemological analysis in the theory of enactive cognition.

Literature

The theory of enactive cognition was first formulated in Varela, Thompson, and Rosch (1991). However, the present paper endorses a radical interpretation of enactive cognition, as discussed in Campbell (1993), Campbell (2001), Handscomb (2007), and Campbell and Handscomb (2007). The present paper will offer additional clarification and analysis of the theory of enactive cognition, especially with a view to its application as a theoretical framework for the image-based reasoning in geometry discussed in Handscomb (2005) and Handscomb (2006). The assumption of transcendental realism makes reference to Schopenhauer (1847/1997, 1859/1969), Whitehead (1929/1981), and Brown (2002). The notion of organism draws on Maturana and Varela (1980), although the interpretation herein is broader. Whitehead (1925/1967) clarifies that the world of scientific materialism is a high-level abstraction, and Whitehead (1919/2007) demonstrates how scientific materialism can be constructed from subjective experience. Reference is made to James' (1890/1950) notion of the specious present and van Hiele's (1986) developmental theory for geometrical reasoning.

The Lemniscus Model of Enactive Cognition

Realism is assumed. In other words, there is an external material world with which organisms like us interact. Our perception of the external world depends on our interaction with the external world. However, these interactions are constrained by the perceptual apparatus available to us. In other words, the world of perception comes into being from the interaction of an observer with the external world. The discussion does not assume a naïve realism, but rather a transcendental realism: the world exists, but not necessarily in the way that our perception shows it to us. With certain qualifications with regard to the distinction between transcendental *idealism* and transcendental *realism*, Berkeley, Kant, and Schopenhauer all espoused variations of this doctrine.

Typically, of course, the organisms that interact with the world will be human beings. However, the analysis is applicable more broadly to the autopoietic unities of Maturana and Varela (1980), and beyond that to more general partitions of the world. By viewing the theory in its greatest generality, its essential elements are thrown into relief. The term "organism" will be used even with regard to the most general sense of a world partition.

An "organism" consists of a collection of material objects; the remaining objects, constituting the rest of the material world, belong to the "complement" of the organism in the world. Typically, an organism will have a fairly well-defined boundary in the macroscopic sense, although the boundary may be less well defined on the microscopic level. In fact, the world and the organism may be so tightly enfolded that their boundary has fractal characteristics. Interaction between material objects belonging to the organism will be referred to as endogenous activity; interaction between material objects belonging to the organism and material objects in the organism's complement will be referred to as exogenous activity.

Just as the activity of an organism consists both of endogenous and exogenous interactions, so does the activity of its complement. Therefore, the complement can be viewed as having its own endogenous activity as well as exogenous interactions with the organism. The exogenous activity of the organism and the exogenous activity of the complement fit each other like a hand inside a glove—as the hand moves, so does the glove; as the glove moves, so does the hand. Endogenous activity alters the stance of the organism toward the complement, leading to exogenous activity that affects the complement; likewise, endogenous activity alters the stance of the complement toward the organism, leading to exogenous activity that affects the organism. Of course, endogenous activity either of organism or complement does not necessarily lead to exogenous activity, although in practice their deeply enfolded boundary acts to minimize purely endogenous activity. The essential points of this discussion can be represented in the diagram below.

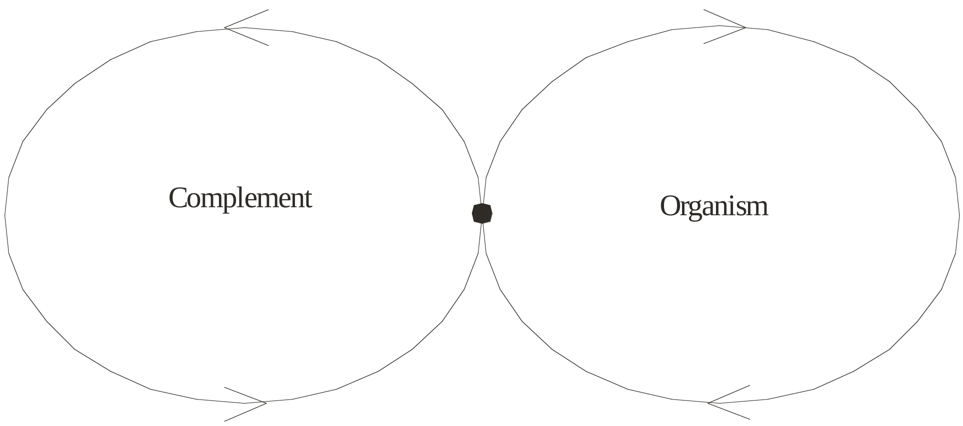

Figure 1: Lemniscus model.

Strictly speaking, the two tangential ellipses do not constitute a mathematical lemniscus, but the term lemniscus will be used because of its evocative nature with respect to the infinity symbol, introduced to mathematics by John Wallis in 1655.

For the time being, it will be assumed that activity takes place in the mechanistic world of scientific materialism, in which material objects move and interact in four-dimensional space-time. The arrows provisionally represent the flow of causal efficacy, although probabilistic quantum effects need not be discounted. Thus the cycle around the organism represents endogenous causal activity of the organism, whereas the cycle around the complement represents the flow of endogenous causal activity in the complement. However, physical causation also flows from the bottom left to the top right and from the bottom right to the top left, and these flows represent exogenous interaction between the organism and its complement. It should be noted that the lemniscus represents the totality of activity, which is continuous, and that the lemniscus evolves as a whole.

Varela et al (1991) originally formulated the theory of enactive cognition. A radical interpretation of enactive cognition was proposed in Campbell (1993), and has been developed in Campbell (2001), Handscomb (2007), and Campbell and Handscomb (2007). The first principle of the radical interpretation is that cognition *is* the interaction between the organism and the world in which it is embedded. In terms of the flow of activity represented in Figure 1, cognition of the organism is constituted of both its exogenous activity and its endogenous activity. With respect to the human organism, and the most characteristically human aspects of cognition, the most significant endogenous activity is electrophysiological activity of the brain.

Varela et al (1991) offer various insights to support the equivalence of exogenous activity and cognition. The equivalence of endogenous activity and cognition, is significantly substantiated by the work of the cognitive neuroscientists on determining neurophysiological correlates for cognitive functions. This evidence is discussed at length in Campbell and Handscomb (2007). As we shall see, however, the equivalence of action and cognition is a position that must be adopted with care.

Another term for cognition that emphasizes subjective lived experience is perception. Perception, therefore, is the subjective correlate of the objective interaction of material objects. Perception and the interaction of the organism within the world are one and the same thing from different perspectives. Perception is even used in this sense for reasoning, imaging, and feeling, which are correlates of endogenous activity.

Schopenhauer (1859/1969) argues that the understanding projects outward, or objectifies, the external world of perception. Brown (2002) contends that perceptual representations are the final stage of a cognitive process that evolves from within the organism, rather than a reconstruction from afferent sensual data. Perception flows into a sensual mould, according to Brown, and may be regarded as an objectification of an internal process. There is a sense that perception has a vector quality in a direction opposite to that of afferent sensual activity. In the flow of Figure 1, the activity arrows from the bottom left to the top right would be reversed from the perspective of perception. Likewise, from the subjective viewpoint of the complement, the arrows from the bottom right to the top left would be reversed. The correlative perceptual diagram, therefore, would be identical with Figure 1, except that all the arrows would be reversed. (It should be remembered that the arrows in Figure 1 do not represent the flow of time, but the flow of causal efficacy.)

It is important to recognize that action and perception are one and the same thing, and that there is no question of causality, in the sense of perception causing action or vice versa. Perception and action are two sides of the same coin, two perspectives on the same ontological substrate. Metaphorically, these two perspectives can be thought of as two ways of reading the same set of letters. Thus, MAY is a month or a vegetable, depending on the point of view, and this is how the dual nature of the lemniscus should be understood.

The lemniscus model implies that the organism has a well-defined boundary, or at least that endogenous activity can be distinguished clearly from exogenous activity. But this is not necessarily true. As mentioned above, the organism and its complement may be so tightly enfolded that their boundary has a fractal nature. In that case, distinguishing endogenous activity from exogenous activity will be difficult. It may be better to think of the organism as characterized by a density of activity that attenuates through a fuzzy "boundary." In this case, there would be a continuum between endogenous activity and exogenous activity, with no clear division between the two. In this respect, the lemniscus model is a conceptual simplification.

All experience is essentially subjective. This was Berkeley's idea, although Aristotle had come to much the same conclusion two thousand years previously, when he wrote, "In a certain sense the mind is all that exists" (De anima, III, 8; cited in Schopenhauer, 1847/1997, p. 210). Kant took up Berkeley's idea, giving us his "Copernican revolution" that resituated philosophical investigation in the knowing subject. Whitehead (1929/1981) starkly claimed, "Apart from the experiences of subjects there is nothing, nothing, nothing, bare nothingness" (p. 18). The extremity of Whitehead's view is mitigated in the present article by a transcendental realism, but at least it can be agreed that the experiences of subjects are all that can be *known*.

Whitehead (e.g., 1925/1967) argues that objective four-dimensional space-time, either of Newton's mechanics or Einstein's general relativity, within which matter and energy interact, is a high-level abstraction. The rationalistic revolution of the Enlightenment overthrew one dogma, that of religion, and replaced it with another dogma, that of scientific materialism, according to Whitehead, and the scientists have subsequently been reluctant to accept alternative world views. Whitehead penned this view in the early twentieth century, but have things changed much? Cognitive neuroscience purports to investigate cognition, but ignores the most immediate evidence of cognition—first-person lived experience, which cannot be reduced to the mechanisms of scientific materialism. Whitehead made the point that scientific materialism has taken as its irrefutable, unquestioned foundation a set of concepts that are high-level abstractions from the fundamental reality of first-person lived experience. And these derivative concepts include the four-dimensional space-time within which material objects move and interact in a mechanistic fashion.

However, given the primacy of first-person lived experience, careful considerations need to be made with respect to scientific materialism. For example, the incommensurability of subjective lived experience and the objective world of scientific materialism is obvious from their respective notions of time. The conceptual construct that is objective time is indefinitely divisible, and isomorphic to the real number line. Subjective time, on the other hand, is not indefinitely divisible, but proceeds from duration to duration, James' (1890/1950) specious present, with one duration shading into the next. In this respect, the two aspects of the lemniscus model do not fit together perfectly, they evolve in fundamentally different ways. The equivalence of action and perception is a principle that should be adopted with care.

The initial assumption was realist, with an independent objective reality of material objects in interaction. Is it necessary to reject this view, to accept that subjective experience is the *only* reality? This path ultimately leads into solipsism and despair. The notion of an objective material reality has great utility, even if this objective reality can only be known conceptually, and cannot be known directly in perception.

It is interesting to compare the Greek discovery of the incommensurability of the square root of two, reputedly by Hippasus, who subsequently was executed for heresy by his fellow

Pythagoreans. It was more than two thousand years after Hippasus' martyrdom before numbers were constructed that could adequately measure the undeniable reality of the diagonal of the square. Does not the undeniable reality of first-person lived experience need a constructed objective reality that is commensurable with it? Scientific materialism will not do. Whitehead (1919/2007) demonstrated the construction of scientific materialism from subjective experience. Perhaps there are other ways to construct an objective reality that will better fit subjective experience. But now we are drifting too far from education. It is time to apply the theoretical model to image-based reasoning in geometry.

Image-based Reasoning in Geometry

The theoretical discussion has been very abstract and general. However, the goal was to devise a framework within which to situate the image-based reasoning in geometry discussed in Handscomb (2005, 2006). By image-based reasoning in geometry is meant geometric reasoning in which inferences are allowed, and even necessary, from the geometry diagram. It is contrasted with formal, axiomatic geometry, in which inferences from the diagram are forbidden. Image-based reasoning in geometry is what is actually required of high school students, at least in North America and the United Kingdom.

The geometer interacts with a geometry diagram. The lemniscus represents the totality of an organism's interaction with its complement in the world within which the organism is embedded. The interaction of the geometer with the geometry diagram will be treated as an autonomous subsystem of the totality of this interaction.

The endogenous activity will consist of the neurophysiological activity of the brain that is correlative to reasoning functions. There are two types of exogenous activity relative to the geometry diagram. Firstly, afferent activity from the diagram to the geometer corresponds to perceptual projection, as understood by Schopenhauer (1859/1969) and Brown (2002), and may be regarded as instantiation of the geometrical diagram by the geometer. Secondly, efferent activity from the geometer to the diagram corresponds to conceptual recognition of the diagram by the geometer, and may be regarded as conceptualization of aspects of the geometry diagram.

The assumption that geometrical reasoning is an autonomous subsystem is an approximation. Endogenous non-reasoning functions, such as affect, clearly impact the reasoning process in humans. Likewise, the geometer typically will instantiate and conceptualize various images and ideas, concurrently with the geometrical reasoning process, that are irrelevant to the geometry. The impact of these extraneous factors on geometrical reasoning, both endogenous and exogenous, would be an interesting study in itself. As a first approximation, however, they must be discounted.

Handscomb (2007) pointed out the similarity between Spinoza's theory of mind and the theory of enactive cognition. Spinoza's categories of knowledge can inform enactive cognition and suggest an epistemological analysis for enactive cognition. Within the context of the lemniscus, Spinoza's *imagination* can be interpreted as knowledge correlative to exogenous activity, the ways in which the organism and its complement physically impact each other. Spinoza's *reason* may be regarded as knowledge correlative to endogenous activity. How does Spinoza's third, and most mysterious category of knowledge, *intuition*, fit within the lemniscus framework? I suggest that there is an evolutionary process whereby one lemniscus state transforms into the next, from one perceptual/conceptual duration to the next. Intuition may be identified as the *structure* of this evolutionary process.

The great task in the study of image-based reasoning in geometry is to clarify the structure of the evolution of the lemniscus states that corresponds to the autonomous subsystem that is image-based reasoning in geometry. This was the idea behind Handscomb

(2005, 2006), which identified instantiation, two types of conceptualization, and five principles that together comprise a logic for image-based reasoning in geometry. There is no space herein to repeat the details of the argument. However, instantiation and conceptualization have already been identified with respect to the lemniscus, Principles 1-3 are correlative to endogenous activity, and in Principles 4 and 5 exogenous activity alters the geometrical image while the endogenous activity remains relatively constant. This model was arrived at by means of observation and introspection. A better understanding of the evolutionary process of neurophysiological activity will allow the reasoning structure to be determined with more accuracy and certainty.

Just as identification of image-based reasoning in geometry as an autonomous subsystem of the lemniscus is an approximation, analysis of the evolution of this subsystem into discrete quanta is also an approximation. All parts of the lemniscus are evolving continuously in a holistic fashion, so that endogenous activity, is always accompanied by exogenous activity, and vice versa. Nevertheless, an analysis such as that of Handscomb (2005, 2006) may be useful for determining effective ways to teach image-based reasoning in geometry.

Geometry Education

The discussion so far has addressed the notion that arguments in image-based reasoning in geometry are structured. Some suggestions were made concerning the nature of this structure (Handscomb, 2005, 2006). The task now is to demonstrate how this is relevant to mathematics education.

Firstly, as Handscomb (2005) points out, image-based reasoning in geometry is the goal of high school education in geometry, rather than formal axiomatic geometry. Given that arguments in image-based reasoning in geometry must inevitably proceed according to certain patterns and structures, the teaching of geometry should facilitate students' thinking processes in flowing according to these same patterns and structures. If students are made aware of the elements that comprise an argument in image-based reasoning in geometry, then it is reasonable to suppose that their own metacognition could facilitate their reasoning. This hypothesis can be tested by means of classroom research once theoretical work on the structure of image-based reasoning in geometry has been completed.

The question arises concerning the order in which the elements of the reasoning process should be taught. Handscomb (2005) makes a close comparison between the proposed model for image-based reasoning in geometry and the van Hiele levels (e.g., van Hiele, 1986). There are significant similarities between the two structures, indicating that there may be a developmental dimension to the model for image-based reasoning in geometry that replicates aspects of the developmental nature of the van Hiele levels. Further research in neurophysiological ontogenesis may help to clarify developmental channels. A developmental dimension indicates that there may be a preferred order in which the elements of the structure for image-based reasoning in geometry are taught.

Ontogenesis aside, Handscomb (2005) notes that the structured process of image-based reasoning in geometry depends crucially upon awareness of the permitted conceptualizations from the diagram. These *schematic* conceptualizations need to be taught early in a student's learning of image-based reasoning in geometry.

The final point to be made is the hypothesis that it is more important for students to be fluent in the structured process of image-based reasoning in geometry than it is for their arguments to be correct. Exactitude can follow later. Pedagogical goals should emphasize fluency before exactitude as a more efficient way for students to learn geometrical reasoning than the converse.

This final section of the paper consists primarily in suggestions for future directions of research, building on the theory outlined in the earlier part of the paper. I hope that these applications, together with their theoretical underpinning, have adequately demonstrated an interdisciplinary approach that is consistent with the goals of PME-NA.

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Mediated Successive Refinement: a Pedagogical Tool for Understanding Mathematical Structures

Background and objectives

In this study, I will first introduce mediated successive refinement, a method that can be used as a pedagogical tool as well as a research tool. Subsequently, I will illustrate how I used this method to examine students' understanding of a combinatorial structure. I will also describe some of the pedagogical advantages of using this method.

This method is inspired by the use of learner-generated examples in the classroom. Watson and Mason (2004) discuss many benefits of using learner-generated examples, among which are promoting reflection on the concept, encouraging creative thought, and helping learner to reason and communicate their understanding in more depth.

In mediated successive refinement, learners are encouraged to generate an example of a situation or a problem whose answer is a particular mathematical structure. Afterward, the problems are gathered as a problem set and redistributed to them. Students are then asked to solve the problems, and reflect on them. In the final step, their responses are gathered and discussed again in the class.

Theoretical framework

Examples have a fundamental role in teaching and learning mathematics. Teachers use examples in mathematics classrooms to help students understand and explore different topics. When learners are invited by the teacher to construct their own examples, it helps them to think about the topic in a different way and it gives them an opportunity to gain a new understanding of the underlying concept.

Dyrszlag (1984), a Polish mathematics educator, has developed a list of "sixty-three abilities grouped in twelve blocks that would account for good understanding" (Sierpinska, 1994, p.118). One of these twelve blocks is dedicated to examples and one is dedicated to non-examples. In analyzing the data, I have considered some of the factors from these two blocks to examine students' understanding of the underlying concept. Since my study was concentrated on combinatorics, I have modified the factors to fit the purpose. Some of these factors are listed below.

Students' ability:

- to give one or more examples of a given combinatorial structure;
- to identify, in a given set of examples, those that are examples of a given combinatorial structure;
- to change a non-example so that an example is obtained;
- to find some common features between designates of different combinatorial configurations;
- to identify the factors that make combinatorial structures different from each other;
- to find and correct a given mistake;
- to identify the relation that accounts for an isomorphism between two combinatorial structures;
- to identify the conditions that make the combinatorial structures non-isomorphic.

Possessing these abilities are by no means sufficient for understanding combinatorics, but they are necessary. By using the method of mediated successive refinement, I was enabled to observe existence or lack of some of the above factors in their responses.

Modes of enquiry and data source

The participants in this study are liberal arts students enrolled in an undergraduate mathematics course called Finite Mathematics. Counting methods and elementary combinatorics is a major component of this course. The students were asked to solve the following problem: "We have 8 people. In how many ways can we choose 2 different groups, each consisting of 3 people?" This question created an interesting dynamic in the class. A few students answered by $2 \times C(8,3)$. When I asked for the reason some replied: "because we want to choose 3 people out of 8, it is $C(8,3)$ and since we are doing this twice (for two groups), it will be $2 \times C(8,3)$." In fact, most students in the class were convinced by this solution. I was inspired to perform this study because of the incorrect but "convincing" answer to this standard combinatorial problem.

The task was for students to design a combinatorial problem whose solution is $2 \times C(8,3)$ or equivalently $C(8,3) + C(8,3)$. My goal for this task was to invite students to think about a particular non-standard combinatorial structure, which was created as a solution to a standard problem by their peers or themselves. After collecting students' responses to the first task, I composed a questionnaire using those responses. For the second task, I asked students to solve and reflect on the problems that were collected from the first task. The goal of the second task was to familiarize them with their collective responses, and to encourage them to reflect on examples that were created by their peers. The data was gathered from the students' written responses to these tasks.

Results and discussion

The major trend in the responses to the first task was problems that could be solved using $C(8,3) \times C(8,3)$ or $C(8,3)^2$. In fact, out of 12 responses, 7 were structurally isomorphic. In the following excerpt, you can read one of these responses to the first task.

Example 1. George wants to go to Mexico. He needs to buy 3 pieces of shirts and pants. He decides to go to The Bay[1] to buy these clothes. At The Bay, he goes to Guess clothing section. The sales person tells him there are 8 types of each. But George only needs 3 pieces. How many possibilities are there for George to choose 3 out of 8 pieces of shirts and 3 pieces out of 8 pieces of pants?

This task revealed that the majority of students could not distinguish between $C(8,3) \times C(8,3)$ and $C(8,3) + C(8,3)$ or $C(8,3)^2$ and $2 \times C(8,3)$.

Two of the students imposed the combinatorial structure through the language of the problem. For example, one of the students responded:

Example 2. What is the sum of choosing 3 people out of 8 people, repetition allowed?

The word "sum" is enforced to ensure the creation of the combinatorial structure $C(8,3) + C(8,3)$.

From the 12 participants, only 3 had a correct combinatorial structure for their examples. One of the correct answers was

Example 3. Student council elections are coming up. There are 2 people running for president. After the president has been chosen, he or she must select 3 councilmen from a group of 8 people who want to be on the council. What is the total number of possible student council arrangements?

This student actually continues to solve her own example. She continues: "You have 2 choices for president and need to choose one so $C(2,1)$, you have 8 choices for councilmen and need to choose 3, so $C(8,3)$. Now we get the total number of choices multiply, so $2 \times C(8,3)$."

[1] The Bay is a Department store in Canada

From this response we can see that the student understands the combinatorial structure $2\times C(8,3)$ and can create an example that represents this structure.

This example construction activity reveals students' weaknesses in understanding this combinatorial structure in particular. However, the scope of this finding extends beyond this particular combinatorial structure. One of the important issues that came up through this study was the fact that students had a hard time constructing examples with additive structures such as $C(8,3)+C(8,3)$. The majority of students' examples had the combinatorial structure $C(8,3)\times C(8,3)$. This shows that students' did not recognize the difference between the additive structure of $C(8,3)+C(8,3)$ and multiplicative structure of $C(8,3)\times C(8,3)$. Some students attempted to construct the examples and were not successful. Therefore, they attempted to use words such as "sum" to artificially impose the combinatorial structure of the task.

In the second task, all students were invited to solve the problems they had generated. In addition to solving the problems, they were asked to determine if the answer to the problem was $2\times C(8,3)$. If yes, they were asked to explain, and if not, they were asked to modify the problem so its solution is $2\times C(8,3)$. One of results was that 10 out of 12 students correctly identified that example 1 (George's dilemma for choosing pant and shirts) could not have been solved by $2\times C(8,3)$, but it should have been solved by $C(8,3)\times C(8,3)$. From the ten participants, two didn't modify the problem, and two modified it incorrectly. Six people modified that problem with the correct structure.

Implications for teaching

According to Batanero et al (1997), one of the important aspects in research in the domain of teaching and learning combinatorics is understanding students' difficulties in this field and identifying variables that might influence this difficulty. By using mediated successive refinement, I encouraged learners to reflect on different aspects of their own understanding of a particular structure and its relation to other structures that they had encountered previously. In addition, I provided them with an opportunity to observe and reflect on the examples that were generated by their peers. I also utilized their responses to identify some of the difficulties and challenges that learners face in understanding combinatorial structures.

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CONSTRAINED BY KNOWLEDGE: THE CASE OF INFINITE PING-PONG BALLS

This report is part of a broader study that investigates university students' resolutions to paradoxes regarding infinity. It examines two mathematics educators' conceptions of infinity by means of their engagement with a well-known paradox: the ping-pong ball conundrum. Their efforts to resolve the paradox, as well as a variant of it, invoked instances of cognitive conflict. In one instance, it was the naïve conception of infinity as inexhaustible that conflicted with the formal resolution. However, in another case, expert knowledge resulted in confusion.

The ping-pong ball conundrum (Burger and Starbird, 2000) is one of many well-known paradoxes that illustrate the counter-intuitive nature of infinity. The question of what happens to an infinite iteration once the process is complete continues to challenge both naïve conceptions of infinity as well as expert ones. This report examines two experts' responses to the ping-pong ball conundrum as well as a variation of it.

Theoretical Background

The counterintuitive nature of infinity, as manifested in students' reasoning, is described in prior research (see among others, Dreyfus and Tsamir 2004; Fischbein 2001; Fischbein, Tirosh, and Hess, 1979; Tall, 2001). Tirosh (1991) noted that there are conflicts between a student's intuitive knowledge of infinity, which generally corresponds to potential infinity, and the formal theory, which comprises actual infinity. As learners are confronted with properties of actual infinity, many go through a state of disequilibrium, or cognitive conflict.

Piaget (1985) described the development of cognitive structures as a cycle that progresses from one stage of equilibrium to another. He referred to equilibrium as the coherence or balance maintained (and sought) by an individual as he or she attempts to construct knowledge. When a learner is confronted with information that is inconsistent with his or her prior knowledge, he or she is said to be in a state of disequilibrium, or cognitive conflict. The transition from cognitive conflict to a new equilibrium compels a learner to refine his or her understanding in order to integrate the new knowledge. Implementing a cognitive conflict framework with regard to learners' conceptions of infinity has been described in mathematics education literature (Tall, 1977; Tsamir and Tirosh, 1999).

In this paper, I examine two cases where cognitive conflict was invoked when participants attempted to resolve paradoxes concerning actual infinity. In one case, the solution to the ping-pong ball conundrum (PP) conflicted with one participant's naïve conception of infinity as inexhaustible. In the other case, it was familiarity with properties of actual infinity that impeded this participant's resolution of a variant to the ping-pong ball paradox (PV).

Setting and Methodology

The participants of this study were two mathematics teachers, Kenny and Eric. Kenny, a high school teacher, had recently completed a Master's in applied mathematics. Eric, who held a Master's in mathematics education, taught Cantor's theory to preservice teachers, including properties of one-to-one correspondences. Data was collected from an interview with the participants, who were asked the following:

PP: You have an infinite set of numbered ping-pong balls and a very large barrel; you are about to embark on an experiment that will last exactly 1 minute. Your task is to place the first 10 balls into the barrel and then remove number 1 in 30 seconds. In half of the remaining time, you place balls 11 – 20 into the barrel and remove number 2. Continue ad infinitum. After 60 seconds, at the end of the experiment, how many ping-pong balls are in the barrel?

PV: Rather than removing the balls in order, at the first interval remove ball 1; at the second, ball 11; at the third, ball 21; and so on. How many balls remain in the barrel?

Results and Analysis

The mathematical resolution to the ping-pong ball conundrum lies in the distinction between potential and actual infinity. The process of putting in and taking out ping-pong balls goes on infinitely. However, after the 60 seconds end, so does the process. Since the balls are removed in order, at the end of the experiment every ball will have been taken out at some point. As such, the barrel ends up empty. Conversely, in the variation although infinitely many balls are taken out, the balls numbered 2 – 10, 12 – 20, 22 – 30, and so on, are never removed. Thus, infinitely many balls remain in the barrel.

Common responses to the ping-pong paradox include the argument that since the rate of in-going balls is greater than the rate of out-going balls, the barrel must contain infinitely many balls at the end of the experiment (Mamolo & Zazkis, 2007). This was also Kenny's initial answer. He connected the concept of infinity with on-going, and had difficulty accepting the argument that the one-to-one correspondences between sets of in-going balls, out-going balls, and time intervals, guaranteed the barrel would end up empty. After some discussion, Kenny reflected that "if you don't think about one-to-one correspondences, the instinct is there are 9 left every time you take one out, so it's 9 infinity." The resolution to the ping-pong variant came much more easily to Kenny, who readily acknowledged there would be balls left in the barrel – although his instinct was that there would be a "bigger" infinity of balls remaining in, than removed from, the barrel.

Eric, who was familiar with Cantor's theory, resolved PP immediately. However, his knowledge of one-to-one correspondences turned out to be an obstacle to his resolution of PV. Unlike Kenny who attended to the rates of in going and out going balls, Eric recognized the one-to-one correspondences in both PP and PV. He concluded that the variant and the "ordered case" should be the same, arguing that "after you go [remove] 1, 11, 21, 31, … 91, etc, you go back to 2." Eric's knowledge of infinite cardinals contributed to his "strong leaning to Cantor's theorem," and although he insisted "at some point we'll get back to 2," he could not justify the claim. During the interview, Eric noticed the conflict between his prior knowledge and the solution to PV, stating "if ball number 2 is there, so is 2 to 10, etc… so, infinite balls there? I have trouble with that." He went on to observe that while "on one hand $\infty - \infty = 0$, on the other it's ∞." After more discussion, Eric recognized the difference between PP and PV, conceding that he was now "convinced" of the solution.

Conclusion

It has been well established that when formal notions are counterintuitive, primary, inaccurate intuitions tend to persist (see among others Fischbein et al., 1979). When a learner recognizes the discrepancy between his or her prior understanding and the new knowledge, he or she is said to be in a state of cognitive conflict. In the case of Kenny, cognitive conflict was invoked when he recognized the inconsistencies between his intuition of infinity and the resolution of PP. Conversely, with Eric, a state of cognitive conflict was invoked when new knowledge was consistent with a naïve interpretation but inconsistent with his expert approach. Eric's understanding of PV was constrained by his knowledge of Cantor's theory, and might have been influenced by the sequence of the tasks. Since the ping-pong variation (PV) was presented immediately after the original paradox (PP), Eric's mindset toward PV might have been swayed by his engagement in the previous task. Future research will attend to the constraints familiarity with formal knowledge might impose on a learner's attempts to make sense of infinity.

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MULTI-LAYERS OF NUMERACY

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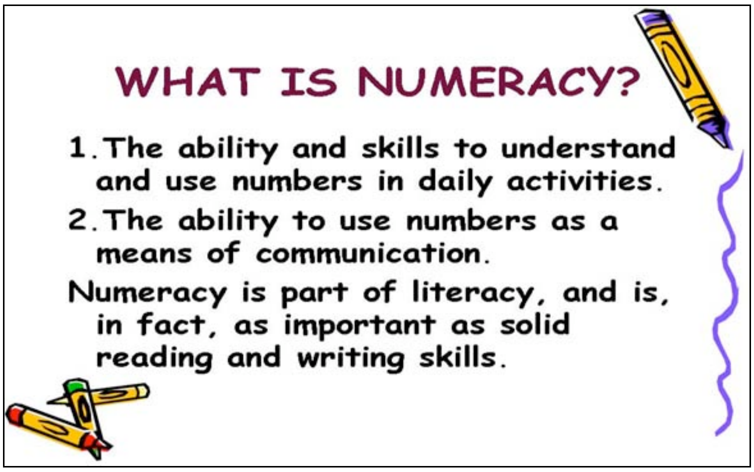
Numeracy means different things to different people. In this paper I will argue that numeracy is multi-layered that requires the ability to integrate mathematics, situated and contextual problem solving, and communication skills. Numeracy is multi-layered: it is an entity, an embodied disposition, a language, a practice, and a cultural activity. Current school reform initiatives acknowledge the importance of connecting school mathematics with students' own experiences in social and cultural contexts. Familiar contexts can make mathematics more accessible to those who have been alienated from it.

Numeracy the Entity (Definition)

Various definitions of numeracy share an emphasis not on mastering specific mathematical content but on the ability to use mathematical knowledge and problem solve. The BC Ministry of Education, for example, has adapted the following definition:

> Numeracy can be defined as the combination of mathematical knowledge, problem solving and communication skills required by all persons to function successfully within our technological world. Numeracy is more than knowing about numbers and number operations. (BCAMT, 1998, p.1)

According to the BCAMT numeracy involves the functional, social, and cultural dimensions of mathematics. Numeracy is the set of math skills needed for one's daily functioning in the home, the workplace, and the community. Beyond daily living skills, numeracy is now being defined as the mathematical knowledge that empowers citizens for life in their particular society (Bishop, 1993). ABC CANADA Literacy Foundation (2005) defines numeracy on its web site as follows:



Source: http://www.abccanada.org/math_literacy/

Figure 1. What is Numeracy?

Such a definition is limiting as it does not take into consideration spatial perception, symbolic reasoning, or graphical representation. In order for today's students to be prepared to succeed as productive members of a society that is profoundly influenced by technology and mathematics it is essential that they have some competence in the area of mathematics. Numeracy is not a case where either one is proficient or not, rather individuals' skills are a continuum of different purposes and levels of accomplishment with numbers.

Hughes, Desforges, Mitchell, & Carre (2000) consider numeracy as more about the ability to use and apply rather than just knowing. Evans (2000) regards numeracy as the ability to process, interpret and communicate numerical, quantitative, spatial, statistical, even mathematical, information, in ways that are appropriate for a variety of contexts, and that will enable a typical member of the culture or subculture to participate effectively in activities that they value.

The International Life Skills Survey defines *quantitative literacy* in a comprehensive manner as follows:

> An aggregate of skills, knowledge, beliefs, dispositions, habits of mind, communication capabilities, and problem solving skills that people need in order to engage effectively in quantitative situations arising in life and work. (ILSS, 2000)

The OECD Programme for International Student Assessment adopts a similar definition, but calls numeracy *mathematical literacy*:

> An individual's capacity to identify and understand the role that mathematics plays in the world, to make well-founded mathematical judgments and engage in mathematics in ways that meet the needs of that individual's current and future life as a constructive, concerned and reflective citizen. (PISA, 2000)

The Mathematics Council of the Alberta Teachers' Association (2004) and other organizations have often interchanged the terms *mathematical literacy* and *numeracy*. The term numeracy tends to be more commonly used in the United Kingdom, Australia, and Europe, while the term mathematical literacy is more commonly used in North America and South Africa (Clarke, Cheeseman, Sullivan, & Clarke, 2000; Wright, Martland, Stafford, & Stanger, 2002). Generating a concise but complete definition for either term is a difficult task. "Numeracy is not the same as mathematics, nor is it an alternative to mathematics. Rather, it is an equal and supporting partner in helping students learn to cope with quantitative demands of modern society" (Steen, 1990, p.115). In some definitions, numeracy is seen as contextualized and situated, and takes into account one's culture, the culture of others, and daily living. Consideration must be given to real-life situations, be it generative computational skills, interpretive skills, or decision making skills (Gal, 2000). Steen (2001) identifies five different dimensions of numeracy, depending on the situation:

- Practical - for immediate use in the routine tasks of life;
- Civic - to understand major public policy issues;

- Professional - to provide skills necessary for employment;
- Recreational - to appreciate games, sports, lotteries;
- Cultural - as part of the tapestry of civilization.

Regardless of the source and difference, however, there exists a common theme in all the definitions, namely, the ability to use, interpret, and communicate mathematics in context. Simply put, numeracy is ‘Math in Action’.

Numeracy the Embodied Disposition (Being Numerate)

What does it mean for a person to be numerate? Withnall (1995) states that a numerate person is one who possesses the math skills required for everyday functioning in the home, the workplace, and the community. However, such skills are “not a fixed entity to be earned and possessed once and for all” (Steen, 1990, p. 214). The Queensland School Curriculum Council (2001) numeracy position paper defines a numerate person as one who is confident in using her own and others’ mathematical know-how to deal with everyday situations. The British Columbia Association of Mathematics Teachers in its brochure on numeracy, suggests that a numerate individual should possess a variety of mathematical skills, knowledge, attitudes, abilities, understanding, intuition, and experience, all of which could be expressed as the number sense, the spatial sense, the statistical sense, and the sense of relationship ((BCAMT, 1998). According to the Australian Association of Mathematics Teachers (1997), to be numerate is to use mathematics effectively to meet the general demands of life at home, paid work, and participation in community and civic life. Johnston (1994) argues,

> to be numerate is more than being able to manipulate numbers, or even being able to succeed in school or university mathematics. Numeracy is a critical awareness which builds bridges between mathematics and the real world, with all its diversity… In this sense…there is no particular level of mathematics associated with [numeracy]: it is as important for an engineer to be numerate as it is for a primary school child, a parent, a car driver or a gardener. The different contexts will require different mathematics to be activated and engaged in” (p. 34).

A numerate person does not need to possess or lack a set of discrete skills, but have the ‘practices and dispositions’ or ‘skills and abilities’ perceived to be needed to meet the numeracy demands of a situation.

In the *BCAMT K-12 Survey*, 99% of the respondents agreed or strongly agreed that numeracy should be valued as a necessary lifelong skill. Though there was some disagreement about the minimal attributes of a numerate person, all agreed that a numerate person should be able to solve problems and confidently negotiate through necessary life skills (BCAMT, 2004). Such skills are usually driven by issues that are important to one’s place of residence and employment.

Numeracy the Practice

Cajete (1999, p.162-179) suggests a model with multidimensional approaches that is centered with creativity as a learning process, and with the perspective of science as a cultural system of knowledge. Such work can also be extrapolated to mathematics education.

The following curriculum model is adapted from Cajete (1999) recognizes and provides for the integration of the intuitive and rational thought processes.

1. An understanding and application of the metaphoric thought process. Metaphoric thinking is closely linked with the process of imaging in creativity. Metaphors allow for expansion and elaboration of creative insights such as synthesis, intuition, and the process of relationships.

2. The understanding and application of appropriate strategies that address the brain patterned learning styles of students. Characteristics and potential of the “whole” brain need to be addressed. The mutual and reciprocal interrelationship between the right and left brain processes needs to be reinforced.

3. Teaching for creativity in science and mathematics by exposing students to creative problem- solving techniques and facilitating their awareness of their own creative abilities and potential.

4. The development and application of situational learning contexts where there is a specific interface between science, mathematics and culture. The identifying of mathematics with the cultural identity of the student is a basic intent of this curriculum approach. Contexts, situations and phenomena in the immediate environment, the home, community or school are all sources for curriculum content.

5. The facilitation of opportunities for student growth and development in their abilities to deal with and adapt to changing environmental influences by setting up a scientifically challenging situation that stimulates creative problem-solving. Establishing learning situations that are experientially based and help students develop their inquiry skills

6. An understanding and application of interdisciplinary perspectives concerning science, culture, and creativity. Activities involving drawing, construction, or artistic exemplification of science or math concepts allow for a fuller expression of culturally-related ideas and for more complete involvement in the learning process.

The above model characterizes teaching and learning, as cyclic, multi-dimensional, multi-directional, and in a constant state of flux. Culture is intimately connected with the nature and expression of the scientific and mathematical thought process. Cajete's curriculum model can be used to complement the prescribed mathematics curricula particularly, though not exclusively, in schools where Indigenous students are enrolled. Such a curriculum model would connect with the NCTM *Principles and Standards* which calls for a common foundation where all students should actively build new knowledge from experience and proper knowledge to be able to understand and use Mathematics. Teachers can help students to recognize that science and math are both creative processes and cultural systems of knowledge.

Numeracy the Language

Steen (1990) implies that numeracy has the same relationship to mathematics as literacy has to language. Each represents a distinctive means of communicating with others depending on the cultural context. There is some tension between narrow and broad interpretations of literacy and numeracy—particularly between their practical benefits and cultural effects—as they must be taught in a realistic context to sustain motivation and to ensure mastery. Just as the term literacy, defined in the Compact Oxford English Dictionary as the ability to read and write, implies the everyday use of letters in the process of communication, numeracy must involve the everyday practical use of numbers. Moreover, the way in which we use these two words influences the way we think.

Literacy is the cornerstone of education. It includes not only reading and writing skills, but the fluency with which one is able to communicate in a language. Literacy has become a cross-curricular concept with shared responsibilities. It is considered the key to lifelong learning where families and communities must be supported and encouraged to promote it. Most areas of the curriculum make demands on students' mathematical knowledge, understanding, abilities, and skills. Hence, numeracy is an intersecting set of literacy practices. A student's level of numeracy is a significant component of his or her literacy level. Like literacy, numeracy is a set of cultural practices that reflect the particular values of the social, cultural, and historical contexts of society. Without the ability to understand basic mathematical ideas, one cannot fully understand polls from contested elections to sports stats, from stock scams and newspaper psychics to diet and medical claims, sex discrimination, insurance, lotteries, and drug testing (Paulos, 1998).

The distinction between numeracy and mathematics creates a curricular and pedagogical challenge for all educators who have a role in developing an individual's numeracy (AAMT, 1997). While numeracy is related to mathematics, for reasons already given, the two terms are not synonymous or equivalent. Having mathematical knowledge such as the system of ideas involving numbers, patterns, logic, and spatial configuration contributes to being numerate, but such knowledge in isolation is not sufficient for learners to become numerate. Numeracy and mathematics should be complementary aspects of the school curriculum, since both are necessary for life and work, and one overlaps the other.

Numeracy is a socially based activity that requires the ability to integrate math and communication skills (Withnall, 1995). Since language is a function of both thought and culture, the ability to use and communicate with mathematics is important. Mathematics has its own culture as well; it possesses a language and a set of norms. It should be embedded in cultural activities that involve everyday tasks and solve everyday problems (Nunes, 1992). People of different cultures and different eras have engaged in mathematical activities to solve the problems they encountered in their daily lives. For example, they devised numbering systems, counted objects, constructed homes, and designed works of art based on mathematical principles. Perhaps they didn't give the name *mathematics* to these activities or attach the title *mathematician* to the people who invented the concepts, but the basic strands of number, spatial, statistical, and patterns that we practice today have resulted from their efforts.

Numeracy the Cultural Activity

Bishop (1988) asserts that mathematics is present across cultures as a human activity, a statement supported by numerous other authors (Ascher, 1991; D'Ambrosio, 1985; Joseph, 1991; Nunes, 1992; Powell & Frankenstein, 1997; Zaslavsky, 1991). Bishop et al. go on to argue that different cultures use mathematics in different ways. The abstraction and the symbolic characteristics that we commonly associate with mathematics have not appeared in every culture. Rather, common mathematics is frequently of an informal nature and part of indigenous knowledge. This can be seen in the mental math used by bazaar merchants in the Middle East, the navigational practices of South Pacific islanders, or the carving of a totem pole in Haida Gwaii. Indisputably, "an enormous range of mathematical techniques and ideas has been developed in all parts of the world" (Bishop et.al., 1993, p. 6).

Bishop (1988) identifies six activities as mathematical practices found in every culture: counting, measuring, designing, locating, explaining, and game playing. This classification scheme could be used to connect school mathematics curricula with cultural practices to form the basis of what we might consider numeracy across cultures. Bishop (1988) has attempted to bridge mathematics in school and culture by connecting a student's experience with mathematics to his or her everyday experiences.

Summary

Numeracy isn't just about mathematics, which is an abstract construct; it is about mathematics situated in culture. Numeracy is multi-layered and should be linked and contextualized. Familiar contexts can make mathematics more accessible to those who have been alienated from it. Relevant learning experiences should be designed to challenge the learner's understanding and extend his or her knowledge to a personal context. Learning activities should be culturally valid and educationally sound. Teachers need to select culturally oriented learning activities that can be used in the study of appropriate topics in mathematics.

Different definitions emphasize different aspects of the term numeracy. For the purposes of this study my working definition of numeracy will be as follows:

> Numeracy is the set of mathematical skills needed for one's daily functioning in the home, the workplace, and the community. It is the willingness and capacity to solve a variety of situated and contextual problems that could be functional, social, and cultural.

Today's mathematics curriculum from pre-kindergarten to Grade 12 is in most cases limited to the major successes of the Eurocentric world. Students of Indigenous and multicultural heritage frequently face the challenge of learning in an environment that may undervalue or ignore their cultural backgrounds. *Principles and Standards for School Mathematics* (NCTM, 2000) calls for a common foundation of mathematics to be learned by all students. It also advocates the need to learn and teach mathematics as a part of cultural heritage and for life. Achieving this goal requires a paradigm shift in the way mathematics is taught and the introduction of culturally inclusive curricula and pedagogy. Trentacosta (1997), in the NCTM Yearbook *The Gift of Diversity*, presents a vision of how classroom practices can embrace and celebrate diversity as well as ensure a powerful mathematics program for all students. Learning activities should build upon a student's prior knowledge and present mathematics in an exciting and inclusive way. Context combined with content should direct teaching in the ongoing cultural quest for knowledge. Partnerships need to be developed with educators, elders, parents, policymakers, and others in the community to promote numeracy and change societal attitudes towards mathematics.

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THE USE OF MATHEMATICAL IMAGERY IN COMPLEXIVIST ACCOUNTS OF EDUCATION AND EDUCATIONAL RESEARCH

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Complexity thinking has been applied to educational contexts for about the past fifteen years. Of the many disciplines that have contributed to the development of complexity thinking, mathematics has played a key role – especially through recent advances in mathematics in the fields of nonlinear dynamics and fractal geometry. Some complexity thinkers have taken to using this "new" mathematics as a source of metaphors for talking about education. This paper documents and reviews a number of these efforts. The paper also discusses the value of such metaphors to educational discourses.

INTRODUCTION

The use of mathematics in educational research has historically been in support of positivist, quantitative experimentation. Mathematics was a tool for the statistical analysis of data or experimental models. Mathematics in the classroom, besides occupying a privileged (and often unexamined) position in the school curriculum, was employed in evaluation schemes while assessing student progress. The role of mathematics in education and educational research has been utilitarian; the content of mathematics has not been used to generate new insights into the nature of learning or to suggest advice about teaching.

Over the past three decades there has been a swing of the educational research pendulum from quantitative to qualitative studies. At the same time as qualitative inquiry was gaining respect in faculties of education a myriad of diverse intellectual initiatives were coalescing into "a new attitude toward studying particular sorts of phenomena". (Davis and Sumara, p.4). This new attitude is an umbrella of discourses known variously as complexity theory, complexity science, or, for the purposes of this paper, complexity thinking. Complexity thinking, for the educationist, begins with recognizing educational phenomena as being complex rather than merely complicated. Educational phenomena in their complexity resist explanations through reductionist analysis; they are not mechanical in the sense that behaviours can be accounted for as an aggregate of component interactions. Not all, but much of the use of complexity thinking in educational research has been qualitative in nature.

Complexity theory is informed by the mathematics of non-linear dynamics (chaos theory) and of fractal geometry. Some educational researchers with a disposition towards qualitative studies have begun to use this "new" mathematics of complexity in a novel fashion: as a source of metaphors to enrich the conversations about education.

This paper will critically review some of the extant work in this area and discuss the potential of using mathematical metaphors in enhancing educational discourses.

THEORETICAL FRAMEWORK

There is such a wide range of sensibilities and activities subsumed under the rubric complexity thinking that it is useful to identify the intent and epistemological stance adopted here. Davis and Sumara (2006) build on previous work to identify three broad categories of utilizing complexity concepts. Hard complexity science "maintains the same desire as analytic science to uncover and *understand the nature of reality*" while soft complexity science sees complexity as "*a way of seeing the world*, an interpretive system rather than a route to or representation of reality." (p. 18). The third approach, denoted as complexity thinking, is a *via media* "concerned with the philosophical and pragmatic implications of assuming a complex universe." (p. 18). The first school has "modern" correspondence theories of knowledge for its epistemological underpinnings; the latter two schools are more comfortable with coherence theories of knowledge and epistemologies situated in postmodern sensibilities. The metaphorical use of mathematics in educational phenomena belongs with "soft complexity science" and "complexity thinking". It is the intent of this author and the authors reviewed here less to uncover accurate facts and create predictive models and more to engender conversations leading to richer understandings.

Placing metaphors within the interpretive and logic within the analytic positivism, we situate ourselves in this context in the interpretive. This is not to argue that logical analysis has no place in educational research, but is more an explicit foregrounding of our stance and purposes here. In the words of William Doll, "Metaphors are generative; they *help us see what we don't see.* Metaphors are open, heuristic, dialogue-engendering. Logic is definitional; it *helps us see more clearly that which we already see.*" (1993, p. 169)

Gray and Macready (2004) discuss the value of *cognitive* metaphors in revealing connections and generating insights essential to innovative thinking. Metaphors, used properly, become part of a continuing metaphor-model cycle. Unlike Wittgenstein, we do not throw away the ladder that metaphor has provided us to get to clearer thinking, but we draw it upward and use it climb to even greater heights. Metaphor has a source and a target. Analogy, in the Gray and Macready model, refers to the congruences between source and target. These correspondences allow the metaphor to be applied. "A metaphor, however, goes beyond analogy by including all the ill-fitting facets of the linked entities – the fractures and the fault lines – in the picture." (Gray and Macready, 2004, p.39). The intention of the metaphor is also considered. Rhetorical metaphor is meant to be persuasive; to convince the reader of the aptness of the linkage. Cognitive metaphor, the one we emphasize here, is meant to be an invitation to explore the strengths and weaknesses of the bridge between source and target. The reader is in dialogue with the cognitive metaphor and may extend or trim the original trope as she grapples with the images. The open ended nature of such reader/metaphor entanglements is a key to creative thinking.

MATHEMATICAL METAPHORS IN EDUCATION

Fractal Geometry

Davis (2005), with his usual attentiveness towards language and how it influences thought, discusses the pervasiveness and the transparency of imagery from Euclidean geometry in everyday situations in general and educational contexts in particular. This account is expanded in Davis and Sumara (20060. Much of this imagery is as odds with fractal geometry, the "geometry of nature". Specifically, Euclidean imagery is linear and decomposes into "simpler" parts. Euclidean metaphors are appropriate for only a limited number of phenomena in the world. As Davis argues "there have been decisive changes in perspective with the recognition that nature is relentlessly nonlinear." (2005, p. 123).

Three qualities of fractal geometry are identified and used as metaphors by Davis (2005): recursivity, scale independence, and self-similarity. A fractal image often begins with a simple figure and a rule to modify that figure. The procedure is iterated recursively (that is, the procedure is applied to the resulting figure of the previous iteration). That such recursions can result in such complex forms arising from simple beginnings is a metaphor for student learning that Davis and others (e.g., Doll (1993)) are unable to ignore.

Complex phenomena in general and fractal images in particular are *scale independent*. Scale independent phenomena do not "simplify" under magnification. The components are as complex as the whole. Unlike the continuous, differentiable function that is locally linear, the complex will retain its unevenness at all scales. Davis (2005) relates the scale independence of fractals to educational research by inviting us to look at the differing foci of educational discourses such as neuroscience, psychology, and sociology as different levels of magnification when viewing a single complex phenomenon.

Self-similarity is the quality of a fractal in which a small, appropriately chosen part of the whole has a structure closely resembling that of the totality. Scale independence and self-similarity lead to the metaphor of nested organizations – complex subsystems self-organizing into larger complex systems and the subsystems themselves emerging from interactions of complex agents. The mathematics of fractals – the geometry of nature – leads to an organic image of education.

Smitherman (2005) pulls another metaphor from fractal geometry – that of a bounded infinity. The von Koch snowflake's infinite perimeter within a finite area is attached metaphorically to a classroom bounded by institutional constraints yet still with infinite possibilities for student learning.

Network topologies

Very closely related to the mathematics of fractals is the study of network architectures. Three topologies for networks are identified in the complexity in education literature (Mowat, 2007, Davis and Sumara, 2006): centralized, decentralized, and distributed. It is the decentralized network with its scale independence and self-similarity that is fractal in nature. Davis and Sumara (2006) note the efficiency of the centralized network in distributing information, but also its vulnerability to catastrophic collapse if the centre node fails. The distributed network (complete graph) is resilient to node failure, but highly inefficient in transmitting information. Consequently, the distributed network is inflexible and slow to adapt. In other words, it is unintelligent. Centralized and distributed networks are simple (or, at most, complicated) in their structure. It is the decentralized network (of which the internet is the paradigmatic exemplar) with its "nodes nodding into nodes" fractal complexity that is the Aristotlean golden mean between robustness and adaptability.

Mowat (2007) uses the decentralized network to describe a possible structuring of (personal) mathematical knowledge. Accepting the desirability of a "healthy" network of mathematical metaphors, Mowat offers some early thoughts on how the metaphor of decentralized networks could be of value in the mathematics classroom. Mowat's characterization of mathematical knowledge as a network is in contrast to a linear, hierarchical structure and has implications for structuring curriculum.

Davis and Sumara (2006) venture that teacher centred classrooms and student centred classrooms are educational examples of centralized and distributed networks. The challenge is to envisage the classroom as a decentralized network and to consider under what conditions such a network might arise.

Nonlinear dynamics (chaos theory)

Much is implicit in the names. Davis (2005) offers contrasting images of learning. The linear image is a straight road leading to the horizon; the nonlinear image is water cascading down a rough rock surface. Prediction and control fit well with the roadway; the myriad of possibilities of patterns that the water could make lends well to turbulent, far-from-equilibrium circumstances of a nonlinear phenomenon. The linear image hints at progress along a predetermined path – a transmission of knowledge view of education. The waterfall suggests many potential pathways within proscribed limits – a transformation of the student attitude towards education.

Chaos theory is a name popularly ascribed to the mathematics of nonlinear dynamics. Chaos, as detailed by Doll (1993), currently has perjorative overtones in ordinary language. Certainly, a classroom teacher would look askance if directed to seek chaos in his or her classroom. Yet seeking the opposite in the classroom – equilibrium – is equivalent to seeking death in the system. Doll (1993) speaks of the two arrows of time emphasized in the nineteenth century – entropy and evolution. As teachers, do we seek evolving classrooms or Carnot engines inexorably dissipating energy into the universe?

The metaphorical lessons on chaos theory are sensitivity to initial conditions, unpredictability, the possibility of sudden and dramatic change (bifurcations and catastrophes), and the possibility of "order for free" – self-organization or emergence. (cf. Stanley, 2005). Smitherman (2005) suggests the possibility of embracing the unpredictable in the classroom. "Linking pedagogical goals with the unpredictable behaviour of students generates a curriculum that is emergent, generative, and open." (p. 161).

Nonlinear systems are characterized by feedback loops. Negative feedback dampens "noise" in the system; positive feedback amplifies turbulence. Davis and Sumara (2006) detail the need for balance between negative and positive feedback in dynamic systems. Classroom control and unchecked reward systems are educational analogues of negative and positive feedback loops.

Phase spaces and attractors

The images of the mathematics of chaos theory are phase spaces. Phase spaces separate in categories by attractors: point, periodic point, periodic, and strange. (Gilstrap, 2005). If the Mandelbrot set is the iconic fractal, then the Lorenz "butterfly" is the same for strange attractors.

Attractors as metaphors have evident applicability. Gilstrap (2005) offers educational metaphors for each class of attractors. A point attractor moves to a state of extreme equilibrium. Some educational tasks that are quickly completed may be considered analogous to point attractors. The periodic point attractor is a repeated task, but one that still seeks equilibrium. Such tasks are ubiquitous in formal educational settings. Periodic attractors model such behaviour as a planet circling the sun. There is continual modification of the trajectory, but repetition at the macroscopic level. An educational example of an activity that would have a periodic attractor is modification of a syllabus each year while preserving the overall content. Alternatively, the setting of annual budgets (without significant changes) would be an example. (Gilstrap, 2005, p. 59).

It is the strange attractor, however, that is of most interest here. "Systems operating within a strange attractor framework move in chaotic patterns of bounded instability." (Gilstrap, 2005, p. 60). Such systems are, simply put, complex. Although Gilstrap addresses the use of strange attractor metaphors to higher education leadership, it is intriguing to speculate what metaphorical insights could be derived from the "phase spaces" of classrooms. Are the trajectories of our classes point attractors or a strange attractors? What sort of attractor do we seek?

Conclusions and suggestions

We begin with a caveat. Stanley, discussing the misuse of the term nonlinear, writes, "Here, some coherence is lost, and some misunderstanding is created with importing a mathematical idea into a different setting." (2005, p. 144) Mathematics is much concerned with precision; metaphor, by its boundary expanding nature, is often imprecise. It may be true that mathematicians will look upon the metaphorical uses of their subject matter and shake their heads sadly for the distortions and misuses they see. But the mathematical metaphor is not for doing mathematics; it is for traveling from the known and familiar to new, previously undreamt of realms. Mathematical metaphors should not be judged solely on the correctness of their usage, but on their fecundity in provoking new insights and questions.

The use of mathematical metaphors in educational discourses is definitely a work in progress. Historically, we have seen metaphors such as the brain as a computer of cognitive science play itself out. But it generated many interesting investigations during its lifetime. Even (or especially) if the objections to the metaphors have added considerably to thinking about learning.

We may come to understand classrooms as nonlinear dynamic systems where far-from-equilibrium social collectivities of teachers and students develop emergent learning. We may look at linearly ordered, hierarchical curricula and wonder about the potential of curriculum with a fractal structure. We may object strenuously to the metaphor of a successful class having decentralized network and uncover our own unrecognized metaphors for class structures. But engaging these metaphors and interrogating them will give us a better understanding of our understandings.

A concluding and some quixotic observation about the value of mathematical metaphors in educational discourses is that allows those of us with backgrounds in mathematics to talk about mathematics with colleagues who would otherwise have nothing to do with the subject. Educational researchers and teachers who would rather not discuss the basin for the attractors for a particular logistic equation may be far more willing to converse about the attractor metaphor. And this is a conversation that might not otherwise have happened.

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ELECTROOCULOGRAPHY: CONNECTING BRAIN AND BEHAVIOR IN MATHEMATICS EDUCATION RESEARCH

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Abstract

Connecting brain and behavior is gaining increasing interest in mathematics education research, in areas as far ranging as mathematical problem solving and mathematics anxiety. Electrooculography (EOG) enables accurate measurements of eye-related behavior (e.g., blinks & movements) by detecting and recording changes in voltage potentials generated by contractions of extraocular and ocular muscles encasing and enclosing the eye. This paper reports on the potential roles and importance of EOG for mathematics educational neuroscience research. Here we identify and discuss three main uses of EOG. First, EOG provides insights into cognitive function and performance. Secondly, it can be used to attenuate eye-related artifacts in electroencephalography (EEG). Thirdly, when used in tandem with EEG, eye-tracking (ET), and audiovisual (AV) data sets, EOG serves as a helpful means for calibrating covert brain activity with overt behavior. Here, we provide an overview of the first two application areas, and we illustrate the third application using an example data set capturing an "aha moment" in our research in the area of mathematical problem solving. We conclude this paper with a summary discussion on these various uses of EOG in connecting brain and behavior in educational research.

Introduction

A major reason for growing interest in educational neuroscience in mathematics education research, aside from the now widespread recognition that cognitive constructs are embodied, is a need for better empirical grounds for developing theories of mental functions and processes (Campbell, 2006 a, b). This, in turn, can reconstitute the conceptual basis for more effective forms of mathematical learning and instruction. According to Byrnes (2001), brain research is relevant to the field of psychology and education to the extent that it fosters better understandings of mind, development and learning. The validity, reliability, and relevance of psychological theories of teaching and learning developed from traditional psychological experiments may variously be corroborated, refined, or refuted through neuroscientific studies or the use of neuroscientific tools and methods to test hypothesizes of any particular theoretical account (cf. Byrnes, 2001; Kosslyn & Koening, 1992). Conversely, research in the neurosciences can also benefit from the more situated and ecologically grounded insights into cognition and learning that typifies the concerns and aims of educational research. Accordingly, educational research that informs and is informed by neuroscientific research, while incorporating methods of cognitive neuroscience with the further aim toward corroborating, refining, or refuting certain models of cognition and learning — that is to say, educational neuroscience — should become of abiding interest and concern to educators and educational researchers alike. This interest includes analyses of data sets measuring brain electrical activity during cognitive processes, such as those involved in mathematical thinking. Theoretical considerations aside, there are also methodological challenges in isolating brain activity with electroencephalography (EEG) and in calibrating EEG with more advanced behavioral data sets, such as eye-tracking (ET), and established behavioral data sets, such as audiovisual (AV) recordings. Here we discuss and illustrate ways in which electrooculography (EOG) can help address these matters.

Electooculography (EOG)

Electrooculography (EOG) is a physiological technique that uses pairs of electrodes placed symmetrically around the eyes, usually to the sides and below, to measure potential voltage differences generated by extraocular muscle contractions, resulting in eye-movements (Fig. 1).

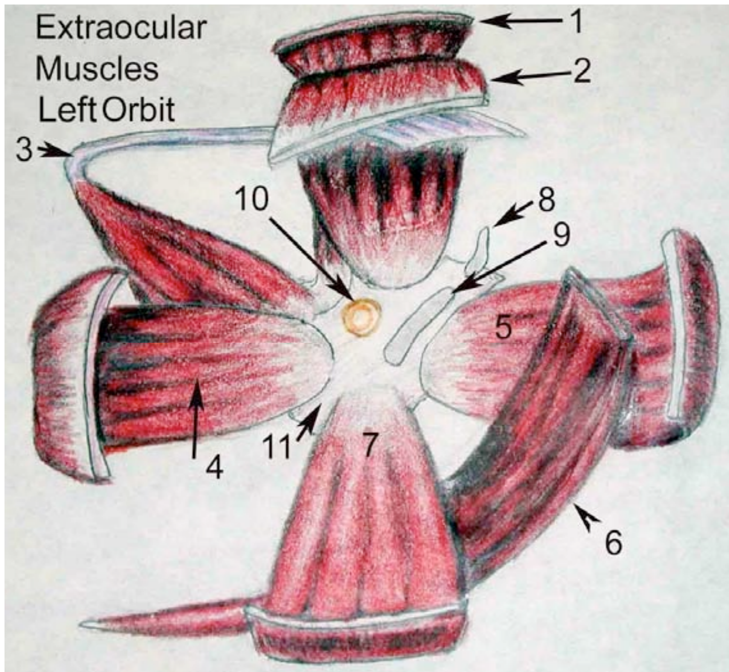


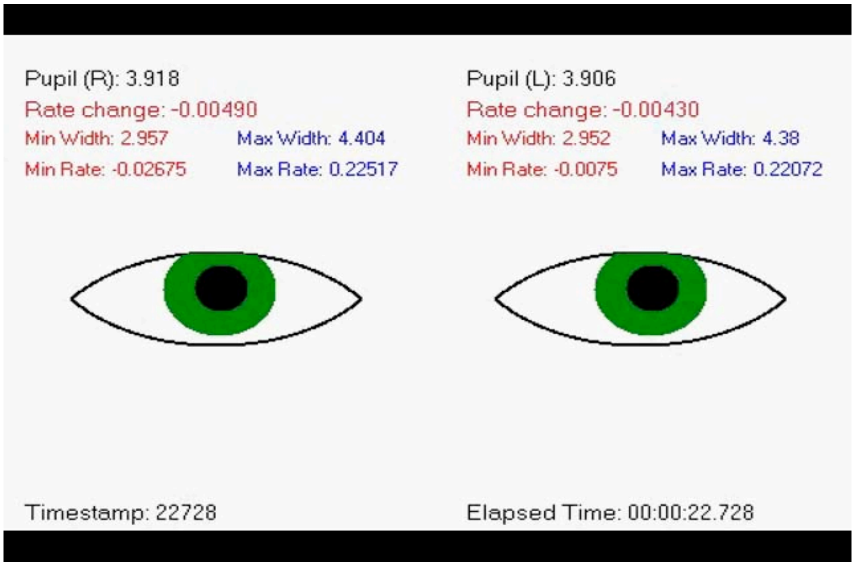
Figure 1. Extraocular muscles: 1) the *Levator Palpebrae Superioris* muscle elevates the eyelid; 2) the *Superior Rectus* muscle makes the eye look upwards or medially or wheel-rotates it medially (intorts); 3) the *Superior Oblique* moves the eye downwards or laterally or wheel-rotates it inwards (i.e. makes twelve o'clock on the cornea move towards the nose); 4) the *Medial Rectus* is the largest of the ocular muscles and stronger than the lateral and is a pure adductor; 5) the *Lateral Rectus* makes the eye look directly laterally in the horizontal plane; 6) the *Inferior Oblique* makes the eye look upwards or laterally or wheel-rotates it; 7) finally, the *Inferior Rectus* makes the eye look downwards or medially or wheel-rotates it laterally. This muscle also serves to depress the lower lid. Other ocular muscles (not shown here) actually enclose the eye.

Taking the voltage potential as constant when the eyes are at rest, changes in EOG potential will then correspond mainly to extraocular muscle contractions, which in turn manifest behaviorally as movements in the eyes and lids. These potentials measured by EOG, however, also tend to interfere with potentials generated by the brain measured by EEG. There are other, much smaller ocular muscles, namely the iris dilator and ciliary muscles affecting pupil dilation and eye focusing, but their EOG effects are virtually indiscernible compared with the effects of extraocular muscles. All eye-related behavior (ERB), i.e., eye-movements, eye-blinks, saccades, dilations, and focusing, voluntary or not, are based on muscles encasing and enclosing the eyes.

Voltage potentials resulting from eye muscle contractions are what are mainly measured using EOG. However, voltage potentials from eye muscle contractions can interfere with EEG, which monitors brain activity. We shall see that EOG enables improved correction of these interfering potentials in EEG data, thereby improving the quality of EEG data and subsequent EEG analysis and interpretation. EOG data also enables time synchronization and integration of EEG with ET and AV data sets by calibrating EOG data with these data sets. First, though, we consider ways in which EOG provides helpful insight into cognition and performance as well.

EOG as a cognitive indicator

It has long been said that the eyes are the windows of the soul. Poetics aside, insofar as the soul is an antiquated term for the mind, it seems reasonable to consider that ERB could provide some indication of various cognitive states, functions, and behavior as well. For instance, ERB, particularly eye-movements, have been studied as an indicator of attention and memory (Kramer & McCarley, 2003), and pupillary response has been studied as an indication of cognitive load (Just, Carpenter, & Miyake, 2003). EOG only records eye muscle activity. Eye-tracking data provides a further avenue for studying ERB, as demonstrated in Mov. 1 below (double click).



Movie 1. Data from an eye-tracker using infrared emitters and sensors to generate and capture low levels of infrared light reflected off the back of the retinas of the eyes, thereby measuring various aspects of ERB.

Many kinds of mathematical activity are connected with ERB, which can variously be voluntary or involuntary. Reading serves as an example of voluntary eye movements, such as reading mathematical texts, reading mathematical 'word problems', or looking at geometrical objects for solving correspondent geometrical problems. In the first case (reading) eye movements measured by EOG can be predictable to some extent (mainly back and forth in horizontal directions). In other cases, eye-movements remain predominantly voluntary, but are more complicated as well, as they can involve virtually any change of direction and speed. Less deliberate to involuntary eye-related behavior, such as eye-blinking (to keep eyes moist) and saccades (rapid macro and micro eye-movements that enable the brain to perceive differences — see Mov. 1 above), are on-going deliberate and pseudo-random phenomena, independently of whatever it is that the eyes happen to be looking.

Behaviorally, the specific character of eye movement during reading is summarized in Duchowski (2002). His study is based on eye-tracking research and its application. Duchowski found that eye fixations usually last 200-250 ms when reading English (non-academic texts). He hypothesizes and cites evidence that as the text becomes conceptually more difficult, fixation durations increase and saccade lengths decrease. Duchowski argues that eye movements are also influenced by textual and typographical variables such as quality of print (including lighting, and colours of fonts), line length, letter spacing. He also found differences in reading silently from reading aloud from the viewpoint of fixation duration.

Eye movement and eye fixation analyses have also been efficiently used as a method of research into strategies of successful and unsuccessful arithmetic word problem solvers (Hegarty, Mayer, and Monk, 1995). According to Hegarty, et al's study, less successful problem solvers fixate their eyes on numbers and relational terms when they re-read parts of an arithmetic word problem, whereas more successful problem solvers fixate eyes on variable names. They propose that successful problem solvers construct problem models and concentrate their attention on appropriate variable names, whereas unsuccessful problem solvers attempt to directly translate key propositions of these problems into a computational procedure, and thereby remain predominantly focused on numbers.

Applying Duchowski's reasoning (2002), we assume that mathematical reading for comprehension should be considered as a conceptually demanding task involving both long eye fixations and different directions of eye movement, including not only the usual horizontal directions, but also backward, upward, downward and different angle directions (especially taking into account that mathematical text contains formulas where understanding them typically requires both reading and re-reading). The same assumption applies to eye movements when reading and solving word problems and geometrical problems. In these cases the direction of eye movements can hardly be predicted and close analyses for identifying, pre-selecting, isolating, and eventually extracting and/or removing EOG signals in EEG data is in order.

EOG for artifact correction

According to Kierkels, Boxtel, and Vogten (2006), before brain activity measured by EEG through scalp voltage potentials is analysed — especially, we would add, in single trial (non-replicable) conditions — eye-related behavior artifacts should be identified and attenuated using signal processing techniques. Fig. 1 illustrates eye-related behavior artifacts in EEG data.

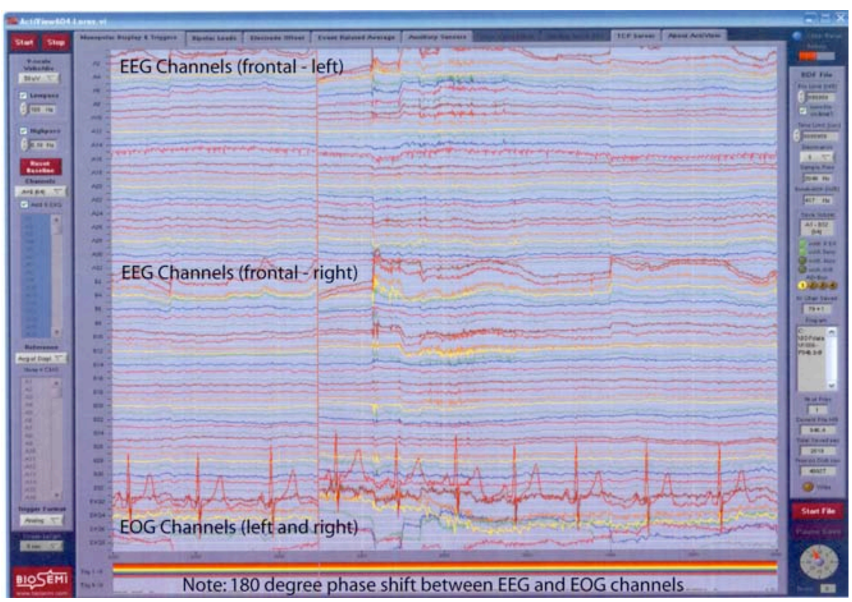


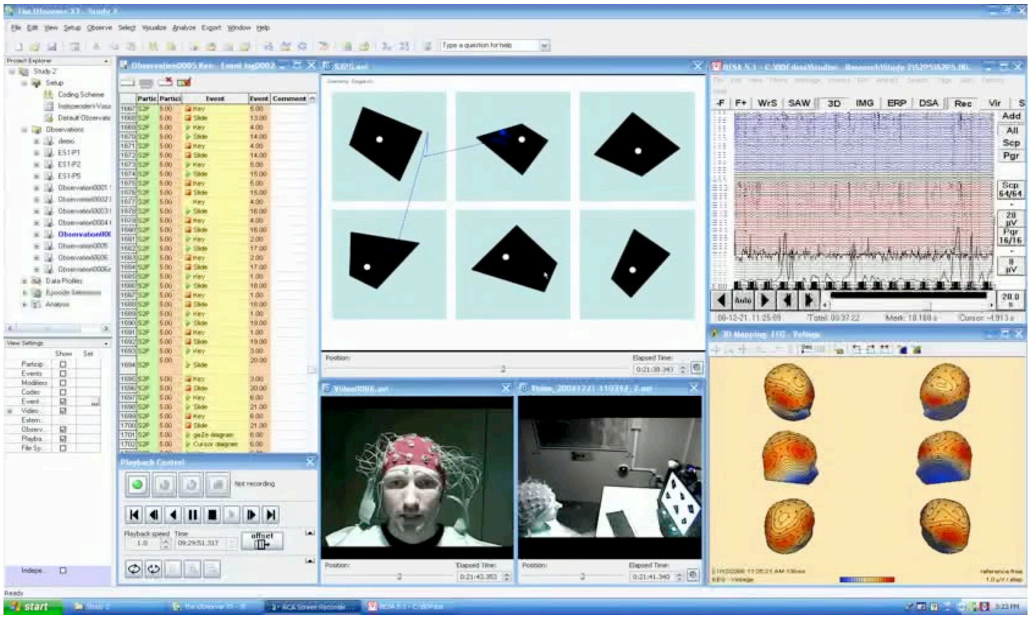
Figure1. This raw data set illustrates strong eye-related artifacts, evident in the EOG channels in the bottom of the figure, in the rightmost two thirds of the figure in the frontal EEG channels, which are closer to the eyes. Although the EEG data toward the leftmost third of the figure is more quiescent, some eye-related behavior artifacts are still evident in the EEG. (The vertical line is the "present moment" of the sweep line of the real-time display at the moment of screen capture. Thus, the quiescent period on the left is actually later in time than the data that is in the process of being overwritten to the right of the sweep line.)

It is helpful to mention here that when retrieving EEG data, the signals generated by eye-related behavior, as well as any other voltage potential signals of non-cerebral origin, should be treated as artifacts (i.e., undesired signals). Puthusserypady (2005) notes that EOG signals are typically the most significant and common artifacts in EEG, and that removal of EOG signals from EEG data forms an important part of preprocessing of EEG data prior to EEG analysis. Generally speaking, EOG signals, eye blinks typically more so than eye movements per se, are of high amplitude (as evident in Fig. 1, they can be as much as ten times the amplitude of signals due to brain activity recorded from the frontal EEG channels) and they are low frequency in nature and affect the low frequency range of EEG signals. Kierkels et al (2006) point out that EOG signals have large disturbing effects on EEG recordings because the eyes are located so close to the brain. There are many methods for eye movement correction in EEG recordings reported in literature, with varying degrees of success for removing EOG signals (Kierkels, et al, 2006; Croft, Chandler, Barry, Cooper, & Clark, 2005).

Ille, Berg, and Scherg (2002) distinguish between methods that remove EOG signals from EEG data without considering brain activity and techniques that attempt to separate artifact and brain activity. These authors consider signal processing methods, such as Principle Component Analysis (PCA), Independent Component Analysis (ICA), Multiple Source Eye Correction (MSEC), that are designed to separate ERB artifacts (i.e., EOG signals) and EEG brain activity, as the most promising methods. According to Croft et al (2005), however, the main limitation of EOG correction methods is that as well as reducing extraocular muscle potentials in the EEG they may also attenuate brain-related potentials. The real difficulty, however, is in determining the accuracy of any such method, because as voltage potentials from both cerebral activity and eye-related behaviors are recorded to some extent in both EEG and EOG, it is not clear *a priori* what a corrected EEG waveform should look like. Kierkels et al (2006) provide a representative study that attempts to overcome this difficulty by employing independent simulations of EEG and EOG data. Using eye tracker data (to model eye movements) along with conductivity properties of the human head, these authors simulated ERB artifacts (i.e., EOG signals), and artifact free EEG. The simulated EEG and EOG waveforms were used then as references when evaluating how well different algorithms are at removing EOG signals. On the basis of designed simulations these authors evaluated six algorithms including PCA and ICA. Using EOG and EEG modeling and simulation methods should be considered as one of the most effective and progressive approaches not only for choosing an appropriate algorithm for EOG artifact removal. In our research, we are certainly interested in removing EOG signals from EEG data, but more germane to our focus in this paper, we are also interested in using EOG data to synchronize brain activities manifest in EEG with eye tracking and other audiovisual behavioral data, and it is this latter point that we turn to in what follows.

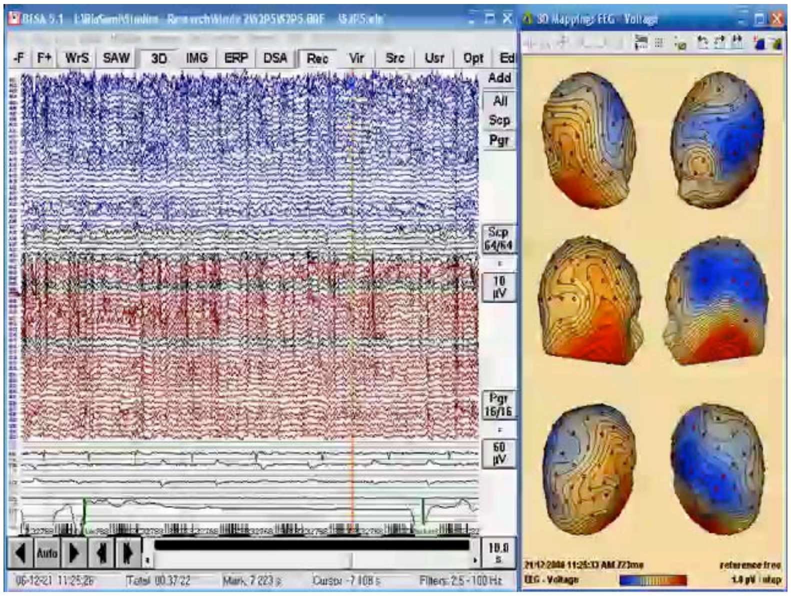
EOG for calibrating datasets

We now demonstrate how EOG data can be used in tandem with ET and AV data sets to synchronize with EEG data sets, which are indirect recordings of the electrical component of the biopotential field generated on the scalp by brain activity. Getting accurate calibrations are crucial to properly associating covert brain activities with overt problem solving tasks. Once these data sets are synchronized, a more integrated approach can be taken in connecting brain and behavior. We use the first part of the sample data set illustrated in Mov. 2 below, from just prior to the participant's utterance of "diagonals" to just after the participant's utterance of "ahh" [once again, please double click on the movie to play it].



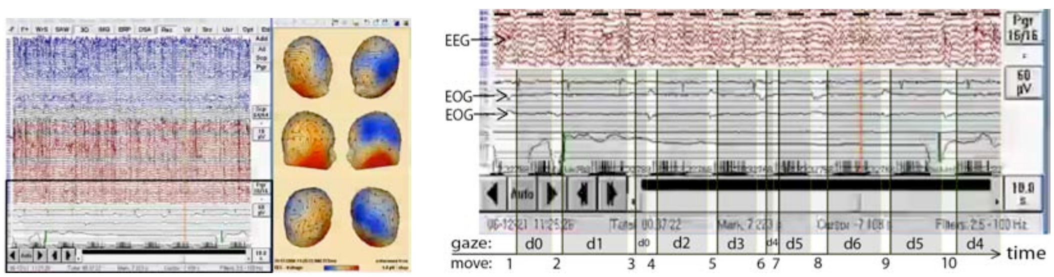
Movie 2. Eye-tracking data overlaying screen capture (upper centre), and audiovisual data sets of the participant (lower centre). Click on the movie to play it. The leftmost side is coding and keyboard capture data. The rightmost side is EEG. The problem here is how to connect EEG data with the eye-tracking data?

Playing the movie helps to reveal the nature of the problem solving task, which in this case is drawing upon one slide from a paradigm of Dehaene, Izard, Pica, & Spelke (2006), which presents participants with six diagrams on each slide. Five of the six diagrams are connected by a common mathematical concept, and the task of participants is to identify the one diagram in each slide that does not conform. The participant in this case with this slide had not been successful initially, and so his attention was drawn to the term "diagonals" in the upper left corner of the screen and asked if awareness thereof would have affected his previous exposures to this task. Mov. 3 illustrates the EEG brain activity about 7 seconds into Mov. 2 [double click to play]:



Movie 3. Three seconds of EEG data (1/40th actual speed), about seven seconds into the data set in Mov. 1.

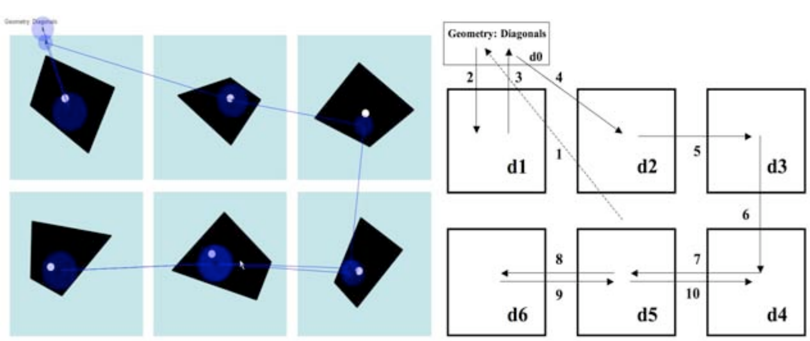
Mov. 3. illustrates approximately three seconds of EEG data, beginning about seven seconds into the data set in Mov. 2. The leftmost side of the movie displays the raw EEG voltage potentials (64 channels worth, with the 32 blue channels representing scalp voltage potential data obtained from the scalp over the brain's left hemisphere and the 32 red channels representing the data over the brain's right hemisphere). The rightmost side illustrates those data being rendered at 1/40th actual speed on the surface of the scalp, where at any given moment, the state corresponds to the EEG data set demarcated by the vertical yellow line. An initial calibration of these EEG brain data with the behavioral data in Fig. 2 and Mov. 2 was made using respiration data on the bottom channel (where the small green vertical markers indicate the onset of the expiration associated with the participant's uttering of the terms "diagonals" and "ahh" respectively). It is evident that these are extremely rich data sets, and that the brain activity is quite dynamic and complex. It is crucial in interpreting this brain activity to very accurately calibrate, i.e., time-synchronize, these brain data with the behavioral data illustrated in Mov. 2. Although time synchronization pulses (TTL pulses) are also used (i.e., the little comb like structures beneath the respiration channel), the EOG data is more intuitive in this regard, and serves to double check the TTL pulse data.



(a) (b)

Figure 2. Initial frame of Mov. 2 on the left with ancillary channels expanded in greater detail to the right.

Fig. 2(a) above is the initial frame of Mov. 3, and the area outlined therein is expanded in Fig. 2(b) for greater detail. Fig. 3(a) below illustrates actual screen captured eye-tracking movements corresponding the first ten seconds of Mov. 2, and Fig. 3(b) is a schematization of those 10 sequential eye-movements, beginning with the initial eye movement "1" terminating at "d0", which is the text associated with the slide, followed by eye movement "2" down to diagram "d1" and so on. The expansion of the ancillary channels in Fig. 3(b) shows data obtained from the EOG electrodes, attached just off the corners of the participant's left and right eyes, highlighting "gaze" regions (d0, d1, ..., d6), and "move" values (1, 2, 3, ..., 10). These annotations in Fig. 2(b) correspond to the annotations indicated in Fig. 3(b). The shaded areas in Fig 2(b) denoted by the gaze regions indicate when the participant was attending to the six diagrams in the slide in Fig 3(a), with "d0" indicating the text region to the upper left. These annotations are most clearly indicated in Fig 3(b).



(a) (b)

Figure 3. Actual eye-movements on the screen stimulus (a), and a schematization of those movements (b).

Conclusion

Eye-related behavior (ERB) can be used to infer cognitive function and to operationalize human performance. From a cognitive perspective, it can be variously considered as sensory-driven, emotionally-responsive, or goal-directed. In terms of performance, eye-related behavioral indexes, like fixation patterns and gaze durations, can operationalize efficiency and workload, for instance, with respect to design and usability. Ocularmotor activity manifest in EOG signals may serve as indicators, therefore, of various aspects of cognition and performance, such as those aspects of attention and memory evident in mathematical problem solving. ERB evident in EOG can be a blessing, and a curse, insofar as ERB activity interferes with EEG recordings that may provide further insights and indicators in this regard. Fortunately, EOG signals can be used with signal processing methods such as ICA to help attenuate those artifacts from EEG data. Finally, it is evident that EOG data serve as a bridge in calibrating EEG and other physiological data sets with more overt behavioral data sets such as eye-tracking and audiovisual data sets. Indeed, EOG serves well in a variety of ways in connecting brain and behavior. As such, it has important methodological roles to play in establishing educational neuroscience, and mathematics educational neuroscience in particular, as a viable new area of educational research.

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A TALE OF TEACHER INITIATED INQUIRY INTO INSTRUCTIONAL PRACTICE: THE CASE OF DIVISION OF FRACTIONS

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This paper reports on the lesson study experience of a group of five mathematics teachers at Southpointe Academy. Teachers participated in collaborative six-week lesson study cycle on the topic of division of fractions. Two lessons with two different approaches to the topic were implemented in two Grade 8 classrooms in the form of "Lesson Study Open House". Outside participants, who took part in the lesson observation and post lesson discussion phases included a mathematics educator, a mathematician, teachers from other schools, and preservice teachers. Here we examine, using the notion of critical lenses, some of the aspects of embedding teacher's learning in their everyday work, or that of their colleagues, and the potential of this work to provide the conditions for increasing the effectiveness of mathematics instruction in schools.

BACKGROUND

There is an increased attention to professional development initiatives that rely on building communities of teachers. While researchers may not yet fully agree on what mathematical knowledge, skills, and habits of mind teachers need to have to teach mathematics effectively, there seems to be a growing consensus that embedding teachers' learning in their everyday work, through a careful examination of their practice and classroom artifacts, increases the likelihood that this learning will be meaningful (Lampert & Ball, 1998; Lieberman, 1996; Stein, Smith and Silver, 1999).

It is believed that the Japanese succeeded in transforming on the national level something as complex and "culturally embedded as teaching" (Stiegler & Hiebert, 1999). Further, it has been speculated that one of the main routes by which such systemic change is accomplished is the well-defined and established practice of lesson study (Chokshi & Fernandez, 2005; Lewis, 2000; Stigler & Hiebert, 1999; Watanabe, 2002). In recent years, aided by the publication of *The Teaching Gap* in 1999 which portrayed lesson study as an effective method for systematic improvement of classroom instruction, the practice swept the United States as a grass root movement, springing up in 335 schools across 32 states and became the focus of dozens of conferences, reports, and published articles (Lewis et al., 2006).

Sparked by such reports, indicating the effectiveness of this form of professional development, a team of teachers at Southpointe Academy decided to immerse themselves in the experience of lesson study. With the general goal of the school in mind, which is to raise the level of student achievement, and within this, also of mathematics instruction, the team organized and conducted, supported by the school administration and other teachers from the school, the Lesson Study Open House.

In a broadest sense, lesson study is a long-term professional development process, which is centred in the classroom and focused on students' learning. The centrepiece of lesson study is the *research lesson*, developed collaboratively, taught by one team member while observed by others, and finally discussed and reflected upon by the whole team. It should be noted that the term "research" in this context means teacher-initiated, practice-based inquiry. This paper does not aim to give a full description of the activities of the team, nor a comprehensive account of teacher learning. Rather, the goal is to provide some snapshots from this concrete instance of lesson study from which to consider the potential of lesson study as a possible way for all teachers to become experts in what they do.

SUBJECTS AND CONTEXT

The team consisted of five teachers, specializing in mathematics teaching at Grade 6 to 12 levels, from Southpointe Academy. They engaged in lesson study activities over a period of six weeks. The activities begun with a session devoted to shaping and articulating the overarching goals for student development as a result of mathematics instruction at the school. The team members had not previously considered the question about what common vision they have for their students. This conversation created a foundation, a sense of unity and purpose for the team's work. The chosen research lesson was on the topic of division of fractions, which was implemented in two Grade 8 classes, using two different approaches, each taught by a different member of the team.

The two lesson plans were designed according to the students' developmental level, but with instructional approaches that would give students a deep understanding of mathematical reasons behind the standard procedure "invert and multiply". The lessons were designed to investigate the effects of two different approaches, one using "mathematical proof" and the other "guided discovery through problem solving". Special care was given to create lessons that could be used as samples of this kind of work, and which closely resemble the structure and level of detail found in the Japanese tradition. For example, some main features of these lesson plans are great detail in concept development along with anticipating student thinking. This is exemplified later in the paper.

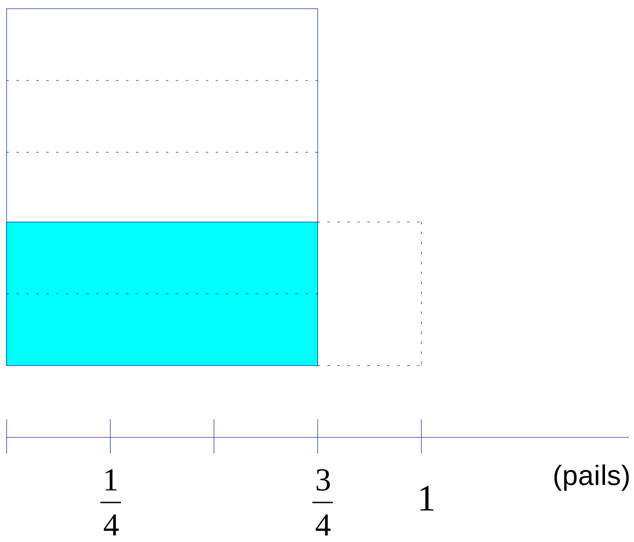
Besides the team, 12 other teachers took part in the lesson observation and post lesson discussion (2 teachers from other schools, 5 preservice teachers, 5 internal teachers who teach other subjects at the school), as well as 2 guests from university who acted as "outside advisors" to provide insights for the group's lesson study process.

THE TWO APPROACHES

One of the goals of this lesson study was to come to understand, based on the observations of student reactions and effects on their learning, which of the two approaches is more effective. It should be noted that the team members chose this particular topic because of its inherent difficulty, and in order to investigate ways of teaching it so that it would be meaningful for the students.

After the team members studied instructional materials from a number of curricula and resources and considered a variety of approaches, two distinct approaches were chosen for the implementation. The first approach started with a formal derivation of the rule "invert and multiply" using an elementary proof, which was then followed by a number of demonstrations of the rule in a geometric context. The intention of this part was twofold - first, to give students a way to come to the realization of the necessity of the rule in a natural way, where they could see and verify it concretely, and second, to reinforce what was found out earlier through deduction. This lesson differed from the traditional instructional approach in that it was full of lively and intellectually stimulating teacher-student interactions, marked by a high degree of student attention on the mathematical content under consideration. However, it resembled the traditional approach in its teaching of symbolic computations first. We refer to this lesson as Lesson A. The second approach was based on the idea of teaching through problem solving, using the method of cognitively guided instruction (CGI). While in the first approach the rule was given upfront, albeit proven and derived with student participation, the second approach aimed at having students come up with the conclusion independently, using the task presented in Figure 1, which is a task adopted from Tokyo Shoseki's Mathematics for Elementary Schools.

With $\frac{3}{4}$ pail of paint you can paint $\frac{2}{5}$ of the wall. How much of the wall can you paint with 1 pail of paint?



To find the area you can paint with 1 pail, you can divide the same way as with whole numbers even though the amount is expressed as a fraction.

Let's think about how to calculate!

Figure 1: The task

The CGI, as described by Fennema et al. (1992), assumes that students already posses intuitive mathematical knowledge, prior experience and a certain level of problem solving skill from which new concepts are to be built through a problem context, designed to elicit that which is to be learned. This approach values the different thinking paths that students use to come up with their conclusions, and uses that as grounds for formalization of new mathematical concepts. The intent was for students to “discover” the rule for division of fractions by solving the problem independently. The teacher then selected several students and invited them to present their solutions on the board for a full class discussion from which the rule for division of fractions was established. Here we refer to this lesson as Lesson B.

Unanticipated, one interesting finding that emerged from the post lesson discussion was that both approaches worked in terms of achieving the goals of the lesson and in terms of student engagement. Participants concluded that both were just as effective. Although some participants may have preferred one approach over the other, they did not, for the most part, make their personal preference interfere with their observations and judgments of the lessons.

It should be noted that before the lesson observation, participants were given a short presentation about the lesson study and they were briefed on the lesson study observation etiquette. They were instructed to observe the teaching and learning itself, without interfering with the environment (say, by helping a student who is found to be struggling), document student reactions and emerging understandings, collect data of student work, and note classroom interactions and supports that seem to bring about student understanding. The discussion was then based on that evidence.

It should also be noted that the team succeeded in developing a sense of collective ownership of the two lessons, and none of the team members were attached to any particular way. Rather, they were genuinely interested in finding out which approach was more effective. It seems that the instructional approach in and of itself (“formal deductive” vs. “cognitively guided”) does not carry the answer to effectiveness of practice. It is not a new finding that there is no single right way to teach. Still, perhaps too much hope and resources have been put into promoting various programs based on the fad of the day that may not be the key for student learning at all.

CRITICAL LENSES

Lesson study can be viewed as an integration of research, teaching and learning with the aim of advancing effective teaching practices. However, having these good intentions is not sufficient. Several authors have already described a number of pitfalls and misconceptions about lesson study that can render the whole process ineffective. Fernandez et al. (2003) suggest that to benefit from lesson study teachers will first need to learn how to apply critical lenses to their examination of lessons. They describe three such lenses and their role in making lesson study powerful. We now consider these lenses in the context of our experience together with their potential for refining the practice of lesson study in general.

The researcher lens

According to Fernandez et al. (2003), it is critical for the success of the team to "see themselves as researchers conducting an empirical examination, organized around asking questions about practice and designing classroom experiments to explore these questions". In particular, they propose four critical aspects of good research: the development of meaningful and testable hypotheses, the use of appropriate means for exploring these hypotheses, the reliance on evidence to judge the success of research endeavors, and the interest in generalizing research findings to other applicable contexts. One such question that the team attempted to answer was the one mentioned earlier, related to the effectiveness of the approach. There were several other hypotheses that were tested and explored during this lesson study, but which cannot be treated fairly in the scope of this paper. One conclusion, however, was that of the importance of maintaining the layers of goals (overarching goal for student development, goals of the mathematics program of school, goals of the unit of study, goals of the lesson) in sight, and consistently relating the planning, observation foci, discussion and results to these goals.

The curriculum developer lens

This perspective is concerned with how to sequence and connect children's knowledge by skillfully orchestrating children's learning both across and within lessons. In this process, careful thought is given to developing what the Japanese call the "lesson progression". The team considered how to relate instruction to prior and future learning through contemplating the development of mathematical content within a lesson. The presence of this lens is exemplified in Figure 2. Note the references such as "we already know", which are placed as explicit links which students are expected to use to deduce the new rule. Curriculum developer lens sensitizes the teachers to continually ask themselves, what kind of knowledge and understandings students will need to bring to the lesson to enable the mathematical ideas to develop coherently.

Lesson Progression:

We already know how to expand the division (we called this the "division property"), for example:

$$15 \div 3 = (15 \times 2) \div (3 \times 2)$$

We will use this to try to divide two fractions. Let us say that we need to divide:

$$(2/3) \div (5/7) = ?$$

Using the rule for the expansion of division, we can multiply both fractions by the reciprocal of the second fraction (the divisor):

$$(2/3 \times 7/5) \div (5/7 \times 7/5)$$

We know how to multiply two fractions: We will multiply numerators of these fractions and write the result in the numerator, and then multiply denominators and write the result in the denominator of the resulting fraction. Now we will apply that rule:

$$[(2\times7)/(3\times5)] \div [(5\times7)/(7\times5)]$$

In the dividend we can write: 14/15,
and in the divisor we can write: 35/35.
But, 35/35 is equal 1, and now we can write our division in this way:

$$(14/15) \div 1$$

We know that every number divided by 1 gives the same number, so we can say:

$$(2/3) \div (5/7) = (2/3 \times 7/5) \div (5/7 \times 7/5) = [(2\times7)/(3\times5)] \div [(5\times7)/(7\times5)] =$$

$$= [(2\times7)/(3\times5)] \div 1 = (2\times7)/(3\times5) = 14/15$$

What did we do here? In order to divide one fraction by another, we multiply first fraction by the reciprocal of the second fraction. This is visible in the last two

Figure 2: Section of Lesson A plan.

The student lens

Another perspective employed in the lesson study process is that from the eyes of the students. Through this lens, teachers attempt to understand students' thinking and anticipate their responses. They use this lens to determine how to use what students already know and are able to do to build students' understanding. The student lens is employed during the stages of planning, implementation and observation. Here we present a part from the lesson plan of Lesson B, which exemplifies how teachers employed the student lens, in the context of anticipating student responses. The team came up with the following six ways in which they anticipated students could come to the solution of the task shown in Figure 1 in particular, and to the rule for division of fractions in general. While only three of the six anticipated student responses emerged in the actual lesson, the team concluded that this in itself was a worthwhile exercise as it expended the pedagogical content knowledge of all team members.

(A) Thinking can be supported by the diagram, using a two step process (divide by 3 then multiply by 4).

$\frac{3}{4}$ pail → $\frac{2}{5}$ wall

$\frac{1}{4}$ pail → $\frac{2}{5} \div 3$ wall (based on previous discussion this is $\frac{2}{5\times 3} = \frac{2}{15}$ wall)

$4\times\frac{1}{4}$ → $\frac{2}{15}\times 4$ wall (based on the results established in previous lesson on multiplication of a fraction by a whole number, and multiplication of fraction, this is $\frac{2\times 4}{15} = \frac{8}{15}$)

When formalizing this process, we look at it as:

$$\frac{2}{5} \div \frac{3}{4} = (\frac{2}{5} \div 3)\times 4 = \frac{2}{5\times 3}\times 4 = \frac{2\times 4}{5\times 3} = \frac{2}{5}\times\frac{4}{3}$$

(B) Thinking can be supported by the property of division (when you multiply both the dividend and the divisor by the same number, the quotients will remain the same). The student can change the division into a (fraction)÷(whole number), and pursue the calculation that way.

$$\frac{2}{5} \div \frac{3}{4} = (\frac{2}{5}\times 4) \div (\frac{3}{4}\times 4) = \frac{8}{5} \div 3 = \frac{8}{5\times 3} = \frac{8}{15}$$

(C) Thinking can be supported by the property of division (when you multiply both the dividend and the divisor by the same number, the quotients will remain the same). The student can change the division into a (whole number)÷(whole number), and pursue the calculation that way.

$$\frac{2}{5} \div \frac{3}{4} = (\frac{2}{5}\times 20) \div (\frac{3}{4}\times 20) = 8 \div 15 = \frac{8}{15}$$

(D) Similar to (C) only this time using the idea of fraction expansion, and postponing the evaluation of numerator and denominator.

$$\frac{2}{5} \div \frac{3}{4} = \frac{\frac{2}{5}}{\frac{3}{4}} = \frac{\frac{2}{5}\times 5\times 4}{\frac{3}{4}\times 5\times 4} = \frac{2\times 4}{3\times 5} = \frac{8}{15}$$

(E) Using the reverse of the procedure for multiplication of two fractions. We have $\frac{a}{b}\times\frac{c}{d} = \frac{a\times c}{b\times d}$, so it also holds that $\frac{e}{f} \div \frac{g}{h} = \frac{e\div g}{f\div h}$. For this to work out expanding the dividend is necessary, to make the numerator of the dividend a multiple of the numerator of the divisor, and likewise, the denominator of the dividend a multiple of the denominator of the divisor.

Expanding $\frac{2}{5}$ so that the numerator is a multiple of 3 and the denominator is a multiple of 4, we must expand by the LCM(3,4)=12.

$$\frac{2}{5} \div \frac{3}{4} = \frac{2\times 12}{5\times 12} \div \frac{3}{4} = \frac{2\times 12\div 3}{5\times 12\div 4} = \frac{8}{15}$$

(F) Combining the approaches (D) and (E), students can think about dividing the numerators and dividing the denominators directly, and then expanding the complex fraction to annihilate the two denominators.

$$\frac{2}{5} \div \frac{3}{4} = \frac{2\div 3}{5\div 4} = \frac{\frac{2}{3}}{\frac{5}{4}} = \frac{\frac{2}{3}\times 3\times 4}{\frac{5}{4}\times 3\times 4} = \frac{2\times 4}{5\times 3} = \frac{8}{15}$$

Figure 3: Anticipating student thinking from Lesson B plan.

Deprivatizing the classroom and peer review of teaching

What happens in the classroom directly impacts student learning. Therefore, it is not a surprise that lesson study, with its features of a real classroom setting and authentic conditions, provides an excellent setting for study, research, examination, training and consequently also for the transformation of teachers' instructional practices. This vision of lesson study as a powerful means for the development of new ideas and practices must include room for knowledgeable coaches who can stimulate the thinking of the teacher teams, so they can rise beyond their own limitations and push their lesson study practice into reach arenas (Fernandez et al., 2003). However, for this to happen it is necessary for teachers to open up their classrooms to their peers, administrators, and outside observers such as mathematics educators and mathematicians who have an interest in mathematics education. This is probably the greatest challenge for many teachers, as it requires trust, confidence, and a belief that the benefits of doing so far outweigh the individual concerns that teachers may have (Sztajn at al., 2007). Given that most North American teachers are used to working in isolation, this kind of "deprivatizing of the classroom" may be perceived as a threat (Armstrong, 1994). In the experience of the Southpointe team, two of the five teachers were willing to teach a public lesson, while the other three expressed some concern over this kind of exposure. As this is a realistic issue, and a possible obstacle to building in-service mathematics education communities, further studies are needed to shed light on what factors contribute to building trust. Teaching is a public practice and should be open to peer review – not as a form of assessing teacher effectiveness, but as a form of professional development.

SUMMARY

Lesson study, as a professional development process, in which teachers systematically examine their practice with the goal of becoming more effective seems to have great potential for the transformation of mathematics education in general, and at any school in particular. One of the strengths of the lesson study is that there are multiple observers who each contribute their observations of student learning. When these individual observations are combined a complete picture of students' experience of the lesson emerges. To provide focus and direction to this work, collaboration with researchers from university is important. In Japan, the outcomes of this process are used to refine curricula and textbook contents, and shape teacher training programs at universities. Lesson study has been credited for having transformed education in Japan into one of the world's best, where student achievement is exceptionally high.

The post lesson discussion is a key component of the process, where participants engage in critiquing the lesson. At Southpointe, these discussions were very rich, insightful and worthwhile, and they certainly contributed to the deepening of pedagogical and the content knowledge, and of the knowledge of how students learn mathematics, of all involved. Instructors and observers concluded they learned a great deal in the process of such “research in action” where all the elements of a real classroom situation, from planning to implementation, come into full view. Regardless of whether the research lessons received critique or praise, the discussions seemed to indicate a satisfaction with the fact that the practice of Lesson Study is being introduced in the school.

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