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Complexity thinking as a pragmatic pedagogical and investigative tool

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Downloadable material can be found at: http://www.cust.educ.ubc.ca/faculty/davis.html

Students' papers are reprinted as submitted with no further editing.

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LOGARITHMS: SNAPSHOTS FROM TWO TASKS

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Our study addresses the understanding of logarithms and common difficulties which high school students encounter as they study this topic. We focus on two tasks: one standard and one non-standard that involve logarithmic expressions or require the use of logarithms in a solution. For the purpose of analysis we have modified the interpretive frameworks developed by Confrey in her study of exponents and exponential expressions, to the study of logarithms. Our results indicate students' disposition towards a procedural approach and reliance on rules, rather than on the meaning of concepts. We conclude with pedagogical considerations.

The miraculous powers of modern calculations are due to three inventions: Arabic notation, decimal fractions and logarithms (Cajori, 1919). The first two of these inventions have been investigated in great detail by researchers in mathematics education, while logarithms have received very limited attention. This is rather surprising, given the centrality of the concept in applied mathematics, as well as in the secondary school mathematics curriculum. This article presents a part of an on-going study that aims to describe and analyse issues involved in the understanding of logarithms by high school students. It is best described as a series of snapshots highlighting students' perceptions, rather than an attempt to draw a comprehensive picture.

As an introduction, let us consider the following excerpt from a classroom interaction between a teacher and a group of grade 12 students. The conversation took place as part of a review after logarithms had been studied for several weeks.

Teacher:	Can you find the exact value of $5\log_3 9$?
Ryan:	You should calculate the log, the log is 2.
Teacher:	Would anybody explain why $\log_3 9$ equals 2?
Bob:	Because it equals log9 divided by log3, and it equals 2 (answer was given using calculator).
Teacher:	Why does $\log_3 9$ equal log9 divided by log3?
Bob:	Because of the change of base law.
Sharon:	Somehow I got 1.2756.
Bob:	O, you just forgot the bracket after 9.
Sharon:	Why do you need that bracket?
Bob:	If you miss it, then you are finding a logarithm of 9 over log3, not a quotient of two logs.
Sharon:	I see it works now. Thanks.

Ryan:	But it is not fair, you've used a calculator!
Teacher:	Is it possible to get the answer without a calculator?
Becky:	I did. I can show it. First, I took 5 under the log, so it became $\log_3 9$. Then, I knew I had to find an exponent of 3 that equals 9 to the power of 5. Basically, I solved the equation $3^x=9^5$. Converting 9 to 3^2, I got x = 10.
Students:	Cool, nice
Teacher:	Can someone suggest a different approach?

Several observations from this exchange are warranted. The immediate and "trivial" solution – that involves 5×2 once the value of $\log_3 9$ is recognized – is missing. Further, there is unnecessary reliance on computational procedure and incompetent use of a calculator to implement this procedure. We wonder, what influences students' choice of approach? What is their understanding of logarithms? How can their understanding be enhanced beyond pushing the "log" button in a calculator? We address these questions in our study.

INTERPRETIVE FRAMEWORKS FOR LOGARITHMS AND LOGARITHMIC FUNCTION

The Frameworks used in this study are developed in analogy to the interpretive frameworks used by Confrey (1991) in the analysis of students' understanding of the exponents and exponential function. The three Frameworks – labelled below as A, B, and C – attend to logarithms considering numbers, operations and functions. Though presented in order of increased complexity, these Frameworks are to be viewed as a system, rather than a linear progression.

Framework A: Logarithms and Logarithmic Expressions as Numbers

In Framework A we investigate to what degree logarithms are understood as numbers and whether the value of a logarithm influences this understanding. In the traditional curriculum, the concept of logarithm is presented as an inverse of the exponent. A novice operating in this framework may correctly interpret, for example, the value of $\log_3 9$, by using the definition ($3^2=9 \Rightarrow \log_3 9=2$), but fail to interpret $\log_3 1/9$, or $\log_3 1$.

Framework B: Operational Meaning of Logarithms

The main issue explored in our second interpretive framework is the students' understanding of the operational character of logarithms. While focusing on operations with logarithmic expressions we were interested in students' awareness of the isomorphism between multiplicative and additive structures that determine the "rules" by which logarithmic expressions are manipulated, as well as students' ability to imply the isomorphic relationship in both directions. Furthermore, students' ability to imply the isomorphic relationship in both directions can be examined. For example, the approach that students take in simplifying $\log_3 90 - \log_3 10$ or in expanding $\log_c a^2 b$ may provide insight into the operational meaning students assign to these expressions.

Framework C: Logarithms as Functions.

In investigating students' understanding of logarithmic functions we consider how students' relate the definition of logarithm to logarithmic function, and how they use its properties and different representations in constructing graphs and solving problems. In the current analysis, this framework is mentioned only in passing.

Research Setting

Participants and course

The participants in this research were 19 secondary school students who were enrolled in the Principles of Mathematics 12 course. It is important to mention that the course Principles of Mathematics 12 is not a required course for high school graduation in our site, so students enrolled in it are a self-selected and motivated group. Generally, the achievement level of these students ranges from middle to high. Many of them chose to enrol in this course because of their future plan to attend post-secondary programs in which this course is required for admission.

The unit *Logarithms and Exponents* was taught as a part of the course. In terms of recommended instructional time, this is the second largest unit accounting for about 17% of the course. While exponents and exponential notations were familiar to students from previous studies, this unit was their first introduction to the concept of a logarithm. The topics addressed in the unit include algebraic representations of exponents and logarithms, main laws and applications, logarithmic and exponential equations, the relationship between the graphs of the exponential and logarithmic functions, number *e* and natural logarithms. Further, the curriculum included modelling situations such as compound interest, radioactivity, continuous growth and decay.

Tasks

As a snapshot from our research, we focus here on two tasks:

(1) Simplify the following expression: $\log_3 54 - \log_3 8 + \log_3 4$.

(2) Which number is larger 25^{625} or 26^{620}? Explain.

These tasks were chosen as they illustrate a variety of tasks students faced in their learning of logarithms. Task 1 is considered "standard" as students approached similar tasks during their class sessions and in their homework. Task 2 is non-standard; it presents novelty in its level of difficulty and in providing no explicit reference to logarithms. Students' work on these tasks resulted in a variety of approaches and provided insight on how they view logarithms.

RESULTS AND ANALYSIS

Task 1: Simplify the following expression: $\log_3 54 - \log_3 8 + \log_3 4$.

This task was part of a quiz administered after the students completed the section on operations with logarithms (n=17). Table 1 presents a quantitative summary of students' solutions, where C, IC and PC indicate "correct", "incorrect" and "partially correct", respectively.

C/IC/PC	Examples of Solutions	#of students presenting this solution
C(1)	$\log_3 54 - \log_3 8 + \log_3 4 = \log_3 \frac{54}{8} \times 4 = \log_3 27 = \frac{\log 27}{\log 3} = 3$	9
C(2)	$\log_3 54 - \log_3 8 + \log_3 4 = \log_3\left(\frac{54}{8}\right) + \log 4 = \log_3(6.75) + \log_3 4$ $= \log_3(6.75 \times 4) = \log_3(27) = 3$	3
IC(1)	$\log_3 54 - \log_3 8 + \log_3 4 = \log_3(54 - 8 + 4) = \log_3 50 = 3.5609$	3
IC(2)	$54^3 \div 8^3 \times 4^3 = 19683$	2

Table 1: Quantitative Summary of Students' Solutions to Task 1

Students' responses to Task 1 are best viewed through Framework B – operational meaning of logarithms. Students' implementing IC(1) and IC(2) experienced difficulty in carrying out the operations. The IC(1) responses can be considered as a case of "misapplication of linearity" (Matz, 1982) or incorrect application of the distributive property. This tendency towards linearity is well documented in mathematics education literature, usually being exemplified with interpreting $(a+b)^2$ as a^2+b^2 or sin(a+b) as sin(a) + sin(b). According to Matz (1982) these errors may be explained as reasonable, though unsuccessful attempts of students to adapt previously acquired knowledge to a new situation. As such, these students performed symbol manipulation overgeneralizing familiar procedures.

IC(2) can be seen as a misinterpretation of the definition of logarithms. Indeed, logarithms are defined based on the exponential relation. It could be the case that the abbreviated phrase "logarithm is the exponent", which is often used in an attempt to interpret the definition, was memorized by these students and interpreted literally, by substituting 54^3 for $\log_3 54$. Considering Framework A, it appears that students presenting this solution did not view logarithms as numbers; as such, they attempted to isolate what they perceived as numbers in order to carry out a calculation. Considering Framework B, another interesting feature of IC(2) is the change of subtraction to division and of addition to multiplication. Though this transformation is appropriate in the context of logarithms, its implementation, as simple substitution, results in an error.

Students implementing C(1) and C(2) solutions demonstrated proficiency in manipulating an expression involving logarithms, as such it is reasonable to conclude that the operational character of logarithms was familiar to these students. In both cases the expression $\log_3 27$ appears in students' solutions, and it is of interest here to observe how students proceed from this point. In C(2), this expression is followed by the answer 3. As participants in this research were instructed to record every step in their solution, we believe that this answer was reached by recognizing that $3^3=27$. In C(1), we make a note of unnecessary application of change of base and the use of a calculator to reach the answer. This approach is similar to the approach recorded in our introductory vignette. It appears that the ability to manipulate algorithmic expressions overshadows students' ability to interpret them as numbers. Symbolic manipulation followed by calculation, wherever possible, is a preferred choice. Therefore the optimistic interpretation of the results will point out that 12 out of 17 students who attempted to solve this problem implemented correct procedures and reached a correct answer. A pessimistic interpretation notes that only three students were able to apply their understanding of the meaning of logarithm in their solutions.

Task 2: Which number is larger 25^{625} or 26^{620}? Explain.

This task was administered as part of a written questionnaire after the completion of the instructional unit on logarithms. To answer this non-standard question, students required a conceptual understanding of logarithms rather than memorization of a learned algorithm or technique. Table 2 provides a summary of students' responses.

C/IC/PC	Examples of Solutions	# of students presenting this solution
C(1)	26^{620} is larger because the logarithm of this number is larger (claim only)	5
C(2)	26^{620} is larger because the logarithm of this number is larger (claim followed by explanation)	2
C(3)	$26^{620} = \left(25^{1.012}\right)^{620} = 25^{632.6}$ $25^{632.6} > 25^{625}$	1
IC(1)	25^{625} is larger because it has a larger exponent	10
IC(2)	25^{625} is larger as it can be written as $5^{2^{5^4}}$	1

Table 2: Quantitative Summary of Students' Solutions to Task 2

Eight students gave the correct answer: however, only three of them – identified on Table 2 as C(2) and C(3) – presented mathematically sound solutions. Others followed their intuition, or just made a lucky guess. Eleven out of 19 students answered incorrectly.

The solution IC(1), produced by ten students, was justified by the claim that a larger exponent determined a larger number. Their guess was based on the premise that the exponent indicates the number of self-multipliers of 25 or 26, and so the "longer" product is the larger one. These participants did not connect this problem to the concept of logarithm whatsoever. Students' response IC(1) may be explained as the intuitive rule of the form "More A – More B" identified by Stavy and Tirosh (2000). A popular exemplification of this rule is in students' intuitive beliefs that a shape with a larger perimeter will also give a larger area, or that a taller container has larger capacity. "Larger exponent – larger number" is yet another example of this rule.

The result IC(2) was unique in this group. The number 625 attracted the participant's attention, since it is a power of 5, and the base of the first number is 5^2. The student tried to use this information, but his conclusion has not been justified. We believe that it was our unfortunate choice of numbers that created this distraction, since noticing powers of 5 does not help to reach the solution in this case.

Among the eight students who correctly identified the larger number, five presented the argument exemplified in Figure 1. They simply found the logarithm in base 10 for both numbers and concluded that a larger logarithm corresponded to a larger number.

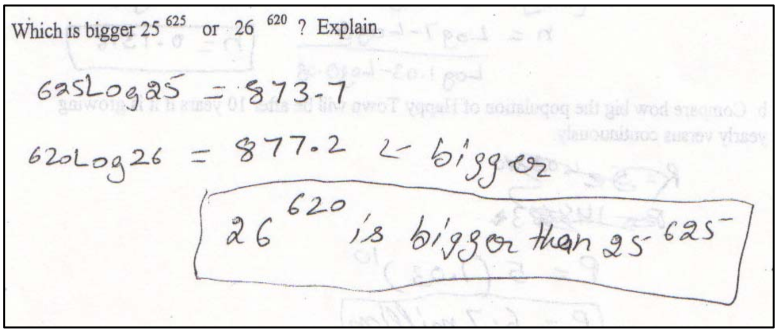

Figure 1: Example of solution C(1)

Since no additional explanation was provided as to why this was the case, it is hard to know whether the conclusion these students drew was based on their understanding of an increasing exponential function (within Framework C), or whether it was a "lucky" implementation of the intuitive "More A – More B" rule. After all, this rule is robust in people's intuition because experience shows that it "works" in a large number of cases.

As mentioned earlier, three students produced the complete and mathematically sound solutions that are shown in Figure 2. The solution in C(2) was used by two participants. Unlike their classmates who produced solution C(1), these two students explained why a larger logarithm corresponded to a larger number by looking at the exponents of 10. The solution C(3) was demonstrated by one student only, who presented both exponential expressions with the same base of 25. From the perspective of Framework A, these solutions illustrate that the students understand not only that logarithms are numbers, but also that any real number can be presented in the form of a logarithm.

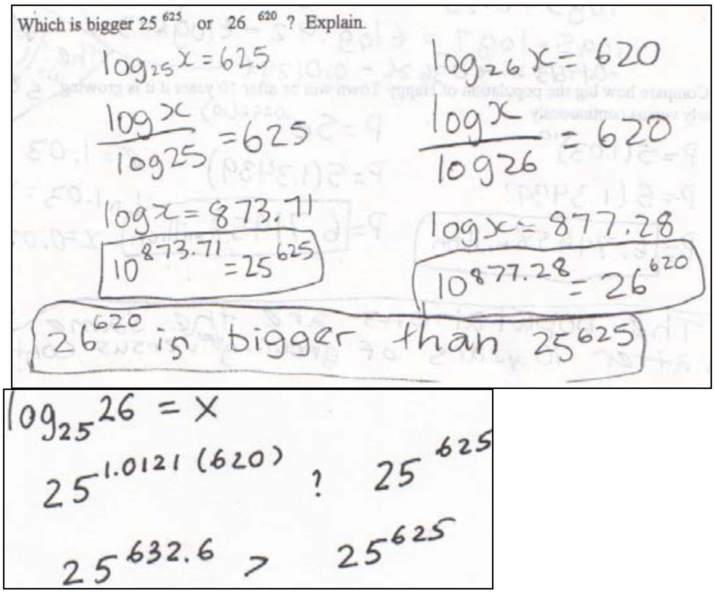

Figure 2: Left: Example of solution C(2), Right: solution C(3)

DISCUSSION AND PEDAGOGICAL CONSIDERATIONS

We believe that understanding the challenges students experience in a certain mathematical content, and determining the source of their difficulties are a necessary steps in an attempt to overcome these difficulties. In this study, we explored students' difficulties with logarithms by attending to a limited number of tasks that students performed. In an attempt to examine several issues involved in students' understanding, we proposed a system of interpretive frameworks. We also wondered whether the frameworks are helpful as the means to this end.

In considering Framework B, we observed that the ability to operate with logarithmic expressions should not be taken as understanding of their operational meaning. The introductory vignette, as well as solutions for Task 1 (labelled C(2)), provide a clear indication that operations can be performed successfully when the meaning is overlooked. The degree to which students' procedural fluency correlates with the operational meaning that students constructed requires the attention of further research. Framework B could be subdivided into "operational fluency" and "operational meaning".

Focusing on Framework A, our results suggest that students' ability to interpret a logarithmic expression and indicate its value does not indicate that logarithms are understood as numbers. We have reported instanced of overgeneralised linearity in working with logarithms, likely derived as an extension of previous experience with whole numbers. Understanding logarithms as numbers could present a greater difficulty. So, the pedagogical question is: Is it possible, and if so, how is it possible to help students understand logarithms as numbers?

It is reported in research that students often consider as numbers only standard decimal representations, and have difficulties in interpreting different representations of numbers as numbers. That is, while 25 is definitely a number, 27-2 or 5^2 are seen as exercises, operations or instructions to follow (Zazkis and Gadowsky, 2001). In order to treat a logarithm as number it should be perceived as an object. Treating mathematical concepts as objects supports the construction of corresponding mental objects in the mind of students (Dubinsky, 1991; Sfard, 1991). One possible way to treat concepts as objects is to involve them as inputs in mathematical processes: that is, to act on them or to perform operations on them. However, as our results show, following the prescribed curriculum and performing operations that implement the laws (for division, multiplication and change of base) does not necessarily serve this purpose. We wonder whether additional tasks integrated into students' experience could enhance understanding of logarithms as objects. For example, the task of ordering and placing on a number line the following set of numbers -log2, log5, log1/2, log1, -log3/4 may promote the understanding of these expressions as numbers. A further task may require the ordered placement of logarithms with different bases. Another example of a task that may support number/object construction is an equation of the form similar to $x\log_7 15 = (x+2)\log 20$. In our experience a task like this introduced confusion, and students attempted a variety of manipulations in order to present the expressions with a common base. However, once logarithms are perceived as numbers, the task in hand is just a linear equation.

As in any research that explores a novel area, we end up with questions rather than definite answers. Focusing on a series of snapshots is the first step in identifying the areas of further attention with the long-term goal of drawing a detailed and comprehensive account of the learners' conception of logarithms. As illustrative snapshots, we described students' work on one standard and one non-standard and challenging task, and provided several pedagogical considerations. Further research will examine the effect of implementing these suggestions on the understanding of logarithms, and will provide a more refined account of what this understanding does or might entail.

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CONVERSATIONS ABOUT CONNECTIONS: A SECONDARY MATHEMATICS TEACHER CONSIDERS QUADRATIC FUNCTIONS AND EQUATIONS

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This paper reports a single case (Robert) from a larger study in which secondary mathematics teachers were interviewed about their understanding of mathematical connections. Teachers participated in three interviews, each progressively more structured and focussed on their explicit connections related to a particular mathematics topic. A model for categorizing mathematical connections is presented and used as an interpretive tool. Robert is a knowledgeable and experienced mathematics teacher who values making connections in general terms. However, he finds making explicit specific mathematical connections difficult and makes connections of limited types.

The National Council of Mathematics Teachers' (NCTM) document, *Principles and Standards for School Mathematics* (2000) and its earlier versions in 1989 and 1991 establish a framework to guide improvement in the teaching and learning of mathematics in schools. The documents identify "mathematical connections" as one of the curriculum standards for all grades K to 12. In this framework, "… mathematics is not a set of isolated topics but rather a web of closely connected ideas" (NCTM, 2000, p. 200). Making the connections is taken to promote students' understanding of new mathematical ideas (NCTM).

In the mathematics education literature, a "mathematical connection" is conceptualized in a variety of ways ranging from mappings between equivalent representations (Hines, 2003) to links between mathematical concepts (Zazkis, 2000), to unifying themes that cut across several domains (Coxford, 1995). There are researchers who never use the term "connections" while they employ the idea; similarly, there are writers who use the term "connections" without definition. In both the research and pedagogical literature the notion of what a mathematical connection is, often remains implicit.

In this study, I take a mathematical connection to be a true relationship between two mathematical ideas. I draw on Skemp's notion of a person's mathematical knowledge as a set of hierarchical schemata (Skemp, 1987), composed of mathematical concepts and the connections among them. These connections may be "vertical", in which a concept can be thought of as a composite of simpler concepts, or "horizontal", in which a concept can be thought of as a transformation of another. I propose the following as a starting list (definitely not exhaustive), of specific types of mathematical connections where A and B represent two mathematical ideas:

1. **A is an alternative representation of B (horizontal)**. The alternative is a different category of representation, for example, symbolic (algebraic), graphic (geometric), pictoral (diagram), manipulative (physical object), verbal description (spoken), written description. For example, the graph of a parabola is an alternative representation of $f(x) = ax^2 + bx + c$ (geometric/algebraic). One of the McDonald's golden arches is an alternative representation of the graph of a parabola (physical/geometric).
2. **A is equivalent to B (horizontal)**. Concepts that are represented in different ways are equivalent too. I use A is equivalent to B for an equivalence within the same form of representation. For example, 3 + 2 is equivalent to 5; $f(x) = ax^2 + bx + c$ is equivalent to $f(x) = a(x-p)^2 + q$. In these examples, both A and B are symbolic. ("5" and 5 cookies on a plate, I would consider alternative representations).

3. **A is similar to B (A intersects B) (horizontal)**. A and B share some features in common. For example, a square is similar to a rectangle. This is ambiguous and imprecise by itself and should be qualified, for example, A is similar to B because they are both… For example, a square is similar to a rectangle because they are both quadrilaterals (or both have 4 sides).
4. **A is included in (is a component of) B, B includes (contains) A (vertical)**. This is a hierarchical relationship between two concepts. For example, a vertex is a component of a parabola.
5. **A is a generalization of B; B is a specific instance (example) of A (vertical)**. This is another kind of hierarchical relationship. For example, $ax^2 + bx + c = 0$ is a generalization of $2x^2 - 7x + 3 = 0$.
6. **A implies B (and other logical relationships)**. This connection indicates a dependence of one concept on another in some logical way. For example, the degree of an equation determines the maximum number of possible roots.
7. **A > B; B < A (and other order relationships)**. For example, 7>2.

Problem Statement and Research Question

In an earlier study (Businskas, 2005), secondary mathematics teachers, interviewed about their views of mathematical connections, spoke favorably of the importance and value of attending to making connections, but found it very difficult to present examples from their own teaching or their own thinking of specific instances of mathematical connections. While students might make some useful connections spontaneously, the accepted position in the pedagogical literature, is that teachers' interventions are necessary if students are to deal with mathematical connections in a systematic and meaningful way (Weinberg, 2001, Thomas & Santiago, 2002). If teachers really don't have a coherent view of how mathematics ideas relate to each other themselves, then they can't very well help their students to recognize those connections either.

In this follow-up study, I try to probe deeper into teachers' understanding of specific topics in mathematics, to try to make their notions of mathematical connections explicit. The study addresses the following question:

- what kinds of mathematical connections can teachers describe in their knowledge of a mathematical topic?

Research Setting

Participants in the study from which this case is derived were ten secondary mathematics teachers who volunteered for the study. Teachers were interviewed individually three times over a three-month period in their schools.

Interviews

The first interview was an acclimatizing interview and drew out information about teachers' background and general views about teaching mathematics.

The second interview was a semi-structured interview about a mathematical topic chosen by the teacher. Prepared questions included:

- Please tell me about your own understanding of this topic… What are the important concepts/ideas and procedures that make up your understanding of this topic?
- Please tell me how the ideas and procedures that you've identified are related to each other or related to other topics in mathematics.

- From your point of view as a teacher, what are the most important concepts and procedures that you want your students to learn?
- What are the important concepts that your students must already know before study this topic? How is each of these "prerequisite" concepts related to the ideas/procedures that you will teach?

Responses that teachers made that included some reference to connections were followed up with questions asking them to elaborate. If teachers did not naturally make any references to connections, they were asked further probing questions, and sometimes even leading questions in an attempt to get them to voice an opinion. After the interview, teachers were asked to show the researcher their lesson plans/planning notes for their topic. These notes were examined for any references to making connections.

The third interview was a task-based interview in which all teachers dealt with the same topic – quadratic functions and equations. Teachers were given a set of 82 cards (see Appendix) containing mathematical terms, formulae and graphs related to this topic, gleaned from a selection of high school mathematics textbooks. They were asked to organize the cards in some way that showed the relationships among them. They were instructed "Please group them in a way that will show how they are connected". After they completed the task, they were asked to explain their organization, and were constantly pushed to elaborate their statements about connections.

Coincidentally, one of the teachers, Robert (a pseudonym), chose "quadratic functions" as his topic, thus providing me with three sources of data (2 interviews and his planning notes) about his thinking about this topic. This paper considers his views.

Analysis

All interviews were transcribed.

For the "chosen topic" interview, I read the text to extract the particular statements made by Robert, that dealt directly with the topic of quadratic functions and equations and with connections, or the teaching of these topics. From the extracted statements, I compiled a summary of Robert's description of his understanding of the topic, and a list of specific references to connections. In these, I looked for ideas, that because of their repetition or his emphasis, might be indicators of consistent themes in Robert's thinking.

For the task-based interview, there is also a visual record of the groupings (see attached photo). From these data, I compiled another list of explicit connections that he made.

I read through Robert's planning notes primarily to identify any aspects that indicated a planful way of attending to connections in his teaching. The data also provided some insight into Robert's thinking about the topic, but its reliability as a measure of his own understanding is questionable because of his reliance on the textbook in developing his lesson sequence.

Data and Interpretations

Robert is in his ninth year of teaching. Teaching is his first career and he is at his second school. Robert has a Bachelor's degree with a major in Mathematics. Of the Mathematics courses that he took towards his degree, he identifies about half of them as being relevant to his work as a high school mathematics teacher. He is confident in his knowledge of the BC curriculum, and is familiar with the NCTM Standards. He also has a Master's degree in education.

He chose the topic of quadratic functions and equations because "I think it has a lot of interesting connections, like visual and algebraic connections and lots of connections to real physical problems". It became clear right from the beginning that Robert could not separate his own personal understanding of the topic from his understanding of it for teaching. His own content knowledge and his pedagogical content knowledge are conflated (Shulman, 1986). In fact, he sees himself as teaching his students everything that he knows about the topic: “I think I try to give them everything I know... I try to tell them everything I know about it”.

Here’s what Robert had to say about quadratic functions and equations in each of the tasks. His comments are summarized and paraphrased, but all mathematical terminology used is his. Even though Robert was asked to talk about his own understanding of the topic, he switched almost immediately to talking about what his students know (or should know).

“Chosen topic” interview

Content summary. Quadratic functions, or parabolas, describe the motion of objects in space. Solving quadratic equations is necessary in many different applications like finance. Important ideas are the concept of the zero, x-intercept, y-intercept, maximum and minimum, “what they mean and what they represent on a graph, how to find them, how they’re related to zeros”. The max/min, or the vertex, is on the line of symmetry which falls between the zeros. The obvious connection is between the equation and the graph.

There are three forms of the equation – the expanded form, the factored form and the vertex form. The exponent shows that the graph is not linear. Equations can be solved “by graphing or by algebra or using the quadratic formula”. The factored form can help you find zeros by plugging in y=0. And if you know the zeros, or two points on the parabola, you can figure out the graph.

Students have to be able to interpret a word problem, come up with an equation and solve it. In problems, we’re usually looking for a max/min or the zeros. Students generally know how to graph, but have trouble with problems when fractions or decimals are involved.

Explicit connections. In this interview, Robert made statements that explicitly referred to mathematical connections. These statements are listed below (chronologically) in his own words, and categorized according to the system outlined earlier. Types of connections not included in the model are shown in *italics*.

Statement	Type of Connection
If you throw a baseball, it’s a quadratic function, a parabola	*Real world* Alternate representation
The connection between obviously an equation and a graph	Alternate representation
The exponent is not going to make it linear	A implies B
Max/min which is the vertex	Equivalent representation
the important relationship is to understand that, that a graph, the picture view of the equation or the formula is the same as the algebraic	Alternate representation
I draw the picture, I do the algebra	Alternate representation
the graph just shows them all the different solutions to the equation, all the different pairs of x and y that work in this equation	Alternate represenation
these two things are the same things, one’s expanded, one’s factored [referring to equations]	Equivalent

those with a value up front; typically they have to find zeros that are not going to be whole numbers [referring to coefficient “a” in the equation $ax^2 + bx + c = 0$]	A implies B (non-zero a implies that zeros will not be whole numbers)
it’s a quadratic function, I’m looking for the maximum point, so I need to find a vertex	A contains B (quad function has max) Equivalent representation (sort of)
need to understand how to solve a quadratic in order to solve those other ones [i.e. cubic and quartic]	*Procedure*

Robert’s explicit connections are pretty sparse. Moreover the connections he identifies are overwhelmingly alternate representations. In fact, he mostly repeats the same connection – that quadratic functions and equations can be alternately represented algebraically and graphically, and makes the specific connection between the max/min and the vertex. Actually these are not the same thing, in that the max/min is equivalent to only one coordinate of the vertex, but Robert never made that distinction.

Task-based interview

After quickly examining all the cards, Robert proceeded to lay them out in groups with little hesitation, completing the task in 15 minutes. He used 57 of the 82 cards. He made six groups (see photo), which are identified below by the leading/top term in the group. The explicit connections that Robert made involving terms in each group are listed in his own words, sometimes slightly paraphrased. Connections among groups are described later.

Group	Explicit connections	Type of connection
1. function…	two different forms of the equation	Equivalent representation
	There’s the algebra side and the geometry side	Equivalent representation
2. algebra…	factor, complete the square, quadratic formula, guess and check – represents the algebraic skills they need	*procedure*
	under factor, you’ve got the remainder theorem, factor theorem, zero property of multiplication, these are all ways to solve	*procedure*
	complete the square is… an algebraic method, but it helps you find the vertex, which makes it easy to graph	*Procedure* *procedure*
	the coefficients are a, b, c… you can find the zeros by plugging into the quadratic formula	A is contained in B
3. geometry…	a parabola is part of a conic section and it’s a curve	A is contained in B
	vertex with (p,q)	Alternate representation
	table of values give you co-ordinates that are points on the curve	Equivalent representation
	maximum and minimum… they help me to find the range	Alternate representation
	I just had some of the transformations grouped together, expansion, compression… translation	A is an example of B
4. [zeros of	zeros and intercept I put in between the major	Alternate representation

a function, root, intercept]	headings [algebra and geometry] because algebraically that's what you're trying to find, and graphically that's also very easy to see	
5. inverse	inverse which is a relation and an example of it	A is contained in B
6. focus	another example of a conic section	Ais an example of B

The connections listed above are within-group connections. Of the explicit connections that Robert offered, only one was offered spontaneously, namely "there's the algebra side and the geometry side". The others were the result of probing, and sometimes repeated probing. In fact, I consciously decided to limit some of the probes because it was clear that Robert was becoming frustrated by not being able to provide an answer.

I've called these connections "within-group" connections; they include both vertical and horizontal connections, but the ideas being linked as fairly simple concepts. I also asked Robert "… can you see some extensions where some of the groups that you've identified here might be related to othe topics in math?" Again, I report the explicit connections that Robert made in his own words or close paraphrases.

Explicit Connection
Function is not just for parabolas, but for any type of function like linear or cubic, doesn't even have to be a polynomial function.
Remainder theorem, factor theorem… you could use it for solving quadratics but it's mainly for solving polynomials that are like cubic or higher.
Translations, or the transformations are not just moving parabolas… you could move any type of graph.
Domain and range, max and min, are important concepts in anything… calculus and beyond.
Conic sections, there are lots more conic sections we can look at, not just the parabola
Table of values used in any type of graphing.
Symmetry and transformations… not just moving graphs around, but moving objects, symmetrical objects and natural objects

I note that Robert offered all the statements above in response to a single question, without any additional probing. Structurally, all the connections are similar in that they are generalizations. Robert identifies a concept or procedure as one that has a broader scope than the topic of quadratic functions and equations, and then offers examples of other math topics to which the general idea applies. This type of connection is similar to Coxford's themes that run across mathematical topics (Coxford, 1995).

Cards left out. Finally, I asked Robert about the cards that he left out. Initially, he dismissed all the ones that he left out as "I didn't think that they added anything new to what I had". When asked further to consider them one by one, he was able to provide a little more detail.

Card(s) left out	Reason
Substitute	It could be a method you use to guess and check, substitute values in, but I didn't think it added anything much.
Derive	If I put derive in… with quadratic formula… but it didn't feel like it added anything to the understanding of the concept.
Variable	Again, I'm going to give the same response.

Perfect square, exponent, value, square root, square, radical	None of these things were key. Well, I think they're all kind of ideas that kids need to know before they start looking at parabolas, but I don't think it adds anything to the understanding… this is basic skills and then there's the understanding of quadratic functions beyond that.
Expression	Didn't fit in because we're looking at equations
Inequality	You could solve inequalities, but that one comes off on its own
$x= 2c/(-b \pm \sqrt{}(b^2 – 4ac)$	Don't really know what that formula's for at all… I assume that it doesn't make any sense here
$x^2 – (r_1 + r_2)x + r_1 r_2 = 0$	Is a quadratic formula but I'm not sure what r_1 and r_2 stand for. [When I told him they stood for roots]… I guess I could have added that with factor.
Associative, commutative and distributive properties	Basic algebra skills but it doesn't add to the understanding of the concept of quadratic
$p \pm \sqrt{}(-q/a)$	Not really sure what that's for
Directrix	I don't remember exactly… it's got something to do with conics
$x = -b/2a$	Don't know what that one's for either
Properties of roots	I don't really know how that fits in… I guess I could say the zeros, the intercept
$(-b/2a, (c - b^2/4a))$	Not really sure what it's for

Robert's explanations for the cards he left out fall into two categories. Some he left out because he did not recognize them. Robert left out 5 of the 14 cards that showed symbolic expressions or equations (36%). While he repeatedly emphasized the importance of the connections between algebraic and geometric forms of quadratic functions and equations, he had difficulty making the connection when the algebraic form was the starting point (for example, $(-b/2a, (c - b^2/4a))$ are the co-ordinates of the vertex, $x = -b/2a$ is an equation for the axis of symmetry).

He left out other cards because he didn't think they were important. When pushed further, he was able to offer up a connection (for example, substitution is a method used in "guess and check"). It appears, that in these cases, he recognized some connections but saw them both as far removed and obvious.

Planning. Robert's planning notes for this topic consisted of 15 hand-written pages organized into five sections following the textbook. His notes contained definitions, examples, homework assignments, and mostly model solutions for exercises in the text. The only statement that might be interpreted as referring to making a connection is "review completing the square – need it in 2.4 [section 2.4]".

Robert's pedagogical content knowledge and the curriculum

The British Columbia curriculum guide for mathematics (2000) lists learning outcomes -

It is expected that students will:

- determine the following characteristics of the graph of a quadratic function: vertex, domain and range, axis of symmetry, intercepts
- connect algebraic and graphical transformations of quadratic functions, using completing the square as required

- model real-world situations, using quadratic functions
- solve quadratic equations, and relate the solutions to the zeros of a corresponding quadratic function, using: factoring, the quadratic formula, graphing
- determine the character of the real and non-real roots of a quadratic equation, using: the discriminant in the quadratic formula, graphing
- describe, graph, and analyse polynomial and rational functions, using technology (p. A-35)

Comparing Robert's descriptions of the topic to the curriculum guide clearly indicates that his view of the topic has become that of the guide. His descriptions contained the organization and language of the curriculum, save for characteristic of roots, which he barely touched on.

Summary

In his teaching, Robert tries to promote his students' understanding of the topic while acknowledging that "not all the kids are going to pick up on all the conceptual knowledge, a lot of kids just learn procedural skills". He speaks positively about connections in general. His ability to explicitly describe mathematical connections seems related to the "grain size" of the concepts being considered. For example, he was able to list a variety of ideas that are common to quadratic functions and equations and to other topics in mathematics. At a finer grain size, dealing with simpler concepts, he found it easy to group them, but quite difficult to describe the relationships among them. At the risk of inferring too much, it seemed that he was blocked by the belief that the specific connections were so obvious, they didn't need to be stated. Explicit connections that he did make were mostly alternate representations, in particular, algebraic and geometrical.

The proposed interpretive model worked well to categorize Robert's responses. Neverthe less, it seems useful to add another category, A is a procedure used in B.

My next step is to analyze the data for the other nine teachers with two ends in view – first, to further refine the interpretive model by using it with a larger data set, and second, to find common themes, which might lay the groundwork for further studies.

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Extending the Descriptive Powers of Heuristics and Biases

Egan Chernoff

This study investigates prospective teachers' content knowledge of elementary number theory from a new perspective, which utilizes the heuristics and biases framework of subjective probability developed by Tversky and Kahneman (1974). In this report, the framework is adopted for analysis of undergraduate students' use of prime numbers. The results suggest that participants' struggles associated with elementary number theory originate in the use of subjective probability.

Shaughnessy (1992) states: "The research of psychologists Daniel Kahneman and Amos Tversky, and many of their colleagues, has provided mathematics educators with a theoretical framework for researching learning in probability and statistics…there is little doubt of the importance of their perspective for diagnosing the psychological bases of subjects' misconceptions of probability and statistics" (p. 470). Are these misconceptions restricted solely to the domain of stochastics? Jones, Carol and Thornton (2006) note: "[H]euristics are strategies that statistically naïve people use to make probability estimates or in the words of the authors [Tversky and Kahneman], judgments under uncertainty" (p. 74). Thus, the heuristics, of the heuristics and biases framework, can be read: probability estimates. Probability estimates are often made and they do not manifest themselves solely in the stochastic domain.

Studies on learning number theory have paid specific attention to students' understanding of prime numbers and prime decomposition (Zazkis & Campbell, 1996; Zazkis & Liljedahl, 2004). Researchers observed that students' awareness of the existence of infinitely many prime numbers and of very large prime numbers co-exists with their belief that every large composite number should be divisible by a small prime. This belief was witnessed, for example, in students' conclusion that a (large) number was prime after checking its divisibility by a small number of small primes. Furthermore, in their attempt to identify factors of a number represented as a product of "large primes," students checked this number's divisibility only by small primes. This was considered as students' implicit belief in "alternative prime decomposition." In this study I questioned whether similar phenomena would emerge when students are engaged in a more familiar task, the one of reducing a fraction.

Tversky and Kahneman (1974) introduced a framework of Heuristics and Biases of Subjective Probability to explain the judgement people make during times of uncertainty. According to this framework, people evaluate phenomena according to their representativeness, availability of instances and adjustment from the anchor.

Representativeness is a heuristic that is used to determine the probability that a particular object (A) belongs to a given set (B). Tversky & Kahneman found that the probabilities are evaluated by the degree to which A would resemble B. The probability that A originates from B or that B generates A is high when the resemblance is strong and low when the resemblance is weak. Judgement based on a stereotype is at the heart of this heuristic. Availability refers to the ease with which a person can bring to her or his mind instances of occurrences of an event. Instances of large classes are usually more easily recalled than instances of less frequent classes. As such, ease of recall leads to higher availability thus a higher subjective probability for an event. Adjustment and anchoring is the heuristic that people use starting with some initial values and making adjustment from them. This implies that results are greatly influenced by the initial value(s) and that different initial values may produce different results.

Participants in this study were 13 undergraduate prospective elementary school teachers enrolled in a course called "Principles of Mathematics for Teachers," which is a core course in the teacher certification program. At the time of the study they had completed the unit on elementary number theory, which included topics of divisibility and divisibility rules, prime numbers and prime decomposition, as well as the Fundamental Theorem of Arithmetic. The task analysed in this report invited students, in a clinical interview setting, to simplify the fraction $\frac{448188}{586092}$. Calculators were available as my attention was focused on students' choice of possible common factors rather than on their ability to perform calculations. The numbers in the numerator and denominator were carefully constructed, in fact, 448188 = $2^2 \times 3 \times 13^3 \times 17$ and 586092 =$2^2 \times 3 \times 13^2 \times 17^2$. For a student this choice assured an easy accessibility to the task (Schoenfeld, 1982) by providing success in the first few steps in finding a common factor.

Tom C.: 7, didn't work as well, and I'll try 11 as **a last resort**…

Katie H.: So here I am again, um, 37,349/48,841 um, can't be divisible by 2, can't be divisible by 5, can't divide by 3 again maybe to divide by 3 again I'll sum…12, 16, 19, 26, that doesn't work, um, so it wouldn't be divisible by 3 again. I don't know, I think at this point I'd probably just try a few numbers like maybe 7 might look like a number, that would work just because of this 49 here. I'll just (pause) divided by 7 equals decimal so that doesn't work. Um, I'm trying to think of any other number that might have a 9 in them (pause), I don't know, **maybe it's simplified**, I think maybe it is. (pause)

Nicole K.: I randomly picked the number 17 only because it's one of those prime numbers, this is true that it's one of those prime numbers that I don't really think about, like I'll go from 0, or I'll go from 2 to 9, and then your instinct is to stop…

The representativeness heuristic was witnessed in students' choices of primes, as possible factors. The "stereotypical" list of primes included: 2,3,5,7,11. Tversky & Kahneman (1974) showed that using the representativeness heuristic to evaluate probability leads to *insensitivity to prior probability of outcomes*. Students were aware that there were more primes greater than 11, than those less than 11. However, this fact was not taken into consideration in an attempt to reduce the given fraction. Numbers like 13 and 17 did not conform to the students' image of the stereotypical primes and were not taken into consideration.

Another bias that results from the representativeness heuristic, *the illusion of validity*, develops a false sense of confidence in predictions that are based on redundant input. It was shown that as redundant input continues the accuracy of prediction decreases, even though at the same time confidence about the prediction is gained. This bias also helps to explain why we saw overuse of the first few prime numbers in participants attempts to simplify the fraction. The redundancy of the stereotypical primes that participants are exposed to through their schooling manifested itself in their increased confidence in using these primes for simplification. Further, the illusion of validity lead participants not only start the task with 2 and 3, but also return to these numbers after higher primes (7 and 11) did not work. That is to say, participants eventually tried larger primes in their simplification, yet they always worked their way back down to lower primes. From the perspective of number theory this is seen as "intuitive belief in alternative prime decomposition", that is, we recognise in students' division attempts the fact that even though the number itself is not divisible by some small prime p, its factor may still be divisible by p. However, from the heuristics and biases perspective, it is the illusion of validity that leads students to try the same primes over and over again, during their progression of trying larger and larger primes. From this perspective, they were being lulled in by a false sense of security from years of exposure to the same small primes.

Further analysis of the task was conducted with the other heuristics and biases of the framework and the results suggest that participants' struggles associated with elementary number theory may originate in the use of subjective probability.

As a researcher of mathematics education, more specifically stochastics education, I take on a role quite different from those researching stochastics in the field of psychology. Shaughnessy (1992) makes a clear distinction between psychologists and mathematics educators: "The psychologists are, therefore, mainly observers and describers of what happens when subjects wrestle with cognitive judgmental tasks…[r]esearchers in mathematics and statistics education are, however, natural interveners" (p. 470). Using this psychological framework to explain phenomena within mathematics education, attempts to bridge the gap between the fields. With this approach, I embrace the influence of the field of psychology, yet at the same time cannot rid the shackles of my natural (mathematics education) tendency to intervene. As such, the heuristics and biases framework of subjective probability is adopted to examine mathematical misconceptions in other domains. The first step of description (presented in this report) will subsequently be followed by reports on the prescriptive power of the framework and the ability to impact educational practices. To the best of my knowledge, this is the first study using the framework to further understand prospective elementary teachers' content knowledge of mathematics. I suggest that the framework is applicable to provide alternative, additional and more refined explanations for undergraduate students' probability estimates – beyond the situations of probability– and to analyse common misconceptions. Further research will examine the extent of the applicability of this framework in analysing students' work on problems in additional content areas and the use of framework as a "misconception meta-crawler". Adoption of the framework in a prescriptive sense will be investigated for implications in teaching practice.

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TEACHING/LEARNING AND BELIEFS – THE RE-EDUCATION OF THE EDUCATOR

Elena Halmaghi

This paper covers one semester of course work in Designs for Learning: Elementary Mathematics Course. A reflective journal, kept by the researcher and her students forms the starting point for this empirical paper. The results suggest that the ambiguity, uncertainty, tension, problem solving, and group projects that were part of the course created the circumstances for reflection on beliefs about mathematics and how mathematics should be taught. This paper mostly reports on instructor's beliefs that where challenged while orchestrating the activities, designing the milieu for challenging and changing students' beliefs, and actively participating in the entire "dance of agency" that was the coursework.

INTRODUCTION

The music starts and I engage in dancing. I let my body feel the rhythm and move. I enjoy myself and I think this is great. Then, a friend whispers (in fact she cries in my ear because the music is loud) "Elena, this is not a rumba. It is a cha, cha." I try to adjust my steps, and to involve my hips in this dance. Clumsy at first, after some moments of concentration, I feel comfortable again with my new dance. The music fades and we all stop dancing. Gathered together at the bar, people who were dancing, along with some others who were watching the dance from the margin, tell me: "When dancing, you should be more relaxed. You look like you've just swallowed a broom and some of your steps were not attuned with the music."

The dancing floor is a room in the Simon Fraser University Surrey campus. The dancers are the instructor –me, and 24 students, participants in a Designs for Learning: Elementary Mathematics course. The music, which started first, was the package I received a few months prior teaching the course. That means, I have started teaching a course with a predetermined course description, an already chosen text book and a prescribed teaching philosophy. The course was designed for prospective and practicing elementary school teachers willing to explore the fundamentals of the learning/teaching process as it applies to mathematics. As the received course description stated, the course was adjusted with the latest research in mathematics learning and was intended to show how such findings could be applied in the classroom. I, as the instructor, have lots of meaningful teaching experience and a well defined teaching philosophy that is very much aligned with Polya's view about teaching, which states that "[t]eaching is not a science; it is an art. If teaching were a science there would be a best way of teaching and everyone would have to teach like that. Since teaching is not a science, there is great latitude and much possibility for personal differences."

I consider myself to be such an artist; able to spontaneously adjust my step in this 'dance of agency' that is the teaching. I stepped into dancing with some fear, for I didn't previously listen to the entire tune, but with belief that my teaching experience and my artistic nature will help me adjust my steps during the performance. How successful I was and what other meanings I derived from this experience is the purpose of this paper to get some light on.

SUBJECTS AND CONTEXT

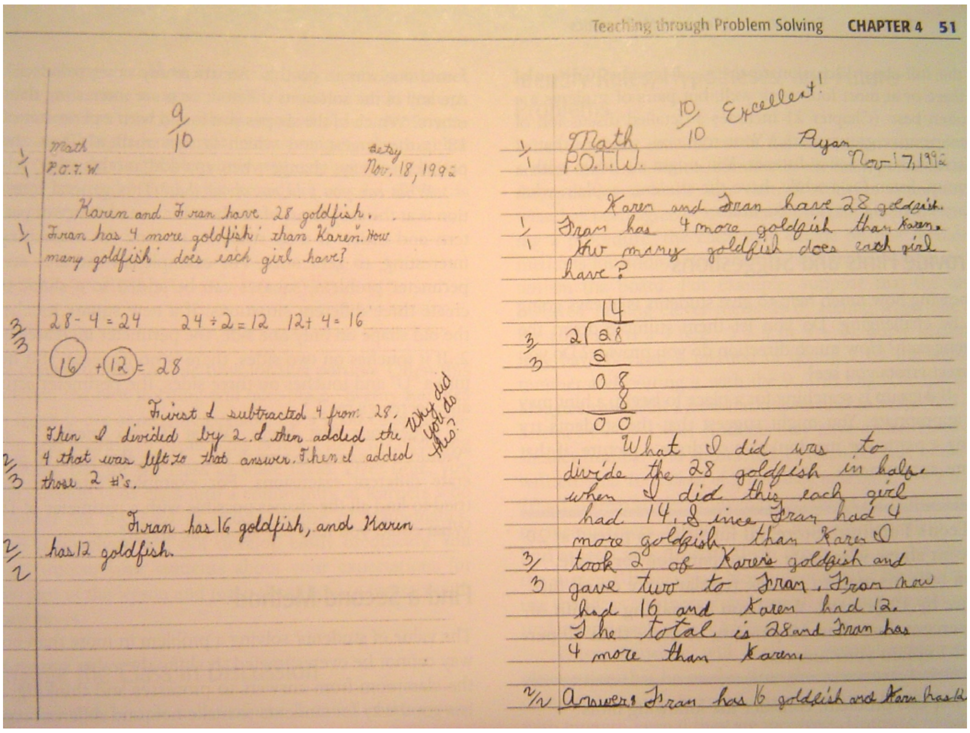
The study took place in the summer of 2006, in a Designs for Learning: Elementary Mathematics Course at SFU Surrey. The participants were 24 preservice elementary teachers, some of them enrolled in the course with the desire of learning how to effectively teach mathematics, some of them being compelled to take the course because they didn't pass the mathematics teaching of the practicum component, most of them fearful of math and having a very weak mathematics knowledge, willing to get recipes for a fast accumulation of mathematics facts that will help them in the future, half of them willing to teach K to 4, the other half oriented toward middle school teaching: a heterogeneous group from many perspectives.

I was to teach a course using a constructivist approach, that is I had to first consider the knowledge and experiences my students bring with them to the learning task. Researchers on preservice teachers found that "teachers begin to learn how to teach long before their formal teacher education begins." (Writes & Tuska, 1968, as cited in Grouws, p. 211) The course should then be built so that my students would expand their previous knowledge of mathematics, curriculum, as well as pedagogical content knowledge and develop their new teaching experiences in the light of the latest research in learning of mathematics.

To the field of mathematics education I am relatively new, though I have an extensive teaching experience. I posses sporadic reflections on my teaching style, teaching methods that work better for me, what to expect from my students…, but my mathematics education discourse is still immature. The landscape of mathematics education is slowly forming in my brain. I have my beliefs about teaching/learning derived mostly from my teaching practice and, in this process of becoming fluent in the mathematics education field, I come across research studies that are constantly challenging my beliefs.

Given that, I stepped into teaching this course with a tension within myself. At the beginning of the course, my idea of constructivism was that as an instructor I am in charge of creating a learning community where students are engaged in problem solving, selected ones, leading to some heuristics that I was prepared to supply. The problems for solving would be either handed in on paper or introduced by a story. The students will try a problem. During this process, if their struggle is too intense, I will scaffold the process for them, by posing questions, or suggesting alternative approaches toward solving it. At the end of the struggle my role is to provide structure, to bring a mathematically formal closure to it. As Sfard argues, my belief is also that the problem solving process "consists in an intricate interplay between operational and structural conceptions of the same notions. … In the process of concept formation, operational conceptions would precede the structural, [and] the absence of structural conception may hinder further development." (Sfard, 1991) In this light, I see my role emerging – the one assigned to help students' structural understanding develop. Reading the text book in preparation for the first class, my beliefs were challenged by the idea that a problem is a task where there should be no perception that there is a "correct" solution method. In my own problem solving process I will start with an activity for which students have no prescribed or memorized rules or methods, (Hiebert et al., 1997) but at the end solutions will be shown and some appreciation of the method or structure of the solving process will be derived. To accept student's solutions without any evaluation, as book suggests, sounds incompatible with my teaching philosophy.

With this self challenge I have designed one my first class activities. I have given my students a problem from the same "category" with the problem presented in the following figure scanned from the text book:


Teaching through Problem Solving CHAPTER 4 51

1/1 Math P.O.T.W. 9/10 Betsy Nov. 18, 1992

Karen and Fran have 28 goldfish. 1/1 Fran has 4 more goldfish than Karen. How many goldfish does each girl have?

3/3 28 - 4 = 24 24 ÷ 2 = 12 12 + 4 = 16

16 + 12 = 28

First I subtracted 4 from 28. Then I divided by 2. I then added the 4 that was left to that answer. Then I added 2/3 those 2 #'s.

Why did you do this?

2/2 Fran has 16 goldfish, and Karen has 12 goldfish.

1/1 Math P.O.T.W. 10/10 Excellent! Ryan Nov 17, 1992

Karen and Fran have 28 goldfish. 1/1 Fran has 4 more goldfish than Karen. How many goldfish does each girl have?

3/3 28 ÷ 2 = 14

What I did was to divide the 28 goldfish in half. when I did this each girl had 14. Since Fran had 4 more goldfish than Karen I 3/3 took 2 of Karen's goldfish and gave two to Fran. Fran now had 16 and Karen had 12. The total is 28 and Fran has 4 more than Karen.

2/2 Answer: Fran has 16 goldfish and Karen has 12

My students solved the problem and I called some of them to show the solutions on the board. There were four solutions chosen – a guess and checking, draw a diagram, algebraic, and a "thinking analyzing" one. While the volunteers were presenting their work, I asked them to speak while writing on the board. I rephrased the majority of their discourses using "fancy" mathematics words, a formal verbal mathematical language.

At the end of the class, one of the homework tasks was an entry journal where they have to comment on the grades given by the teacher on the above picture. My own mental image on the comments I will be reading from my students, given the applied treatment (the problems solved together and the emphasis on structure and method) was that at least some of them will have an appreciation of Betsy's method of solving the given problem, and will give both kids the same grade… No such findings in the journals. Did I get everything wrong from the beginning? My feedback to the mentioned entry on one of my student's journal was: "At this moment I am also inclined to believe that Ryan would figure out a way to solve the problem with big numbers; his way, not the standard method that Betsy exhibited in her paper."

WHERE THE 'DANCE OF AGENCY' STARTS

While I had the revelation that nobody in the class has noticed that Ryan's "method" of solving the given problem may hinder him when dealing with the same problem with bigger numbers, some of my students have other findings. Here is what Lucy recorded in her journal related to our issue: "It was a revelation to me when I saw papers on which students had not only performed their calculations but had also recorded their thinking. It's sad to say, but I have never been asked to do this and I have never asked my students to do this. What a simple, but profound way to evaluation [sic] for understanding!" The appreciation of teaching for understanding and the way to evaluate it started.

In a study conducted by the National Center of Research on Teacher Education at Michigan State University, Ball documented that prospective teachers possess limited knowledge of mathematics, inadequate for teaching. She argues that “subject matter knowledge should be a central focus of teacher education programs.” Loaded which such ammunition I began my ‘dance of agency’ with a step where my partner was the disciplinary agency. A huge part of my teaching experience is a dance between the human and the disciplinary agency. By the human agency, I mean I am the class authority. In general I design the rules of the game and, when I want my students to see fewer constraints from the instructor, I yield the authority to mathematics. I like so much to play a game of changing the agency when I insert proofs in my class by telling them “You don’t have to believe me that I am right when I show this result. Here is the proof for it.”

At a Changing the Culture Conference on the topic “Obstacles in Understanding Mathematics,” one of the presenters argued that mathematics for her has always been a safe ground. In mathematics, she could find tools to help her distinguish right from wrong, truth from false. She could find the answer, while in humanities, there was too much ambiguity, unanswered questions, too many opinions… Those were obstacles for her understanding or pursuing non-mathematical subjects. There is comfort in knowing the rules, there is comfort in finding the correct answer, but is not this only some instrumental understanding that Skemp alluded to? How should I move my students to relational understanding?

Ironically, to engage in a ‘dance of agency’ where teaching for a relational understanding means yielding the human agency to students, there is no safe ground for the teacher, at first. Duality, ambiguity, uncertainty, and tension are part of this complex game. Knowledge of mathematics, curriculum, pedagogy, didactics and class preparation should be combined with spontaneous action. “Did Elena teach today?” was the prompt for one of my students’ journal entry. Jenny writes: “I loved the way you just gave us the manipulatives today and asked to play with them. It was so different. At first, our group had no idea how to “play” with it, but when you came around and sort of guided our play by asking us to make number 14 with our base 5 blocks, we were stumped!” Here is a journal account which acknowledged that instructor’s discrete intervention helped a group of students start a play with manipulatives. After giving them a starting point, the players engaged into discovering way more than what the instructor knew and imagine that was going to happen when she planned the activity. It takes courage to let students explore and present their findings when the authority from the discipline is absent. Is this a glimpse of good mathematics teaching? Is good mathematics teaching a path where getting lost and finding ones way is more important than following a precise map that will deliver everybody safe at the destination?

Perry’s research on mathematics teacher education indicates “that both experienced and novice teachers tend to rely on external authority in making decisions regarding course content and pedagogical strategies.” (Grouws, p.229) While trying to help my students move from the authority of prescribed learning outcomes and traditional way of teaching/learning mathematics, I find myself yielding to the new authority in this course – the constructivist ideas promoted by the textbook. This is not without tension and frustration. While lots of students were happy with the text book and happy to have it as reference when their real teaching begins, some of them were constantly reporting that they “don’t buy into this whole new reformed mathematics.” Frustration was one of key words in my students’ journals. They commented that a big part of the new program is exploring and experimenting – a frustrating experience for students who have spent their first years of school learning to do things “correctly.” What about teachers who have spent a life teaching “correctly”? I would add to this.

Using Pickering metaphor of ‘dance of agency’ in teaching mathematics, Boaler’s human agent in the classrooms where students’ active participation is encouraged is not the teacher anymore, but the student. Drawing on Boaler’s metaphor, Wagner (2004) let his students use their own voices in mathematics activities, therefore engaging in their own ‘dance of agency.’ Phil, one of the participants in my course, unhappy with the score that we are dancing by, notes: “Today I was reminded that the one area that I always had trouble with was solving word problems. … Like I said in class today, I am great if you give me the rules and tell me how to do it. But if you ask me to figure out the process on my own and understand the concept and why we solve the questions the way we do, I get frustrated and annoyed because I know I can’t do it.” In the same manner, Troy writes: “Problem solving makes me feel angry, frustrated, and depressed.” I read the comments. I felt the frustration. I perceived my class as becoming less and less enthusiastic. I saw some dancers leaving the dance floor and taking a seat in the boring crowd on the margin. Paulos argues that “innumeracy is widespread even among otherwise educated people” because “[m]athematics as a useful tool or as a way of thinking or as a source of pleasure is a notion foreign to most elementary-education curricula.” (Paulos, p. 75) Mathematics as a source of pleasure seems to be something very odd. Was I the only one in that crowd that found delight in problem solving? “Knowledge of subject, curriculum, or even teaching methods, needs to combine with teachers’ own thoughts and ideas as they engage in something of a conceptual dance.” (Boaler, p. 1-12) It seems that this was not enough for my class. Should I have invited mathematics agency to perform a solo again in our dance?

I was aware from the beginning that this course is not about teaching mathematics. It was about engaging the elementary teachers in a ‘dance of agency’ where they should live a multilevel experience – as students active participants in designing and living an education course, as teachers willing to give birth to a coherent teaching philosophy. In this scenario the agency of discipline should be the last one to enter the dance floor. The agency should be given to some other domains – the research in mathematics education and to the collective voice of the 25 people involved in this work. My lack of teaching non-mathematics courses made me clumsy on the dance floor. How could I promote changing in beliefs when the intentionally excluded agency of the discipline made me unconvincing when experiencing the new steps? Or maybe, along the way, I embraced a constructivism that conflicted with my teaching beliefs in many aspects?

CONCLUSIONS

Boaler argues that as dancers do not learn their craft by reading the steps from a book or looking from the margin, but by stepping on the dance floor, in the same manner teachers are learning how to teach by practicing teaching. I did my practice with the preservice teachers. I stumbled. I almost fell when I read my course evaluations. Did I not expect them to be looking the way they were? I did. In this course I let the disciplinary agency leave the dance floor, and my voice became whisper. As Sfard argues, by leaving too much of the play to the power of improvisation my students brought into the course, I have deprived my students of the chance of seeing me fluent, enthusiastic, and skillful in bringing structure, mathematical structure, into it.

> [T]he teacher who requires the learners to work on their own, who keeps from 'telling,' and who never demonstrates her own ways of doing mathematics, deprives the student of the only opportunity they have to be introduced to mathematical discourse and to its meta-rules. Mathematics teacher who abstains from displaying her own mathematical skills may be compared to a foreign language teacher who never turns to her students in the language they are supposed to learn. The historical reasons because of which mathematical discourse developed the way it did would not convince today's student. Thus, it is naive to think that either mathematical discursive habits or the ability to speak a foreign language could be developed by children left to themselves. (Sfard, 2000)

In the light of this quote from Sfard, thinking again at Paulos argument, I cannot refrain from asking myself a simple question: Did I show my students that I really find pleasure in doing and teaching mathematics?

Brent & Sumara note that "as one moves into the uncertain atmosphere of a pre-service teacher education class, considers the apprehension associated with practicum experiences, or examines the uncertainties announced by even the most seasoned of teachers, it is clear that there exists a stubborn dissatisfaction with our understandings of how one learn how to teach." The 'dance of agency' in a classroom of preservice teachers with a sessional instructor is not entertainment. This dance has some of the most complex steps, steps that the instructor should practice way before moving on the dance floor, steps that should be backed up by meaningful teaching experience and sound research. And, how can someone practice it if not by being completely immersed in one the most complex experiences – teaching a course for preservice teachers?

An enactivist theory of cognition "requires teachers and teacher educators to reconceive the practice of teaching by blurring the lines between knower and known, teacher and student, school and community." As I reflect on my teaching experiences, especially the one with preservice teachers, I agree that there should be some blurriness in the teaching setting, but this fogginess should be superficial. Ambiguity is good as long as the class is warned about and convinced that this is temporary and at the end the tension resolves into some rewarding outcome. I also become more convinced that, to enable the construction of knowledge, the teacher should live a double teaching life – as a dancer, with no noticeable difference between her and the other dancers, and as the DJ behind the dance floor, the person who selects the music and takes charge of the steps to follow. "Research on beliefs, although fraught with pitfalls to avoid and difficulties to surmount, has great potential to inform educational research and practice and therefore worth the effort." (Leatham, 2006) I am ready to repeat the experience of teaching another section of Designs for Learning: Elementary Mathematics course and I believe that next time I will be able to provoke and notice the changes in beliefs I would like my students to experience.

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ENACTIVE COGNITION AND SPINOZA'S THEORY OF MIND

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The purpose of this paper is to highlight the remarkable similarities between enactive cognition and Spinoza's theory of mind. Both frameworks entail a single ontology and a dual epistemology. The two systems can inform and support each other. According to Spinoza, mathematical knowledge develops from common notions, ideas which are universally present in the world. Common notions can be interpreted to be recurrent patterns of enaction. In both frameworks mathematics reflects the deep structure of the world. However, human mathematical knowledge is limited because a human being, and in fact the whole of human society, is only an embodied fragment of a very large and complex universe. There are implications for the teaching of mathematics.

INTRODUCTION

According to the theory of enactive cognition, cognition is the interaction between an organism and the world in which it is embodied. It is closely aligned with the phenomenology of Merleau-Ponty. Campbell (2001) has proposed a radical enactivism, with Merleau-Ponty's notion of *flesh* as the ontological primitive. This interpretation of enactive cognition has remarkable similarities with Spinoza's theory of mind. In both systems the monist ontology supports two epistemologies. Spinoza can illuminate and inform some aspects of enactive cognition.

In Spinoza's epistemology the mind is "the idea of the body," which is altered by impact with other bodies. "Common notions" are ideas that are present in all interactions between bodies. According to Spinoza, common notions are the foundation of mathematics. From the enactive perspective, common notions are recurrent patterns of enaction.

Spinoza's metaphysics implies a deeply structured world. If it is assumed that there *is* a deep structure to the world, then the objectivity and utility of mathematics is guaranteed. However, humans are only finite, situated fragments of a large and complex universe. For that matter all of human society constitutes only a small part of the whole. Human mathematical understanding is limited by its fragmented, situated engagement with reality.

Enactive cognition and mind as idea of the body have implications for the teaching of mathematics, especially with regard to constructivist thinking.

LITERATURE

Enactive cognition is developed in Varela, Thompson, and Rosch (1991). Damasio (1994) provides some neurobiological support for the theory, and Campbell (2001) proposes a radical enactivism that utilizes Merleau-Ponty's notion of *flesh* as an ontological primitive.

The primary source for Spinoza is the *Ethics* (1677/1996). An exposition of Spinoza's ideas, particularly as they can be related to mathematics, is given in Hampshire (2005). Parkinson (1954) provides a useful clarification of Spinozistic epistemology. Changeux and Ricoeur (2000) and Damasio (2003) engage Spinoza for the contributions he can make to understanding current issues in sociology and neurobiology, respectively.

The notion that the universe is deeply structured is a feature of Spinoza's philosophy. Tarnas (2006) and Campbell (2002) make strong arguments for accepting the idea of a deeply structured external reality.

Bohm (1980/1983) and Lakatos (1976) provide interesting perspectives on the limitivist conception of mathematics that is a feature both of enactive cognition and Spinoza's theory of mind.

THE EMBODIED MIND

Cartesian dualism rigorously separates the two worlds of mind and body. Various monist formulations, on the other hand, claim that all is mind (the idealists) or all is matter (the materialists). The theory of enactive cognition of Varela, Thompson, and Rosch (1991) exists in a middle ground between the extremes of monism and dualism. The residual dualism of enaction is expressed in the fundamental idea of *double embodiment*. Thus, we are physical beings existing in the world, but also we perceive the world, the world exists in us. According to Merleau-Ponty, whose phenomenology is the philosophical foundation of enactive cognition, "The world is inseparable from the subject, but from a subject which is nothing but a project of the world, and the subject is inseparable from the world, but from a world which the subject itself projects" (cited in Varela et al, 1991, p. 4). Varela et al write that "*embodiment* has this double sense: it encompasses both the body as a lived, experiential structure and the body as the context or milieu of the mind" (ibid., p. xvi, authors' italics).

The embodied point of view is active rather than passive. The very fact of being embedded in the world means that the organism receives external stimuli that change the internal milieu. But a change in the internal milieu will change the way the organism acts, which in turn will alter the stimuli that it receives. Varela et al (1991) describe this in poetic terms as "organism and environment enfold into each other and unfold from one another in the fundamental circularity that is life itself" (p. 217). The circle spins at the flickering pace of phenomenal time. It is impossible to identify which comes first, stimulus or response, and the enactive perspective is to regard organism and the world in which it is embodied as a single, interactive structure. According to Merleau-Ponty, "When the eye and the ear follow an animal in flight, it is impossible to say 'which started first' in the exchange of stimuli and responses" (ibid., cited p. 174).

A possible break in the circle could occur if the organism ceased acting. Would it not still receive stimuli that changed the internal milieu? According to Damasio (1994),

> Perceiving the environment, then, is not just a matter of having the brain receive direct signals from a given stimulus, let alone receiving direct pictures. The organism actively modifies itself so that the interfacing can take place as well as possible. (p. 225)

Damasio's arguments are based on neurophysiological data. In a similar vein, Varela et al (1991) demonstrate, for example, that the experience of colour depends on active perception. But they would go further and claim that *no* perception is possible without action. This insight is fundamental to the enactive approach: perception *is* perceptually-guided action. According to the Varela et al, "Perception is not simply embedded within and constrained by the surrounding world; it also contributes to the enactment of the surrounding world. . . . The organism both initiates and is shaped by the environment" (p. 174).

Cognition is defined by Varela et al (1991) as "Enaction: A History of structural coupling that brings forth the world" (p. 206). The *history of structural coupling* is the dance between organism and the world. They move together in perfect synchrony, neither taking the lead, but both moving to the same melody. The two are more closely intertwined than any lovers. The boundaries of the organism do not stop at the physical shell of the body, but include organs, blood, and nerves, all of which are guided directly by the features of the world or indirectly through other structures within the organism that *are* directly affected by the world. Mind *is* this interaction. Consider the two aspects of embodiment. Firstly, the external manifestation of mind is physical activity. This physical activity includes, but of course is not delimited by, the electrical activity of the brain. Secondly, the internal manifestation of mind is everyday lived experience. And this world of lived experience is that which is brought forth by the history of structural coupling.

Campbell (2001) argues that the residual dualism of double embodiment is "somewhat redolent of more traditional 'interactionist' concerns with mediating between the Cartesian real and ideal worlds than bypassing them altogether" (p. 7). He proposes instead a *radical enactivism* based on Merleau-Ponty's *flesh* as a single ontological primitive encompassing both mind and matter. Accordingly, the "objective 'real' world we are in and the subjective 'ideal' world within us [are] manifestations of the same world" (ibid., p. 3). The "real" and the "ideal" are therefore distinguished in epistemology *not* ontology.

There are remarkable correspondences between enactive cognition and Spinoza's theory of mind. The agreement is even closer for radical enactivism.

SPINOZA'S THEORY OF MIND

Spinoza's philosophical system has been applied recently to discussions in sociology and neurobiology (e.g., Changeux and Ricoeur, 2000; Damasio, 2003). Spinoza's theory of mind also has implications for theories of cognition.

Spinoza defines *substance* as "what is in itself and is conceived through itself, that whose concept does not require the concept of another thing, from it must be formed" (E1 D3)[1]. Substance, in other words, is not contingent. Spinoza demonstrates that each substance must be its own cause (E1 P6 C) and the cause of no other substance (E1 P8 S2). Moreover, since these self-caused substances can be conceived, they also exist (E1 P7). This implication follows from Spinoza's first definition: "By cause of itself I understand that whose essence involves existence, or *that whose nature cannot be conceived except as existing*" (E1 D1, my italics). There can only be one such substance (E1 P14), which, along with its modes (discussed below), must therefore consist of all that there is. Spinoza calls it God or Nature.

Substance can be known through its *attributes*, which "the intellect perceives of substance, as constituting its essence" (E1 D4). Two attributes of substance are *Thought* and *Extension*: "Thought is an attribute of God, or God is a thinking thing" (E2 P1); "Extension is an attribute of God, or God is an extended thing" (E2 P2). Interestingly, Spinoza argues that although substance must have an infinite number of infinite attributes, Thought and Extension are the only two conceivable by humans.

Thought and Extension are intimately related. A good way to understand this connection is to consider their manifestation on the human level. Firstly, however, it is necessary to determine what kind of thing a human being is in Spinoza's philosophy. Spinoza defines *mode* as "the affections of substance, or, that which is in another through which it is also conceived" (E1 D5). According to Spinoza, "Particular things are nothing but affections of God's attributes, or modes by which God's attributes are expressed in a certain and determinate way" (E1 P25 C). Human beings as *extended* things are finite modes under the attribute of Extension; human beings as *thinking* things are finite modes under the attribute of Thought. These two finite modes correspond to body and mind, respectively.

Spinoza defines *idea* as "a concept of the mind which the mind forms because it is a thinking thing" (E2 D3). The fundamental connection between mind and body is expressed as follows: "The object of the idea constituting the human mind is the body, or a certain mode of extension which actually exists, *and nothing else*" (E2 P13, my italics). Moreover, "*The human mind does not know the human body, nor does it know that it exists, except through the ideas of the affections by which the body is affected*" (E2 P19, my italics). Changes in the body, in other words, are precipitated by interaction with other finite modes. The mind is simply the procession of ideas of the body as the body itself is changed through these interactions. Moreover, "*The idea of any mode in which the human body is affected by*

[1] The quotes from Spinoza's *Ethics* are Curley's translation in Spinoza (1677/1996). According to the usual practice, they are referred to by book (E), proposition (P), definition (D), corollary (C), and scholium (S).

external bodies must involve the nature of the human body and at the same time the nature of the external body" (E2 P16, my italics), meaning that external affects, in other words perceptions, are necessarily integrated and transformed by the human receiver.

Spinoza's construction of mind appears to be very similar to enactive cognition. However, the theory of enactive cognition requires an *active* engagement between organism and world, whereas Spinoza implies that ideas can correspond to a *passive* body that is acted on by the world.

A crucial point is that the "*order and connection of ideas is the same as the order and connection of things*" (E2 P7, my italics). In other words, it is better not to think of mind and body as two separate entities, with somehow a causal relationship between them, but rather as *one and the same thing under two different attributes*.

The idea of a single substance in which mind and body are finite modes of the two ways in which this substance can be known, Thought and Extension, is closely related to Merleau-Ponty's notion of flesh, discussed above. Extension corresponds to the objective point of view, in which we are embodied in the world; Thought corresponds to the subjective point of view in which the world is embodied in us.

There are therefore substantial similarities between the two ways of understanding the relationship between mind and world. The next task is to discuss the ways in which Spinoza and enactive cognition can inform each other concerning the nature of mathematics and mathematical thinking.

LIMITIVIST MATHEMATICS

Spinoza's psychology has three categories of knowledge: Imagination, Reason, and Intuition. Imagination refers to knowledge of particulars resulting from sensory data. Reason refers to knowledge obtained by deduction from common notions, ideas that are omnipresent in the world. In other words, common notions are present as ideas whenever one body affects another body. Spinoza's common notions can be interpreted in the framework of enactive cognition as *recurrent patterns of enaction*. Spinoza's view of Intuition, his highest category of knowledge, is more difficult to define, and not even Parkinson's (1954) extensive investigation of Spinozistic epistemology provides a clear explication of Intuition.

The primitive concepts of mathematics, number and form, are common notions, ideas that are obtained by abstraction and generalization from sensory impressions, and Reason would therefore include mathematical knowledge. Hampshire (2005) interprets Spinoza's understanding of mathematics to be glimpses of a deep underlying structure of the universe:

> Adequate [i.e. reasoned] explanation is only found when we are dealing with an aspect of the material world which is everywhere the same, the same in our own bodies as in all bodies. Then we are concerned only with the laws that prevail throughout Nature and that are systematically related. Along this path we arrive at the order of the intellect and we approach the infinite idea of God or Nature. (pp. 10-11)

It is important to note in this passage that Hampshire refers to *approaching*, rather than reaching, the "infinite idea of God or Nature." I interpret this to mean that the mathematical knowledge that is achievable by humans in their capacities as limited fragments of a very large and complex universe, is necessarily limited. It is worth quoting Hampshire in full on this point:

> The infinite idea of God or Nature must be the true mathematical representation of the deep structure of Nature and of its most general laws, everywhere and at all time in operation. Our mathematical knowledge, as it develops and unfolds, gives us glimpses of

this; but the knowledge must always remain fragmentary, because our powers of thought are finite, and because our active thought must always be subject to interruptions. Lastly, the inputs that we receive in interaction with objects in the environment are limited both by our position in the common order of nature and by our sensory equipment and brains, that is, by the limitations of our bodies. Our minds are correspondingly limited. The mind is the active part of the whole person, who is both mind and body; the body attaches the mind to a particular place in the order of time and space and in the common order of nature. (ibid., p. 11)

There is nothing in this discussion of Spinoza's view of mathematics that is inconsistent with enactive cognition. Spinoza, in his "infinite idea of God or Nature," clearly implies a deeply structured world. Is it unreasonable to postulate the same within the enactive perspective, and that human mathematical thinking is limited because of the fragmented, situated human engagement with this structure?

Tarnas (2006) writes forcefully of the alienation that must result if external structure is denied of the world:

> For is it not an extraordinary act of human hubris—literally, a hubris of cosmic proportions—to assume that the exclusive source of *all meaning and purpose in the universe* is ultimately centered in the human mind, which is therefore absolutely unique and special and in this sense superior to the entire cosmos? . . . To base our entire world view on the a priori principle that whenever human beings perceive any . . . suggestion of purposefully coherent order and intelligible meaning, these *must* be understood as no more than human constructions and projections, as ultimately rooted in the human mind and *never* in the world? (p. 35, author's italics)

An embodied perspective on the existence of a structured external reality is provided by Campbell (2002). He argues that *the very embodiment of human beings within the noumenal realm* in itself assures a degree of correspondence between human ideas and an external reality:

> When one considers that we are conscious, reflective, and free embodied beings constituted of and embedded within the noumenal realm, we are lead to the realization that, subjectively, our autonomous *actions* can be seen to arise through us from within that very realm. (p. 432, my italics)

If this correspondence does exist, then a structured human cognition implies a structured external world.

Provided that a world with a deep structure is allowed, the objectivity of mathematics follows as a matter of course. The universal utility of mathematics in modelling aspects of the world is also assured, because aspects of that deep structure that are already present will themselves embody mathematics.

Extending the metaphor that cognition is the intimate dance between individual and the environment, it can be seen, therefore, that *we all dance to a common melody, a melody that emerges from the groundswell of our being as embodied fragments of a deeply structured universe*.

Human mathematical knowledge will always be limited because humans are only finite, situated fragments of a large and complex universe—in fact, all of human society is only a finite, situated fragment of the whole. However, human mathematical knowledge does increase, and the increasing utility of mathematics implies that the direction of increase is toward greater knowledge of the deep structure of the world.

The question of *why* human knowledge of the deep structure of Nature increases is a deep one. It is possible to approach it either from the perspective of an organism's adapting to survive and flourish, or from the perspective of the Spinozistic idea of *conatus*. On the other hand, the method of proofs and refutations of Lakotos (1976) may provide an answer to the question of *how* human knowledge of the deep structure of the world increases. In either case, it is beyond the scope of this paper to do justice to these ideas.

Bohm (1980/1983) uses the hologram as a metaphor for the implicate order. This metaphor can be used to understand the limited nature of human mathematics. A hologram is a way of encoding data such that the whole is encoded in each part of the hologram, no matter how small. However, when data is retrieved from a fragment of the entire hologram, it loses resolution, and the loss of resolution increases as the size of the fragment decreases. By analogy, the very small fragment of reality that is a human being retrieves only a fuzzy understanding of the deep structure of the world. The method of proofs and refutations offers a way of improving the "resolution."

IMPLICATIONS FOR MATHEMATICS EDUCATION

The viewpoint discussed herein is constructivist in that recurrent patterns of enaction create *individual* mathematical understanding. To the extent that society is a significant environmental factor for cognition that is characteristically human, mathematical understanding is also a *social* phenomenon. However, the radical constructivism of von Glasersfeld (1995) and the social constructivism of Ernest (1998) are not completely compatible with the views discussed in this paper because they do not allow that mathematical knowledge can originate in a deeply structured external reality rather than in the individual or in society, respectively.

The existence of a structured external reality as the basis for mathematics indicates that the source of knowledge for the student is the same as that of the teacher. Just as cognition arises from the interaction between individual and world, mathematical understanding can arise from the interaction between student and teacher, in which the teacher is a significant component of the student's environment. In Spinoza's language both are finite modes that affect each other. The teacher structures this interaction to facilitate student learning. Constructivist models of learning in which the teacher arranges situations for the student to construct the student's own understanding may be regarded as an *indirect* interaction between student and teacher. Either way, the student's construction of mathematical knowledge is likely to be similar to that of the teacher's because of the shared features of their situated embodiments as human beings in deeply structured reality.

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New Definition, Old Concepts: Exploring the Connections in Combinatorics

Shabnam Kavousian

Background and objectives

Definitions are one of the most integral parts of mathematics. There is an abundance of research about the role of definitions in teaching and learning mathematics (Edwards, 1999, Vinner, 1991, Leikin & Winicki-Landman, 2000). Vinner (1991) examines the pedagogical and epistemological role of definitions in mathematics. Edwards (1999) concentrates on students' understanding of the importance of definition. Leikin & Winicki-Landman (2000) explore the relations between different definitions in detail. Moreover, they all emphasize the importance of understanding formal definitions for students and teachers alike.

In this report, I will examine the ways that undergraduate students attempt to understand a new definition, and how they use this new definition to solve a problem.

Conceptual framework

In this study, I am going to use *concept definition/concept image* framework, which was developed by Tall and Vinner (1981). This framework was further refined by Moore (1994) by adding *concept usage*. The term *concept-understanding scheme* is consisted of these three aspects of a concept (Moore, 1994). According to Tall and Vinner (1981), concept definition is "a form of words used to specify that concept", and concept image is "the total cognitive structure that is associated with the concept". Moore (1994) describes concept usage as "the ways one operates with the concept in generating or using examples or in doing proofs". I will use the term *concept usage* to include the ways one operates with the concept in solving problems in addition to the ones described by Moore.

Using this conceptual framework, I will examine students' understanding of the general concepts in elementary combinatorics. I will furthermore observe the ways that students deal with a new definition and their efforts to understand it using their existing knowledge.

Modes of enquiry and data source

The participants in this study were undergraduate students enrolled in the first year discrete mathematics course in Simon Fraser University. The students enrolled in this course are generally mathematics, science or engineering majors. Eight students volunteered to participate in this study. It is important to mention that the interviews were not a part of the course evaluation and I was not the instructor of the course.

A few days before the interviews, I emailed the participants a definition, instructions, and a short questionnaire about their mathematical background. I designed the definition specifically for the purpose of this research. However, participants had seen similar definitions before, such as the definitions of combination, permutation, and multinomial coefficient.

The main part of data was collected during clinical interviews, where students were presented with a set of tasks, which was designed to examine their understanding of combinatorics in general, and of this new definition in particular. For the purpose of this report, I will concentrate on students' initial understanding of the definition, and the first task in the interview.

> **Definition**. *Trization* of a set of n distinct elements is a placement of these elements into 3 different cells, with k_i objects in cell i ($i = 1,2,3$), and $k_1 + k_2 + k_3 = n$. The order of objects in each cell does not matter.
>
> The number of trizations of a set with n elements with k_i objects in cell i ($i = 1,2,3$) is denoted by $T(n : k_1, k_2, k_3)$.
>
> **The first task**. How many different 6-digit numbers can be made with digits 1, 2, and 3, if 1 can be used only once, 2 can be used only twice, and 3 can only be used three times.

The interviews were semi-structured, conducted individually, and were audio-recorded. Each interview lasted between 40 minutes to an hour. The students were encouraged to think out loud, and to write as much as possible. The combination of transcription of the audio files and students' writings during and before the interview created the set of data.

Results and discussion

The results can be organized in two parts. The first part is the initial stage, which includes the examination of students' understanding of trization before the interview. The second part is the intermediate stage. In this stage, I will look at the student's attitude towards the first task, and will examine their understanding of the relation between this task and trization.

One interesting trend during the initial stage was students' use of drawing to assist them in understanding the definition. Five out of eight students used some sort of drawing in their initial stage. This shows the important role of the "images" in student's formation of concept image when they first encounter a new definition.

Another trend, during the initial stage, was attempting to find a formula that corresponds to the number of all possible trizations. Although students were not asked to find any formula, four of them showed efforts (unsuccessfully) to find a formula. Without the formula, they claimed, they did not understand the definition. This showed that without knowing the formula, students did not feel confident that they had some understanding of this definition. Furthermore, it was evident that students felt that having a counting formula shows a better understanding than recognition of the objects that exemplify the definition. In fact, only one student attempted to create an example of a combinatorial structure which qualified as a trization.

The major trend during the intermediate stage was students' ability to use a correct formula to find the answer to the first task. In fact, five of the participants found the correct answer to the first task very quickly. One of the participants, Danny, claimed it is just like "UNUSUAL" problem. When he was asked to explain more, he added, their instructor asked them how many words can be made with the letters in the word "UNUSUAL". He quickly spotted the similarity between the two problems and solved the first problem successfully. When Danny was asked to explain why the formula works, he didn't know. He claimed that he is sure that it works, but didn't know why it works. This showed that he accepted the formula as a fact, and he is able to use it effectively in similar conditions.

Conclusion

This investigation revealed that figures and pictures are an essential part of students' formation of concept image. Another observation was that most students did not try to understand or examine the objects that the definition exemplifies, rather they looked for a formula that counted the total number of those "unknown" objects.

However, it has been shown that learners need to be familiar with the objects that a definition defines to be able to use the concepts in creative ways. In combinatorics, understanding the combinatorial structures is essential. One implication for teaching can be that combinatorial structures and their connections be emphasized. If the structures are understood, the formula, at least at the elementary level, can be derived easily. If learners do not become familiar with the objects that exemplify a concept, they can fail to see all the interesting connections between different structures, and this will potentially turn a creative concept into a set of rote memorizations of different formulae. The ability to use the formula, combined with the initial resistance to use trization for the first task, showed participants' tendency towards the use of formula as opposed to (not in conjunction with) familiarity with the combinatorial structures identified in the definition or by the first task.

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POINTS OF MISCONCEPTION CONCERNING INFINITY

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This study explores views of infinity of first-year university students enrolled in a mathematics foundation course, prior to and throughout instruction on the mathematical theory involved. A series of questionnaires that focus on geometrical representations of infinity was administered over the course of several weeks. Along with investigating students' naïve conceptions of infinity, this enquiry also examines changes of those views as beliefs, intuition, and instruction are combined. The findings reveal that students' conceptions about the nature of points, for instance, prevented them from drawing any correlation between numbers and points on a number line. Furthermore, a preliminary theoretical analysis using an APOS framework asserts that participants conceive of infinity mainly as a process, that is, as a potential to, say, create as many points as desired on a line segment to account for their infinite number.

The study of students' notions of infinity is a rich area of current research. In this report I address issues regarding the tensions between intuition, beliefs, and actual infinity. My research explores questions such as what are students' responses when comparing the number of points on different geometrical objects, what ideas do they maintain about numbers as points on a number line, and what difference does instruction make on student perceptions.

Background and Objectives

The counterintuitive nature of infinity has provided researchers with an opportunity to observe inconsistencies in students' reasoning as they confront well-known paradoxes or issues of cardinality (Dubinsky et. al. 2005; Dreyfus and Tsamir 2004; Tall 2001; Fischbein 2001; Fischbein et al. 1979). To the best of my knowledge, only a few studies examine students' conceptions with regard to infinity in a geometrical context (Dreyfus and Tsamir 2004; Tirosh 1999; Tsamir and Tirosh 1996, Fischbein et al. 1981). Tsamir and Tirosh (1996), for instance, explored students' intuitive decisions when comparing geometrical objects such as squares of different sizes, supplementing the claim of Fischbein et al. (1981) in a similar study that an intuitive leap is necessary to establish meaning about infinity. Extending on these topics, my study examines student responses to tasks such as drawing a comparison between the number of points on line segments of different lengths with a "broken line segment" and its "missing points," or comparing the number of points on a line or open line segment with those on a circle, two geometrical objects with significantly different topologies.

Methodology

Participants in this research are first-year university students enrolled in a mathematics foundations course. Data collection relies on a series of written questionnaires, group discussions, and clinical interviews. In addition to unearthing students' conceptions about infinity, the questionnaires explore whether intuitions and beliefs change as a result of further learning experiences and personal reflection. Specific questions and tasks for the data collection instruments include the items listed below. The first item was posed in order to establish what connections students made about numbers and points on a number line, whereas the last four items relate more specifically to issues about infinity.

- How many fractions can you find between the numbers 1/17 and 1/19? How do you know?
- What is the relationship between the number of points on a line segment and the length of that line segment? How do you know?

For the following questions, students were given a diagram of the geometrical objects figured below.

- Compare the number of points on A and B. What conclusions can you make? (likewise, students were asked to compare A and C, B and C)
- What can you say about the number of points missing on B? How do you know?
- Consider line segments A and C again. Suppose that the length of A is equal to the length of C + x, where x is some number greater than zero. What can you say about the number of points on the portion of A whose length is x?

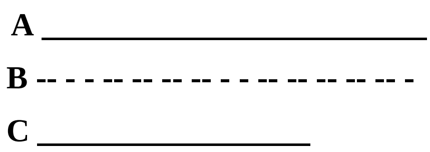

Figure 1

Results and Discussion

An analysis of the data based on APOS Theory (Dubinsky and McDonald, 2001) suggests how participants conceptualize infinity in the different contexts suggested above. Furthermore, the data reveal adjustments students make in their concept images and definitions (Tall and Vinner, 1981) as they respond to inconsistent results in their responses as well as counterintuitive properties of infinity. Survey responses indicate that subjects are able to interiorise the actions of finding a number between two others and partitioning a line segment into parts, to form process structures that Dubinsky et al. (2005) claim correspond to potential infinity. The data suggest participants have not developed the concept of "actual infinity [as] the mental object obtained through encapsulation of that process" (Dubinsky et al. 2005, p. 12). For instance, after a group discussion and some instruction, many infinite responses to the first item were because "the possibilities are endless," or "numbers after [the] decimal point can be added infinitely." These participants may have interiorised the action of listing numbers to a process without encapsulating it to a complete object that exists between any two numbers.

Similarly, the justification for infinite responses to the last three items relied on reasons such as having the ability to "create as many points on the line as we want." Infinite responses were less frequent than for the first item: nearly half of the participants gave finite responses despite instruction. A clear lack of connection between points on a real number line and numerical values was observed. In response to the second question, which was posed at an earlier date than the last three items, 70% of participants revealed beliefs that points were either the places that a line segment starts and ends, or else they were markers that partition a line segment into equal units. Half of these participants adjusted their conceptions of a point by the time the last three items were posed, however their understanding seemed vague and there was no indication that a correspondence between number and point on a number line was made.

This study opens the door for further investigation in some issues that may be overlooked or taken for granted, such as the connection between points on a number line and numbers. Moreover, it provides further understanding of one naïve perception of infinity and how it may evolve or change with instructor intervention. Finally, these results may unearth an added component to the interplay between action and process when a complete execution of the action is impossible.

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SITUATED NATURE OF NUMERACY IN HAIDA GWAII: KEY ISSUES AND CHALLENGES TO LEARN MATHEMATICS FOR ABORIGINAL STUDENTS.

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This study looks at the situated nature of numeracy. Members of the Haida Role Model Program in Haida Gwaii, BC Canada were interviewed. Many participants initially had difficulty identifying how mathematics is applied in their daily lives. As the conversations on community of practice evolved, it became clear that one's ability to learn mathematics increased if the context, personal and cultural relevance was meaningful. The results of situated understanding may assist teachers to find ways of using this work to make mathematics school curriculum and pedagogy more meaningful for Aboriginal students.

Introduction

Numeracy is a socially based activity that requires the ability to integrate math and communication skills (Withnall, 1995). Mathematics should be embedded in cultural activities that involve everyday tasks and solve everyday problems (Nunes, 1992). People of different cultures and different eras have engaged in mathematical activities to solve the problems they encountered in their daily lives.

Since the performance and participation rates of Aboriginal students in mathematics in BC are significantly lower than those of non-Aboriginal students; this study investigates if one's ability to learn mathematics increases if the context, personal and cultural relevance is meaningful. In this study members of the *Haida Role Model Program* on the islands of Haida Gwaii were interviewed to determine how they "Do the Math" in the daily lives through the frameworks of community of practice and ethnomathematics. Haida Gwaii is a collection of islands situated off the northern coast of British Columbia and south of Alaska. Half the population on the islands belongs to the Haida nation, and the students of Haida heritage have lower performance and participation rates in mathematics (Province of British Columbia, Ministry of Education, 2005). Though the Haida are a small nation and it is not necessarily representative of all the First Nations or Aboriginal people in the country, some of the numeracy practices of this nation are similar to those practiced by other nations. The students need to be proud that their culture and daily living has mathematical practices embedded for centuries, even though in schools the mathematics is taught in non-contextual abstract way. If Aboriginal students see themselves included and represented in the curriculum, then their learning of mathematics may improve.

Many Aboriginal students find themselves participating in two cultures – the culture of the home/community and the culture of the school. Students see little connection between the two cultures; hence many rich learning tasks from the community are lost in the school. This is particularly true in mathematics, where Aboriginal students feel alienated from the de-contextualized mathematics curriculum. Teaching mathematics with the use of familiar situations and examples can help students attach meaning to the concepts that they learn in school mathematics. Eventually, with the implementation of such strategies the participation rates and achievement of Aboriginal students in mathematics would increase.

Theoretical framework

Ethnomathematics has been identified as the study of mathematics that takes into consideration the culture in which mathematics arises (Ascher, 1991; Bishop, 1988;

D'Ambrosio, 1985; Zaslavsky, 1991). D'Ambrosio (1990) also defines ethnomathematics in the following way: "Resorting to etymology, the term ethnomathematics is introduced as the art or technique (tics) of explaining, understanding, coping with (mathema) the socio-culture and natural (ethno) environment" (p. 22). Ethnomathematics seeks to identify the diverse ways in which cultural groups quantify, compare, classify, measure, and explain day-to-day phenomena in their own environment. D'Ambrosio (1990) acknowledges the need to consider a holistic view of mathematics that includes one's culture, the culture of others, language, and the algorithms used and combined to construct individual abilities or even disabilities in mathematics.

Lave and Wenger (1991) argue that many activities are learned through mutual engagement in a joint participation, called a *community of practice*, where the participants are involved in a set of relationships over time in a context of lived experience (p. 98). On the contrary, many classroom learning activities involve abstract and context-free knowledge. Social interaction is a critical component of situated learning—learners become involved in a community of practice, which embodies certain beliefs and behaviours to be acquired. As the beginner or novice moves from the periphery of this community to its centre, he or she becomes more active and engaged within the culture and hence assumes the role of the expert or elder. Lave and Wenger (1991) call this process *legitimate peripheral participation*. Hence, school mathematics is a form of situated learning, which needs to take place within a context. The context needs to be mathematically meaningful to the learner, and the curriculum should make sense of the local social and cultural situations.

Methodology

After some brainstorming with members of the Haida Education Council it became clear that the participants for the study should be members of the *Haida Role Model Program*. The *Haida Role Model Program* consists of elders, professionals, and community members that assist teachers in schools by integrating Haida knowledge and perspective with the school curriculum. The Role Models provide a vital connection between the school district community and the Haida community. Since the Role Models had been screened and identified by the school district, I knew that this group would be representative of the community. Even though I stayed in Haida Gwaii for six weeks during my field work I still didn't get a chance to interview some of the people listed in the program. In total twenty three members of the community were interviewed.

The primary data for this study consists of excerpts from the interviews. All interviews were digitally tape recorded, and verbatim transcripts were produced. According to Patton (1990), full transcriptions of interviews are the most desirable data to obtain. I developed an "open coding" system which enabled me to analyze the interviews and document how the participants "Do the Math" in the daily lives. The mathematical practices in the community life of Haida Gwaii are unique to its people, land and context. One should be careful in generalizing such an experience for Aboriginal students in other regions.

Sample Interviews

During the field work of this study a number of participants indicated that they didn't use mathematics in their daily lives. However, as the interviews progressed it was evident that they did use mathematical activities such as: counting, measuring, designing, locating, and explaining. Below are two sample interview transcripts about making a bracelet, and a button blanket.

The making of jewelry, uses mathematical concepts such as symmetry, congruency, and transformations. The bracelets are representative of a transitional time when the Haida were not allowed to tattoo themselves anymore. One of the solutions that the Haida found was to actually put the design on bracelets. James Sawyer a master jewelry maker explains the process of making a bracelet.

> I will take a common size 6 inch bracelet put a centerline on it, measure on each side of it, maybe half an inch to an inch—depending on if I am going to do two designs in the middle—like a split design. I would then leave an inch on each side of the centerline and draw that in. You only have to draw it on one side because after you are done carving out your formline, you then transfer or trace it. You draw it on tracing paper, and then flip it over and then you've got the same carving on the other side. You can measure it out if you really want to but it's on such a small space, I've gotten to the point now where I'm just doing the secondary lines just by eye.
>
> -James Sawyer

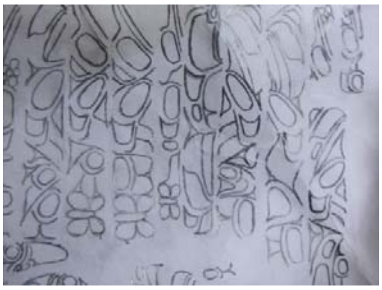

Figure 1: Designs created on tracing paper before making silver bracelets by James Sawyer

Button blankets have been used by a number of First Nations in the Pacific Northwest for hundreds of years as a representation of ones family lineage and crests. Crests and clans were inherited from the mother and were displayed on a number of artifacts. At one time, the blankets were mostly used for trade, today they are used for ceremonial purposes.

In making a blanket you use various math concepts such as patterns, measurement, estimation, congruency and symmetry. Some blankets might have vertical and horizontal symmetry, others might not be symmetrical as the top and bottom might have different geometric shapes to represent the head or feet. Irene Mills, a master button blanket maker, talks about the process of making a blanket.

> When I wanted to do a new blanket I asked a young artist, Tyson Brown, if he would do a design for my blanket. I decided on the size of the blanket and marked out where the border would be so he could ensure that the design filled the blanket. He wanted to make sure that the design was mostly on the back so if you stand with your blanket most of it is on the back. I enlarged the design to three feet in length— the integrity of the design would fill most of the blanket and made sure that it wasn't too small on the back. And then for centering—once you get the design laid out—you just measure how much room you have on either side and there needs to be twice as much room on the bottom as there is on the top.
>
> I learned how to put buttons on a blanket from my mother. For example, she showed my how to start out in the corners and then measure how much room four buttons takes up and then divide that in the length to see how many spaces are needed so that you end with four buttons. Sometimes your calculations are

> incorrect, and you need to make minor adjustments as you get closer to the bottom—maybe, a fraction of an inch, so that it will still look like you followed the pattern. And it's changed so subtly that you won't notice.
>
> -Irene Mills

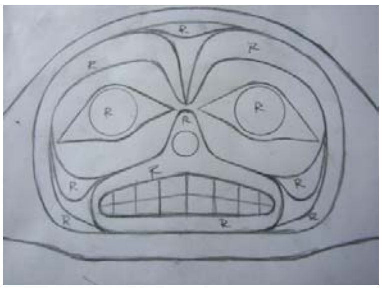

Figure 2: Initial design and the four button border of Irene's button blanket

Excerpts from both the artists show how their use of mathematics is in a personal and culturally relevant context. My questions in the interviews not only addressed the purpose of my study but also investigated some of the factors that shaped a person's understanding and use of mathematics in their daily living, and how this relates to the mathematics learned in school. Nunes (1993) noted that the street mathematics used by child vendors in Brazil and the mathematics taught in school were comparable "in terms of the mathematical properties they implicitly used, the properties turned out to be the same" (p. 94).

Key Issues

The above-mentioned interviews show that there are a number of mathematical concepts embedded in the daily living. As the interviews were analyzed many issues and challenges surfaced. Four key issues: Teacher Training, Early Intervention, Outreach / Relationships, and Cultural and Traditional Knowledge emerged and will be discussed further. Over the years, similar issues have also been identified by various task forces, position statements, and other documents (*Indian Control of Indian Education*, 1972; *Early Math Strategy,* 2003; *NCTM Position Statements,* 2005; *Education Action Plan,* 2005).

Teacher Training

The NCTM Position: Closing the Achievement Gap (2005) outlines that "to close the achievement gap, all students need the opportunity to learn challenging mathematics from a well-qualified teacher who will make connections to the background, needs, and cultures of all learners." The quality of instruction is a function of how well teachers know and understand mathematics for which they are assigned to teach (NCTM, 2000). For teachers to be well-qualified and effective, they must have a "profound understanding of fundamental mathematics" (Ma, 1999).

Irvine and Armento (2001) identify the significance and urgency of implementing culturally responsive pedagogy. This term implies that teachers should be responsive to the students' culture in their teaching. Educators must be child centered and aware of the prior knowledge, language, and experiences of the students in their classes. Moreover, having well-qualified math teachers who are mathematically competent and pedagogically proficient would be beneficial for all learners: Aboriginal and non-Aboriginal.

Nichol and Robinson (2000) suggest that teachers in the methodology courses should be equipped with a range of teaching strategies reflecting the diverse learning needs and preferred ways of learning for Aboriginal students. In any case, the teacher's underlying beliefs are also of fundamental importance in the effective teaching of mathematics.

Most teacher education programs are centered on teaching from a Euro-western cultural perspective. Cajete (1999, p.162-179) recommends an indigenous curriculum model with multidimensional approaches that is centered with creativity as a learning process, and with the perspective of mathematics as a cultural system of knowledge.

Early Intervention

The Results & Discussion of National Roundtable on Aboriginal ECD (2005) reported that Euro-western approaches often do not fit the needs, interests, or development and learning styles of Aboriginal students. Thus, community-based strategies should be developed to address the full spectrum of early childhood services based on Aboriginal principles and benchmarks. Early intervention to address student learning difficulties in mathematics is more successful than responding to accumulated deficits at a later date (MOE, 1999).

Young children are naturally inquisitive about mathematics, and teachers can build on this inquisitiveness to help students develop the positive attitudes that often occur when one understands and makes sense of a topic (Expert Panel on Early Math in Ontario, 2003).

The most important connection for early mathematics development is between the intuitive, informal mathematics that students have learned through their own experiences and the mathematics they are learning in school. All other connections – between one mathematical concept and another, between different mathematics topics, between mathematics and other fields of knowledge, and between mathematics and everyday life – are supported by the link between the students' informal experiences and more formal mathematics. (NCTM, 2000, p. 132)

Outreach / Relationships

A key concept shared by many Aboriginal people is that of *relationality*, which is the belief and understanding of the interconnectedness of our world and all within it. In addition, relationality encompasses other realities that we cannot see, but of which we are aware (Wilson, 2003). The Alaska Native Knowledge Network (1999) developed "cultural standards" on the belief that a firm grounding in the heritage language and culture, indigenous to a particular place, is a fundamental prerequisite for developing culturally-healthy students and communities associated with that place. This would then act as an essential ingredient for identifying the appropriate qualities and practices associated with culturally-responsive educators, curricula, and schools.

When home and school work together, students have increased opportunities to gain numeracy skills necessary for success in school and beyond. Opening channels of communication with the home sends the message to children and their parents that the mathematics at school is worthy and important. Teachers also benefit from strong home-school partnerships. Better communication is needed with parents, who often do not understand why their children fail to learn math the way they used to. The trauma and issues associated with residential schools still affect many members of the Aboriginal community, especially parents, in BC. Some parents choose not to visit a school because it brings back the emotional trauma associated with the past.

Like reading stories, the parents could do interactive math activities such as counting, measuring, designing, estimating, locating, or even game-playing. Parents want their children to be successful but sometimes do not know how to help. By involving parents in a friendly environment, greater self-confidence, self-discipline, and teamwork is built with the student, home, and school. Many Aboriginal students lose interest in school, especially in mathematics, around grade 8. Students need to see how mathematics is relevant, and how it could assist them in finding solutions to their everyday problems.

Cultural and Traditional Knowledge

Canada must affirm indigenous knowledge as an integral and essential part of the national heritage of Canada, to be preserved and enhanced for the benefit of current and future Canadians (*Education Action Plan*, 2005). Aikenhead (2002), Davison (2002), and Nichol and Robinson (2000) suggest that when culturally inclusive curricula and pedagogy are delivered in a way that accounts for learner diversity, then Aboriginal students' achievement improves significantly. Bishop (1988) indicates that the cultural background of students is rich in terms of the resources from which mathematics concepts can be developed. Aboriginal students participate in two cultures – the culture of the home and the culture of the school. Many of these students see little connection between these two cultures; consequently, many potentially rich situations from the native culture are lost to the school (Davison, 2002). A paradigm shift must occur to allow students to bridge the two worldviews: indigenous and Euro-western, which are both equally valid and important.

According to the indigenous tradition, all children have gifts. Mathematics should not be taught by deconstructing the students' own traditional values and knowledge, but by making connections. Since protocols are different in each Aboriginal community, coming up with a generic system that will serve all Aboriginal communities is a complicated task. Many Aboriginal communities and jurisdictions, such as the Government of Nunavut, have produced documents that outline their guiding principles that are culturally specific to their region. Culturally–responsive educators must take the time to find the indigenous way of knowing, and recognize the validity and integrity of the traditional knowledge system.

Final Thoughts

A change in the beliefs, attitudes, and policy will require sustained effort, time, and resources. Since public education institutions have not yet implemented many of the calls for change, educational stakeholders must periodically review the key issues outlined above: Teacher Training, Early Intervention, Outreach / Relationships, and Cultural and Traditional Knowledge.

Davison (2002) asserts that the use of cultural situations can improve the learning of mathematics by Aboriginal students in several ways. When the teaching of mathematics uses ideas from the culture, students value their cultural heritage more. The integration of the students' experiential mathematics with their school mathematics can help them make new connections at a personal level.

> Cultural approaches start from the belief that if youth are solidly grounded in their Aboriginal identity and cultural knowledge, they will have strong personal resources to develop intellectually, physically, emotionally and spiritually. The ability to implement culture-based curriculum goes hand in hand with the authority to control what happens in the school system. (Royal Commission on Aboriginal Peoples, 1996, p. 478)

Teachers of Aboriginal students must be encouraged to adapt their teaching to match the learning styles of their students. When planning to include Aboriginal cultural concepts and traditional knowledge in the mathematics curriculum, educators must seek the guidance of the local communities to fully understand and address the concepts that are unique to those communities. Many teachers of Aboriginal students are not members of that community.

Therefore, extensive pre-service and professional development sessions would be needed to help teachers gain the knowledge they need. Teachers also need to be mathematically competent and pedagogically proficient, as many do not feel comfortable with the content at the elementary level.

Success in mathematics in the early grades has a profound effect on mathematical proficiency in later years (Expert Panel on Early Math in Ontario, 2003). Parents, teachers, and children should all be partners in the learning process and should work together to support a child's educational journey. Outreach programs are needed to bridge the gaps between the school, community, and home, and mentors, role models, and elders are need to guide Aboriginal students during and after school, all the way to graduation.

Principles and Standards for School Mathematics (NCTM, 2000) calls for a common foundation of mathematics to be learned by all students. It also advocates the need to learn and teach mathematics as a part of cultural heritage and for life. Achieving this goal requires a paradigm shift in the way mathematics is taught and the introduction of culturally inclusive curricula and pedagogy. Partnerships need to be developed with educators, elders, parents, policymakers, and others in the community to promote numeracy and change societal attitudes towards mathematics to reflect the fact that mathematics is inherent in everyday living and anyone can do mathematics. Eventually, with the implementation of such strategies the participation rates and achievement of Aboriginal students in mathematics would increase.

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USING A REFLECTIVE READING OF HISTORY TO SCRUTINIZE PERSONAL BELIEFS ABOUT THE NATURE OF MATHEMATICS

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Considerable research has been done on the impact of beliefs about mathematics on the teaching and learning of mathematics. It has been pointed out that it is often difficult for individuals to identify and articulate their personal beliefs. These difficulties are problematical for research in which clear and accurate statements of beliefs are a significant component of the data. It is proposed here that a hermeneutical reading of episodes in the history of mathematics with the specific intent of eliciting personal beliefs about mathematics may be helpful in obtaining clear and accurate belief statements. The author experiments with this method by reading through historical vignettes in the lives of Descartes and Galileo.

All mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics. (René Thom)

But what bothered me was that I didn't know what my own opinion was. What was worse, I didn't have a basis; a criterion on which to evaluate different opinions, to advocate or attack one viewpoint or another. (Philip J. Davis & Reuben Hersh, 1981)

Background and Objectives

The underlying premise of this exercise is that personal beliefs about the nature of mathematics are of crucial importance to all participants in the mathematics education endeavour: teachers, learners, and researchers alike. A second assertion is that it is a difficult task for an individual to identify and articulate these beliefs. As Leatham puts it:

> One prevalent pitfall of research on teachers' beliefs is to take a positivistic approach to belief structures, assuming that teachers can easily articulate their beliefs and that there is a one-to-one correspondence between what teachers state and what researchers think those statements mean. (2006, p.91)

Granting the inherent challenges and value in uncovering and making intelligible our beliefs, a possible mechanism for facilitating such tasks is offered. It is posited that a critical, reflective reading of episodes in the history of mathematics can uncover beliefs about the nature of mathematics that hitherto had not been transparent to the reader. The reading is critical to the extent that the reader is sceptical of conventional interpretations of history; it is reflective in its intent to understand one's own conceptualization of mathematics. Effectively, it may be regarded as a hermeneutical reading of historical accounts.

If this use of history does indeed assist individuals in understanding and expressing their beliefs, then the technique would be of value in research on mathematical beliefs and practice. One of the challenges in such research is for the researcher to come to an "accurate" understanding of a subject's beliefs. A reflective reading of history, if effective, can also be of worth in mathematics classrooms where an objective is for the student to develop a "coherent philosophy of mathematics".

Conceptual Framework and Method

The study of history illuminates the present as much as the past; our interpretations tell as much about ourselves as they do about historical personages. The approach to history taken in this study is that of a Foulcaudian archaeology to the extent that it seeks to uncover the "positive unconscious" that eludes the scientist. The reference to actual persons is less in the spirit of Foucault's "history of thought". (Foucault, 1970). Similarly the project here is less to comprehend the epistemes of Descartes' and Galileo's Europe and more and attempt to uncover the 'positive unconscious' of today and of the individual reader.

The method employed here is a reflective reading of history with the intent on elucidating personal beliefs about the nature of mathematics. (To some extent, this is evocative of Cartesian introspection!) Since such beliefs are indeed personal, the actual conclusions reached will vary among the 'reflective readers'. The technique is successful to the extent that the reader had clarified his or her beliefs and is able to communicate them.

The author revisited historical accounts of Descartes as the "Father of Modern (pure) Mathematics" and Galileo as the "Father of Modern Science (applied mathematics)" with the intent of clarifying his understanding of the distinction between pure and applied mathematics. The readings and the reflections extended both forwards and backwards in history. Surprises and challenges to pre-existing "knowledge" were encountered. To borrow vocabulary from constructivism, disequilibration was followed by accommodation and assimilation. The third stage was to write about the specific issue at hand. The "read, reflect, articulate" cycle could be iterated indefinitely.

Illustrative vignettes

Motivation and emphasis

Both Descartes and Galileo extolled the power and grandeur of mathematics. Yet they chose different purposes for mathematics. Galileo wrote famously in "The Assayer" that the grand book of the universe "is written in the language of mathematics" and that it would be "humanly impossible to understand a single word of it" without mathematics. Descartes, in "A Discourse on Method" delighted in mathematics "because of the certainty of its demonstrations and the evidence of its reasoning", although he was "astonished" that mathematics had "no loftier edifice" constructed on it than mechanical arts. It was Descartes purpose to *use* the methods of mathematics to explain that "all those things which fall under human cognisance". Using mathematics to explain nature or to organize thought. Thinking of mathematics as a language and as a source of certainty. Is it the use and user of mathematics that determines its nature?

Science and mathematics

Both Descartes and Galileo have been titled as "the Father of Modern Science". Descartes articulated the concept of an impersonal, mechanical universe and had a well developed physics. As Garber points out, however, "there was a curious lack of any substantive role for mathematics" in his physics." Ross (1996) credits Galileo with inventing mathematical physics. "There is no math in Aristotle's *Physics*. There is nothing but math in modern physics books. Galileo made the change. It is inconceivable now that science could be done any other way." Is there the possibility of a new use of mathematics, an as yet untried marriage of mathematics and another discipline, that seems incongruous now, but will seem inconceivable that it could be otherwise in the future?

Descartes commented on Galileo in a letter to Father Mersenne. (I have found no indication that Galileo ever commented on Descartes.) Eaton quotes Descartes, "In this [his 'attempts to examine physical matters by methods of mathematics'] I am in entire agreement with him, and I believe that there is absolutely no other way of discovering the truth." Yet he fails to do so in his own science. If a genius of Descartes' stature misses such an "obvious" connection between mathematics and science, then what connections might we be missing today? And, in light of this, how can mathematics ever be considered an ahistorical, immutable discipline?

Galileo is perhaps best known for his role in the contretemps between church and science resulting from his inspired endorsement of a heliocentric Copernican model of the heavens. It was not, however, his support for the Copernican model. The Copernican model was well liked for its mathematical elegance and how it simplified calculating positions of heavenly bodies. This allowed for the casting of more accurate astrological horoscope (which was one of Galileo's responsibilities as a court mathematician). The church took umbrage at Galileo's insistence that the Copernican model *described reality*. Predictive mathematics transformed to descriptive. And the transformation of the role of mathematics was central to a social upheaval. How can mathematics be deemed asocial and acultural when it has such a significant impact?

Homo Faber

Proclus, writing in the fifth century, described various divisions of mathematics. One such division is given as "mathematics on the one hand as concerned with things conceived by the mind, and on the other hand as concerned with and applied to things perceived by the senses." Descartes seemed preoccupied with things conceived by the mind. It is likely that Descartes worked in optics for a time and was a military engineer. He was able to work with tools, but he *chose* to focus on matters of the mind.

Galileo, on the other hand, was a tool-maker and tool-user *extraordinaire*. His sector compasses could be used for computation and measurement. It was a mathematical tool designed for *practical* purposes. While Galileo and Descartes both worked with and theorized about optics, it was Galileo who improved the telescope and used it to earn yet another paternity title, "the Father of Modern Astronomy".

When Galileo was rolling balls down inclined planes, it must be remembered that he had to invent and fabricate water clocks to "measure time". The experimentalist in the laboratory would also develop and employ the mathematical techniques (foreshadowing the integral calculus!) necessary for his mathematical explanation (cf. Struik, pp. 198- 209) of uniformly accelerated motion. This, of course, is not a mathematician of the twenty-first century (until I recall accounts of Mandelbrot experimenting/ "playing" with rudimentary computer images in the 1970's). Has mathematics moved completely beyond the realm of sensible world? What of game theory, fractal geometry, and non-linear dynamics? Perhaps pre-occupation of mathematicians in the late 19th century and 20th century with rigour and deduction was only a swing of the historical pendulum, rather than a "progression towards ideal mathematics".

Economic imperatives

What motivates a mathematician to do the work that he or she does? Descartes was a man of moderate, but independent means. He had few familial obligations (due to unfortunate deaths). And he accepted his patronage appointment possibly to avoid conflict with theological authorities. By reputation, Descartes was a man who slept in late and spent much time thinking. This is a man who had training as a military engineer, a comprehensive education, and had worked successfully with optics and mechanical devices. Would his use of mathematics have changed if he had found it necessary to "make his way in the world"?

Galileo, on the other hand, was driven by financial concerns and seemed at times the consummate entrepreneur. Galileo began as a professor of mathematics and astronomy and later had patronage appointments as a court mathematician. He was awarded patents and given raises for his inventions and was given to effusive dedications in his books. At one time, Galileo housed students who paid him room and board, purchased the sector compasses that Galileo had invented, purchased the text he had authored on their use, and charged tuition. As well, he rented room and board to an in-house instrument maker.

What constraints on the lives of mathematicians today modify the work they choose to do?

Results and Conclusions

An intentional reading of history did serve to clarify my beliefs about the nature of mathematics. It reinforced my understanding of mathematics as a human activity and that the mathematics of today is not necessarily the mathematics of yesterday or tomorrow.

The "read, reflect, articulate" cycle will enable be to state and defend my opinions with greater clarity. By excavating the "positive unconscious" I can at least be aware of the extent to which my practices and beliefs are consistent. This will inform my research and my teaching practices.

It is use of the history of mathematics that could be of value to anyone concerned with the relationship of practices and beliefs.

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Mathematics Anxiety among preservice teachers in the professional development program: A factor and reliability analysis

Radcliffe Siddo

Abstract

The Mathematics Anxiety Rating Scale (MARS) developed by Richardson and Suinn (1972) is a measure of mathematics anxiety for use in treatment and research. A reduced version, the Revised Mathematics Anxiety Scale (RMARS) was developed by Alexander and Martray in 1989. Using a large sample (N=815), Hopko (2003) revised the RMARS due to inadequate support for either one- or two- factor model (learning mathematics anxiety and mathematics evaluation anxiety). The latter scale was subjected to factor analysis using 25 preservice teachers in the Professional Development Program (PDP). Three major factors accounted for 73% of the variance, confirming the instrument as a multidimensional measure. Internal consistency reliability coefficients of the revised measure were moderately strong. The revised version may represent a more fruitful measurement approach for using a small sample for assessing mathematics anxiety.

Objectives of the Study

According to Richardson and Suinn (1972), the original 98-item Mathematics Anxiety Rating Scale (MARS), was developed to measure a phenomenon: "feelings of tension and anxiety that interfere with manipulation of numbers and the solving of mathematical problems in a wide variety of ordinary life and academic situations" (p. 551). This instrument has been acknowledged as a multidimensional measure focusing on anxiety concerning mathematics testing and numerical manipulation (Kazelskis, Reeves, Kersh, Bailey, Cole, Larmon, Hall, & holliday, 2000); also, dissimilarities in factor structure have been reported for varying sample sizes and/or populations. For instance, Rounds and Hendel (1980) populations were from a university of only female participants in a mathematics treatment program, and identified two factors and labeled them "math test anxiety" and "numerical anxiety". Resnick, Viehe, and Segal (1982) found three factors from a population of freshmen college students: evaluation anxiety, social responsibility anxiety, and arithmetic computation anxiety that accounted for 32%, 5% ad 4% of the variance, respectively. There was no internal consistency reliability coefficients reported.

Alexander and Martray (1989) developed a reduced version of the 98-item original, the 25-item RMARS. The modified instrument decreased administration time and added to relieve scoring, while maintaining psychometric properties equivalent to MARS. There were three factors identified among the undergraduates' students in psychology, labeling these "math test anxiety", "numerical task anxiety" and "math course anxiety". This accounted for only 24%, 4%, and 3% of the total variance, respectively. The internal consistency reliability coefficients for factor 1 (mathematics test anxiety) were .96, factor 2 (numeric test anxiety) were .84, and factor 3 (mathematics course anxiety) were .97. Furthermore, Bowd and Brady (2002) found three factors form a population of undergraduate education majors who were completing their final year of a B.Ed. program: mathematics test anxiety, mathematics course anxiety, and numerical anxiety. This accounted for 34%, 22% and 17% of the variance, respectively. The internal consistency reliability for the entire instrument was Cronbach alpha= .97.

There are two purposes for this study. Firstly, factor analysis were performed and investigated from the 12-item revised MARS-R (Hopko, 2003). A number of researchers have pointed out (i.e., McMorris, 1992; Weinberg, 1992), the literature implies that the nature of mathematics anxiety may vary differently among populations and sample sizes, so it may be misleading to assume that the shorter version of the 12-item revised MARS-R comprises of one-, two- or three- primary factors without further investigation. Secondly, internal consistency reliability coefficients were examined for this revised version of MARS-R which may yield differing reliability scores of assessing mathematics anxiety amongst preservice teachers in the professional development program.

Perspectives or Theoretical Framework

Gronlund and Linn (1990) stated, "Reliability refers to the *results* obtained with an evaluation [psychometric] instrument and not to the instrument itself. Therefore it is more appropriate to speak of the reliability of 'test scores' or the 'measurement' than of the 'test' or the 'instrument' (p.78). Many researchers have failed to report reliability results in their articles. Establishing reliability in the interpretation of data is very important. This ensures strong statistical significance within the results (Henson, 2001; Reinhardt, 1996).

Moreover, poor reliability will reduce statistical power (Onwuegbuzie & Daniel, 2000) and possibly lead to inappropriate conclusions in research findings (Thompson, 1994). Reliability may fluctuate depending on the sample size and population; researchers should always examine the reliability of their data in hand and report these results even in nonmeasurement studies (Capraro, Capraro, & Henson, 2001).

One of the primary considerations in using factor analysis involves deciding how many factors to extract from the data (Bessant, 1995). There are two forms of factor analysis to consider: *exploratory* and *confirmatory*. Exploratory factor analysis is an a posteriori technique for reducing the data through clustering, helping to render the data more manageable. Confirmatory factor analysis postulates, a priori, that one already knows what the measurements represent, and analyzes the data accordingly. These analyses may provide evidence-based, quantifiable indications as to how to group, measure, evaluate, and analyze this sample accordingly.

Methods, Techniques, or Modes of inquiry

Participants included 25 preservice teachers in the Professional Development Program who completed 12-item revised MARS-R. PDP is a three-semester program that involves 12 months of study. This program is designed to prepare teachers for careers at either elementary or secondary school levels. Participation was voluntary, however, none declined.

The instrument of inquiry was the 12-item revised MARS-R. This instrument measures anxiety in math-related situations and have been known to have two subscales: Learning Math Anxiety (LMA), which relates to anxiety about the process of learning and Math Evaluation Anxiety (MEA), which is more directly related to testing situations (Hopko, 2003). Participants record their responses on a 5-point Likert-type scale ranging from 1 (none at all) to 5 (very much). Item scores are summed to give a total range of 12 to 60, with higher scores reflecting higher levels of mathematics anxiety.

Results and/or conclusions/point of view

The mean score was 21.52 (SD=14.27). Internal consistency reliability for the entire instrument was moderately high (Cronbach alpha=.91), but not as high as the Cronbach alpha=.97 reported by Bowd and Brady (2002). Principal component factor analysis yielded three factors which accounted for approximately 51%, 11%, and 11% of the variance, respectively. The first factor, labeled Mathematics Course Anxiety, loaded primarily on items dealing with classroom activities in mathematics (i.e., thinking about math, preparation for math, listening to a lesson, and receiving a textbook). The factor was defined by 6 items that loaded highest on it (ranging from .40 through .79). The second factor labeled Mathematics Test Anxiety, loaded primarily on items dealing with taking mathematical assessments, e.g., aptitude test and 'pop' quiz. The factor was defined by two items that loaded the highest on it (ranging from .85 to .91). The third factor, labeled Numerical Task Anxiety, loaded on items concerned mostly with performing numerical activities, e.g., having to use tables in a textbook. The factor was defined by four items that loaded the highest on it (ranging from .55 to .87). The internal consistency reliability coefficients for factor 1 (mathematics course anxiety) is .86, factor 2 (mathematics test anxiety) is .90, and factor 3 (numerical task anxiety) is .86.

Significance of the study

These factors, identified for preservice teachers in the professional development program, closely resembled those obtained by Brady and Bowd (2002) with senior undergraduate students in education. However, the first factor structure being mathematics course anxiety accounted for the vast amount of variance which was almost twice that was reported by Bowd and Brady (2002). Mathematics Course Anxiety was the most significant factor for preservice teachers, probably reflecting anxiety attached to the real prospect, for most of them, of teaching mathematics after graduation. On the other hand, this study accounted for lower variances of both math test anxiety and numerical task anxiety. The data confirm observations that the revised MARS-R gives measure of three meaningful constructs and its factorial structure and may vary somewhat with different populations and sample sizes. In addition, this study shows that the revised MARS-R reduces reliability scores compared to previous studies, however, provides a greater variance amongst the factors. Furthermore, the internal reliability coefficient serves as an example of how reliability scores can vary from sample sizes and/or populations.

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MATHEMATICS EDUCATION IN CHINA

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From the characteristics of Chinese mathematics education, this article introduces the Chinese way of mathematics teaching. It is demonstrated that the "paradox of Chinese learners" might originally be a misperception.

A paradox of Chinese learners

Since 1980s, many international comparative studies on mathematics achievement, which include Chinese students at primary and secondary schools, have repeatedly shown an apparent contradiction between the teacher-dominated learning environment (i.e. large classes, whole-class teaching, examination-driven teaching, focus on content rather than process, emphasis on memorization, etc) which is generally perceived to be non-conducive to learning, and the fact that the outstanding performance of students in comparative studies-such as those carried out by the International Association for the Evaluation of educational Achievement (IEA) (Mullis et al., 2000).This contradictory situation was called paradox of the Chinese.

In recent years, this phenomenon has been discussed by a number of authors (e.g. see Biggers, 1996; Leung, 2001; Mok et al., 2001). It has also led to many studies on the psychological and pedagogical perspectives of Chinese teaching and learning.

Research findings seem to suggest that a teacher-dominated lesson may not be necessarily bad for learning, just as a student-centered lesson may not always be positive. It is obvious that simple social interaction labels such as, "teacher-dominated" or "student-centered" have not explained the heart of the matter. Explore the characteristics of Chinese mathematics education will help us to uncover the paradox of Chinese mathematics learning.

Characteristics of Chinese mathematics education

The examination system

China's civilization had a great impact on education in China. For many centuries, Chinese education was characterized as scholar-nurturing education. Education was equated with moral superiority that justified political power and high social-states. One of the distinctive features of this form of scholar-nurturing education was the dominance of the state, which grew steadily with the elaboration of the examination system (Pepper, 1996).

Chinese examination system, civil service recruitment method and educational system employed from the Han dynasty (206 B.C.–A.D. 220) until it was abolished by the Ch'ing dowager empress Tz'u Hsi in 1905 under pressure from leading Chinese intellectuals. The concept of a state ruled by men of ability and virtue was an outgrowth of Confucian philosophy. The examination system was an attempt to recruit men on the basis of merit rather than on the basis of family or political connection. Because success in the examination system was the basis of social status and because education was the key to success in the system, education was highly regarded in traditional China. If a person passed the provincial examination, his entire family was raised in status to that of scholar gentry, thereby receiving prestige and privilege. The texts studied for the examination were the Confucian classics. In the T'ang dynasty (618–906) the examination system was reorganized and more efficiently administered. Because some scholars criticized the emphasis on memorization without practical application and the narrow scope of the examinations, the system underwent further change in the Sung dynasty (960–1279). Wang An Shi reformed the examination, stressing the understanding of underlying ideas and the ability to apply classical insights to contemporary problems. In the Ming dynasty (1368–1644) the commentaries of the Sung Neo-Confucian philosopher Chu shi were adopted as the orthodox interpretation of the classics. Although only a small percentage of students could achieve office, students spent 20 to 30 years memorizing the orthodox commentaries in preparation for a series of up to eight examinations for the highest degree.

China has a long history in examination system that drives curriculum instruction. As Ashmore and Cao(1997) observe, "examinations are crucial feature of Chinese education. They determine whether an individual is eligible for more advanced training and what form that training will take. In recent decades, primary school graduates have been required to take an examination to determine which middle school (Junior-high school) they will attend. Students are admitted into different schools according to their scores. Those with high scores are admitted to the key middle schools. After finishing three years in middle school, students have to take an examination to determine whether they will enter a key high school or a regular light school. Those who fail the entrance examination are placed in vocational high school. Once graduated from high school, students have to take rigorous entrance examination to be able to qualify for university. Some Chinese feel that there are some advantages to this system. For example, students will have a strong basic foundation in all subjects and have strong capability to enter the competitive world.

And examinations also determine the designing of curricular, the use of textbook, and teaching method. Under the examination driven system, Chinese mathematics education pay more attention on basic mathematics knowledge and skills, we call them "two basics".

Two basics

Any theory of mathematics education would likely concern two aspects: first, help students gain the basic mathematics knowledge and skills; secondly, let students realize full individual development and foster their create mathematics thinking. Success is ensured if both of them can be equally emphasized and appropriately interwoven. (zhang)

Two basics means the principle of "basic knowledge and basic skills", it was explicitly put forward for the teaching of mathematics. Due to historical and cultural reasons, mathematics education in China emphasizes the importance of foundations.

Historical root and social environment for the "two basics" principle in mathematics teaching

1) China is sense one of few countries where human's ancient civilization has had a continuous existence. Thousand years of agricultural culture, especially culture developed from plantation of paddy-field crops, required detailed and crafty artifice. Given a small land area, farmers had to rely on well-practiced and efficient techniques to obtain maximum outputs. It was very different from nomadic society's culture where people can make a living through the extension of rearing area. Thus, in the Chinese society, to be equipped with effective and efficient "skills" is of vital importance for survival.

2) Confucianism is the orthodox tradition of Chinese culture. In a very long time, Confucianism endorses a clan system, where obedience to the unified emperor is universally required. Individuals have less room for their self-creativity, compared to the west. This is also uniformity of teaching contents as well as requirements. The result of the existence of unified foundation but a lack of individual development among students (Brand,1987; Murphy, 1987. Wong, 1998).

3) The strict and unified examination systems have driven students to only learn the contents that will be tested in exams.

4) In the 1950s, the mathematics education in China was heavily influenced by former Soviet Union's mathematics education, which further increased the emphasis on "mathematics foundation" and "basic mathematics".

Features of mathematics teaching under the "two basics":

1) Teachers play a central role in classroom.

Since the class size can reach as many as 40-50 students, it is impossible for teachers practice individualized teaching. Therefore, individualized teaching is not a primary goal. What topic to teach in every period and how long to let students stay at every state in a class are decided by teachers. The pace of classroom teaching is led by teacher's judgment based on most students' learning ability in the class. Students are required to follow the pace of progress. The central role demands teachers to have profound understanding on the topic and to find the best way to organize own teaching.

2) Effective teaching is emphasized.

It requires teachers to present the main mathematics contents as quickly as possible so that students won't spend too much time in a winding path. One of the criterions to evaluate classroom teaching is to examine whether teacher completes the objectives that are set in advance. In this aspect, investigation, group discussion and real-life application would be considered a waste of time if such activities are relatively far away from the main topic, "discovery", "constructivist teaching", "group discussion", "real-life mathematics" could only be practiced in a very limited amount of time in classroom, certainly not on a daily basis. However, it does not mean zero. Oral questioning is very popular interaction in practical teaching. Teacher usually asks a series of relatively easy questions about the topic and students answer them individually, in a group or teacher provides answer. Questioning process guides students to reach the learning objectives step by step instead of their own discovery.

3) The pattern of "teach only essential and ensure plenty of practice" is used in teaching.

It does not support the idea of "understanding first", but insists that both understanding and manipulation are equal importance. This means that what teacher explain and demonstrate should be essential in order to save time for student to do more exercises and problem solving. It is not necessary for them to spend much time to make students understand, as it is believed that this is unlikely to be accomplished through first explanation. It is better that students do exercises after they have understood. However without their thorough understanding, students could practice first and then develop their understanding through their plenty of exercises.

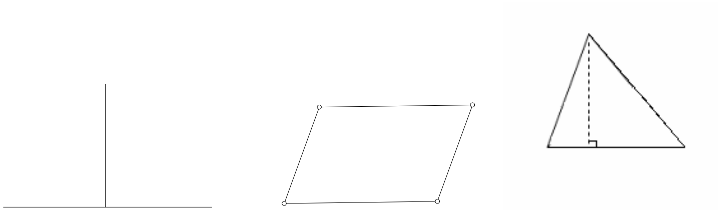
Teaching with variation

Based on the experience and some longitudinal experiments in China and heavily influenced by cognitive theory and constructivism, a theory of mathematics teaching and learning, called theory of variation, has been developed. This theory is derived from the phenomenography(Bowden and Marton, 1998; Huang, 2002; Marton, 1999; Marton and Booth, 1997; Rovio-Johansson, 1999). According to this theory, meaningful learning is to enable learners to establish a substantial and non-arbitrary connection between the new and their previous knowledge, and classroom activities are to help students establish this kind of connection by experiencing certain dimensions of variation. This theory suggests that two types of variation are helpful for meaningful learning. One is called Conceptual variation, and the other is called Procedural variation.

1. Conceptual variation consists two parts. One part is composed of varying the connotation of a concept: Standard variation and nonstandard variation.

For example, a certain figure is the standard representation for the relevant concept due the visual perception or initial awareness for the learners, as shown in figure 1:

Standard figures



Non- Standard figures

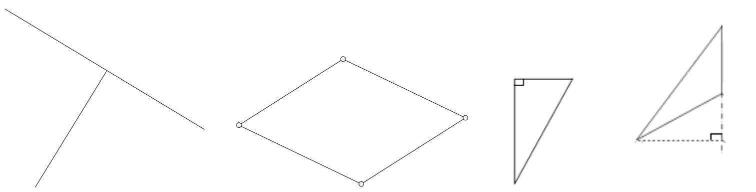
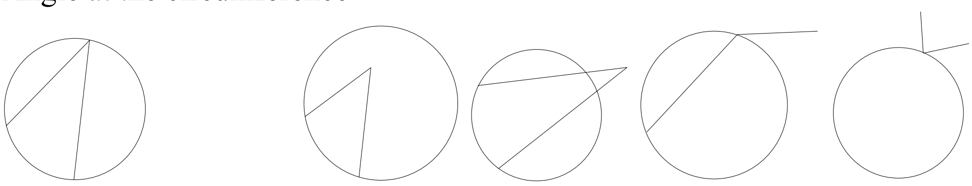


Figure1: geometrical standard figure and non-standard figure varation

The other part consists of highlighting the substantial features of the concept by contrasting with counterexamples or non-conceptual diagrams. The function of this variation is to provide learners with multiple from different perspectives.

For instance, in two-dimensional geometry, through comparing nor-concept figure and concept figure, the essence of a concept can be clarified and highlighted visually, as shown in figure2:

Angle at the circumference



Concept figure Non- concept figure

Figure2: non-concept figure variation for discerning the essence of concepts

There are many forms of non-concept figure variations. Demonstrating counterexamples is one commonly used method of non-concept figure variation, as illustrated in figure3:

"Is the line perpendicular to the radius the tangent of the circle?"

"Is the quadrilateral where the two diagonals are perpendicular a rhombus?"

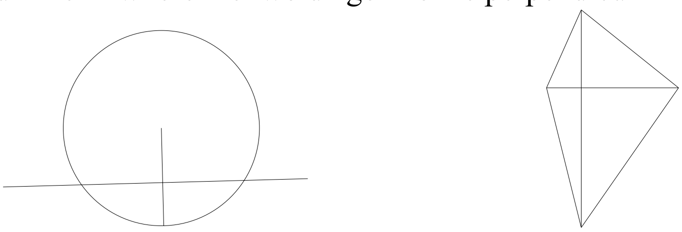
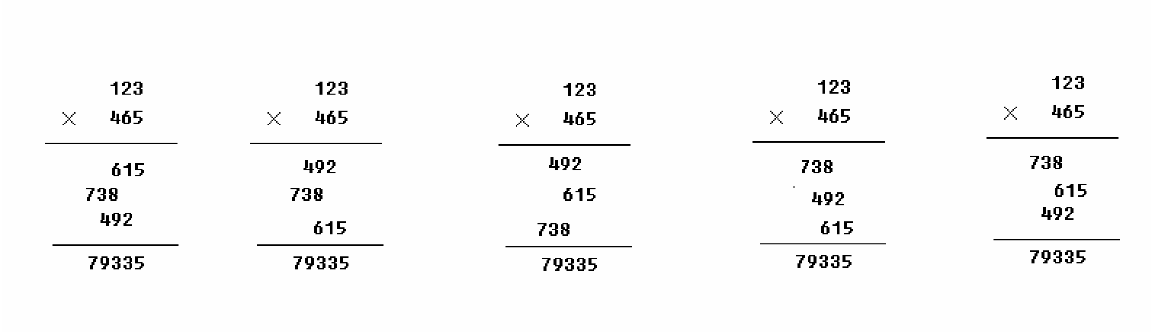


Figure3: counterexample

The figures clearly demonstrate that the line perpendicular to the radius is not necessarily a tangent of the circle, while the right convincingly indicates that the quadrilateral where the two diagonals are perpendicular is not necessarily a rhombus.

2. Procedural variation is concerned with the process of forming a concept logically or historically, arriving at solutions to problems (scaffolding, transformation), and forming knowledge structure (relations among different concepts). The function of procedural variation is to help learners acquire knowledge step by step, develop learners' experience in problem solving progressively, and form well structured knowledge. Ma (1999) described a typical example in her research. Chen. An experienced teacher could propose five different

“non-conventional” strategies to help students solve the problem of 123×465. He proposed that there may be five ways other than the conventional way of lining up:



Those five strategies create a framework for students to understand the “standard” procedure from different angle. All these serve to make students understand the essential theory-the place value system-underlying multi-digit number multiplication procedure. This kind of sophisticated way of teaching is much more effective than the simple repetition of tasks. With teacher’s profound understanding, subject matters presented in classroom are carefully chosen and well organized so that it has meaningful variation in different aspects.

3. Furthermore, these notions are supported by Dienes’(1973) theory of mathematics learning, Vygorsky’s (1978) notion of zone of proximal development, and Sfard’s (1991) duality of mathematical concept. According to this theory, the space of variation consists of different dimensions of variation in the classroom, and they form the necessary condition for students’ learning in relation to certain learning objectives. For the teacher, it is critical aspects of the learning object through appropriate activities. For the teacher, it is critical to consider how to create a proper space of variation focusing on critical aspects of the learning object through appropriate activities. For the learner, it is important to experience the space of variation through participation in constituting the space of variation.

Changes in teaching

Many Chinese teachers are reducing routine problem solving and emphasizing the variation of problems in teaching. Meanwhile, mathematics teachers are paying more attention to students’ thinking process in the study of basic knowledge and skill. In particular, open-ended problem has been introduced to the curriculum standards, textbooks and college entrance examination as well.

Conclusion

In Chinese classrooms where emphasis is placed on constructing subject knowledge systematically (Zhang, S. Li, and J. Li, 2003), it is critical to set a suitable “potential distance” and space of variation in order to implement effective teaching. Probably, the superficial phenomena such as large size of classroom in China, where the teacher controls the class activities and prefers to explain the content clearly and effectively, would reduce the researchers to characterize Chinese classrooms as being teacher-centered with students learning passively. However, when investigating how the lessons are organized and how students involve themselves in the process of learning, it was found that teaching by the right way, even with large classes, students could still actively involve themselves in the process of learning and achieve a meaningful learning. Moreover, it is possible to avoid rote learning. Thus, the “paradox of Chinese learners” might originally be a misperception.

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