SIMON FRASER UNIVERITY | FACULTY OF EDUCATION

# MEDS-C 2015

PROCEEDINGS OF THE 10th ANNUAL MATHEMATICS EDUCATION DOCTORAL STUDENTS CONFERENCE

**OCTOBER 17, 2015**

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# CONTRIBUTIONS

MEDS-C 2015 was organized by members of the Mathematics Education Doctoral Program. The conference would not be possible without the following contributions:

- Conference Coordinators: Darien Allen and Minnie Liu
- Program Coordinators: Lyla Alsalim and Tanya Noble
- Review Coordinators: Sheree Rodney and Mina Sedaghatjou
- Proceedings Editors: Peter Lee, Andrew Hare, and Judy Larsen
- Technology Support Team: Oi-Lam Ng and Mina Sedaghatjou
- Timer: Jeffrey Truman
- Lunch Coordinator: Melania Alvarez
- Snack Coordinators: Milica Videnovic and Annette Rouleau

# PLENARY SPEAKER

**CYNTHIA NICOL**

## SLOW PEDAGOGY, RESEARCH AND RELATIONS: BUILDING RELATIONSHIPS FOR RESEARCH THAT MATTER

*This presentation explores mathematics education research that interrupts, intervenes and is intentionally participatory. I consider how theories of place and land-based participatory research methodologies can come together to better understand culturally responsive mathematics education. At the core of any participatory, activist research is the need for relationship building. Drawing upon examples from various projects I consider the concept of slow pedagogy and research as a way of building relationships for research that ultimately makes a difference in the world.*

# ABSTRACTS

## DARIEN ALLAN

TEACHERS TEACH, STUDENTS LEARN?

*This paper approaches the issue of student learning from the student perspective. Leontiev's activity theory is used to analyse observed student behaviours and student goals to deduce student motive. In this study three student amalgams are analysed in the particular activity setting of taking notes. Although the observed actions are similar, the results show that the primary motives are different, and that this ultimately had consequences for their learning. Thus, what teachers notice in the classroom may not provide enough information to determine whether or not learning will occur.*

## LYLA ALSALIM

AN EXPLORATION INTO A SAUDI MATHEMAICS TEACHER'S FIGURED WORLDS

*In this paper, patterns-of-participation theory serves as a lens to interpret and understand Saudi high school mathematics teachers' practices. This framework focuses mainly on understanding what practices and figured worlds are significant for the teacher and how the teacher engages in those figured worlds. The data presented is about Abeer, a high school mathematics teacher in Saudi Arabia. The data generated suggests that there are five significant practices or figured worlds to Abeer's sense of her practice as a mathematics teacher. The paper discusses and explains these figured worlds.*

## MELANIA ALVAREZ

A PHENOMENOLOGY PERSPECTIVE TO STUDY PROFESSIONAL DEVELOPMENT

*The purpose of this phenomenological study is to provide a phenomenological framework to study professional development.*

## ANDREW HARE

"WHAT WE NEED TO SHOW IS THAT T IS WELL-DEFINED": GESTURE AND DIAGRAM IN ABSTRACT ALGEBRA

*Proving that a mapping is well-defined is a key step in many results in group theory. Such an argument is required in the Orbit-Stabilizer Theorem, as well as the theorem that gives a necessary and sufficient condition for the set of cosets of a subgroup to form a group (namely, that the subgroup is a normal subgroup). It can be challenging for students to even realize that such an argument must be provided, as it is the first* *time they have been exposed to functions defined on equivalence classes. This paper studies the gestures and diagrams that one professor uses to explain the concept of well-definedness.*

**JUDY LARSEN**

NEGOTIATING MEANING: A CASE OF TEACHERS DISCUSSING MATHEMATICAL ABSTRACTION IN THE BLOGOSPHERE

*Many mathematics teachers engage in the practice of blogging. Although they are separated geographically, they are able to discuss teaching-related issues. In an effort to better understand the nature of these discussions, this paper presents an analysis of one particular episode of such a discussion. Wenger's theoretical framework of communities of practice informs the analysis by providing a tool to explain the negotiation of meaning in the episode. Results indicate that the blogging medium supports continuity of discussions and can allow for the negotiation of meaning, but that a more nuanced treatment of the construct is necessary.*

**PETER LEE**

DISCURSIVE PRACTICES USED IN DEFINING MATHEMATICAL LEARNING DISABILITIES: A BRIEF TEXTBOOK ANALYSIS

*This paper compares two separate descriptions of mathematical learning disabilities given in a textbook on learning disabilities. One description is from an early edition of the textbook and the other description is from a later edition of the same textbook. Critical discourse analysis is applied to both descriptions to investigate the changing discursive practices used in defining mathematical learning disabilities. While a "medical discourse" dominates both the early and later editions, there is evidence of other competing discourses which suggests some instability around the boundaries of what constitutes a "mathematical learning disability."*

**MINNIE LIU**

STUDENTS' MODELLING PROCESS – A CASE STUDY

*This article presents a case study in which three grade 8 students worked collaboratively to solve a modelling task. Data indicate that students viewed the problem situation from increasing levels of realistic perspective. At the beginning of the modelling process, there was a disconnect between students' mathematical solution and the original situation and operated with numbers with a minimal understanding of the problem situation. After multiple interventions from the researcher, students were eventually able to draw on their lived-experiences and to*

*view the problem situation from an increasingly real world perspective and generated a realistic mathematical solution.*

**MASOMEH NEJAD**

UNDERGRADUATE STUDENTS' PERCEPTION OF TRANSFORMATION OF SINUSOIDAL FUNCTIONS

*Trigonometry is one of the fundamental topics taught in high school and university curricula, but it is considered as one of the most challenging subjects for teaching and learning. In the current study, two different theoretical frameworks have been used to examine undergraduate students' perception of transformation of sinusoidal functions. The results show that students did not grasp fully the concept of transformation of sinusoidal functions.*

**OI-LAM NG**

COHERENT AND DIVERGING DISCOURSE IN MATHEMATICAL ACTIVITIES WITH DYNAMIC GEOMETRY

*This paper discusses the detection of coherent and diverging discourse in mathematical exploratory activities using dynamic geometry environments (DGEs). The data is drawn from a larger study examining patterns of communication within pair-work mathematical activities using touchscreen DGEs. A thinking-as-communicating approach is used to analyse communication involving pairs of high school calculus students' exploration of the area-accumulating function using touchscreen DGEs. Results show that the students integrated speech, gestures and touchscreen-dragging synchronically and diachronically to engage in coherent and diverging discourses. In particular, it is shown that conflicting ideas can be communicated non-linguistically. This paper raises questions about new forms of communication mobilised by touchscreen DGEs.*

**TANYA NOBLE**

IDENTITY AND MATHEMATIS: A PRELIMINARY EXPLORATION

*This preliminary exploration considers the Affinity-identity students have with mathematics. The relevance of this became apparent as I noticed my own Affinity-Identity reaffirm my growth into the role of a mathematics education researcher. I explore possible ways to identify affinity and develop a picture of how identity of the learner and the subject to be learned can be identified. Development of identity is a communicative practice situated in cultural experiences and therefore an analysis of both praxis and individual communication could offer deep insight into* *influences on student identity as they seek affinity with the identity projected onto mathematics.*

**ZAKIEH PARHIZGAR, HASSAN ALAMOLHODAEI**

STUDENTS' FLOW EXPERIENCES IN THREE TYPES OF MATHEMATICAL PROBLEMS AND TWO DIFFERENT EDUCATIONAL ENVIRONMENTS

*In this study, flow experiences of 244 students in three types of mathematical problems (intra-mathematical problems, word problems and modelling problems) were examined. Pretest and posttest groups were used in this study. The treatment was that the students, in two groups with different training formats (teacher-centered and student-centered), attended 6 sessions of modelling problem solving. The results of this study show that students' flow experiences don't have a significant difference in these three types of mathematical problems. Teaching modelling problems increased students' flow experiences in reality-related mathematical problems, but it didn't have any effect on students' flow experience in intra-mathematical problems. Also, the student-centered group had positive effects on flow experiences in word problems.*

**SHEREE RODNEY**

"THE OTHER TEN": AUDEN'S SENSE OF NUMBER

*This paper presents a case study of a student (age 5 years 6 months) name Auden, who interacted with a touchscreen App called TouchCounts. This App was designed to support children's activities around counting. I use Sfard's commognitive framework to show how Auden thinks and learns about numbers. I show how Auden's exploration of numbers helps me understand the challenges children face when moving from identifying subsequent number as they appear in the natural counting sequence to identifying numbers that appears before and after each other.*

**ANNETTE ROULEAU**

TEACHER TENSIONS: THE CASE OF NAOMI

*Tensions are endemic to the teaching profession. Viewed as dichotomous forces, tensions shape the experiences of mathematics teachers, affecting both their practice and professional growth. In this article, I use Berry's (2007) framework to identify and examine some of the tensions experienced by Naomi in her practice of teaching mathematics. While previous research presents the image of teachers as dilemma managers who accept and cope with continuing tensions, my research suggests that a desire to resolve these tensions may impact teaching practice and professional growth needs.*

**MINA SEDAGHATJOU**

A BLIND UNDERGRADUATE STUDENT'S JOURNEY TOWARD AN UNDERSTANDING OF PRE-CALCULUS CONCEPTS

*In this paper we argue that although mathematical communication and learning are inherent multimodal and embodied subjects, sight disabled students are also able to conceptualize visuospatial information. Adapting Vygotsky's mediation theory, we show that the lack of access to visual fields in an advanced mathematics course does not obstruct a blind student in visualization, but rather modifies it. We argue proper precise tactile materials might empower blind students to better visualize mathematical functions.*

**JEFFREY TRUMAN**

MATHEMATICS LEARNING AMONG UNDERGRADUATES ON THE AUTISM SPECTRUM

*This study examines the mathematical learning of an undergraduate student on the autism spectrum, as well as some views of autism in the Vygotskian framework. I aim to expand on previous research, which often focuses on younger students in the K-12 school system. I have conducted a series of interviews with one student, recording hour-long sessions each week. The interviews involved a combination of asking for the interviewee's views on learning mathematics, self-reports of experiences (both directly related to courses and not), and some particular mathematical tasks. I present some preliminary findings from these interviews and ideas for further research.*

**MILICA VIDENOVIC**

OVERVIEW OF RESEARCH ON STUDENTS' VIEWS OF ORAL ASSESSMENT IN MATHEMATICS

*With the increased emphasis on closed book written examinations, there is a critical need for implementing the oral assessments in mathematics courses. Based on the overview of research on students' views of oral assessment in their mathematics courses, most of the students had positive views towards its use. In this paper, the research on oral assessment in mathematics was summarized with a specific focus on students' views.*

# TEACHERS TEACH, STUDENTS LEARN?

Darien Allan

Simon Fraser University

*This paper approaches the issue of student learning from the student perspective. Leontiev's activity theory is used to analyse observed student behaviours and student goals to deduce student motive. In this study three student amalgams are analysed in the particular activity setting of taking notes. Although the observed actions are similar, the results show that the primary motives are different, and that this ultimately had consequences for their learning. Thus, what teachers notice in the classroom may not provide enough information to determine whether or not learning will occur.*

## INTRODUCTION

Substantial research has and continues to look at how to improve teaching, and how to effect better student learning. What is often overlooked is a vital element: student motive. Though not sufficient in itself, motivation is a critical factor in the determination of the nature of the learning that occurs. The intent of this paper is to approach the issue of student learning from the student perspective. Using Leontiev's activity theory observed student behaviours and student goals are analysed to deduce student motive. The motives students hold affect whether or not learning occurs, and the nature of that learning that does occur.

The aim of this particular study, and of the larger research project, is to investigate and answer two questions: *What are the behaviours that students exhibit in different activity settings in the mathematics classroom*; and *What are the motives that drive their behaviours?* In this particular paper the observed behaviours take place in the particular activity setting of 'taking notes'. To address the first question, regarding student behaviour, it is necessary to introduce the notion of 'studenting'.

### Studenting

The term 'studenting' was coined by Gary Fenstermacher in 1986. Initially, he describes this concept in terms of a cohort of student behaviours including "*getting along with one's teachers, coping with one's peers, dealing with one's parents about being a student, and handling the non-academic aspects of school life*" (p. 39). In essence, Fenstermacher describes studenting as what students do to help themselves learn. A later definition encompasses other behaviours such as "'*psyching out' teachers, figuring out how to get certain grades, 'beating the system', dealing with boredom so that it is not obvious to teachers, negotiating the best deals on reading and writing assignments*" (Fenstermacher, 1994, p. 1) and other similar practices.

Analyses have shown that the process of schooling produces a number of unintended consequences, some desirable, but also many that are patently objectionable (Engeström, 1991) and counterproductive to the goal of student learning. This appears to be especially true within the mathematics classroom. Preliminary studies have shown that across the board students are finding ways to subvert the expectations of the teacher in ways that the teacher is not aware of (Liljedahl & Allan, 2013a; 2013b).

**Behaviours within the mathematics classroom**

Although some student behaviours are ubiquitous across subjects, certain actions (or inactions) may be more or less prevalent, or even unique to the secondary mathematics classroom. Aaron and Herbst (2012) use 'instructional identities' to define and discuss the particular attitudes students hold towards specific subjects and the behaviours they enact within the confines of different instructional settings.

> Within particular instructional situations, students may differentiate by taking on different identities with different stances toward the work required by or toward the stakes of the situation. … Because different students understand the figured world in different ways these students feel that different actions are appropriate when faced with a mathematical task. (p. 6)

Students come to the mathematics classroom with particular expectations about mathematics and school. These expectations are reflected in attitudes towards and behaviours within the mathematics classroom. I see instructional situations as a subset of activity settings – one in which instruction is taking place, and one in which learning is a presumed outcome.

If we entertain the idea that students behave differently in different instructional settings (or activity settings) within the mathematics classroom (and in comparison to other subjects), we are compelled to investigate the nature of these particular behaviours, and in what specific settings they occur (or don't occur). Furthermore, how can we analyse what is driving students' actions?

## THEORETICAL FRAMEWORK

According to Leontiev, all activity is driven by some motive, whether the subject is aware of that motive or not.

Activity Theory was chosen as a framework for analysis for its ability to describe what is rather than what is ideal. Rather than categorizing students by their behaviours (e.g., on-task or off-task), presuming students behave rationally or ideally (e.g., game theory), or assuming the students' primary goal is learning (e.g., didactic contract (Brousseau, 1997)), activity theory allows for a description of what is observed and said without overlaying pre-existing assumptions or judgments. These observations, taken together, can then be used to develop a hypothesis for what is driving student action: their motive, which may be something other than a desire to learn. Individual behavior and motive within the classroom collective is best viewed through a theoretical lens primarily comprised of Leontiev's Activity Theory (1978).

For Leontiev, "[a]ctivity does not exist without a motive; 'non-motivated' activity is not activity without a motive but activity with a subjectively and objectively hidden motive" (1978, p. 99). The object of an activity is its motive, and is something that can meet a need of the subject. Motives arise from needs, which are the ultimate cause of human activity. Figure 1, below, illustrates the relationship between the elements in Leontiev's development of activity theory.

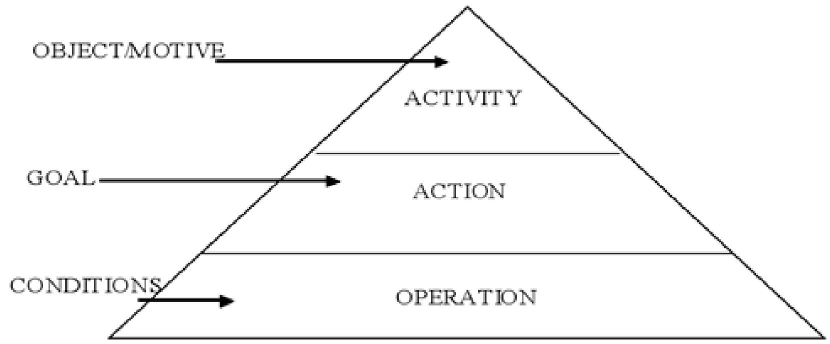


Figure 1: Pyramid Representation of Leontiev's Three-Level Model of Activity (1974)

Motives drive activity, and activities are directed at goals. People have many goals, which shift in importance and in content on the basis of both contextual and intrapersonal factors. At the very top of this hierarchy is the motive for activity. At any time an individual has a hierarchy of these motives, the order of which is determined through and as a result of one's activity.

Actions are the many steps that comprise an activity, although not all are immediately related to the motive (Kaptelinin & Nardi, 2012). Actions are directed towards specific targets, called goals. Goals are conscious, in contrast to motives, of which a subject is not usually aware.

The fact that motive is often hidden from the subject suggests difficulty in determining the ultimate motive. This obstacle can be overcome by utilizing an "actions first" strategy (Kaptelinin & Nardi, 2012). The strategy begins at the level of goals, which people are generally aware of and can express, and the analysis is subsequently expanded up to higher goals and ultimately the motive. Given the primary interest of this study is the students' actions/goals and activity/motive only the top two levels of Leontiev's pyramid will be considered in this paper.

Cataloguing the observable aspects of studenting (actions) and analysing these together with students' goals through the lens of activity theory can offer new insights into student motives and provide researchers and educators with evidence to better understand student behaviour.

Whether the behaviours discussed in the later section are unique to mathematics is beyond the scope of this paper. Regardless, the issue of exclusivity with respect to mathematics is not what is significant – what is important is that these behaviours are occurring within the mathematics class, arguably much more frequently and pervasively than the teacher perceives or suspects. These actions together with the associated goals, point to particular motives that have significant impact on the nature of the learning that occurs, or whether learning takes place at all.

Although considerable research can be found looking at students' motives and behaviours, what is lacking is a catalogue of student behaviour across activity settings in the mathematics classroom. This is the first objective of the larger research study, the second is to investigate what drives this behaviour by analysing it using Leontiev's activity theory. This particular study looks at student behaviour in the activity setting of 'taking notes'.

## METHODOLOGY

The nature of the research questions requires a particular approach – an ethnographic study, though the study itself is not ethnography. In accordance with this approach, I spent significant time immersed in the classes under study observing and interacting with students, taking fieldnotes, and asking questions. Analysis occurred throughout the process of data collection whereby what was observed and recorded in one lesson provoked questions and shifted focus for the next observation/interviews. Both an etic and emic perspective were utilized. In line with Mason (2011) I endeavour to provide a strict 'accounting of' student behaviour rather than 'accounting for' observed actions.

### Participants

The data for analysis are taken from a larger study conducted in three secondary school mathematics classes in British Columbia.

### Data collection and analysis

Data were collected during the 2013-2014 school year. Throughout the fall semester the class was observed for twelve periods, each period ranging from 60 to 75 minutes. Classroom lessons and informal interviews were audio recorded and transcribed for later analysis and comparison with field notes taken during the class. The data discussed here has been subjected to an analysis using Leontiev's activity theory in order to determine the likely primary motive underlying the students' behaviour.

I have constructed profiles of three fictional students in order to illustrate potential uses of Leontiev's Activity Theory applied to studenting in the mathematics classroom. These accounts are not fictional in the sense that all behaviours and goals were present in the data, but do not belong to one unique participant in the study. I use this amalgamation of cases because it is not possible due to space considerations to describe all students who displayed particular types of behaviours. The amalgamation of cases has been used previously by Piaget (1923/2001) and in mathematics education by Leron and Hazzan (1997), Zazkis and Koichu (2014), and Liljedahl, Andrà, Di Martino, and Rouleau (2015).

The profiles are created from a composition of student behaviours across the three classrooms to demonstrate the similarities of the behaviours and goals across teaching methods and schools.

Although the students who comprise the profiles are from both genders, I have used feminine names for the fictional students to ward against any gender stereotyping or comparison.

## RESULTS AND ANALYSIS

### Profiles: Actions and goals

The three composite students described in this section are named Jenna, Kyla, and Laura. All three students demonstrated the behaviour of taking notes in the activity setting of a lesson. In what follows I first provide excerpts from informal interviews with the students, then describe each student's goals based on these conversations. After this I analyse their motive(s) in the subsequent section. I chose the context of taking notes, and consider only students who take notes because the observable behaviour of taking notes is often perceived as a proxy for learning.

Jenna

Jenna copies down what the teacher writes on the board. When asked why she takes notes, Jenna replied that it keeps her engaged. If she didn't she might "*lose focus and doze off in class*" and she has nothing else to do because "*I'm not allowed to use my electronic devices*". Jenna admitted "*sometimes when I'm writing stuff down I feel like I'm just writing it down, not actually learning it*".

Jenna appears to take notes because that is what is expected and to do anything else might get her into trouble with the teacher.

Kyla

She writes notes so that she can "*know how to do it for the booklet*" if she forgets she can refer to her notes "*over and over until you get it*". Kyla says for review she "*just sits there and reads over the notes*" but will also "*refer to it if I have a question that's similar*". But if she has "*homework that doesn't really have to do with the lesson then the notes won't really help*". She feels that she "*won't remember anything if she doesn't write it down*" and she believes that "*the more you write the more you remember because you've physically actually written the words*". Kyla looks at her notes before the test to help her 'remember'.

Kyla's rationale for taking notes is characterized by an emphasis on memorizing or remembering.

Laura

Laura stated "*I feel like I understand it more if I write it down*". She writes comprehensive notes including, but not limited to, what the teacher writes and says. She later rewrites them outside of class: "*When I take my notes I take them in complete, exactly, everything the teacher says and then some, and then I add my own things to them. And that's how I comprehend most of the stuff … just if I can think of a way to relate this to something else, or if there's just an easier way of me thinking about it, I'll put that up by the side.*"

Laura writes down the teacher's explanation (not just what is written on the board) and later synthesizes her notes and adds to them, bringing in other ideas that she feels relate to the topic and will help her to better understand it.

**Motive**

Looking at each student's actions and goals, it is possible to expand upwards and suggest her motive.

Considering Jenna's stated reasons for taking notes, it appears that her goal during an activity where she might be expected to take notes is to avoid getting into trouble. These types of goals are associated with avoiding negative attention. Jenna's primary motive in this activity setting is not to learn, but to stay out of trouble and to 'get through' the course.

Kyla appears to want to learn. However, it is not the same type of learning that one can see in the description of Laura. Kyla is concerned with remembering, memorizing, and not forgetting. She uses notes to help her complete similar homework questions and to refresh her memory before a test. While one might suggest that her primary motive is learning, the description of Kyla brings to mind Skemp's 'instrumental understanding'. Kyla's actions are most likely to result in her having procedural knowledge - knowing only what to do or having "*rules without reasons*" (Skemp, 1987, p. 153). Her primary motive is most likely to want to be seen as a good student and to get a good grade.

Laura, like Kyla, wants to learn. However, in contrast to Kyla, Laura is focussed on understanding, in the sense of Skemp's 'relational understanding'. Rather than talking about remembering or memorizing she tries to build connections between ideas. Laura's primary motive is learning.

For me, learning and understanding *is* relational understanding – knowing not only *what* to do, but *why* to do it. In this vein, a student can only have a primary motive of learning when it arises from goals linked to relational understanding. Only Laura has a primary motive of learning and thus only Laura is likely to have an outcome of learning.

## DISCUSSION AND CONCLUSION

The goals and motives students hold have consequences for their learning. On the surface, with respect to their observed behaviours within the classroom, each of the three students performed similar actions. So to the teacher, each of these students is doing what is expected (whether it was explicitly required or not). Underpinning their observable actions are students' reasons for taking notes, their goals, and this is where the students diverge. The reasons for taking notes that the students offered in the informal interviews point to the students' goals, which can in turn be used to expand upwards to ultimately determine motive. Jenna's motive was to stay out of trouble, Kyla's was to get a good grade, and Laura's was to learn. The particular motive a student holds is important because if a student does not have a primary motive of learning, the outcome will be something other than learning. Although students' achievement was not explicitly measured during the study, it can be reported that Jenna performed poorly on assessments, Kyla had satisfactory results, and only Laura demonstrated real understanding of the concepts.

It is important to remember that all of these students' observable actions appeared the same. From the teachers' perspective, things like taking notes, doing homework, and coming for help are indicators or stand-ins for learning and understanding – even if they are not explicitly required. Although teachers may see students' actions as indications of learning, they are often just proxies for learning. Thus even when students are conforming to (possible) teacher expectations, as in the three student amalgams, there appears to be a gap between actual student activity and (relational) understanding, the ideal outcome associated with a primary motive of learning. Frameworks like the didactic contract (Brousseau, 1997) may apply when the desire to learn drives student activity but when learning is not the primary motive, other approaches must be explored, such as activity theory.

Other preliminary results show that there are a variety of goals that manifest in similar actions, suggesting that considered alone, a student's actions are insufficient to deduce his or her motive. These results also show that even when students do not conform to the teacher's expected behaviours, from the student's perspective there is a certain rationality to their actions. Although the analysis of the results is only in the initial stages, it is evident that in many cases non-conforming studenting does have a valid justification. It is anticipated that extended analysis will provide further evidence to support this claim.

What these results suggest is that a deeper understanding of the perspective of the student within the classroom unit could serve to provide teachers with cause for reflective thought on their policies and practices that influence student learning.

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# AN EXPLORATION INTO A SAUDI MATHEMAICS TEACHER'S FIGURED WORLDS

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*In this paper, patterns-of-participation theory serves as a lens to interpret and understand Saudi high school mathematics teachers' practices. This framework focuses mainly on understanding what practices and figured worlds are significant for the teacher and how the teacher engages in those figured worlds. The data presented is about Abeer, a high school mathematics teacher in Saudi Arabia. The data generated suggests that there are five significant practices or figured worlds to Abeer's sense of her practice as a mathematics teacher. The paper discusses and explains these figured worlds.*

## PURPOSES OF THE STUDY

In Saudi Arabia, the *education system has undergone* major changes in the past decade. Government agencies involved in education have introduced new policies, standards, programs, and curricula. One of the major reform initiatives directly addresses existing mathematics curriculum. In 2011, the Ministry of Education introduced new mathematics textbooks, which is usually the primary, and sometimes only, resource for teachers. The new mathematics textbooks are based on the curricula published by McGraw Hill Education Learning Company. This is the first time that the Ministry of Education introduced textbooks that were developed outside the country. These changes are accompanied by high expectations that teachers will incorporate the changes seamlessly without consideration of their existing practices. This paper is part of an ongoing study that intends to gain a better understanding of how high school mathematics teachers in Saudi Arabia are coping with recent education reform, including how their practices are evolving in response to the changes that are happening in the education system.

## THEORETICAL FRAMEWORK

In this paper, patterns-of-participation (PoP) (Skott, 2010, 2011, 2013) approach serves as a lens to interpret and understand Saudi high school mathematics teachers' practices during the current reform movement. The PoP framework identifies teachers' practice as being how teachers narrate and position themselves in relation to multiple, and sometimes conflicting, figured worlds (Skott, 2013). Figured worlds are imagined communities that function dialectically and dialogically as if in worlds. They constitute sites of possibility that offer individuals the tools to impact their own behaviour within these worlds (Holland, Skinner, Lachicotte, & Cain, 1998; Skott, 2013).

Traditionally, most research in education that focuses on studying teachers' practices adopt an acquisitionist approach, especially those studying teachers' beliefs and knowledge in relation to teachers' practices (Skott, 2013). Recently, more researchers, including Skott (2009, 2010, 2013), adopt participationism as a metaphor for human functioning to understand teachers' practices. "The origins of participationism can, indeed, be traced to acquisitionists' unsuccessful attempts to deal with certain long-standing dilemmas about human thinking" (Sfard, 2006, p.153). Skott presents PoP as a coherent, participatory framework that is capable of dealing with matters usually faced in the distinct fields of teachers' knowledge, beliefs, and identity. Therefore, PoP is a theoretical framework that aims to understand the relationships between teachers' practice and social factors. Skott (2010, 2011) initially developed the patterns-of-participation framework in relation to teachers' beliefs. However, in order to develop a more coherent approach to understand teachers' practices, Skott (2013) extended the framework to include knowledge and identity.

The social approach of research in mathematics education has progressively promoted the notion that practice is not only a personal individual matter; it is in fact situated in the sociocultural context. Although the relationships between individual and social factors of human functioning have generated much debate in mathematics education, it is mainly in relation to student learning (Skott, 2013). Therefore, PoP is a theoretical framework that aims to understand the relationships between teachers' practice and social factors. To a considerable degree, PoP adopts participationism as a metaphor for human functioning more than mainstream belief research. Therefore, PoP draws on the work of participationism researchers, specifically Vygotsky, Lave & Wenger, and Sfard.

"The intention of PoP is to take this one step further by limiting the emphasis on acquisition and include a perspective on the dynamics between the current practice and the individual teacher's engagement in other past and present ones" (Skott, 2013, p. 557). This framework focuses mainly in understanding what practices and figured worlds are significant for the teacher and how the teacher engages in those figured worlds. A teacher's engagement with these figured worlds inform and adjust the interpretations s/he makes to her/himself and the way s/he engages in on-going interaction in the classroom. These figured worlds work in a very complex system where they could support and sometimes, restrict one another as the teacher contributes to classroom practice.

## METHODOLOGY

This paper is part of an ongoing study that intends to develop more coherent understandings of Saudi high school mathematics teachers' practices during the current reform movement. For data analysis, I used a qualitative analysis approach based on grounded theory method as used by Skott (2013).

The data presented in this paper comes from Abeer, a high school mathematics teacher. Abeer has six years of experience working at the same school. She graduated from university with a Bachelor of Education degree with a specialization in mathematics. The education courses Abeer had in university focused on general issues related to teaching, such as lesson planning and classroom management. She does not have any experience taking educational course related to teaching mathematics specifically. Abeer was not among the teachers who used the new textbooks in the first year of implementation since she was teaching grade 11 at that time. She has been teaching from the new textbooks for three years. When I met Abeer she was teaching 21 lessons per week to students in grade 11.

I conducted two semi-structured interviews with Abeer. The first one before I observed her teaching two lessons and the second interview was conducted after the classroom observations. In the first, I asked her to reflect on her experiences with mathematics teaching and learning in school, at university, and during her practicum year. I also asked her questions related to different aspects of the new reform movement in education system in Saudi Arabia. I also asked Abeer to reflect on her experience teaching mathematics using the old and the new textbooks. The second interview focused on her experiences with teaching mathematics at her school and on her relationships with the school, her colleagues, and the students. I asked her to reflect on the lesson planning process she had in order to prepare the lessons I observed. During the second interview, I also used a stimulated recall technique by playing audio recordings of parts from the lessons to facilitate her conversation about her own teaching practice in the classroom.

During my visits to Abeer's school, I was also able to collect some data from informal observations of staff-room communication between Abeer and her colleagues. I also have a copy of Abeer's lesson planning notebook and some of her worksheets and tests samples.

## DISCUSSION

The aim of this paper is to develop a deeper understanding of the participant teacher's significant practice and figured worlds and how she engages with these figured worlds.

As a teacher positions herself in relation to her profession as a mathematics teacher, she draws on several, often incompatible, figured worlds. Her engagement with these figured worlds does not only appear in her verbal communication, but also by the choices she makes in her all other actions related to her profession, such as her immediate reaction to certain student behaviors or the way she expresses her view when engaging in a conversation with her colleagues.

Teachers' engagement with figured worlds inform and adjust the perceptions they make and the way they engage in on-going interaction in the classroom. These figured worlds work in a very complex system where they could support, and sometimes restrict, one another as every teacher contributes to classroom practice.

### Abeer's classroom

Abeer has 30-33 students in every class. Her students sit in groups that do not remain static; she changes who is in each group every week. New groups are formed every Sunday (the first day of the school day in Saudi Arabia) by assigning students randomly. The groups are arranged in square formation with four to five students in each group. In her classroom, students work individually as well as in groups. The textbook is always a part of her lesson. She usually starts her lesson with an activity that students do within their groups. She often chooses the activity from the textbook, but sometimes she comes with a different activity than what is in the textbook. Abeer rarely assigns her students homework, but when she does, her students know that they are not required to work on it at home. Every Thursday, which is the last school day of the week, the lesson starts with a quick quiz. The quiz is often related to what they have been doing in class during the week.

## RESULTS

The data generated from Abeer's interviews suggest that she has five significant practices or figured worlds in her practice as a mathematics teacher after six years of teaching. These figured worlds are mathematics, textbooks, responsibility for students' achievement, reform or change, and her relation with other in school. It is important to clarify that I am not claiming that these are the only figured worlds that contributes to Abeer's sense of her practice as mathematics teacher.

### Mathematics

Abeer's sense of her practice has developed as a result of her own experiences as a student and a teacher of mathematics. From her experience as a student, she was always told that learning mathematics is about finding the right answer to a problem. Her own notion of mathematics learning as a teacher is quite different. "One of the most important aspects of mathematics learning is encouraging students to focus more on the thought processes being used rather than focusing completely on finding the correct answer". When Abeer decided to become a mathematics teacher, she knew there were two ways to do so, the "easy way" and the "hard way". "It would be an easy choice to teach mathematics as a subject of right and wrong answers and that's all there is to it". Abeer decided to take the hard challenging way.

Doing mathematics in Abeer's classroom is not about finding the right answer, it is about applying thinking skills and being able to explain how to find an answer. She endorses students' ability to think reflectively about their answers. Abeer's ultimate goal for her students learning experience is to get her students to get to a point where they are sure of their answer and they can explain how they got it.

In her classroom, Abeer often invites her students to think of multiple approaches to solving mathematical problems because, as she indicates, it is very important to student learning of the mathematical concepts. Having students look for multiples ways of finding solutions and using multiple strategies provides Abeer with more opportunities to encourage her students to talk about their work and explain how they reached a solution.

## The textbook

Abeer has a strong appreciation for the textbooks she is currently using in her teaching. The implementation of the new textbooks, according to Abeer, was a necessary step. For her, the old textbooks were insufficient in providing the knowledge and incentives for teachers to reflect on their practice. Abeer uses the new textbooks as a tool to reflect on her practice. She indicates that when she plans her lessons, and before she chooses a particular part from the textbooks to use in her lesson, she asks herself, "What do I actually want to get away from this? What do I need my students to take away from this? How can I add to enrich students learning experience? By answering these questions, I can imagine different scenarios." It seems that Abeer engages deeply with the textbook during her planning for the lesson phase. Also, in her classroom Abeer often invites her students to engage with the textbook.

In her classroom, the textbook has an active presence. Abeer often invites her students to engage with the textbook. At the beginning of the lesson, she asks the students to read the parts of the lesson, which are previously covered skills and concepts, learning outcome of the lesson, and the major mathematical vocabulary used in the lesson. She gives them few minutes to discuss what they read in their groups. Then, she asks them to read the purpose of the lesson section, which presents information usually related to real life situations and sometimes requires the students to answer questions that follow the information. She usually discusses this part with the students and this discussion leads to the introduction of the main concept of the lesson.

During the lesson, Abeer also refers her students to the textbook many times. The students read most of the instruction part of the lesson. Sometimes, Abeer reads certain parts of the textbook and the students follow along in their textbook; other times, she asks one student to read aloud from the textbook, and in some cases the students are asked to read some parts and discuss in their groups.

## The reform or change

Abeer is very enthusiastic about the current reform movement in Saudi Arabia. Her enthusiasm shows from her involvement in any professional development activities offered to her. She indicates that most teachers don't like to participate in any activity related to professional development. For Abeer, these activities provide her the opportunity to find new ideas, meet different teachers, and talk about mathematics teaching. Abeer is excited and strongly supports the new curriculum, which is considered one of the major steps in the reform movement to the education system in Saudi Arabia. For Abeer, the new mathematics curriculum created a new language to talk about mathematics teaching and learning.

Reform for Abeer means change. When she talks about reform she usually explains how the current reform movement has changed some aspects of her practice. Abeer described the first two years of her teaching career as being very traditional. "I was the person who had the mathematics and my job was simply to present what I know in the class. I used to start my lesson by presenting the concept and writing on the board the definition and the formula, and then I would work few examples and then give them a worksheet and tell them to do just as I did then I would correct them really quick". Abeer re-defined her role as teacher in the classroom as a result of her inspiration of the reform ideas.

**Abeer's relationship with other staff members in her school**

Abeer appreciates the experience of working in her school. She is dedicated to the development of her school. She realizes the importance of a school community to create a positive learning environment for everyone. She indicates that schools are not only for sitting and learning for students; "everyone who works at this school learns from this experience". Abeer values communication with other teachers in the school and considers it an important source for her personal learning experience as teacher. Abeer also draws on the principal's support for a learning community among teachers. According to Abeer, although the principal of the school was not a mathematics teacher, she appreciates mathematics as a subject and realizes its importance.

A school inspector is a part of the education system in Saudi Arabia. . For Abeer, the inspector's visit is usually a challenging part of her practice. Her general view of what is valuable in her teaching does not seem to match the inspector's view. According to Abeer, "the inspector cares about how much I give and whether I am following the plan we received from the Ministry of Education. I don't follow their plan; I usually make my own plan which I always adjust during the year."

**Responsibility of students' achievement**

In the interviews, Abeer appears strongly committed to her students' achievement. Sharing students' main interest and understanding their biggest concerns is significant to her philosophies about her role as a high school mathematics teacher. For Abeer, teachers are meant to meet the needs and address the concerns of their students. Abeer constantly engages with her students' emotional state of being under pressure to achieve in high school and then move on to college. Abeer relies on the collaborative working environment she has at her school to reflect on students' achievements. During her group meetings, Abeer talks with other teachers about improving students' achievement and helping them achieve not just at school but on other standardized tests they take during high school. She and the other teachers organize workshops for students to help them prepare for these tests.

## CONCLUSION

Abeer has a strong dedication to her job and a willingness to deal with the challenges and the responsibilities involved. She has always wanted to become an educator and was greatly influenced by many of her experiences learning mathematics at school. Her main motive to become mathematics teacher was that she wanted to be able to share with her students the amazing rewarding experience learning mathematics can have. According to Abeer, interacting with the students is the best part of being a teacher. The most exciting moments are when you can actually see students start to understand mathematics concepts.

As Abeer positions herself in relation to her profession as a mathematics teacher, she draws on several, often incompatible, figured worlds. Her engagement with these figured worlds does not only appear in her verbal communication, but also by the choices she makes in all other actions related to her profession such as her immediate reaction to certain students' behaviour, or the way she expresses her view when engaging in a conversation with her colleagues. It is also important to clarify that I am not claiming that these are the only figured worlds that contributes to Abeer's sense of her practice as mathematics teacher. It is very challenging to get access to all the practices and figured worlds that are possibly significant for Abeer's classroom interaction. For instance, challenges could occur if the figured worlds are related to the teacher's experience in schools and university (Skott, 2013). I plan to collect more data to gain a deeper understanding of Abeer's teaching practices.

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# A PHENOMENOLOGY PERSPECTIVE TO STUDY PROFESSIONAL DEVELOPMENT

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*The purpose of this phenomenological study is to provide a phenomenological framework to study professional development.*

## INTRODUCTION

Extensive research has been conducted about students and pre-service teachers learning in a variety of classroom settings where their interactions with instructors or facilitators are examined, but this is not so much the case with in-service teachers as they are engaged in learning opportunities provided by professional development. As I was doing research regarding successful professional development, I found no descriptions about the lived experiences between professional developers and teachers. What I usually found were survey results from teachers' program evaluations, which provided information about teachers' individual perspectives regarding the effectiveness of professional development sessions, or statistical analysis of school performances after the application of a particular professional development opportunity. Not much if anything has been done to show what happens during those learning opportunities and how teachers' and professional developers' predispositions and the setting allow for a fruitful learning opportunity and when they will not (Garet, Porter, Desimone, Birman, & Yoon, 2001; Grammatikopoulos, Gregoriadis, & Zachopoulou, 2013; Linder & Simpson, 2014). This study presents a framework that attempts to provide an initial tool for a description and analysis of this experience.

## FRAMEWORK

The use of the phenomenological perspective seems to be quite suitable for the task at hand, given that the purpose of this research is to analyze the "lived experience" of professional development sessions and describe teachers' reactions to a variety of approaches and activities put forward by the professional developer. This study explores and describes "lifeworld" learning experiences, and look for the meaning of a phenomenon by uncovering, as much as possible, the many layers that socially and culturally influence a person's experience in their lifeworld, where lifeworld is defined by Van Manen (1997) as "the world of immediate experience", the world as "already there" (p. 182). According to Woodruff (2013) phenomenologist practices consist of three different methods:

- Rich descriptions of lived experience
- Use of relevant features in the context to interpret the experience
- Analysis of the form of a type of experience

One can combine all three to analyze the description of the lived experience, and to interpret it by assessing and using the relevant features in the context and analyzing structures which resonate with our own experience—that which one can be conscious of. This study focuses on engagement, and how through engagement teachers show their motivation, wants, needs and learning. In this study, engagement is defined as different ways in which individuals and groups come into action, or become involved with an activity, where these different ways of acting/engaging are influenced by different levels of motivation, motives, and goals.

What is the difference between motivation and motive? According to the experts, motivation is a theoretical construct that can be used to explain behaviour (Elliot & Marti, 2001; Pardee, 1990), whereas motive is something that moves a person to do something (Ryan & Deci, 2000). According to Ryan and Deci's Motivation Theory (2000), there are different levels and different types of motivation which are closely related to attitudes, goals, and/or wants that prompt an individual to particular actions, and depending on the source of attitudes, goals, or wants we would be able to distinguish between intrinsic and extrinsic motivation. Motive and wants are concepts that point to elements within and outside of an individual, and they seem closely connected. Motive corresponds to a want or a preference that is sufficiently strong enough that it moves us to action or deliberate inaction (Braybrooke1992). In this study a want is something we want to be given or we want to obtain and motives will be something that would drive us to actions, with this action cast back to engagement, which is usually an indicator of learning.

According to Woolfolk's (1998) motivational theory, people's motives and wants are behind the level of engagement, which is geared towards specific intentions or objectives. Because motivation and motives are deeply guided by teachers' wants, these key elements will hinder or encourage teachers' engagement and learning during the professional development session, and the facilitators must tend not only to their own motivation, motives or wants but also those of the teachers, while still planning and delivering the professional development sessions and keeping as much as possible the original objectives of the session. Research has shown that the motives and wants that teachers bring into a professional development environment affect teachers' levels of engagement during the session (Liljedahl, 2014). Consequently, engagement is an indicator of wants and motives and vice versa. So, teachers' engagement in an activity can provide a good source of information regarding their learning and their practice. As mentioned before the literature mentions level of engagement as a way to find out the success of a particular opportunity for learning. However, in the literature the term "level of engagement" is used but not really defined. Woolfolk's (1998) motivational theory mentions intensity of involvement in an activity, but does not provide a definition for level of engagement. But how is intensity of involvement defined? There are two elements that usually allow us to sense how intensely involve participants are in an activity, these two elements are flow and mood and that usually will provide professional developers with signals as to how to proceed.

This study will consider engagement as mainly composed of two factors: flow and mood; where I define flow as the fluidity of action. Flow indicates a measure (intensity) of action from an individual or a group. I define mood as state of feeling or being at a particular time and place. Flow and mood are helpful and relevant tools in analyzing engagement in trying to recognize when learning is happening through communication and cognition and when it is not and why. Levels of flow in communication and the mood among participants during professional development sessions will indicate different types or modes of engagement, some of which are conducive to learning and some which are not. Through engagement one can witness moments of reflection, moments of enlightenment, exploration, rejection, resistance, confusion, fear, rejection, doubt, participants' motives and wants, and ultimately their learning. Changes in flow and mood, can bring about points of inflection, which in some instances will indicate a commognitive conflict, which can lead to or produce learning occurrences for an individual or the whole group or where an opportunity for learning can be lost. The commognitive adjective crafted by Sfard (2007, 2008) implies that thinking is not a self-contained mechanism separate from speaking or more generally communication but as a form that "have developed from a patterned collective activity" (p. 571), and she views thinking as "an individualized form of interpersonal communication" and as such "a product of collective doing" (Sfard, 2008). She sees commognitive conflict as the source of mathematical learning, and which she defines as "the phenomenon that occurs when seemingly conflicting narratives come from different discourses – from discourses that differ in their use of words, in the rules of substantiation, and so on. Such discourses are incommensurable rather than incompatible, that is, they do not share criteria for deciding whether a given narrative should be endorsed… cannot automatically count as mutually exclusive even if they sound contradictory… whereas acquisitionist views conflict resolution as making sense of the world, commognitivist regard it as making sense of other people's thinking (and thus talking) about this world. This means gradual acceptance, 'customization', and rationalization—figuring out the inner logic – of other people's discourses." (Sfard, 2007, pps. 577-578).

This framework which is based on a modified version of Remillard's (2012) analytical perspective, which connects positioning and engagement in order to analyze the various ways a professional developer tries to position teachers in an experience that is engaging to them. Professional developers want to position teachers in a way where interaction is possible and furthermore this is something that they want to do of their own volition. Remillard classifies positioning and interactions by using the following terms, which I redefine as follows:

- mode of address: how a group of people are position to engage in an activity.
- forms of address: artifacts/resources used in order to position people.
- modes of engagement: the way people react/engage
- forms of engagement: what artifacts/resources, ideas result from the activity.

The framework provides a circular model (Figure 1). It starts when the professional developer plans and applies particular modes and forms of address during the professional development session. The particular forms and modes of address run within a particular background of goals, motives and wants from all the participants involved including the professional developer. Particular mode of engagement follows with particular moods and levels of activity or flow, and forms of address are re-sourced into particular forms of engagement. As the professional developer observes and reflects upon the resulting modes and forms of engagement, she responds accordingly either by continuing or modifying the current mode and form of address. Modifications depend on how much the professional developer is able to perceive a level of learning where hopefully the teachers' and professional developer's motives, wants, and goals are addressed and realized.

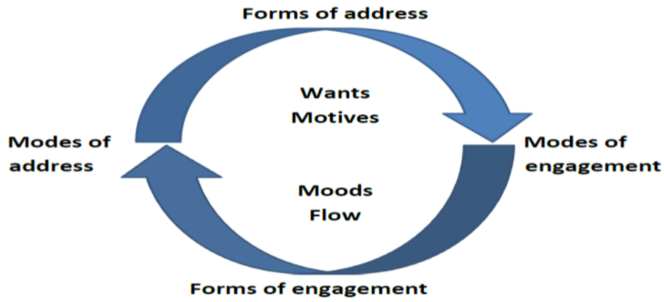


Figure 1

## METHODOLOGY

So, how can I describe and put together all of these elements without simply creating long, boring narratives? A visual analytical tool would be helpful and I was directed by my advisor to look at interactive flowcharts. Flowcharts were introduced by Sfard (2001) and Sfard and Kieran (2001) in order to allow the reader to experience meta-discursive intentions in conversations (ibid. p. 58), and in our case it also would allow us to visualize the fluidity or flow of the conversation. In Sfard and Kieran's (2001) charts interactions among participants are indicated by re-active, pro-active arrows, in our model we have added arrows that indicate implicit agreement, support or reflection. Additionally, arrows come in different colors to indicate the kind of preoccupations of motivations behind the interaction. Data regarding the actions of teachers in a professional development setting were gathered in order to describe their participation and engagement throughout the sessions. This was done by audio-recording most of the professional development sessions, surveys, and through notes made by the facilitator during and after each session about teachers' engagement. After every session notes were made by the professional developer reflecting on the effectiveness of the session, which concerns were addressed, the level of engagement, her feeling and thoughts as well as questions that she wished she had asked during the session and questions that she hoped to ask at the next session with the group. This reflective process served as a preliminary analysis of the data (Glesne, 1999). Most observations of teachers' behaviour were made during the professional development sessions. For the data analysis the following phenomenological protocol was used, which is a simplified version of Hycner's (1999) process used by Groenewald (2004) together with some steps delineated by Van Manen (1997): First, we investigated the experience as we lived-it, then we bracketed the assumptions being made, as we looked at the data we delineate units of meaning, then we clustered them in themes and we create a composite. The participants in this study were the teachers involved in professional development, and the facilitator. The professional developer, is also a participant because her reflections and responses to teachers' engagement as well as my motives and wants are part of the study.

## PRELIMINARY RESULTS

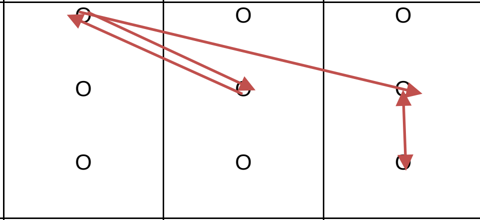
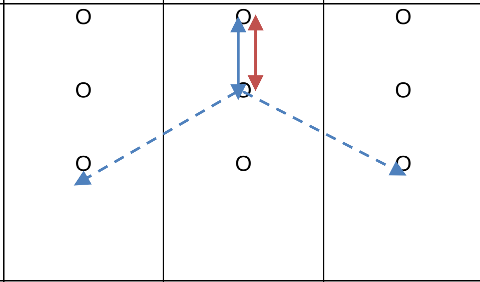
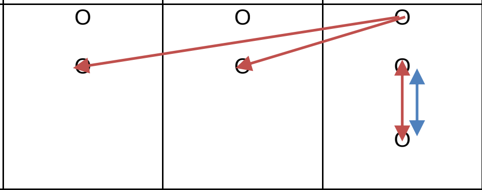
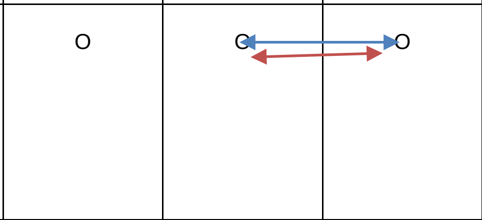
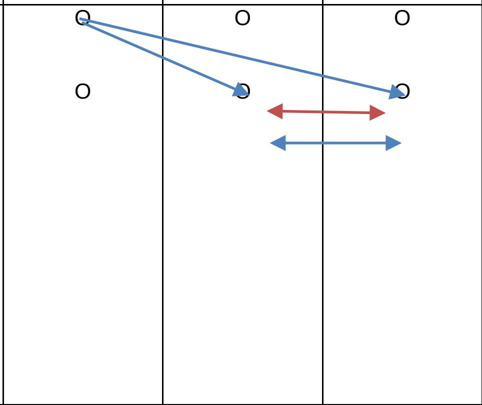
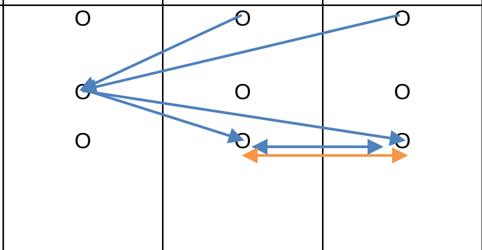
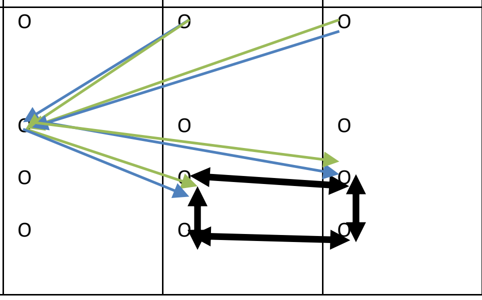
The following chart depicts an interaction with a group of Kindergarten teachers. Before the following interaction the professional developer spent a substantial amount of time discussing with them the resources and tried to make them look at particular lessons and have a discussion about them.

**Mode of address** the professional developer asked teachers a questions that would help them reflect on their previous practice and compare it with what was done in the program that we were looking it. In this particular case, they have been looking at lessons and activities within them to teach students words and their mathematical meaning and use; words like same, different, more, less, etc. This particular program emphasizes the importance of teaching one term at the time to make sure that students will fully understand before using words together to describe an object.

**Flow and Mood of engagement**: The flow of the conversation change from being slow as they were reflecting on the professional developer questions and then the conversation became more fluid, the teachers became actively engage in analyzing the lessons they were looking at. We can see this change as the arrows change, we first have active pro-active and re-active together arrows, which indicate that there is not much interaction among the teachers, however those arrows change to supportive and reflective arrows which indicate a more fluid engagement. Jane's and Cosima's mood is first rejecting (Jane first reaction was to say no) but then with Cosima's reflective answer it changes first to reflective and their engagement became mainly exploratory.

**Units of meaning**: Asking about comparing their current practice with what is expected in the current program had an effect with these teachers because they could see how this program could address some issues they were not aware of and that possibly was interfering with their students learning.

The first two point of inflection are an AHA moment that will further trigger moment of reflection making sense individually of a new realization about their practice and how to improve it. From there an exchange of ideas took place among the teachers and then I saw pertinent to ask a second question so that they will focus on the math. From here a deep discussion between Jane and Cosima started where they discussed possibilities, changes and adaptations to their practice

| Table K.2.4 | Facilitator | Jane | Cosima | Notes |
|---|---|---|---|---|
| Facilitator: Up to now each lesson explores one concept. How busy are usually your lessons? Do you teach one or several concepts at the same time?<br>Jane: No, uh [not] really.<br>Cosima: (reflecting and commenting) I do. I do because to me is so easy |  | | | Point of inflexion. |
| Jane (reflecting): I was not always aware. I don't think I was aware that I am doing more than one concept, but this is so specific that it makes me recall right back from maybe.<br><br>Jane is reflecting on her previous practice versus what she sees in the new program. She is talking to us but not really (this is marked by the dashed arrow) |  | | | Point of Inflection |
| Cosima: Even when we are doing addition...I am not following concrete, pictorial, abstract… I never realized it is a different thing for them, to see it rather than touch it. (She was going through the same process of reflection as Jane) |  | | | |
| Then a conversation ensued between Jane and Cosima about doing too much during a lesson and getting the children confused and how to watch out for that and check each other's lessons for a while for busyness. They were looking at the manual, discussing the topic in unit one. |  | | | |
| Then, while looking at patterns I made the following comments:<br>Facilitator: In order to be able to see patterns, perhaps we should look and describe the design.<br>A discussion ensued about patterns and how to discuss different attributes in class. I did very little prompting afterward. Jane and Cosima were discussing a variety of ideas and they discussed what to do to help students articulate what they actually see. They also compared previous practice with what they saw of the new program. |  | | | Point of inflexion Excerpt 1 |
| Finally, they asked me several questions about how to use the resources for the lessons we just reviewed, but this time it was not about classroom management. Rather, we actually looked at the mathematical ideas, and also there was a discussion between Jane and Cosima about how to get the parents involved. |  | | | |
| Afterwards without much prompting they looked at the following two lessons and they mainly had a discussion among themselves. They were carefully reading the teacher's guide and asking questions. But the general diagram of the following 26 minutes looked like this. Either Jane or Cosima would ask me a question about terminology and/or the program and I would |  | | | Excerpt 2 |

<table>
<tr><td>reply and then they would have a discussion on their own. The thick black arrow indicates previous practice, new program, mathematical ideas and terminology.</td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="5"><ul><li>Blue arrow: solid means we are discussing the mathematical part, and the dashed arrow means we are discussing classroom management with resources.</li><li>Red arrow: previous practice</li><li>Solid black arrow: New program, previous practice, and terminology</li><li>Green arrow: terminology and definitions</li><li>Orange arrow: parent opinions.</li></ul></td></tr>
</table>

**Wants:** Cosima and Jane wanted a program that made sense to the children and had a logically sound development and to be able to have a full understanding of mathematical connections and sequencing. I wanted them to look carefully at the lessons and have a discussion about the pedagogy and the mathematics involved which they did.

**Themes:** Connecting a new program with previous practice can be a good way of learning about teacher's previous practice and position them in a way where they actively become engage with the new program. A reflective mode can lead to an actively exploratory mode.

## CONCLUSION

Analyzing professional development under the phenomenological paradigm is a task that has to take into account many aspects that are part of this activity; this framework is a first step into that direction. In analyzing how the professional developer positions participating teachers into an activity and analyzing their reaction or "level of engagement" by looking into the flow, mood, the possible wants, the units of meaning and themes it is our hope that we can better answer the question that many of us ask as we develop our professional developing sessions: what works and how?

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# "WHAT WE NEED TO SHOW IS THAT T IS WELL-DEFINED": GESTURE AND DIAGRAM IN ABSTRACT ALGEBRA

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*Proving that a mapping is well-defined is a key step in many results in group theory. Such an argument is required in the Orbit-Stabilizer Theorem, as well as the theorem that gives a necessary and sufficient condition for the set of cosets of a subgroup to form a group (namely, that the subgroup is a normal subgroup). It can be challenging for students to even realize that such an argument must be provided, as it is the first time they have been exposed to functions defined on equivalence classes. This paper studies the gestures and diagrams that one professor uses to explain the concept of well-definedness.*

## INTRUDUCTION

The first abstract algebra course that an undergraduate math major takes can be a turning point for the student (Weber & Larsen, 2008). Before this point, "algebra" may have been a synonym for the sorts of manipulations one makes to transform equations into other equations that happen to be more useful for one's purposes. An example might be factoring equations in order to solve for the unknown. After this point, "algebra" refers to structures like groups, rings, fields and modules; to standard constructions like subgroup, quotient group, Cartesian product of groups; and most of all, to proofs of all the important results, as opposed to calculations of answers. Student difficulties with handling this transition are well-known, and researchers have examined key concepts in these courses and how it is students understand them or fail to understand them (Dubinsky, Dautermann, Leron, & Zazkis, 1994; Asiala, Dubinsky, Mathews, Morics, & Oktac, 1997). More recently, instructional sequences have been designed that guide students toward reinvention of some of these key concepts, namely that of group, isomorphism, and quotient group (Larsen 2013; Larsen & Lockwood, 2013). We concentrate here on the concept of well-definedness, which plays a critical role in the study of group theory.

## THEORETICAL FRAMEWORK

Mathematician and philosopher Châtelet (1999) highlights the following features of mathematical gestures. They have the power to awaken other gestures in their wake. They are not just a movement in space; rather they bring into being a new manner of movement. They are not decided in advance and then carried out; "one is infused with the gesture before knowing it" (p. 10). By contrast with two other mathematical devices, "functions" and "acts", where a function can only express how to go from one state to another, and an act, when finished, has no more to offer, a gesture is "elastic, it can crouch on itself, leap beyond itself and reverberate" (p. 10). Perhaps most importantly for Châtelet, a gesture can "store up all the allusion's provocative virtualities, without debasing it into abbreviation" (p. 10). In turn, the diagram, which "can transfix a gesture, bring it to rest, long before it curls up into a sign" (p. 10), can go on giving and evoking and suggesting without emptying itself.

## DATA COLLECTION

35 lectures in a third-year undergraduate mathematics course (*Groups*) at a mid-sized University in Western Canada were video-recorded. The video camera was focussed exclusively on the lecturer "J" (and not the students). A transcript was made of all of the video-recordings.

## DATA ANALYSIS

### Well-defined: Episode 1

J begins writing the statement of the Orbit-Stabilizer Theorem (Theorem 7.3 of the Gallian's *Contemporary Abstract Algebra*): Let $G$ be a finite group of permutations of a set $S$. Then $|G| = |orb_G(i)| \cdot |stab_G(i)|$ for each $i \in S$. In words this says that the number of elements of $G$ (the order of $G$) is equal to the product of two numbers. The first number counts how many different elements of the set $S$ are reachable when all the permutations in $G$ are each applied to the element $i$ in $S$. The second number counts the number of permutations in $G$ which, when applied to the element $i$ in $S$, give the result $i$ (in other words, these permutations 'fix' $i$). The professor begins the proof with what he says is the "key idea" right at the start. He writes: "Let $i \in S$. We claim that there is a 1-1 correspondence $T$ from the left cosets of $stab_G(i)$ in $G$ to the elements of $orb_G(i)$" and then writes out the short argument that shows that the Orbit-Stabilizer Theorem follows from this claim by a use of Lagrange's theorem.

J then says: "There's only one thing that requires any thought in this proof. I mean once we have that, which is not entirely obvious, once we have that, the entire proof is like turning the handle." It will turn out that the "one place where we need to think" is to show that a certain map is well-defined.

He proceeds to define a map $T$ as follows: $T\big(\varphi stab_G(i)\big) = \varphi(i)$. He demonstrates how obvious this choice is by asking the class for what the right hand side ought to be, and gets the answer from a student, commenting "what else could it map into".

Then he asks them to leave three lines blank, and leaving this space blank on the board himself, he titles two lines "$T$ is 1 – 1" and "$T$ is onto". He says that the 1 -1 part will be a one-liner in a moment, and then proves that the mapping $T$ they've defined is onto. Once finished, he returns to the empty space, saying "What's this missing proof- missing part of the proof doing? I mean what- what else can- why don't we just show it's one one and onto and go home". A student answers: "show that it's like well-defined":

For the following lines I will also keep track of the hand movements he makes (any hand movements will be enclosed in curly parentheses):

> We said well just pick a coset {touches the term phi stab G of i} phi stab G of i {holds out hands, palms up, shaking them slightly}. And the problem is {hands are still out, together, not moving} let's suppose that I picked- {touches the term phi stab G of i again} there are different ways of calling the same coset right? {returns to his earlier pose, hands together, palms up, holding an imaginary coset} You can have different coset reps, {hands move apart slightly} I can have some psi that lives inside the coset, {writes 'psi' in the air with his index finger, then points to the phi stab G of i term on the board} I could have some psi stab G of i {holds out hands, palms up} that could mean the same coset {the first half of the key gesture WD1: the hands draw apart dramatically, separated by a foot}. But let's suppose that phi and psi send me somewhere different {the finale of the key gesture WD1: he alternates moving each hand a little, weighing imaginary cosets, and then immediately on 'send' both hands turn into index fingers that fly out towards the class, splayed each at 45 degrees to the forward direction}.

Here, in the hands, are all the reverberations of 'well-definedness'. The one object, that is really two, and might therefore go to two different places; they must show it really goes only to one place. The function, that they have defined as taking an input and giving an output, hides almost as a secret that it is taking an input with a special name out of a bag of objects each with different names. They have decided to consider all objects in the bag as being the same. In the hesitation in his hands between the hands being together, and the hands being separated, is the hesitation of even realizing that they must check that their definition "makes sense".

J turns to the board in order to write what he has gestured:

> So let's see what the picture- what's the problem. The problem is that we start off here {draws a circle right after "problem"} and let's suppose we have two different names for the same coset {writes out the equation phi stab G of i = psi stab G of i}. Right, that's the same coset with two different coset representatives. OK and now T tells me - well in one case {puts dot in the middle of the circle} if you use the first name {draws line emerging from center, touches left hand side of equation, adds arrow to line, labels line with T} I'm gonna map you to phi of i {writes phi of I at the end of this labelled line}. But if you use the second name for the same object {draws second line emerging from center, slanted down and to the right; adds arrow to line, holds right hand side of equation} I'm gonna map you to psi of i {labels line with T, writes psi of I at the end lf this labelled line}. That would not be good {walks away from the board}. OK so actually what we need to show {walks back to diagram} is that if this happens {uses index and pinky finger to simultaneously hold both sides of the equation} these are the same {carries his gesture over to the ends of the two labelled lines, and holds simultaneously phi of i and psi of i}.

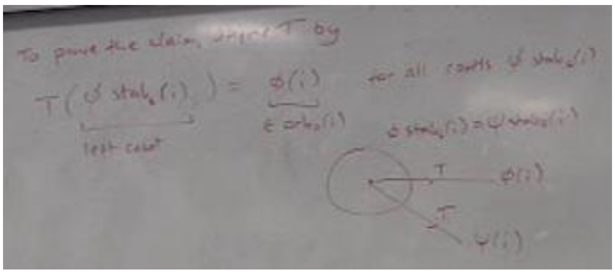

Figure 1: Well-definedness diagram in the Orbit-Stabilizer Theorem

Note that the source here is one circle, with the two names given as a left hand side and right hand side of an equation, whereas with the earlier gesture WD1, the hands which had once been together, diverged in a movement that he can't quite replicate on the board. The "two-ness" of the source, and how it is related to the "one-ness", is more direct and vital in WD1. Note too that the diagram is incomplete; it requires the powerful finishing gesture of holding the equality and then holding the two separate endpoints of the arrows to meld them into one unique outcome. However, WD1 was also incomplete, as the two fingers splayed out in two different directions, and they really ought to wind up at the same place. Now he is ready to finish the proof:

> So what we need to show is that T is well-defined {writes "T is well-defined"}. And by all that discussion {waves his index finger over the diagram and the equation above it} we just need to show that this implies this {touches equation on first "this"; touches in between the ends of the two arrows on second "this"; realizes this is not what he means} sorry, this implies that these are equal {touches equation on "this"; holds both ends of the two arros with index finger and pinky finger simultaneously}. And what I'm gonna say there {points to the line in his proof that says "T is 1-1", maintains the double finger hold} is reverse the above argument {still holding}. Because one to one says that if these are equal then these are equal ok {lifts index and pinky at "if", retouches with them at first "these", carries the gesture over to the equation and simultaneously holds both sides on the second "these"}.

So as promised earlier, proving that T is 1-1 is indeed a one-liner. We see now in this situation that the concepts of injectivity and well-definedness are very close to one another; reversing the gesture of one gives you the gesture for the other. The gestures have evoked each other.

**Well-defined: Episode 2**

WD2 occurs just after his proof of Theorem 24.1, which states that for any element $a$ of a finite group $G$, the order of the conjugacy class of $a$ is equal to the index of the centralizer of $a$ in $G$. Again he has defined a map T (from the set of left cosets of the centralizer of $a$ to the conjugacy class of $a$), and now that we are near the end of the course, he simply asks them to "check" that T is well-defined, 1-1 and onto, from which the theorem follows. In his post-proof comment phase, he observes:

> Well-defined is if you like, the converse of one one. Well-defined is if you take two things with the same name {puts notes down, holds both hands up in the air in front of him and above his head} and then you apply the map {holds hands there, shakes them at "apply the map"} you end up in the same place {hands dive down, end up clasped together at waist level, see Figure 2}. So two things with the s- {repeats: hands back up in the air in the earlier front and high position, shaking them} with the same name end up equal to each other {shakes them, hands dive down to lower clasped position}. One one is the images are equal to each other {looks down at this hands, shakes them a little} the things that you started off with are equal to each other {hands go up, separate, eyes track them, then he brings the hands horizontally together until they are joined, see Figure 3}. So if you like {one hand at the bottom, one hand sweeping the space up and down between the lower position and the upper position} one is the reverse process of the other OK.

 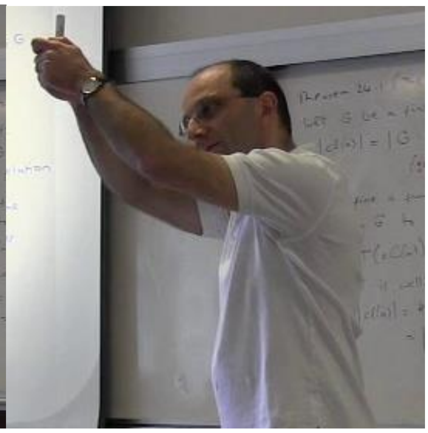

Figure 2: Well-definedness gesture WD2, bottom position. Figure 3: Well-defined gesture WD2, top position.

WD2 is the most complete well-definedness gesture in the course, and although I have named it "well-definedness", it includes in a natural way a 1 – 1 gesture.

**When must we check for well-definedness?**

It is important at this point to discuss the "function" gesture. J sometimes indicates the result of performing a mapping using his hands and fingers. There's a place to start (the input), a little shaking or movement to activate it, a clear line or direction of movement, and then a final resting place. It is exactly this gesture that the well-definedness gestures must both *approach* (because if a function is indeed well-defined, then the "function" gesture is the one that is appropriate!), and also *avoid* (because if a function were obviously well-defined, we would not have any skepticism or doubt, and there would be nothing to check!). This is an interesting site, because it is not often that students are in a position in an undergraduate mathematics class where they are asked to do something, but aren't given an explicit prescription of exactly when to do the thing asked. The issue comes up on yet another occasion:

1 S: How do you see that?

2 J: Ah, how do you see that it needs to be well-defined.

3 S: I understand that things need to be well-defined-

4 S: When you're looking at something for the first time?

5 J: That's why I'm giving it as an example.

Perhaps the student wants to know why on so many occasions mappings are defined and they don't bother to check that they are well-defined, and on other occasions it becomes an issue. What is the deciding factor that makes it an issue? J's next attempt at a response is to say that the "politically correct" answer is that "you should always check that a mapping is well-defined as a matter of principle" while immediately acknowledging that "most of the time you just don't do it because it's kind of just tedious and most mappings are". The student presses further, asking "so should we do that every time?", to which J answers:

> In this- like I said, I've drawn particular attention- the coset stuff- when you've different ways of naming the same object you gotta be careful with- in this chapter on homomorphisms we're gonna see this idea several times. So just sort of heighten your sense of- your awareness or sense of danger around this.

There is a lot of experience in play in each of J's lines. And yet we might feel with the student that a clear-cut answer is being avoided.

I suspect that a clear-cut answer is impossible for a few reasons. Let's first cite a standard description of failing to be well-defined:

> A proposed function $f: X \to Y$ can fail to be well-defined in two ways: (1) some $x$ in the domain $X$ can fail to have a $y$ in the co-domain to which to map or (2) some $x$ in the domain can be mapped (ambiguously) to two different $y$'s in the co-domain. (Hunter, 94).

The first way of failing to be well-defined occurred once in the course. It is the second way that occurs most frequently in group theory. So this might be a reason why J couldn't give a clear-cut answer in the time he had available: he would first need to explain the two sorts of ways functions can fail to be well-defined, and also begin to give examples of the first way because it has less commonly been seen by them. A second reason is that it is very hard to enumerate all the ways in which you might be faced with a situation where (2) applies. So J does the best he can: he mentions the issue every time it comes up, and he solves examples where the issue emerges so the students can see it come up in the sorts of circumstances that it does.

The student's repeated questioning, J's honest and careful replies: the moment is significant. It is perhaps the clearest moment in the course where there remains some irreducible gap between student and teacher that might be bridged only by hands-on experience.

**Well-definedness and quotient groups**

I turn now to the most critical appearance of this concept in introductory group theory. The context is Theorem 9.2 (tagline "Factor Groups"). This is the theorem that gives a necessary and sufficient condition for a subgroup $H$ of a group $G$ to be normal in $G$: $H$ is normal in $G$ if and only if the set of left cosets $G/H$ forms a group under the operation $aHbH = abH$. Before stating the theorem J illustrated using the dihedral group $D_3$ that such a binary operation of multiplication need not be well-defined. Just before he begins the proof he comments:

> And what we'll see as we do the proof right now, we're gonna see that the only way that this operation could fail to be a group is if the operation is not well-defined. Everything else is gonna work just fine. We're gonna get all the properties we would want from a group formally, but the only place that can fail is not well-defined.

J stressed "only" and "fail" in a pronounced manner. J makes clear here, (and will often repeat this important observation), that the only obstruction to having a group structure on the set of left cosets is that the multiplication might not be well-defined.

J now tackles the question at hand. Note that he begins in a very similar way to how he began just before he drew the diagram in Figure 1, asking "what's the problem" (recall that recognizing that the problem exists in the first place was exactly the student's question in the last section):

> So what's the problem. The problem- ok let's- let's say that a H is a prime H {writes the equation a H = a prime H}. We have two different names for the same coset here and we have two different names for another coset {writes "and b H = b prime H"}. Now what could be the problem. Well if this is a H {draws a circle with label a H = a prime H} and this is b H {draws circle below the first one, with label b H = b prime H} then if we form- if we use these two and we get to ab H {draws arrows that end at a new circle, labels it ab H}, by combining them like - we don't want to get to a different place if we used different names a prime b prime H {draws arrows that end at a new circle, labels it a prime b prime H}. So the question is, can that happen, and we hope not, OK? {draws a question mark, points with his pen to his entire diagram}.

So he explicitly reminds them of the diagram in Figure 1. The difference here is that we have yet another doubling in play. Let's get the doublings straight. The first doubling was that of the source object: two names for the same thing. The second doubling was the output object: conceivably our mapping could take us to two places, hopefully it won't. The third doubling here is new, and is due to the fact that our proposed binary operation here is a map from $G/H \times G/H$ to $G/H$. His finished diagram is in Figure 4:

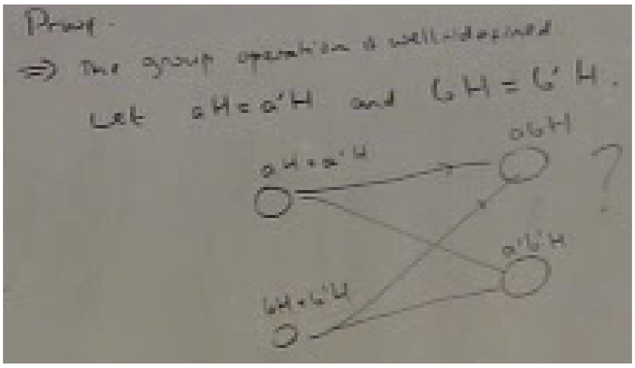


Figure 4: Well-definedness and the Quotient Group Theorem

This is a bit like an image of two eyes that are experiencing double vision, instead of focussing correctly on a single object, which is not unlike the experience of dealing with a well-definedness situation. Like Figure 1, this diagram is incomplete, but this one announces it, in the form of a question mark. Like Figure 1, the input is not doubled, but labels above each input express the doubling. Finally, like Figure 1, the output is doubled.

## DISCUSSION

The quotient group is a central concept in group theory, and plays a role in virtually every development from that moment forward. Since well-definedness of the multiplication operation is the only hurdle that must be cleared in order for the set of cosets to become a group, we can see that well-definedness is a critical ingredient in group theory. It is a crucial realization that a natural attempt at a definition of a mapping or operation might fail to make sense. Negotiating this conceptual leap was a major event in the early history of group theory, performed implicitly by Galois, revealed in an increasingly explicit manner in the work of later mathematicians, culminating in work by Holder (Nicholson, 1993).

Students must negotiate this conceptual leap as well. We have studied here the approach of one lecturer to sharing with his students how he thinks about the well-definedness of mappings. With the movement of his hands he clarifies the connection to what it means for a function to be one to one. He also illuminates the sorts of situations in which one will have to argue that a mapping is well-defined: we must have two objects with different names that we are considering to be the same. For this lecturer, explaining well-definedness – what it is, and when to expect to have to prove it – requires hand gestures, and diagrams born of those gestures.

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# NEGOTIATING MEANING: A CASE OF TEACHERS DISCUSSING MATHEMATICAL ABSTRACTION IN THE BLOGOSPHERE

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*Many mathematics teachers engage in the practice of blogging. Although they are separated geographically, they are able to discuss teaching-related issues. In an effort to better understand the nature of these discussions, this paper presents an analysis of one particular episode of such a discussion. Wenger's theoretical framework of communities of practice informs the analysis by providing a tool to explain the negotiation of meaning in the episode. Results indicate that the blogging medium supports continuity of discussions and can allow for the negotiation of meaning, but that a more nuanced treatment of the construct is necessary.*

## INTRODUCTION

> *I've been throwing around the term "pyramid of abstraction" recently, and there was some great pushback on Twitter this evening. This post is my attempt to clarify what I mean, and why I think it is a useful perspective to building students' knowledge.*
>
> *(Kane, 2015c, para. 1)*

The blog post quote above was written by Dylan Kane, a mathematics teacher, after a series of interactions on Twitter with Dan Meyer and Michael Pershan. The series of interactions were prompted by Dylan's original blog post that he shared on Twitter, in which he explained his ideas about a metaphor for mathematical abstraction. The discussion that resulted from this formulation presents an instance of negotiating the meaning of a metaphor that may be useful in the teaching and learning of mathematics. More importantly, it provides an example of how mathematics teachers are engaging in the practice of blogging, which is the focus of this paper.

Mathematics teachers around the world are choosing out of their own will to create, organize, and manage their own personal blog pages. On these publicly visible virtual pages, they make relatively regular posts, which are most often related to their work as mathematics teachers. These posts can include written and/or media enhanced recounts from their teaching experiences, links to interesting resources, and responses to posts that other bloggers have made.

However, since blog pages are individually managed and no universal blogging platform exists, finding like-minded bloggers may at times be difficult. To mitigate this issue, many bloggers extend their practice through Twitter, a universal micro-blogging platform that allows users to create searchable profiles. There are now 361 profile listings on the Math Twitter Blogosphere directory (MTBoS, 2015), most of whom have both a Twitter handle and a blog page listed on their profile. Although micro-blogging on Twitter is slightly different in nature than blogging due to the 140 character limit on each post, which forces users to communicate ideas more concisely, it is still considered a form of blogging (Ebner, 2013). Blog pages allow users to explore ideas deeply, but Twitter makes it easier for like-minded bloggers to connect. Mathematics teacher bloggers often use both mediums to express ideas, linking between the two when necessary. For instance, some bloggers include snips of Twitter conversations in their blog posts, and other bloggers link to their blog posts within their Twitter posts. In this way, teachers are communicating with peers across the world.

This sort of collegial interaction is unusual for teachers because teaching is generally an isolated profession (Flinders, 1988). Mathematics teachers' typical contact with other mathematics teachers is limited to that of department meetings and teacher lunchrooms, where teachers usually plan lessons and grade papers individually (Arbaugh, 2003). Professional development initiatives require time, funding, and facilitation, and are therefore limited to sparse one-time professional development workshops throughout the year whose benefits are not generally found to be effective in stimulating teacher collaboration (Ball, 2002).

However, there are over 600 mathematics teacher bloggers (Meyer, 2014) who seem to be overcoming these constraints. It is evident that they spend hours writing publicly about their daily practice, posting resources, and sharing their dilemmas with no compensation and no mandate. This unprompted, unfunded, and unevaluated teacher activity is a rich phenomenon of interest that deserves attention. Ironically, this phenomenon is largely unstudied. Empirical investigations related to blogs in mathematics education are limited to studies on the utility of blogging as a pedagogical tool within either a mathematics course (Nehme, 2011) or a mathematics education course (Silverman & Clay, 2010; Stein, 2009). These studies do not account for the autonomous and self-driven nature of the blogosphere and there is no clear work in mathematics education exploring the activities of these teachers.

As such, this study is guided by the overall research question of what the mathematics teacher blogosphere occasions for teachers who engage in blogging in relation to their practice. To this end, I pursue an investigation of one episode that exemplifies the type of interaction that is possible within the mathematics teacher blogosphere.

## THEORETICAL FRAMEWORK

Since blogging is an individual practice that is made public (Efimova, 2009), a mid-level theory that considers situated participation is desirable. C*ommunities of practice* (Wenger, 1998) is one such theory: it is a social theory of learning where learning is considered as increasing *participation* in the pursuit of valued enterprises that are meaningful in a particular social context. *Practice* is at the heart of Wenger's (1998) *communities of practice*, and a key aspect of *practice* is the ability to motivate the social production of *meaning*. The continuous production of *meaning* is termed as the *negotiation of meaning*, and is further defined by the duality between *participation* and *reification* (Wenger, 1998). *Participation* is "the process of taking part and also to the relations with others that reflect this process" and *reification* is "the process of giving form to our experience by producing objects that congeal this experience into 'thingness'" (Wenger, 1998, p. 58). Both *participation* and *reification* shape the participant and the community in which they participate in an ongoing manner.

Wenger (1998) claims that *participation* and *reification* imply each other, require and enable each other, and interact with each other. Wenger (1998) notes that the benefit to viewing the *negotiation of meaning* as a dual process is that it allows one to question how the production of meaning is distributed. He notes that there is "a unity in their duality, [because] to understand one, it is necessary to understand the other, [and] to enable one, it is necessary to enable the other" (Wenger, 1998, p. 62). Various combinations of the two will produce different experiences of meaning, and together they can create dynamism and richness in meaning if a particular balance is struck.

Wenger (1998) notes that "when too much reliance is placed on one at the expense of the other, the continuity of meaning is likely to become problematic in practice" (p. 65). If *participation* dominates, and "most of what matters is left unreified, then there may not be enough material to anchor the specificities of coordination and to uncover diverging assumptions" (p. 65). However, if *reification* dominates, and "everything is reified, but with little opportunity for shared experience and interactive negotiation, then there may not be enough overlap in participation to recover a coordinated, relevant, or generative meaning" (p. 65). In essence, *participation* allows for re-negotiation of *meaning*, and *reification* creates the conditions for new *meanings*. In this interwoven relationship, the two aspects work together to drive the process of *negotiation of meaning*. As such, this framework is suitable to examine the social production of meaning within the mathematics teacher blogosphere with the aim of determining what the blogosphere occasions for mathematics teachers.

## METHOD

In the process of acquiring data from a larger study within the blogosphere, I have acquired a notable list of mathematics teachers I now subscribe to. Every user will see different posts based on who they subscribe to, and it is important to note that my pervasively subjective position influences what I can notice in this ultra-personalized and dense virtual environment. Aside from creating a profile, my activity has been predominantly that of a 'lurker' in that I have not made significant contributions to the blogosphere. Although the position of 'lurking' has been valuable, this position changed slightly after my attendance to the MTBoS conference 'Twitter Math Camp 2015' (TMC15). After the conference, I found myself connected to this group of MTBoS bloggers more than ever before. This shift in position allowed me to be closer to the active conversations these members were having on Twitter.

It was during this time after my return from TMC15 that I encountered a particular conversation that I flagged as interesting based on my theoretical framework. This particular conversation was chosen for its power to illustrate a possible mode of interaction in the blogosphere. After the conversation took place, I used *storify.com* to chronologically reconstruct it by identifying all tweets related to the conversation from the feeds of each of the members who participated, arranging them chronologically.

I then contacted the initiator of the discussion, Dylan Kane, for an interview. I interviewed him by proxy about his experiences in the blogosphere as well as about his reactions to the conversation that had occurred on Twitter in relation to his blog posts. The interview was voice recorded and transcribed completely. The transcription of the interview with Dylan, the reconstructed Twitter conversation, and the blog posts Dylan linked to within his Twitter posts comprise the data set for the analysis of this case.

Each of these sources of data were then coded according to notions of Wenger's (1998) *participation* and *reification*. The data were then arranged into a chronological order so that a complete discussion could be seen with codes indicating moments of *participation* and *reification*. In this case, *participation* was considered to be any action that a blogger took as part of the blogging practice, and *reification* was considered to be any trace that was left from a *participation*. The interplay between these aspects was then considered in relation to the data, and conclusions were drawn about the nature of the *negotiation of meaning* as exemplified in this case of mathematics teacher blogging. For purposes of brevity, a reduced version of the analysis is presented in what follows.

## ANALYSIS

On June 25, 2015, Dylan *participates* in thinking about his past formulations about problem solving. He *participates* in writing this idea out in a blog post (Kane, 2015b). Within the body of this post, Dylan discusses his ideas on how teachers can help students deal with solving difficult mathematical problems by helping them become better equipped to transfer prior knowledge to new contexts in mathematics. At one point, he brings up the idea of a 'ladder of abstraction' that he has heard references to. He explains that the 'ladder of abstraction' refers to "the idea that students' understanding begins with the concrete, and climbs a metaphorical ladder as it becomes more and more abstract" (Kane, 2015b, para. 11). However, he believes that the metaphor is incomplete.

> Based on this perspective, I think the metaphor of a ladder of abstraction would be better replaced by a pyramid of abstraction . . . I worry that the ladder of abstraction metaphor leads me to believe that, once a student understands one concrete example of a function, they are ready for a more abstract example. While some students may be, I want to focus on building a broad base first, and then moving up the pyramid after we have spent time analyzing the connections between the examples and the underlying structure.
>
> (Kane, 2015b, para. 12-13)

Although this formulation of the 'pyramid of abstraction' is built on a history of *participation* and *reification* that Dylan has engaged in, I take this to be the beginning of this particular episode of *negotiation of meaning*. By writing this down and publishing it as a blog post that is publicly visible, Dylan has created a *reification*, and it is likely that this *reification* is influenced by the very process of *participating* in writing the blog post.

A month later, on July 17, 2015, Dylan *participates* in writing a reflection about his experiences at PCMI in relation to mathematical problem solving. Within this reflection, he makes a reference back to his June 25, 2015 post regarding his description of a ‘pyramid of abstraction.’

> This gets at something I wrote about recently that I called the *pyramid of abstraction* – that students build abstract ideas from looking at connections between a wide variety of examples, rather than simply jumping from concrete to abstract. (Kane, 2015a, para. 6)

After *participating* in this reflective moment, trying to recapture his experiences from a professional development encounter, and connecting his experiences with something he had previously written about, he writes his ideas down and publishes them. In this act, he produces another *reification*. This *reification* now has a rich history, and through the use of hyperlinks, one can trace the ideas back to where they evolved from for Dylan. It not only encapsulates the history of ideas, but it also prompts further *participation* for either Dylan or other mathematics teacher bloggers who *participate* in reading his post.

Simultaneously to posting this blog post, Dylan also publishes a link to it on Twitter. Doing this creates a rich *reification* because of the number of traces that it leaves and the visibility it affords. It is also simultaneously a manner of *participating* in the blogging practice. Wenger (1998) notes that *participation* and *reification* are so closely tied that sometimes they may be difficult to pry apart. This is evident here, particularly due to the nature of the blogging practice.

Shortly after publishing the link to his post on Twitter, Michael Pershan responds in agreement with Dylan’s reference to the ‘pyramid of abstraction’ metaphor (Figure 1).

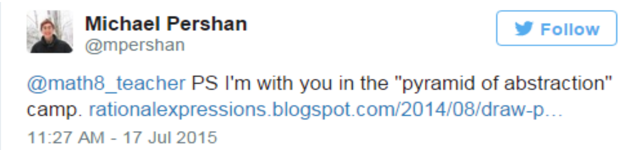


Figure 1 Michael Pershan's response to Dylan's post

In the interview with Dylan, he confides that he and Michael have discussed the idea of a ‘pyramid of abstraction’ for a while and that that they have come to a shared meaning for the term. Hence, this idea of a ‘pyramid of abstraction,’ which is itself a *reification*, has a long history of *negotiation of meaning*.

Fifteen days after Dylan’s Twitter post of the link to the blog post that included a link to his ‘pyramid of abstraction’ post, Dan Meyer poses a question to Dylan and Michael on Twitter about the meaning of the apex of the ‘pyramid of abstraction’ (Figure 2).

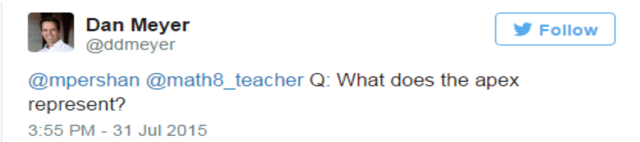


Figure 2 Dan Meyer's question about the pyramid of abstraction

The power of this *reification* is that it is pervasively public. Dan Meyer currently has 39, 320 followers. Michael Pershan has 3,423 followers, and Dylan Kane has 785 followers. The nature of Twitter is that if a person tags another person in a post, only people who follow both of those people see the post. Since Dan, Michael, and Dylan are all mathematics teacher bloggers and share followers, the number of people who see such a post is large. Hence, Dan's *reification* is bound to prompt further *participation*, leading to the continuous evolution of the *negotiation of meaning* around this particular notion of abstraction within mathematical problem solving.

Over the course of a few hours, these iterations of *participation* and *reification* continue, and the *negotiation of meaning* evolves the notion of 'abstraction' and its role in mathematical problem solving. Initially, Dylan and Michael restate their interpretations of the *meaning* of a 'pyramid of abstraction.' To this, Dan responds with a critique claiming that there can be no 'top' to abstraction. This prompts a *re-negotiation* of the term 'abstraction,' and prompts Dylan to re-define what he means by the concept that he has *reified* (Figure 3).

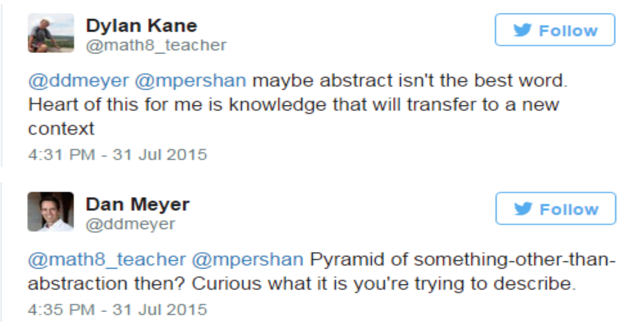


Figure 3 Negotiation of the term 'abstract'

In this interplay of multiple iterations of *participation* and *reification*, the meaning of the term 'abstraction' is *negotiated*, and it is clear that nuances within their interpretations exist. In this interaction, they are realizing that the metaphor can mean different things to other mathematics teachers. In fact, when Dylan reflects on this experience in the interview, he states that the biggest thing he has learned from this *negotiation*, is that metaphors can mean different things to different people.

After a few hours, Dylan writes a blog post in response to the discussion and links to it on Twitter. In this post, he restates his interpretation of 'abstraction' by claiming that he has used the term to refer to something that helps students build "a concept in a way that they can apply it in a variety of different contexts" (Kane, 2015c, para. 6). However, he also expresses an acknowledgement that this may not be how others view 'abstraction.'

> Do I Actually Understand Abstraction? I'm redefining abstraction a bit, and I'm also lumping knowledge transfer as one giant idea, which it might not be.
>
> (Kane, 2015c, para. 19)

This blog post may be seen as a *reification* of a reflective form of *participation*. However, it seems to represent a different form of *reification* than that merely of a trace left from a *participation* in the blogging practice. This *reification* seems to indicate a heightened level of awareness to which Dylan has arrived. However, the construct of *reification* as defined by Wenger (1998) does not allow for this differentiation.

## DISCUSSION AND CONCLUSION

In the analysis, it is found that *participation* in the mathematics teacher blogging practice includes reading, thinking about, and writing blog or micro-blog (Twitter) posts in relation to mathematics teaching and learning. In turn, *reification* in this practice is that of publishing such posts. It is clear that within the mathematics teacher blogging practice, *participation* and *reification* are intimately intertwined, and their close co-evolving and co-implicated relationship drives the *negotiation of meaning* among participants and ensures continuity of the *negotiation*. This points to the powerful nature of the medium itself.

This analysis also reveals a need to redefine *participation* and *reification* for mathematics teacher blogging in particular because various forms of each have surfaced. As noted above, various forms of *participation* exist, and each form implies a different mode of interaction in the *negotiation of meaning*. In particular, some instances of *participation* are prolonged and accompanied by reflective activity, which contrasts the instances where a quick knee-jerk tweet is made. An exploration of the role of reflection in blogger *participation* may be fruitful in future studies.

Further, it is also evident from the analysis that certain *reifications* can be of higher order than others. In the practice of mathematics teacher blogging, each post forms a *reification* because it leaves a trace of a *participation*. However, certain *reifications* indicate that a shift in meaning has occurred. There is currently no terminology within Wenger's (1998) frame to refer to such occurrences. However, it may very well be that a new term needs to be introduced to refer to higher order *reifications*.

Finally, it should be noted that the construct of *negotiation of meaning* also has the power to reveal *what* is being negotiated. In this case, the teachers found it valuable to invest time into engaging in a prolonged episode of *negotiating the meaning* of 'abstraction.' Looking for more of these instances may help identify teaching-related issues that mathematics teachers take interest in. Most evidently, the blogging medium can make it possible to trace the development of ideas in the blogosphere. Each idea has a history and a future. The history is traceable, and the future is quickly unfolding.

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# DISCURSIVE PRACTICES USED IN DEFINING MATHEMATICAL LEARNING DISABILITIES: A BRIEF TEXTBOOK ANALYSIS

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*This paper compares two separate descriptions of mathematical learning disabilities given in a textbook on learning disabilities. One description is from an early edition of the textbook and the other description is from a later edition of the same textbook. Critical discourse analysis is applied to both descriptions to investigate the changing discursive practices used in defining mathematical learning disabilities. While a "medical discourse" dominates both the early and later editions, there is evidence of other competing discourses which suggests some instability around the boundaries of what constitutes a "mathematical learning disability."*

## INTRODUCTION

There has been much research on reading disabilities (e.g. dyslexia) but much less research done in the area of mathematical learning disabilities (MLD) (e.g. dyscalculia). Yet the prevalence rates for dyslexia and dyscalculia are approximately equal (around 5% to 8%). Moreover, much of the research on MLD is from a cognitive psychology and special education perspective and more research needs to be done from a mathematics education perspective. The recent growth in the use of sociopolitical perspectives in mathematics education has introduced new analytic tools that were not traditionally used in the field such as discourse analysis. This research report will apply Norman Fairclough's (1992) version of critical discourse analysis (CDA) in order to analyze some of the changing discourse practices used in the definition of MLD found in a well-established textbook on learning disabilities.

## THEORETICAL FRAMEWORK

The theoretical framework I have chosen for this analysis is borrowed from Gina M. Borgioli's (2008) article examining the practices around the identification and labeling of students with learning disabilities in mathematics. Her framework applies a critical pedagogy (Kincheloe & McLaren, 1994) perspective looking through the lens of ableism (Rauscher & McClintock, 1997). Critical pedagogy "seeks to uncover hegemonic (power) relations, ideologies, and inequities in education, critique instrumental rationality, and inspire a movement for change toward social justice" (Borgioli, 2008, p. 132). One key assumption of critical pedagogy relevant to this paper is "acknowledging schooling as a form of cultural politics that endorses only particular forms of knowledge, thus creating a dominant group of successful knowers and 'others'" (Borgioli, 2008, p. 132). Moreover, Borgioli (2008) interprets ableism to be

> A deliberate act of exclusion and discrimination. It entails those in a position of (political, economic, and/or educational) power narrowly defining what society and educators are to consider as acceptable 'school mathematics' as well as acceptable evidence for students' demonstration of proficiency in school mathematics. (p. 133)

## METHODOLOGY AND METHODS

Norman Fairclough's (1992) CDA is a useful framework for analyzing discourse as social practice and, in particular, the discursive practices used in defining MLD within textbooks. The main objective of Fairclough's approach to language analysis is to study social and cultural change. Shifts in language use play a central role in the understanding of changes in social phenomena. Fairclough's CDA synthesizes two different senses of discourse—the social-theoretical sense (such as Foucault's) and the "text-and-interaction" sense—and forms a three-dimensional model in the following way:

> Any discursive "event" (i.e. any instance of discourse) is seen as being simultaneously a piece of text, an instance of discursive practice, and an instance of social practice. The "text" dimension attends to language analysis of texts. The "discursive practice" dimension, like "interaction" in the "text-and-interaction" view of discourse, specifies the nature of the processes of text production and interpretation, for example which types of discourse (including "discourses" in the more social-theoretical sense) are drawn upon and how they are combined. The "social practice" dimension attends to issues of concern in social analysis such as the institutional and organizational circumstances of the discursive event and how that shapes the nature of the discursive practice, and the constitutive/constructive effects of discourse referred to above. (p. 4)

This synthesis of the socially and linguistically oriented views of discourse is what Fairclough calls a "social theory of discourse."

Fairclough's (1992) multi-dimensional approach emphasizes the importance of text and language analysis (such as systemic functional linguistics) in discourse analysis and has developed an explicit and operational approach for researchers to analyze discourses. There are two key focal points of any analysis in Fairclough's CDA: the order of discourse (the totality of discursive practices of an institution, and the relationships between them) and the communicative event (an instance of language use). Due to the small scope of this paper, the focus will be primarily on the communicative event (the two textbook excerpts), and Fairclough's three-dimensional model of discourse (as text, discursive practice, and social practice) will be applied to it. While my analysis of these three dimensions for each excerpt will be done together, each dimension should be separated analytically. For example, the analysis of discourse as text involves a focus on the linguistic features of a text (linguistic analysis). This may involve an analysis of vocabulary, syntax, and the grammar of sentences. The analysis of discourse as discursive practice concerns how authors draw on existing discourses and genres to create a new text. A key question to ask is: Are discourse types (genres and discourses) used conventionally or creatively? Conventional discourse practice involves a normative use of discourse types and helps to reproduce the relationships in the order of discourse. On the other hand, creative discourse practice often mixes together a number of genres and discourses and helps to restructure the boundaries of the order of discourse. The last dimension to be considered when analyzing a discourse sample is discourse as social practice: What is the relation between this discourse practice and the larger social structure to which it belongs (is it conventional and normative or creative and innovative)? Does it transform or reproduce existing social practices? At this point, theories beyond the discourse analytical type such as social or cultural (or math education) theory need to be drawn upon.

## ANALYSIS AND DISCUSSION

I have selected two short, but related excerpts from two textbooks on learning disabilities for analysis using this three-dimensional approach. The excerpts are from the same textbook but different editions (Lerner, 1989; Lerner & Johns, 2015) and reproduced below. Janet W. Lerner is the primary author of each textbook and her textbook is now in its thirteenth edition. I have selected this textbook because of its longevity and it is representative of many other textbooks on learning disabilities. Also, the overall structure and organization of the textbook, surprisingly, has not changed much, making comparative analysis somewhat easier. The two excerpts I have selected are from the fifth (1989) and thirteenth (2015) editions respectively. Both textbooks have a common chapter related to MLD, and the section of this chapter I have chosen to examine is from the introductory description and definition of MLD.

### Analysis of excerpt from the 1989 edition

The first excerpt from the 1989 edition is entitled "Characteristics of Students with Mathematics Disabilities" and is reproduced below:

> The subject of remedial mathematics has received much less attention than remedial reading. Yet for many youngsters with learning disabilities, mathematics is the major area of difficulty. A severe mathematics learning disability and related conceptual disturbances in learning quantitative elements are sometimes referred to as *dyscalculia*. As with the term *dyslexia*, which is applied to a severe reading disability (see Chapter 12), dyscalculia has medical connotation suggesting a central nervous system involvement. Whatever it is called, severe difficulty in learning mathematics can be a debilitating problem in school and in later life. Not all students with learning disabilities have difficulty with number concepts. In fact, some students with severe reading disability have strong mathematics skills. Yet mathematics disorders do affect a significant portion of the learning-disabled population. McLeod and Crump (1978) found that about one-half of students with learning disabilities require supplemental work in mathematics, although only 10 percent were seriously deficient in mathematics.
>
> Many of the symptoms of learning disabilities can be linked to mathematics difficulties. Factors that play a role in mathematics learning include language, conceptual, visual-spatial, and memory abilities (Johnson, 1987). Difficulties in spatial relationships, visual perception, symbol recognition, language development, and cognitive learning strategies all have obvious implications for quantitative learning. It must be remembered,

> however, that each student is unique, and not all students who have a mathematics problem will exhibit the same characteristics. We need much more research on the subtypes of mathematics disabilities (Keogh and Babbitt, 1986). (Lerner, 1989, pp. 430-1)

Firstly, the author does not clearly define what a mathematics disability is or even clearly state what to call it. She draws on many different discourses in an attempt to describe students who have difficulties with math, but no one discourse seems to be dominant and fix the meaning of a mathematics disability (according to Fairclough, these interdiscursive mixes can be a sign of instability). It seems so little is known about mathematics disabilities that it needs to be juxtaposed with discourses on reading disabilities in order to gain any meaning itself: "remedial mathematics has received much less attention than remedial reading," and "dyscalculia" is compared to "dyslexia."

There also appears to be inconsistencies in what to call or describe the very thing to be defined. The phrase "mathematics disabilities" is used in the title of the section, but appears only once in the body of the text at the very end as part of the phrase "subtypes of mathematics disabilities." The text begins by talking about "remedial mathematics," then the next sentence is about mathematics difficulties in the context of "youngsters with learning disabilities." The following sentence contains three different labels: "severe mathematics learning disability," "related conceptual disturbances in learning quantitative elements," and "dyscalculia." We come closest to the origins of a mathematics disability when dyscalculia is described as having "a medical connotation suggesting a central nervous system involvement." This instability in the label used to describe a mathematics disability is reflected in the modalities "sometimes referred to," "suggesting," and "whatever it is called" indicating low affinity for the statements and descriptions made.

Although there may exist some uncertainty in the text around the label itself (the text suggests several names: mathematics disability, mathematics learning disability, dyscalculia, mathematics difficulties, and mathematics disorders), there is a strong sense that mathematics disabilities have a biological basis because of the medical discourse that dominates the text. For instance, dyscalculia "has a medical connotation suggesting a central nervous system involvement." There is other medical terminology used throughout the text that suggests a medical discourse in relation to mathematics disabilities reminiscent of a doctor remedying a patient's illness: "remedial," "severe," "debilitating," "disorders," "deficient," "symptoms," "factors," and "subtypes." This medical discourse is reinforced by the impersonal and objective nature of the text as if one were reading a medical book. The frequent use of passive clauses and nominalizations aim to obscure agency and the attribution of responsibility as if mathematical disabilities are a natural phenomenon. For example: "The subject of remedial mathematics has received much less attention than remedial reading." Attention from whom? Teachers? Academics? Psychologists? Also, the author writes: "A severe mathematics learning disability and related conceptual disturbances in learning quantitative elements are sometimes referred to as dyscalculia." Again, who is doing the referring? There are many more examples like these that obscure the individuals doing the labeling (diagnosing?) and create a feeling of depersonalization throughout the passage.

Despite the apparent objective nature of mathematics disabilities that the text suggests, there are some hints of uncertainty evidenced by the theme “not all students” that begins the sentence “Not all students with learning disabilities have difficulty with number concepts” and the secondary clause “not all students who have a mathematics problem will exhibit the same characteristics.” The qualifier “not all” suggests the complexity of mathematics disabilities and that it is not homogeneous in terms of causes or characteristics. Moreover, there is hedging suggested by the statements “It must be remembered, however, that each student is unique” and “We need much more research on the subtypes of mathematics disabilities.” This is consistent with research that has been unable to isolate the cognitive variables that underlie MLD due to many confounding factors. For instance, difficulties in math and reading acquisition are often attributed to the same cognitive processes (phonological in nature). However, what’s most notable about these hedges is the personalization of the author. Rather than using the imperative statement “Remember that each student is unique,” she is more passive and indirect: “It must be remembered, however.” The author seems to be taking a softer tone here, one that’s less authoritative, and seems to hint at some empathy towards the students by reminding readers of their uniqueness. The use of the personal pronoun “we” is also quite telling. While somewhat ambiguous in meaning, the “we” seems to be positioning the readers (likely teachers) and the author as equals in the sense that “we” all share the same goal in better understanding the mathematics disabilities of “our” students. This less objective and more “sympathetic” discourse competes with the more dominant “clinical” discourse that runs throughout the text.

**Analysis of excerpt from the 2015 edition**

The second excerpt is entitled “Students With Mathematics Difficulties and Students With Mathematics Learning Disabilities” and is reproduced below:

> Many students have difficulty in acquiring and using mathematics skills. Researchers differentiate 2 different groups: (1) students with math difficulties and (2) students with mathematics learning disabilities. In this book, we offer strategies for teaching both groups. Students with math difficulties perform poorly in mathematics achievement tests. Over 30% of eighth-grade students score below basic math performance on the National Assessment of Educational Progress (NAEP) (Maccini, et al., 2008; Mazzocco, 2007).
>
> In contrast, students with mathematics learning disabilities comprise about 6% of the general population (Mazzocco, 2007). Mathematics learning disabilities is a biologically based disorder and is related to difficulties in cognitive processing and brain functioning. Research with functional Magnetic Resonance Imaging (fMRI) shows these cognitive processing dysfunctions (Mazzocco, 2007; Gersten, Clarke, & Mazzocco, 2007).
>
> Approximately 26% of students with learning disabilities exhibit problems in the area of mathematics. Over 50% of students with disabilities have mathematics goals written into

> their individualized education programs (IEPs) (Kunsch, Jitendra, & Sood, 2007; Miller & Hudson, 2007; Cass et al., 2003).
>
> The term dyscalculia is a medically oriented term that describes a severe disability in mathematics with medical connotations. When an adult suffers a brain injury and loses abilities in arithmetic, medical professionals identify the loss of math skills related to the neurological impairment as dyscalculia. An analogous term in reading is *dyslexia*, the loss of reading skills that has medical and cognitive connotations.
>
> Both mathematics difficulties and mathematics learning disabilities that emerge in elementary school often continue through the secondary school years. Not only is a mathematics disability a debilitating problem for individuals during school years, but it also continues to impair them as adults in their daily lives (Maccini, Mulcahy, & Wilson, 2007; Adelman & Vogel, 2003; Cass et al., 2003). Almost one-half of the children who are identified with severe mathematics difficulties in the fourth grade are still classified as having serious mathematics difficulties three years later (Gersten & Jordan, 2005; National Center for Learning Disabilities, 2006; Swanson, 2007).
>
> It should be emphasized that not all students with learning disabilities or related disabilities encounter difficulty with number concepts. In fact, some individuals with severe reading disabilities do well in mathematics and exhibit a strong aptitude in quantitative thinking.
>
> The identification and treatment of mathematics disabilities have received much less attention than problems associated with reading disabilities (Fuchs, Fuchs, & Hollenbeck, 2007; Gersten, Clarke, & Mazzocco, 2007). For students with mathematics difficulties, the mathematics curriculum in most general education classrooms does not pay sufficient attention to learning differences in mathematics among students. Moreover, the general education mathematics curriculum does not allot sufficient time for instruction, for guided practice, or for practical applications. Further, mathematical concepts are introduced at too rapid a rate for students who have difficulty with math. If students do not have sufficient time to fully grasp a mathematical concept and to practice it before another mathematical concept is introduced, they feel overwhelmed and become confused (Cawley & Foley, 2001; Butler et al., 2003). (Lerner & Johns, 2015, pp. 423-4)

Many changes have been made between the fifth and thirteenth editions of this section of the textbook. Most notably is an explosion in neurological research into learning disabilities that is indicated by the increase in these references cited from two in the fifth edition to fourteen different references cited in the fifteenth edition. While much of the medical discourse that is present in the earlier edition remains in the latest edition, this time it is dominated by this “brain research” which wasn’t present in the earlier edition. MLD is now defined as “a biologically based disorder and is related to difficulties in cognitive processing and brain functioning.” This definition is supported by the latest technology used in neurological research: “Research with functional Magnetic Resonance Imaging (fMRI) shows these cognitive processing dysfunctions.” Moreover, the use of prevalence rates, commonly used in medical discourse for illness and disease, is notably present in the first three paragraphs of the section. Dyscalculia is still mentioned in the latest edition but with an even more explicit medical connotation: “The term dyscalculia is a medically oriented term that describes a severe disability in mathematics with medical connotations" (it is also, strangely, related to brain injury in adults). And like the earlier edition, the word meanings, transitivity and modality give the text an objective and impersonal feeling characteristic of medical discourse. Note also the use of the phrase "identification and treatment of mathematics disabilities" in the last paragraph, as if MLD were an illness to be cured.

Another one of the more notable changes between editions occurs in the title from "Characteristics of Students with Mathematics Disabilities" to "Students With Mathematics Difficulties and Students With Mathematics Learning Disabilities". This change is a reflection of the need by researchers to make a distinction between an inherent (biologically based) mathematics disability versus one caused by predominantly environmental factors (e.g. poor instruction or socioeconomic factors). And the less strict category of mathematics difficulties enables researchers to study a larger group of children who struggle with mathematics (Mazzocco, 2007). This change seems to be an explicit response to clarify some of the inconsistent terminology that is evident in the earlier edition and to acknowledge the challenge in identifying a cause of students' struggles with mathematics.

Another key change is the addition of the last paragraph. Here the authors suggest that part of the cause of students' difficulties in mathematics is institutional in nature: "the mathematics curriculum in most general education classrooms does not pay sufficient attention to learning differences in mathematics among students. Moreover, the general education mathematics curriculum does not allot sufficient time for instruction, for guided practice, or for practical application." It is not teachers who do not differentiate learning enough or do not give sufficient time for instruction, but the curriculum. This is reflected by the grammar of the sentences in the way they personify the curriculum as if it were the agent. It seems the curriculum is to blame for some of the difficulties students have with mathematics. Similarly, the use of the passive in the sentence "Mathematical concepts are introduced at too rapid a rate for students who have difficulty with math" seems to shift some of the responsibility from the teachers (by obscuring agency) to mathematics itself and the nature of the subject as concept laden. Thus, the transitivity of the last paragraph seems to place some of the responsibility for students' struggles with math on the institution of schooling or the discipline of mathematics. This suggestion of "environmental causes" is notably absent from the first excerpt that squarely locates the source of students' struggles within the student. The last clause, "they feel overwhelmed and become confused," seems to position the struggling student as "victim" and mathematics as the "perpetrator."

## CONCLUSION

Although medical discourse is still heavily present in the second excerpt (likely due to recent research on the brain) and is rearticulated as the dominant discourse type, there does seem to be an introduction of some creative discursive mixes. The change in the title and the addition of the last paragraph introduce new discourses that shift some of the boundaries around the discursive practices on MLD—from a medical model that locates the cause of the disability within the individual to one that notes other possible causes located beyond the individual and within institutions. The expansion of the title from "Mathematics Disabilities" to one that includes "Mathematics Difficulties" is a clear acknowledgement that the source of mathematics struggles goes beyond the individual and that environmental factors need to be considered. What was once thought to be an inherent mathematics disability, may well have been caused by something else (e.g. socioeconomic factors) and hence the need for the new category of "mathematics difficulties" to account for this change in thinking. This analysis is thus consistent with the lens of ableism which aims to expose the shifting boundaries often used to box underachieving students (Borgioli, 2008) and reveal the fluid and discursive nature of how a "disability" is defined.

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# STUDENTS' MODELLING PROCESS – A CASE STUDY

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*This article presents a case study in which three grade 8 students worked collaboratively to solve a modelling task. Data indicate that students viewed the problem situation from increasing levels of realistic perspective. At the beginning of the modelling process, there was a disconnect between students' mathematical solution and the original situation and operated with numbers with a minimal understanding of the problem situation. After multiple interventions from the researcher, students were eventually able to draw on their lived-experiences and to view the problem situation from an increasingly real world perspective and generated a realistic mathematical solution.*

## INTRODUCTION

In a 1992 article, Dave Hewitt discusses the concept of "train spotting" – to solve problems by looking for patterns and to predict solutions, to pay attention to the numbers involved in the problem but move away from the original situation. Hewitt suggests that what students seem to lack is not a fluency of their mathematical knowledge but a connection between their mathematical solution and the original situation, and the ability to recognize, access and apply this knowledge to solve everyday problems. This results in problem solvers learning something about the patterns, but not about the original mathematical situation. This disconnection between their solutions and the original problem also contributes to the gap between reality and the world of mathematics.

A widely cited example that has been used to demonstrate this disconnection between students' mathematical solution and the original situation is the bus problem (Carpenter, Lindquist, Matthews, & Silver, 1983):

> An army bus holds 36 soldiers, If 1128 soldiers are being bussed to their training site, how many buses are needed?

Results of the study indicate that roughly 70% of students correctly carried out the division that is required to produce a mathematical solution. However, more than half of these students gave the answer of 31 buses or 31 buses remainder 12, and less than a quarter of these students gave the answer of 32 buses. The results of this study, among others (Greer, 1997; Verschaffel, De Corte, & Lasure, 1994), show that while students often are capable of producing a mathematical solution, they tend to disregard their mathematical solution from a real world perspective. These studies point out students' inability to draw on their lived-experiences to verify the solution from a real world perspective, and the gap between the real world and the world of mathematics.

A possible way to address this gap between the real world and the world of mathematics is through the use of modelling tasks. Modelling tasks are problems situated in the real world. They require students to use mathematics as a tool to produce a mathematical solution to the problem from a real world perspective. The process of which students solve modelling problems can be described by modelling cycles (see Figure 1). Students begin with understanding the situation. They then simplify the situation and create a real model to represent the situation, mathematize the real model in reality into a mathematical model in the world of mathematics, determine a mathematical solution using the mathematical model, and verify the mathematical solution by comparing it to the original situation (Borromeo Ferri, 2006; Mason, 1991). The processes of mathematization and interpretation connect reality with the world of mathematics and are crucial steps in the modelling process. Research has shown that mathematization is one of the steps in the modelling cycle where many students experience difficulty, while interpretation and verification are often lacking in students' modelling processes (Galbraith and Stillman, 2006).

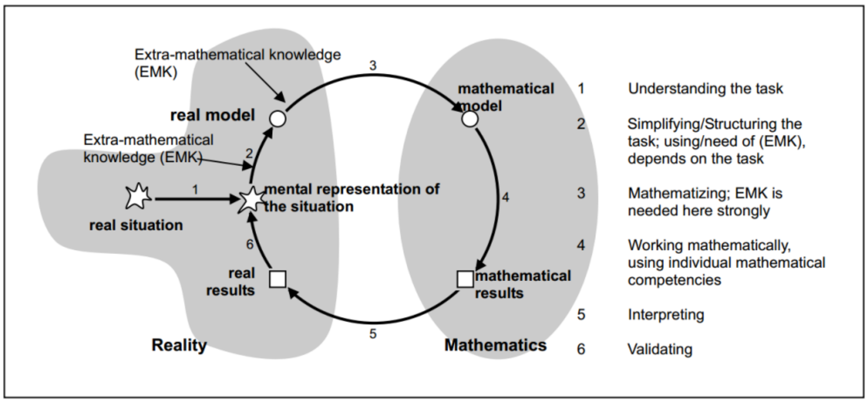


Figure 1: Modeling Cycle proposed by Borromeo Ferri (2006)

This study investigates the process of which students generate a realistic mathematical solution to a modelling task and their ability to draw on their lived-experiences to evaluate the created solution from a real world perspective. In this study, I intend to take a closer look at the mathematization and interpretation steps in the modelling process. Do students collect all information they need prior to building a real model and mathematization? How do students decide what information is required in their real model? Do they modify their real model during the modelling process? If so, how do students decide what modifications are necessary? How do students verify their mathematical solution? How do students decide whether to accept their mathematical solution?

## PARTICIPANTS AND METHODS

This study is part of a larger project which looks at students' developments in mathematical literacy skills through a modelling practice. In this article I present a case study of three grade 8 (age 13-14) mathematics students' modelling process. The three students are Hannah, Kevin, and Betsy[1]. At the time of this study, these students have little experiences with such tasks.

This case study took place in a grade 8 mathematics class, where students were given a modeling task and were asked to work as a group to solve the task, "The Door Problem". Students were asked to pay attention to their thoughts and approaches used as they worked on the problem. Data include in-class observations, field notes, impromptu interviews, post task interviews, and audio recordings of students' work in their group. The following is a summary of "The Door Problem":

> I have recently renovated my home office. As a final touch, I'm going to repaint and decorate my office door. The plan is to paint and then decorate the door with Starbucks gift cards. How much will this project cost me?

The mathematics required for students to complete the task is fairly minimal. Students need to be able to do basic computations, determine the dimensions and surface area of objects, etc. However, in order to be successful, students also need to draw on their lived-experiences and make various decisions and assumptions about the situation. In this case, these experiences include an understanding of the various tools one needs to paint, the area of paint coverage per volume of paint used, etc. In what follows I present these three students' modelling process with a focus on mathematization, interpretation and verification.

## STUDENTS' MODELLING PROCESS

Hannah, Kevin, and Betsy began by assuming the office door has the same dimensions as the classroom door. Instead of measuring the door using a ruler or a meter stick, they measured the door using gift cards, and planned on covering the door entirely with gift cards. This was their first real model.

Upon determining the number of gift cards that fit on the door horizontally (12 cards) and vertically (24 cards)[2], the group quickly determined the number of gift cards that can fit on one side of the door. At this point, students aimed to determine a solution to the problem as quickly as possible. They did not have a very clear direction of where they were going or what exactly they were trying to do. They multiplied 12 by 24 and got 288, and used their estimation/rounding skills to determine the approximate number of gift cards they needed to completely cover one side of the door. This was their first attempt of mathematizing their real model into a mathematical one.

Prior to using the number of gift cards determined to calculate the cost for the project, the group went back to discuss the way the gift cards were placed, and looked up the possible cost for each gift card. Hannah first verified that Kevin measured the door by

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*[1] Pseudonyms are used to protect students' identities.*

*[2] Kevin made a mistake with the measurements. The classroom door is 91.44cm wide and 203.2cm tall. A gift card is 8.57cm wide and 5.40cm tall. Kevin measured the width of the door using the short edge of the card, and the length of the door using the long edge of the card. Based on these measurements, Kevin should be able to fit more than 16 gift cards across the door.*

placing the cards vertically on the door, and soon enough, the group ran into the idea of aesthetic. Since the gift cards have a horizontal design (see Figure 2), Betsy believed that placing the cards vertically on the door would make the design look odd.


Figure 2: A Starbucks Gift Card

The discussion on how the cards should be placed lasted about ten minutes. Meanwhile, students also did some re-measurements. Despite Betsy's concerns, the group decided to place the cards vertically and mathematized their idea and generated a mathematical solution. This was their second attempt of mathematization.

| | |
|---|---|
| Hannah | So we, when put the cards on the door, we think, we are thinking of, we |
| R | So |
| Hannah | Vertical? Yeah. So we did that after that, and then, the bottom line we filled the door. |
| R | Ok, so how many… how many cards? |
| Kevin | Each card is five dollars, and we |
| Hannah | Vertical. This is the price (points at the solution on a piece of paper), so we just put |
| R | Wait a minute, 480[3] times 5 is not 240. |
| Kevin | 2040. |

This is the group's first mathematical solution. At this stage, students believed they needed to fill both sides of the door with gift cards, ignored the cost aspects of the problem, and failed to consider other tools required for this project. They did not verify their solution either. They seemed to care more about finding a possible solution to the problem than a realistic solution to the problem. In order to steer the group to a more realistic solution than the one they have produced, the researcher intervened by pointing out to the group that their current solution is extremely costly and they should reconsider their approach to the problem. This is when Hannah and Betsy realized that they were to decorate rather than to cover the door with gift cards, and that they should consider the cost for painting the door as a part of the solution.

| | |
|---|---|
| Hannah | You don't have to cover the whole door? |
| R | Well I dunno, you decide. |

[3] *The group re-measured the width of the door using the long edge of the card (10 cards), and multiplied 10 by 24 (the number of cards that could fit along the height of the door, measured using the long edge of the card). They then multiplied the answer (240) by 2 (2 sides of the door) to get 480 cards. While they have made a mistake here, this mistake does not affect their final solution to the problem.*

| | |
|---|---|
| Hannah | The paint? |
| Betsy | Wait, it says, huh, paint the door, and then decorate the door. |
| Hannah | Oooohhh… |

As the group gained a better understanding of the problem situation, they went ahead to discuss the possibility of various designs. After about a twenty-minute discussion, the group settled on a particular design and determined the number of gift cards required, the cost of the gift cards, and the cost of the paint required for the project. They then used the information and generated a mathematical solution.

As the group talked to the researcher about their solution, they also wanted to use the opportunity to verify their solution with the researcher. While the cost of the gift cards and the cost of the paint seemed reasonable, the group has not yet considered painting tools and adhesive as a part of the budget. Therefore, the researcher took the opportunity and hinted to the group the needs of these tools.

| | |
|---|---|
| R | Okay, show me, show me the entire details. … show me what colour of paint I'm getting, do I need … br… |
| Hannah | Oooo the brush!!! |
| R | Ahh… the brush … And how am I getting the cards on (the door)? |
| Hannah | Ooooooo |
| Rachel[4] | You need glue! |
| Hannah | Oooooo |

After another ten minutes of discussion, the group added painting tools and a roll of tape to their budget and showed their work to the researcher again. At this point, the students have created a detailed plan and included a can of green paint[5], a paint brush, 28 gift cards, and a roll of tape in their budget. However, they were still missing a few minor details in their plan. During the discussion, Betsy pointed out that they have not yet considered sales tax. In order to cope with this, the group attempted to remove sales tax from their equation by assuming they "live in a tax free country" or "Oregon, USA[6]".

| | |
|---|---|
| Betsy | Did you add tax? |
| Hannah | (In a whiny tone) Why did you… |
| R | Ooo did you add tax? |
| Hannah | Okay… let's say we live in a tax free country? |
| Kevin | Oregon, USA. |

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*[4] Rachel is a student in the same class but in a different group. She overheard the conversation and answered the researcher's question.*

*[5] The can of paint has a volume of 946mL, which is sufficient to paint both the front and the back of the door twice. However, students' solution does not explain how many coats they are painting the door. Therefore, it is difficult to judge whether they fully understood the concept of area of coverage.*

*[6] Oregon, USA is a state in the USA that does not have sales tax.*

After some more discussion, Betsy accepted Kevin and Hannah's idea of removing tax from the real model in order to simplify the situation. In the end, the group further modified their model by replacing the paint brush with a paint roller, and added a paint tray and glue to their budget. They submitted what seems to be a reasonable budget along with a design of the door.

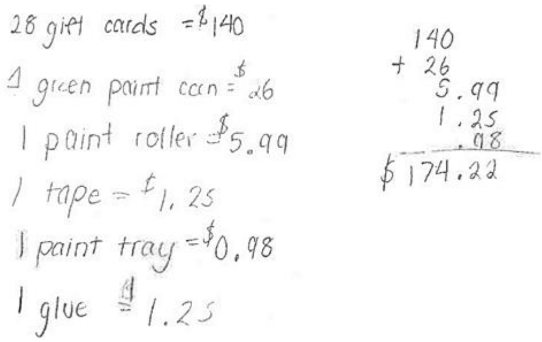


Figure 3: Students' Budget for the "Door Project" – Final Submission

## STUDENTS' MATHEMATIZATION PROCESS

This case study demonstrates novice task solvers' modelling process to create a realistic mathematical solution to a problem. In the beginning, students attempted to create a mathematical solution based on their preliminary understanding of the problem. They began by measuring the dimensions of the door and using these measurements and the cost per gift card to generate a solution. Unfortunately, they focused on a single aspect of the problem (cover the door with gift cards) and were unable to draw on their lived-experiences to help them consider other aspects. This hindered their process and resulted in an unrealistic solution (over $2000). Furthermore, they did not verify their solution by comparing it to the original situation. At this point, it can be said that there was a disconnect between students' solution and the original situation, and students approached the problem from a non-real world perspective. They extracted numerical values from the original problem and operated with these numbers to generate a solution, and ignored all realistic aspects of the problem. Students were disconnected from the original situation. This is similar to Hewitt's (1992) "train spotting" concept.

After the researcher's intervention, students went back to the original situation and reconsidered the situation from a slightly more realistic perspective than before, and created a design and included paint in their real model. After they mathematized their real model and generated a mathematical solution, they verified their solution with the researcher. In this sense, students completed the modelling cycle for a second time, and were able to interpret the situation based on the context provided in the problem, which is a slightly more realistic perspective than before. However, they were not yet able to incorporate things that were not mentioned in the problem into their solution.

As the researcher attempted to push the group to further improve their solution, students modified their real model by adding painting tools to their budget. They were now able to expand their view of the situation to include things that were not mentioned in the original problem in their real model when prompted. The process of modifying the real model, mathematizing, and determining a mathematical solution happened very smoothly and naturally. This may be a result of the nature of the task. Unlike other modelling tasks, the mathematical skills and calculations involved in "The Door Problem" are very rudimentary. Students naturally went ahead and determined the mathematical solution after building the mathematical model and mathematizing the real model. Therefore, there seems to lack a clear distinction between mathematical model and mathematical results in this case study.

As students discussed their solution with the researcher for the final time, one of the students drew on her lived-experiences and pointed out the problem with tax. Afterwards, these students further modified their real model and improved their mathematical solution based on their knowledge about painting. This was the first time the group was able to spontaneously draw on their lived-experiences and added other realistic aspects to their solution without the researcher's help.

To summarize, data in this case study demonstrate students' multiple attempts to generate a realistic mathematical solution to a problem based on their multiple levels of realistic perspectives of the problem. In the beginning, students viewed the situation from a rather non-realistic perspective and extracted numbers from the situation to create an unrealistic mathematical solution. As the researcher intervened and as students' understanding of the situation deepened, students used the context given in the problem to solve the problem from an increasingly realistic perspective. After further interventions, students used some of their lived-experiences to improve their solution. Eventually, students were able to spontaneously draw on their lived-experiences in attempts to create a realistic solution, and were able to view the problem from a real world perspective.

The mathematics required to produce a realistic mathematical solution to this problem is rather rudimentary. However, students are required to bring in their lived-experiences to make proper assumptions about the solution in order to be successful in creating a realistic solution. In this case study, these three students relied mostly on the researcher to help them draw on their experiences in order to produce a realistic solution. When asked about this, students claimed that it is not that they do not have these experiences, but they were rather overwhelmed by the situation and were only able to focus on the mathematical procedures. They claimed that although they all have some sort of painting experiences, they were not able to draw on these experiences to help them solve the problem until they were reminded of these experiences.

## CONCLUSION

Novice task solvers go through the modelling process multiple times to generate a realistic solution. This result is in line with Stender and Kaiser's (in press) study. Data in this study shows students' ability to view the problem from an increasingly realistic perspective as their understanding deepened and as they were reminded of the realistic aspects of the situation. Since this article presents results of a case study and includes only a small sample of participants, further research is required before any generalizations can be made. Further research areas include a deeper investigation into the researcher's role during the modelling process, students' multiple levels of real world perspective, and other possible reasons behind students' inability to draw on their lived-experiences during the modelling process.

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# UNDERGRADUATE STUDENTS' PERCEPTION OF TRANSFORMATION OF SINUSOIDAL FUNCTIONS

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*Trigonometry is one of the fundamental topics taught in high school and university curricula, but it is considered as one of the most challenging subjects for teaching and learning. In the current study, two different theoretical frameworks have been used to examine undergraduate students' perception of transformation of sinusoidal functions. The results show that students did not grasp fully the concept of transformation of sinusoidal functions.*

## INTRODUCTION

Trigonometry has a long history. Many ancient people used trigonometry for different purposes. For example, Egyptians applied trigonometry to determine the correlation between the lengths of shadow of a vertical stick with the time of day (see Figure 1). Astronomers also used trigonometry to find longitude and latitude of stars, as well as the size and distance of moon and sun. However, it was not until a Persian Mathematician named Khwarizmi introduced trigonometric functions to the word. Since then, trigonometry is one of the main topics in mathematics. Students often are required to assign time for learning trigonometry, especially trigonometric functions, because a strong foundation in trigonometric functions will likely strengthen their learning of various mathematical topics, such as Fourier series and integration techniques (Moor, 2010). It is shown that understanding calculus and analysis is dependent on learning of trigonometric functions (Hirsh, Weinhold & Nicolas, 1991; Demir, 2011). However, research studies (e.g. Weber (2005)) show that learning and understanding trigonometric functions is a difficult and challenging task for students, compared to other mathematics functions, such as polynomial functions, exponential and logarithmic functions.

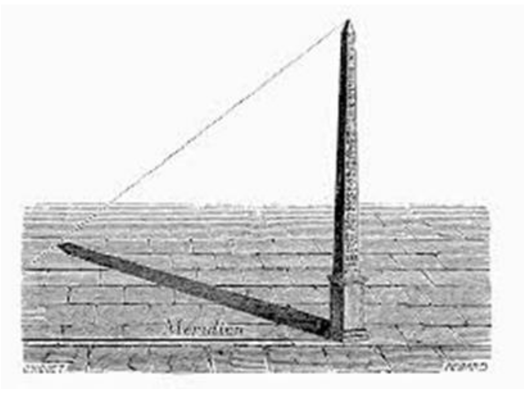

Figure 1: Egyptians' use of trigonometry

Kendal and Stacey (1997), for instance, assessed 178 high school students' understanding of trigonometric functions through two different methods: ratio and unit circle. The research shows that students had a better understanding of trigonometric functions where functions were defined in terms of ratios of sides of right angled triangles (students often taught SOHCAHTOA as a memory aid), rather than with the unit circle method. These results are consistent with those of Palmer's study (1980) in which the students were randomly assigned either a ratio or unit circle instruction to inspect which group grasped more trigonometric functions. The students in the ratio classes outperformed their unit circle counterparts. Burch (1981) also concluded that the participant students had difficulty interpreting trigonometric functions in the unite circle, recognizing that x and y coordinates of a point on the unit circle are cosine and sine values of corresponding angles compared with other determined trigonometric functions in terms of right triangle. However, the results of Kendal and Stacey's (1997) study were in contrast with the finding of Weber's (2005) study in which students performed more successfully when they were learning trigonometric functions in the unit circle than right triangle.

Brown (2005), Demir (2011) and Marchi (2012) suggest that students would better understand trigonometric functions if they have more opportunities to take part in a learning trajectory where both the unit circle and right triangles are utilized, rather than learning trigonometric functions through instructional designs which focused on only one of these two methods. To examine high school students' understanding of trigonometric functions, Brown (2005) developed a model of students' understanding of trigonometric functions within the geometry world of triangles and angles (in degrees), and within the context of the unit circle. In this model, the sine and cosine of an angle can be defined in three different ways: as ratios, as distances, and as coordinates. The results of Brown's (2005) investigations of 120 high school students revealed that those students who were able to use and connect all interpretations ( ratios in right triangle, directed distances and coordinates) and moved flexibly between them, could define trigonometric functions and were also better problem solvers. However, Brown (2005) noticed that the majority of the students could still define trigonometric functions and work effectively with only one view, instead of making connections between all three representations. Brown concluded that most students had an incomplete understanding of trigonometric functions.

In spite of all efforts has been done in the area of trigonometry, especially trigonometric functions, still there has been some gaps in the literature review. There is not research study focus on "how students transform sinusoidal function" and "what students' perception of transformation of sinusoidal functions is". As such, the main goal of this study is investigating answer for the above mentioned questions.

## DATA COLLECTION AND ANALYSIS

The participants in the study were three undergraduate students, three females, from a large North American university. The participants were selected from among students who had either completed calculus I course and were enrolled in calculus II (1 students) and in calculus I course at the time of the interview (1 students) in the Mathematics Department. Another student finished calculus courses (Calculus I and II) and she was enrolled in algebraic courses. The three participants were students who had volunteered to participate in the study after I had made a general request to the whole class.

A 60-minutes task-based interview was conducted and students were required to complete two types of interview tasks:

**1)** Recognizing sinusoidal functions (the tasks were: Task 1: $f(x) = sin(2x)$, Task 2: $f(x) = sin(\frac{2}{3}x)$ and Task 3: $f(x) = cos(\frac{2}{5}x - \frac{\pi}{5})$).

**2)** Assigning coordinates (Task 4: $f(x) = sin(4x)$ and Task 5: $f(x) = cos(3x - \frac{\pi}{4})$**).** For the first type of tasks the sketches indicating the sinusoidal graphs were given and students were required to identify the transformed sinusoidal functions represent the given graphs (see Figure 2). For the second types of tasks, a particular transformed sinusoidal functions were given along with a wavy displace (see Figure 3) and students needed to assign coordinate on the wavy curve in a way that describe the given function.

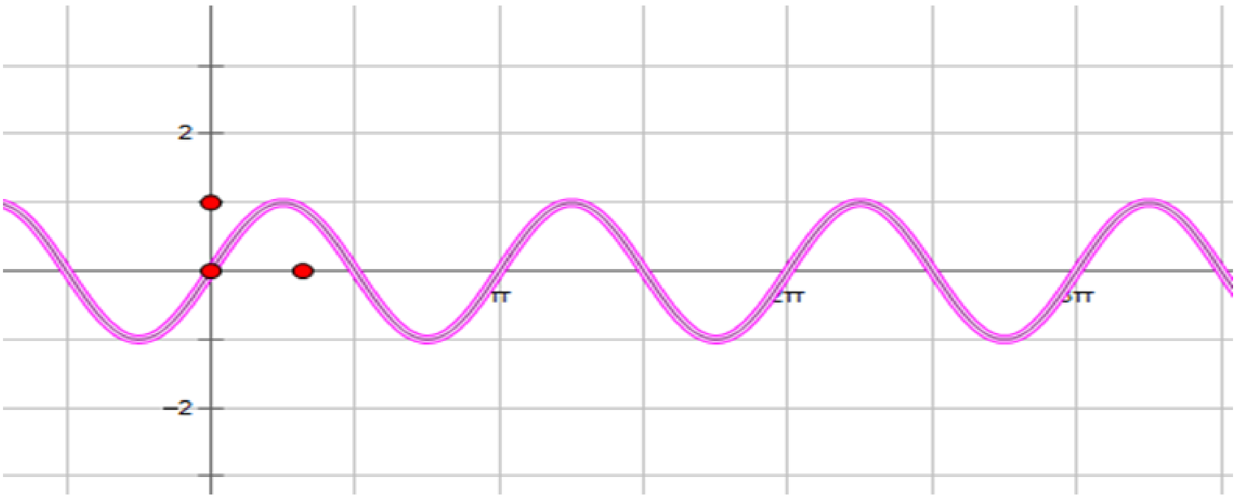


Figure 2: Snapshot of task 2

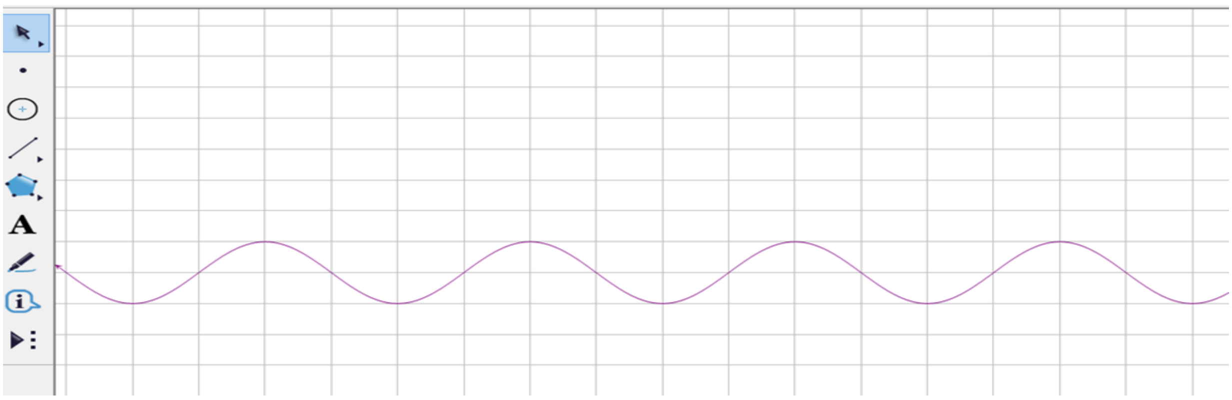


Figure 3: a wavy displace for the second type of tasks

The collected data in this study were analyzed and interpreted according to two theoretical frameworks. Mason's theory of shifts of attention is first framework to be used. Mason's theory provides opportunity for the researcher to study the critical role of attention and awareness in learning and understanding mathematics and in particular the concept of transformation of sinusoidal functions. Mason divided attention structure in to 5 categories: 1) holding wholes; 2)discerning details, 3)recognizing relationships; 4)perceiving properties; and 5)reasoning on the basis of agreed properties. Mason's theory provides opportunities to investigate whether students focus their attention, for instance, on detail of the transformed sinusoidal functions/graph, or they attempted to find relationship between symbolic and graphical representation. The second theory is Presmeg's (1989) visualization theory. She identified five different category of visual imagery: 1) concrete, pictorial imagery; (2) pattern imagery; (3) memory images of formulae; (4) kinesthetic imagery; and (5) dynamic (moving) imagery. As a broad theory, it is used for investigating students' visual mental constructs since the participants applied their imaginary skills in different occasions when they completed the interview tasks. Visualization will allow investigating mental process in the mind of participants which can be identified through drawing on the paper, or manipulating a diagram on the computer screen.

Reviewing students' answers and reading the transcripts lead the researcher into two main themes: "recognizing the period (B)" and "recognizing the phase shift (C)" (. Please note that in all the 5 interview tasks the participants were required to connect the period of the given sinusoidal function or the sine curves to a coefficient of x in a sine or a cosine function. For brevity, we refer to this connection as "recognizing the period" (see Figure 4). In the Tasks 3 and 5 the participants were required to recognize the phase shift/horizontal shift. From the canonical function ( the phase shift is obtained by determining the change being made to the x-value (see Figure 5). The displacement will be to the left if the phase shift is negative and to the right if the phase shift is positive. For brevity, we refer to this connection as "recognizing the phase shift".

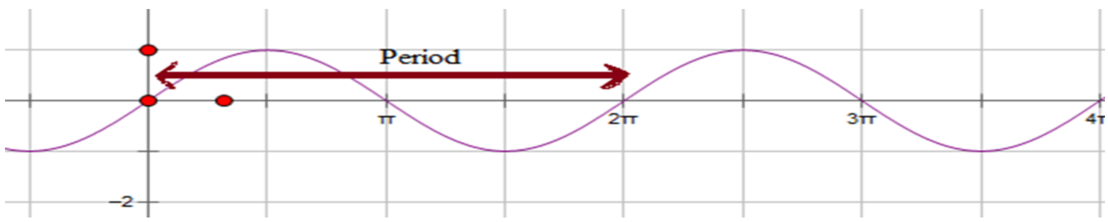


Figure 4: Recognizing the period

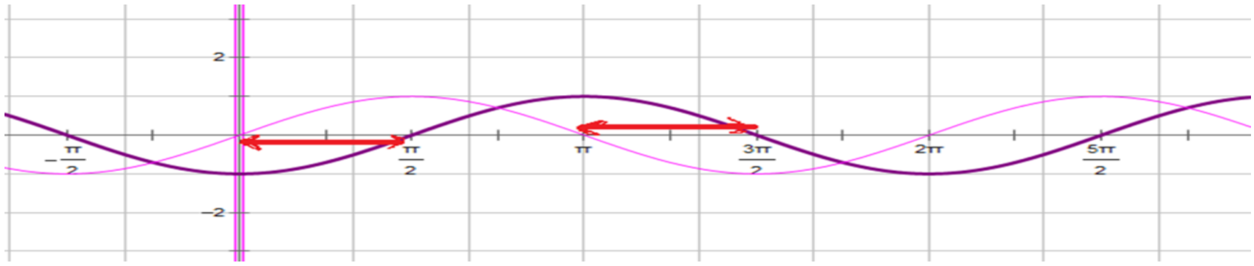


Figure 5: Recognizing phase shift/horizontal shift

**Recognizing period**

When students completed mathematics Task 1, they all initially were confused and they all agreed that the function described the given graph (see Figure 2) was $f(x) = sin(\frac{1}{2}x)$ For instance, Rose expressed:

> "…It is sine graph, because it starts at 0. it should be $f(x) = sin(\frac{1}{2}x)$ . The sine graph starts at 0 and then π and 2π but this one is o, π, 2π /3. This is half of sin graph... The period here is π while it is 2π in the original sine curve."

Emma also stated:

> "….because a normal sine graph is between 0 and 2π but in this one, the graph is compressed by half so it means that it is going to be $f(x) = sin(\frac{1}{2}x)$ "

As it is clear from the above excerpts, the students focused their attention on the details embedded in the given graph and they tried to fin relationship between the graphical representations with the symbolic representation of the transformed function. They were using pictorial imagery of canonical sinusoidal functions *("…because a normal sine graph is between 0 and 2π")*. Sally also used her Kinaesthetic imagery (see Figure 6) while she attempted to described the function, although similar to the other students she was unsuccessful.

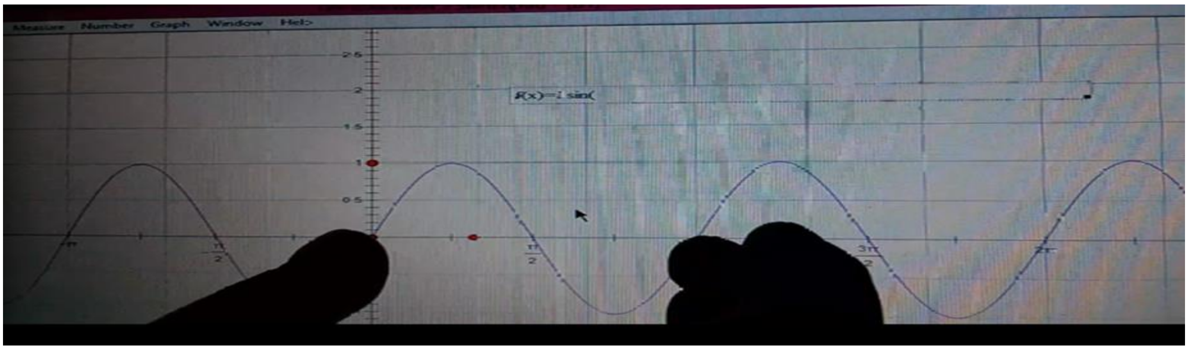

Figure 6: Sally's kinaesthetic visualization

After observing the graph of $f(x) = sin(\frac{1}{2}x)$ students noticed their mistake and they all found a correct transformed function for the given graph. Since then, for the rest of interview tasks (such as Task 4) when the coefficient of x in the transformed sinusoidal function was a whole number, students successfully completed the tasks. However, in other tasks such as Task 2 and 3, only one student was able to recognize the period successfully. Emma stated:

> "…the graph should go over $2\pi$… it is f(x)= sin(2/3x) because the period is $2\pi$/B and the period ends here…here is $3\pi$"

As it is clear from the statement, Emma tried to discern details from the given graph to identify the transformed function. She also used her memory image of formula ($2\pi$/B) in order to find period. Besides, identifying period, students needed to recognize phase shift as well.

**Recognizing phase shift**

Tasks 3 and 5 involved phase shift. However, none of the participants successfully realized phase/horizontal shift. As an example, for task 5 (f($x$)=$cos(3x - \pi/4)$) Sally expressed:

> " … because the amplitude is not changing so here is 1 and -1… original period is $2\pi$…we have transformation on x to the right direction. So we move the graph over by $\pi/4$…"

While she focused on the properties of canonical sine function (*the amplitude is not changing so here is 1 and -1… original period is $2\pi$),* she was unable to recognize $\pi/12$ as a proper horizontal shift. In other words, if one thinks about the standard formula of transformation of sinusoidal function: $f(x)=sin/cos(B(x \pm C)+D)$ and $f(x)=sin/cos(Bx \pm BC)+D$, while BC correspond with phase shift, she recognize it as "C." It was not only Sally who found phase shift incorrectly; the other two participants were unsuccessful as well.

## RESULTS

Result of the study show that the three students participated in this research study focused their attention mostly on the: discerning details from the given transformed sinusoidal function or graph, finding relationships between the graphical representation and symbolic representation of the transformed functions, and perceiving properties of sinusoidal functions while transformed sinusoidal functions. They also relied on concert imagery of the canonical sinusoidal functions, pattern imagery of the transformed functions and kinesthetic imagery. However, data show that students were unsuccessful in realizing period when the coefficient of x was not a whole number in the transformed sinusoidal function, although they were able to identify period when the coefficient of x was a whole number. The data reveled that none of the participants successfully realize proper phase shift from the given transformed function or their corresponding graphs.

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# COHERENT AND DIVERGING DISCOURSE IN MATHEMATICAL ACTIVITIES WITH DYNAMIC GEOMETRY

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*This paper discusses the detection of coherent and diverging discourse in mathematical exploratory activities using dynamic geometry environments (DGEs). The data is drawn from a larger study examining patterns of communication within pair-work mathematical activities using touchscreen DGEs. A thinking-as-communicating approach is used to analyse communication involving pairs of high school calculus students' exploration of the area-accumulating function using touchscreen DGEs. Results show that the students integrated speech, gestures and touchscreen-dragging synchronically and diachronically to engage in coherent and diverging discourses. In particular, it is shown that conflicting ideas can be communicated non-linguistically. This paper raises questions about new forms of communication mobilised by touchscreen DGEs.*

## INTRODUCTION

During the past decade, the evolution of digital technology has resulted in an integrated system for communication and representational expressivity, opening possibilities for social mediation of ideas (Hegedus & Moreno-Armella, 2014). In particular, the introduction of digital technologies in schools has given rise to new ways of doing and representing mathematics. Recently, a number of research has focused upon the haptic and dynamic nature of touchscreen digital technologies and how this affords new modes of thinking and learning (see for example, Falcade, Laborde & Mariotti, 2007; Sinclair & de Freitas, in press). Related to this, my previous work drew on the effect of dynamic geometry environments (DGEs), particularly when presented on touchscreen devices, for facilitating a multimodal communication about calculus. One of my significant findings is that touchscreen-dragging on DGEs enabled students to communicate temporal relationships with their fingers, where verbal communication was also transformed in the presence of dragging (Ng, 2014). As touchscreen technology continues to enhance digital experience of learners, its affordances for mathematical learning, especially as it is complemented with DGE affordances, is an important area of study. There is a need to understand what mathematics is communicated and how it is communicated in an era of touchscreen DGEs. This study focuses the latter, by investigating high school students' discourse in mathematical activities with DGEs. More specifically, this study addresses how coherent and diverging discourses are communicated in mathematical exploratory activities with DGEs.

## THEORETICAL FRAMEWORK

The study adopts a *learning as participation* perspective that considers learning mathematics as a social activity (Lave & Wenger, 1991). This perspective suggests that learning is located neither in the heads nor outside of the individual, but in the relationship between a person and a social world. Sfard's communicational framework (2008) is based upon the learning as participation perspective and highlights the communicative aspects of thinking and learning. Sfard (2008) uses the term *commognition* to encompass thinking (individual *cognition*) and interpersonal *communication* as manifestations of the same phenomenon.

Sfard theorises mathematical thinking as the process of individualising or developing one's mathematical discourse. She proposes four features of the mathematical discourse, *word use, visual mediator, routines, and narratives*, which could be used to analyse mathematical thinking and changes in thinking. For the purpose of this paper, the first three features will be used for analysing the use of language, gestures, and dragging in one's the mathematical discourse. *Word use* is a main feature in mathematical discourse; it is "an-all important matter because […] it is what the user is able to say about (and thus to see in) the world" (p. 133). As a student engages in a mathematical problem, her mathematical discourse is not limited to the vocabulary she uses. For example, her hand-drawn diagram and gestures can be taken as a form of *visual mediator* to complement word use. *Routines* are meta-rules defining a discursive pattern that repeats itself in certain types of situations. In learning situations, teachers may use certain words or gestures repeatedly model a discursive pattern, such as looking for patterns and what it means to be "the same". Hence, gestures can be taken as both a routine for defining a discursive pattern that repeats itself, and a visual mediator in a student's discourse. A student may use gestures to reason the change of tangent slope, in which it becomes a routine, or to mediate the variance of the tangent slope, in which it becomes a visual mediator. This communication can be interpersonal when it is directed to another student or intrapersonal when it is directed to oneself. According to Sfard (2009),"Using gestures to make interlocutors' realizing procedures public is an effective way to help all the participants to interpret mathematical signifiers in the same way and thus to play with the same objects" (p.198).

Sfard conceptualises learning mathematics as a change in one's mathematical discourse. This can be observed through one's word use, visual mediator, routines, and narratives. Diverging discourse arises as inconsistent words, visual mediators, routines or narratives are used in communication. Communicating in diverging discourses may lead to *commognitive conflicts*—the encounter between interlocutors who use the same mathematical signifier in different ways or perform the same mathematical tasks according to different rules.

## METHODOLOGY

The study was part of a larger research project aimed at studying high school calculus students' patterns of communication during pair-work mathematical activities using touchscreen DGEs. The participants are five pairs of calculus students enrolled at a culturally diverse high school in Western Canada. In this paper, I focus on the data concerning an exploratory activity about the area-accumulating functions. The purpose of the present study is to investigate how coherent and diverging discourses are communicated in mathematical exploratory activities with touchscreen DGEs. To answer this research question, the participants needed to have some prior experience working in an exploratory setting with touchscreen DGEs. At the time of study, the participants had just completed the differential calculus component of the course, where key concepts in differential calculus had been taught with consistent use of a class set of touchscreen DGEs in their regular calculus classroom. During these lessons, students were guided by the classroom teacher to explore and discuss calculus concepts as presented on the DGEs in pairs before the traditional lectures are given.

The task took place outside of school hours in the participants' regular classroom. It invited the five pairs of participants, who were regular partners in their class, to discuss a pre-designed sketch presented on the touchscreen DGE application, SketchExplorer (Jackiw, 2011). The sketch contained five pages and was designed to facilitate exploration of the area-accumulating function, $A(x)={}_a\int^x f(t)dt$. As the participants had yet to learn this in their regular classroom, the goal of the sketch was to introduce the idea of "area as a function", and this can be achieved when the students were able to relate the set of "green traces" as the graph of area under 'f' from 'a' to 'x' (Figure 1). The researcher turned on the videotaping function of the camera facing the student-pairs and left the room, until the students finished exploring and discussing. Each student-pair took between 30 to 45 minutes for completing the task.

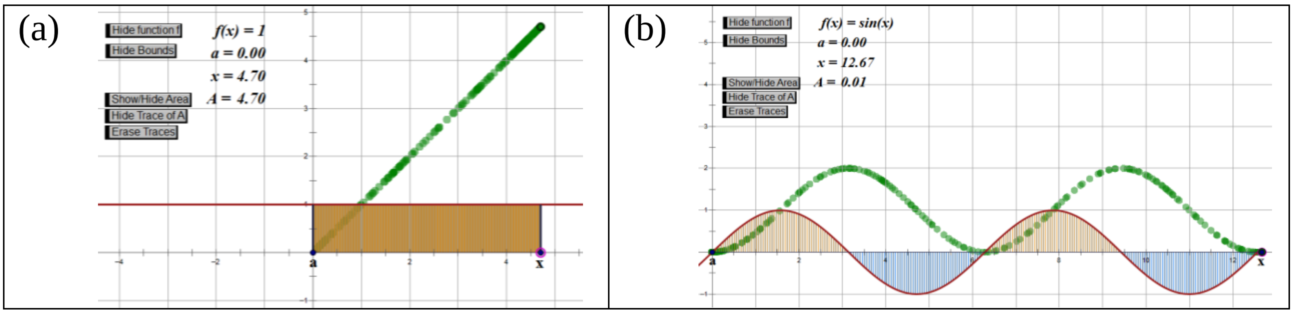



Figure 1(a) A page representing the area-accumulated under f(x)=1, a=0 and (b) under f(x)=sinx, a=0. As 'x' is dragged, area $A(x)=a\int x f(t)dt$ is traced dynamically in green.

As it was necessary for this study to capture student pairs' utterances, gestures, and dragging actions while interacting with DGEs, special methodological decisions are made around the transcribing of data and its organisation. Unlike conventional transcripts which informs only "who said what", I included a record of "who gestured what" and "who dragged what" in the transcript. As a transcript convention, certain parts of the transcript were underlined and double-underlined to keep track of the words spoken while a gesture or dragging action was performed simultaneously by one of the students. In addition, two kinds of analyses, synchronic and diachronic, were carried out in order to address the coherency of discourse in the student pairs' communication. According to Arzarello (2006), these two kinds of analyses are needed to fully understand a semiotic activity. Synchronic analysis enables the study of relationship among different semiotic sets activated simultaneously. Diachronic analysis studies the same phenomenon in successive moments. To this end, a synchronic lens is used to analyse the coherency of the participants' discourse at one moment of calculus communication, and a diachronic lens is used to observe how participants respond to each other successively using speech, gestures and dragging, and whether diverging discourses have arisen over time.

## RESULTS

Data from two student pairs were selected in this paper to exemplify the multimodal communication used generally by all student pairs, and in particular, to illustrate characteristics of coherent or diverging discourse during the activity.

### From coherent to diverging discourses

The transcript below was taken between Larry and Ivy's discussion while they were on the page with f(x)=1. At Turn 15, the students were 2 minutes into their discussion.

| | | |
|---|---|---|
| 15 | Larry: | So, 'A' is probably the <u>dotted line, right</u>? Oh, so like it's like <u>from here, to here.</u> <u><Larry drags 'x'></u> |
| 16 | Ivy: | Oh. Oh, <u>so that's the area.</u> <u><Larry drags 'x'></u> |
| 17 | Ivy: | <u><Ivy drags 'x'></u> <u><Ivy gestures></u> 'Cuz... one and one...one times one is one... <Ivy drags 'x'> one times two is two...one times three is three...so that's... I guess that slope is the area of that. Shaded. <Ivy taps Larry's arm> Try the next one. |
| 18 | Larry: | Oh. Oh, <u>so that's the area.</u> <u><Larry drags 'x'></u> |

At Turn 15, Larry suggested that 'A' was "probably the dotted line," while dragging 'x' around. Then, while Larry was still dragging, Ivy responded verbally, "Oh. Oh, so that's area" (Figure 2a). Her use of pronoun "that" seemed to be referring to Larry's "dotted line" in his previous utterance, which would mean that Ivy was communicating the dotted line as area. At Turn 17, Ivy took on the role of dragging and used a series of gestures while speaking of "area" numerically. She talked of three different moments of calculating area under f(x)=1 from a=0 to 'x' as she dragged 'x'. She first dragged 'x' to x=1. Then, while she said "one times one is one," she used her middle finger to point towards the draggable point 'x', which was at (1,0) at the time, followed by pointing towards the point (1,1) (Figure 2b). Next, she dragged 'x' continuously from x=1 to x=3 and uttered, "one time two is two, one times three is three" (Figure 2c). It was clear that Ivy was stating how the area of the rectangles could be calculated numerically. The timing of her utterance corresponded to the very state of the rectangle as 'x' was dragged continuously, showing that she was coordinating her dragging and speech. She added the word "shaded" at the end of, "I guess that slope is the area of that," to clarify that she was talking about the shaded area. Ivy ended her utterance with "Try the next one. 'Kay". The significance of this utterance was that Ivy asked Larry to turn the page, and Larry agreed to do so. Not only did he turn the page at Ivy's request, Larry uttered the exact same words "so that's area" (Turn 18) in response to Ivy's explanation. All this suggest that the two students were "on the same page" both literally and figuratively—in that they were both ready to move onto the next page.

| (a) Ivy' gesture accompanied by the utterance, "Oh. Oh, so that's the area." | (b) Ivy points at the point (1,0) and then (1,1) while uttering, "one times one is one". | (c) Ivy drags 'x' from x=1 to x=2 and then x=3, all the while speaking of the calculation of area. |
| --- | --- | --- |
|  |  |  |

Fig. 2 (a-c) Selected snapshots of Larry and Ivy's gestures and dragging (Turns 15-18)

A minute later, while they were on Page 2, which showed the function f(x)=x, the students noticed that the sign of 'x' and 'y' affected the sign of the "area". Turns 23 to 24 briefly illustrates the students' discourse about the sign of the "area".

23 Ivy: Mmm... So yeah, the area becomes negative because that's a negative 'x', right? <Larry drags 'x'> <Ivy gestures>

24 Larry: Mmm… So when it's at zero, it'd be zero. <Larry drags 'x'>

The students talked about "area", in particular, the sign of "area" in the above transcript. At Turn 23, Ivy's speech accompanied Larry's prolonged dragging of 'x'. In this utterance, Ivy used the conjunctive "because" to suggest a causal relationship between the sign of the area and the sign of 'x'. Perhaps, she had noticed that when the sign of 'x' was negative, the area under the function f(x)=x from a=0 to 'x' was negative. Turn 24 showed Larry's response to Ivy. He first stopped dragging and said, "Mmm." Then, he resumed dragging of 'x', dragged it towards x=0, and said, "So when it's at zero, it'd be zero." It was suggested that Larry was naming the two mathematical objects that Ivy had talked about previously with the pronoun, "it". This would mean that he was communicating the idea that, "when ['x'] is at zero, [the area] would be zero". Thus, his discourse integrating speech and dragging of 'x' towards zero was coherent with Ivy's discourse.

Recall that Larry and Ivy were in agreement in the beginning about when to turn to the next page. In contrast, at Turn 39, the two students showed disagreement when Larry wanted to turn from Page 3 (with $f(x)=x^2$) to Page 4 (with f(x)=sinx):

39 Larry: I guess. K, and then there's sine. <Larry drags 'x'> <Ivy gestures>

40 Ivy: Wait, wait. <Larry drags 'x'>

As shown in Turn 39, Larry was ready to turn the page when he said "K, and then there's sine," as he placed his finger on the page tab "Area under sine." Apparently, Ivy was not ready to move on, as she said, "wait, wait". This meant that she was hoping to stay on the page and perhaps to find out something that Larry did not. Moments before this, Larry and Ivy had agreed that the green traces on this page were of "third degree". Perhaps Larry was satisfied with this discovery, but Ivy was not. In fact, it was observed that their thinking began to take different turns from this point forward. Larry and Ivy began to develop diverging discourses after Ivy's "wait, wait". More specifically, Ivy seemed interested in finding invariance across all pages, but Larry did not seem interested in pursuing it. For example, towards the end of the activity, Ivy noticed something "strange", while Larry seemed satisfied with their discussion. The students left their discussion on that note at the end of the mathematical activity.

**Commognitive conflicts communicated through dragging**

The previous shows the change of Larry and Ivy's discourse from a coherent to a diverging one at the end of their exploration. The analysis to follow capture the use of touchscreen-dragging as a mode of communication, in particular, that gave rise to commognitive between Huang and George.

At the 5-minute mark of their exploration, Huang and George exploited the multi-touch function of the DGE to drag 'a' and 'x' simultaneously and respectively. After 5 seconds of dragging simultaneously and silently by the two students, Huang broke silence with his question, "What are you doing George?" Since both students had already been dragging for 5 seconds, it seemed unlikely that Huang had just realised that George was also dragging alongside him, and so this analysis did not regard his question, "what are you doing", in a literal sense. The situated meaning of this question seems to be that Huang was in disagreement with what George was doing. There might have been a change in his thinking from 5 seconds ago, since he had not raised this question back then. It also suggests that there existed a conflict in the students' discourse, even though they were not actually speaking. In Sfard's terms, this commognitive conflict, as reflected in the students' dragging routines, led to Huang's question about what George was doing.

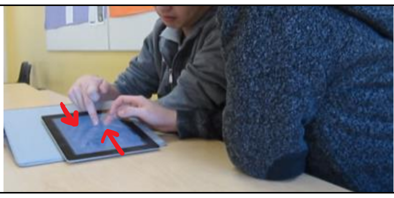
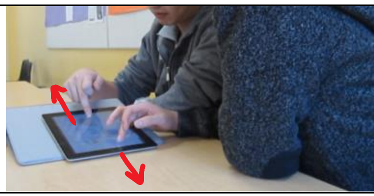
| (a) Huang and George each dragged a point on the iPad towards each other | (b) Huang and George each dragged a point on the iPad away from each other |
|---|---|
|  |  |

Figure 2 (a,b). Huang and George dragged 'a' and 'x' respectively for 5 seconds before Huang broke silence, "what are you doing, George?"

Upon Huang's question, George refrained himself from dragging and, in so doing, let Huang be the sole dragger on the iPad. George's response could be taken as his attempt to resolve the commognitive conflict in the students' dragging routines, by voluntarily refraining himself from actively participating in the mathematical activity. Hence, George watched still as Huang dragged '*x*' for another 9 seconds, twice changing his direction of dragging. Importantly, the present analysis shows that commognitive conflicts could be present in non-verbal communications. The students might have communicated diverging discourse through their dragging routines as indicated by Huang's remark. The analysis shows that George's means to resolve the conflict was also non-verbal. Therefore, it can be said that the students had relied on non-verbal communications for developing their mathematical discourses during this episode.

## DISCUSSION

This paper raises important questions about the way coherent or diverging discourses are communicated in pair-work and group-work mathematical exploratory activities using DGEs. The analysis shows how different attention towards the DGE can serve to diverge two students' discourse. The student pairs were told to discuss what they saw in the sketch containing five page tabs: the different pages designed in the sketch contributed to the student pairs' agreement and disagreement as they had to decide when to move on to a new page or go back to a previous one. For example, in Larry and Ivy's early exploration, Ivy tapped Larry's arm to signal Larry to turn the page, which was agreed by Larry. In contrast, moments after they turned to Page 2, Larry and Ivy showed disagreement when Larry tried to turn to the Page 3 and Ivy's called out, "wait". Using a communicational framework, agreeing on when to turn a page can reveal what the student/pair was attending to, especially by observing the student/pair's discourse after a page was turned. Therefore, these moments can provide useful pointers as to what a student/pair is thinking mathematically. In Larry and Ivy's case, Ivy was likely interested in finding more about what she later called "strange" about the area-accumulating function, yet Larry was not interested in pursuing it but more interested to move on to the next page. Indeed, as shown in a later analysis, Larry seemed quite satisfied with being able to state the sign of A(x) and was not interested in exploring the shape of A(x). This suggests the students' discourses have diverged at the end of their exploration.

Besides the action of turning a page, the student pairs also used particular words to communicate in coherent and diverging mathematical discourse. There were ample evidence showing that the student pairs complemented each other's communication such as finishing their partners' utterances verbally. Besides word use, their gestures and dragging complemented their partners' also. The simultaneous and complementary use of dragging and speech by two different persons indicates a coherent way of mathematical thinking by both discussants. Hence, the student pairs were engaging in one coherent discourse as opposed to two divergent ones. In contrast, at one point, Huang and George were engaging in conflicting discourse while both students were dragging simultaneously. After dragging 'a' and 'x' respectively for five seconds simultaneously, Huang asked George, "what are you doing," implying that he did not understand what George was doing. This means that a commognitive conflict may have arisen.

In summary, I combine Sfard's commognitive approach to enrich the understanding of how student-pairs engage in coherent and diverging discourse within mathematical exploratory activities with touchscreen DGEs. In particular, I identified the kinds of words, gestures and touchscreen-dragging that were characteristics of the coherent and diverging discourses. Consistent with the findings of the larger research that this study was situated, a multimodal view of communication was essential for framing the results of the study. Examining student pairs' speech, gestures and dragging synchronically provides information on whether or not they are communicating coherently at the given moment, while a diachronic analysis helps to see whether or not their discourse has diverged during the course of the activity. Diverging discourse may be detected by a disagreement of when to turn the page, questioning the action of one's partner, non-complementary use of speech, gestures, and dragging, as well as the use of pronouns in singular form ("I" or "you") as opposed to a plural form ("we"). This developed framework for understanding mathematical discourses within exploratory activities with DGEs is especially relevant for classroom teaching because it informs when a pair or group is making progress in their development of the mathematical discourse collectively or individually. In particular, this framework can be used by classroom teachers to decide when intervention is needed to help bring the discourse back in sync, which is what Wells (2014) terms, "teaching from the sideline". Future research should consider adopting a multimodal communicational framework to examine the coherency of discourse in other mathematical activities, and those situated in other tool-based learning environments to address the role of tools in thinking and learning.

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# IDENTITY AND MATHEMATIS: A PRELIMINARY EXPLORATION

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*This preliminary exploration considers the Affinity-identity students have with mathematics. The relevance of this became apparent as I noticed my own Affinity-Identity reaffirm my growth into the role of a mathematics education researcher. I explore possible ways to identify affinity and develop a picture of how identity of the learner and the subject to be learned can be identified. Development of identity is a communicative practice situated in cultural experiences and therefore an analysis of both praxis and individual communication could offer deep insight into influences on student identity as they seek affinity with the identity projected onto mathematics.*

## IDENTITY AND MATHEMATICS

At the intersection of mathematics and student[7] learning of mathematics there exists a boundary where mathematical knowledge is transferred through communication (verbal, physical, written and symbolic). I suggest it critical to study the tangible yet elusive gap between student identity and the identity assigned to mathematics. The praxis (Grootenboer, 2013) of the mathematics classroom and its influence on the formation of student identity with respect to mathematics must be the starting point of analysis.

As I deepen my own insights of the important contribution identity brings to learning I developed a curiosity about the identity students, as a collective, have come to apply to mathematics. It is this identity that students seek affinity with in order to develop deep connections with the subject. *Does mathematics have an identity with which students build affinity? What tools are available for the researcher to recognize the identity students have granted mathematics?*

This is a preliminary exploration of student identity to inform the relevance and importance Affinity-identity brings to the relationship between students and the mathematics. This exploration seeks to reflect on the significant insight different Identity lenses offer to the development of a strong identity framework to explore the importance of Affinity-Identity.

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[7] Throughout this paper I was tempted to use the term learner in place of student. I perceive learners to have a more active role in their studies and that this is a better term for a socio cultural perspective. The term student builds an image of an individual receiving information. I kept with student only because all of the research on identity was “student identity” rather than “learner identity”.

## LEARNING AS COMMUNICATIVE: WEAVING MY EXPERIENCES AS LEARNER AND TEACHER

The nature of learning is communicative (Sfard & Prusak, 2005) and requires a socio-mathematical perspective linked to the social practices people engage in to produce or construct mathematical knowledge and facts (Restivo, 1997, p. 16). Grootenboer (2013) discusses *praxis* as a holistic view of the culture of the mathematics classroom where the students' broader mathematical behaviour is engaged. If a student does not have affinity with mathematics they may perceive an insurmountable gap to successful learning of mathematics.

The study of student identity has coincided with my own transformation from practicing teacher to theoretical observer. Through personal reflection I recognized similar patterns inherent in my own learning experience and in the experience of students in a variety of educational institutions. Through my own self-study I reflected on significant moments in my journey where I witnessed affect coincide with a reification of my own identity as it intertwined with the identity I perceived as that of the mathematics education community.

A high school mathematics student stepping into a mathematics classroom or a high school mathematics teacher attempting to make a move into the theoretical world of mathematics research, both are students who seek affiliation or closeness with a group to share experiences, ideas, and challenges that enhance their learning (Cobb, Gresalfi, & Hodge, 2009). Mathematics curriculum is steeped in a strong history of strict criteria that dominates the identities of scholarly mathematics. This curriculum arguably isolates students who do not follow the clear traditional paths to the study of maths. D'Ambrosio (1987) prompts consideration that time has seen growth of mathematics into two clearly distinct branches: 1) "Scholarly" mathematics, which was incorporated in the ideal education of Greeks; and 2) "practical" mathematics, reserved for manual workers (p.15). These branches are evidenced in both the mathematics curriculum and the mathematics education research community.

The tension existing between student identity and the perceived identity of mathematics is foundational to student's beliefs about learning about their abilities in the mathematics classroom. The identity of the student is formed within an interpretive system seeded with re-contextualization of the students past experiences with mathematics. Nixon (1994) observed that education mediates between the culture of the individual student and the classroom culture. In the mathematics classroom, teachers act as agents of mathematical identity; in this role they make choices that communicate identity of a successful mathematics student. These mathematical traits are reinforced through assessment practices as well as recommendations about student ability and trajectory (Nixon, 1994; Grootenboer, 2013). Through praxis the invisible aspects of the cultural core are conveyed to the student. In order for students to have an affinity with mathematics they must find commonalities between aspect of their own core identity (Sfard & Prusak, 2005) and the identities granted power through scholarly mathematics.

Experience in the mathematics classroom profoundly influences how students identify with mathematics. Identity is constructed through connections to mathematics through praxis. The notion of the external influences on identity appears in works by Sfard and Prusak (2005), and Cobb et al. (2009). Burton (1994) writes that if mathematics were to be experienced as a searching, hesitant, intuitive area of study, open to interpretation and challenge, these would be a much greater identification with its style and ideas by pupils, both males and females, different classes and different races (p.79). In this sense mathematics should be presented in different ways within different contexts.

Nixon (1994) views the changes in mathematics education from three broad perspectives: (a) those related to the nature of mathematics as a discipline; (b) those concerned with research about teachers and the teaching of mathematics, and (c) those concerning pupil perspectives (p. 7). The dominant identity in the mathematics classroom has traditionally been rooted in the Institutional Identities and Discourse Identities of scholarly mathematics (D'Ambrosio, 1997; Gee, 2001). My own reflections through personal experiences have guided my interest in the emphasis of pupil perspectives fostered by the current Reform movement in education. The reform movement advocates for an expansion of a set of shared experiences among students to build interactions in the gap that exists between students' mathematics identity and the identity they perceive is that of mathematics (Nasir, 2002, p. 238). Experiences fostered through the reform movement are where viable possibilities provide for a restoration of balance among the various identity lenses. The reform movement follows classroom cultures that nurture social interactions necessary to access Gee's A-Identity (Affinity Identity).

## PRELIMINARY DATA COLLECTION

*Data was collected through a questionnaire distributed on the first day of school in September to 90 grade nine students. Students were asked to reflect on two questions:* What is mathematics? *and* What does it mean to learn mathematics? *The student responses implied there was a range of mathematical experience in the sample:*

> "Since you're looking at many different ways for a solution to a problem, it helps you look in different perspectives to other questions in life" (Grade 9 student, Vancouver, BC)

> "Learning basic skill to succeed in life, but at this point everything we learned in high school has nothing to do the real world" (Grade 9 student, Vancouver, BC)

> "To learn mathematics means to understand it. You can't learn something that you don't understand" (Grade 9 student, Vancouver, BC)

What is this "it" that students refer to?

## WHERE TO GO FROM HERE?

Researchers such as Sfard and Prusak (2005) and Cobb et al. (2009) are interested in a formalization of research on student identity. My own views about attempts to use tools in prescriptive manners to gain insight into the individuals perceived identity leads to difficulties: 1) The researcher will impose their own experiences onto the experiences of the subjects, thereby reconstructing identity within a context that is not purely specific to the student; 2) The student will be less likely willing to participate in research if they believe that their own identity is erroneously represented.

As I further my own understanding of identity in mathematics education I am curious about the identity students, as a collective, attach to mathematics. A deeper exploration of the identity that exists in the boundary where students move to deeper affinity with mathematics is fundamental to further my own ideas. Despite the cautionary warnings I have with respect to the application of verbal indicators of identity outlined by Sfard and Prusak (2005) I feel the need to utilize them in an analysis of my own data and then re-evaluate their effectiveness and what I view they offer to the field.

As I begin to explore the language contained within the student responses to the preliminary data, I will seek commonalities and possible insight into the identity that students have assigned to mathematics.

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# STUDENTS' FLOW EXPERIENCES IN THREE TYPES OF MATHEMATICAL PROBLEMS AND TWO DIFFERENT EDUCATIONAL ENVIRONMENTS

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*In this study, flow experiences of 244 students in three types of mathematical problems (intra-mathematical problems, word problems and modelling problems) were examined. Pre-test and post-test groups were used in this study. The treatment was that the students, in two groups with different training formats (teacher-centered and student-centered), attended 6 sessions of modelling problem solving. The results of this study show that students' flow experiences don't have a significant difference in these three types of mathematical problems. Teaching modelling problems increased students' flow experiences in reality-related mathematical problems, but it didn't have any effect on students' flow experience in intra-mathematical problems. Also, the student-centered group had positive effects on flow experiences in word problems.*

## INTRODUCTION

English and Sriraman (2010) emphasize that "the more we can incorporate genuinely real-world problems within the curriculum, the better our chances of enhancing students' motivation and competencies in mathematical problem solving" (p. 268).

So, it is expected that by providing activities which has a reference to reality we will be able to strengthen the learning process, improve students' motivation, activate positive emotions, and increase the number of students who enjoy mathematics (Schukajlow et al., 2012). Many students are not motivated to learn mathematics and that is because they believe mathematics is not related to the real world and they are often anxious and impatient while practicing it. Do they like modelling problems more and involve more in problem-solving process?

## THEORETICAL BACKGROUND

### Flow theory

A useful model for evaluating subjective experiences while teaching specific assignments is the flow experience theory (Csikszentmihalyi, 1990). Csikszentmihalyi (1990) describes flow experience as follows: A kind of favorable subjective experience in which the person feels cognitively capable, has deep concentration on the task and is highly motivated to continue it. Also, the person feels a pleasure which is derived from the balance between the skills and challenges (Asakawa, 2004). Creating appropriate challenges and offering opportunities for enhancing the skills (e.g. providing immediate feedback and the gradual teaching of the more complex skills based on the previously learned ones) can be one of the most ideal ways for engaging and attracting the students (Csikszentmihalyi, 1997). In this study, we examine the flow experiences of students while solving mathematical modelling problems and compare them with other types of mathematical problems.

## Mathematical problems based on their connection or lack of connection to the real world

Mathematical problems are often divided into three groups: modelling problems, word problems, and intra-mathematical problems (Niss, Blum, & Galbraith, 2007, p. 12). The following describes each of these problems: (1) Modelling problems. Mathematical modelling is: applying mathematics in solving problems in real-life situations which don't have regular structure (Galbraith and Clatworthy, 1990). One of the models that can be used for describing modelling activities is the modelling cycle proposed by Blum and Leiss (2007). (2) Word problems. Word problems are nothing more than unmasking a purely mathematical problem which is expressed in terms of the real world (Blum and Niss, 1991). (3) Intra-mathematical problems. The third type of the problems is the ones which does not have any connection to reality and are introduced using mathematical propositions. For solving them, only the appropriate mathematical procedures and concepts are needed.

## RESEARCH QUESTIONS

This study has been designed and done to answer the following questions:

- Question 1: In which type of mathematical problems (intra-mathematical, word, and modelling) students experience more flow?
- Question 2: What are the effects of teaching mathematical modelling problems on students' flow experiences on different mathematical problems?
- Question 3: Are students' flow experiences different in teacher-centered and student-centered methods?

## METHOD

This research was conducted as pre-test, educational intervention, and post-test. Two different teaching methods were employed: the direct teaching method (teacher-centred) and the operative strategic method (student-centred) (see Blum, 2011). Because some studies show that learning activities based on student-centered teaching methods can result in increasing motivation and positive emotions in students (see, Hänze & Berger, 2007; Gläser-Zikuda, Fuß, Laukenmann, Metz, & Randler, 2005). The current study was done in nine grade 10 classes. The participants of this study were 244 female students aged 15-16 from three private high schools. The majority of students in those schools had high scores in mathematics. The sample size was estimated using PASS software, considering at least 80% power for t-test in pretest and posttest independent sample t-test (its nonparametric equivalent, the Wilcoxon test) and a significance level of 5% based on the result of Schukajlow et al., (2012) study. So, 72 students were in a student-centred classroom (3 class) during the course of this study while 172 were in a teacher-centred classroom (6 class). The study consisted of a series of three initial math tests that included tasks from all three problem types. After completing the tests, the students completed a flow questionnaire. Next were six teaching intervention sessions during which students received direct instruction in modelling problems. After the intervention, the students again completed three math tests that included all the problem types, followed by completing the same flow questionnaire. In the intervention sessions students were trained using modelling problems about "Pythagorean theorem" and "Linear Functions."

This study was done by three teachers who were familiar with modelling problems beforehand. Also, they were provided with all problems and solutions of the training sessions. It should be noted that one of the teachers who participated in this study is the author of this paper and 7 classes were run by her and the two other teachers were responsible for carrying out the research through teacher-centered method. This method was consistent with the classes, since most mathematics classes are conducted in the traditional way in Iran.

**Measuring Tools**

In this study, the Flow Perceptions Questionnaire (Cronbach's α = .82) substantiated by Egbert (2003) with minor changes was used to examine the flow experiences. It consists of 14 items, but one of them was not understandable for students after being translated to Persian, so it was omitted from the items(During this task, I could make decisions about what to study, how to study it, and/or with whom to study). The Perceptions Questionnaire measures the quality of subjective experience such as level of interest, degree of concentration, enjoyment of the activity, and amount of perceived control of the activity and there are no items directly related to perceived balance between challenge and skills, so two items related to the balance between challenges and skills were added to the questionnaire(My mathematical skills were in par with the provided challenges & I believe that my skills enabled me to overcome the challenges. So the flow questionnaire that was used in this study consists 15 items (see appendix). Respondents fill out the flow questionnaire immediately after they solve mathematical problems in pre-test and post-test. Egbert used this questionnaire to find out whether flow happens in foreign language classes (see Egbert, 2003). This questionnaire has been used in other studies (Sedig, 2007; Azizi & Ghonsooly, 2015). This questionnaire is based on Likert format, having a 5-point scale from 5 (very strongly agree) to 1 (very strongly disagree). An example of one of the items is: This task excited my curiosity.

**RESULTS**

**Data analysis**

In this study, parametric tests (repeated measures ANOVA test type I and type II) were used for answering the research questions. The major assumptions of repeated measures ANOVA were examined: (1) the errors were normally distributed (2) the error term epsilon had constant variance (3) the errors were uncorrelated at lag order one. Reliability of the questionnaires was examined using Cronbach's alpha test: see Table 1.

| Reliability (Cronbach's alpha) | | |
|---|---|---|
| | Pre-test | Post-test |
| Intra-mathematical problems | 0.886 | 0.893 |
| Word problems | 0.880 | 0.901 |
| Modelling problems | 0.918 | 0.935 |

Table 1: Reliability of the flow scales

**Type of task and students' flow experiences**

In order to examine the first question, the repeated measures ANOVA test type I has been used. Statistical analysis shows that there is no significant difference in students' flow experiences among these three types of problems ($F(2,486)=2.9$, $P=.064$); however, their mean (see table 2) shows that the mean of flow experiences in word problems are the highest and the intra-mathematical and modelling problems are in the second and third place respectively.

*The effect of teaching modelling problems on students' flow experiences*

For examining students' flow experiences after teaching modelling problems, the repeated measures ANOVA type II has been used. The results obtained from repeated measures ANOVA type II test shows that teaching modelling problems does not affect the flow experiences in intra-mathematical problems ($F(1,242)=2.02$, $P=.157$), has positive effect on students' flow experiences in word problems ($F(1,242)=18.67$, $P<.001$) and modelling problems ($F(1,242)=28.48$, $P<.001$). Also, comparing the mean of flow experiences in pre-test and post-test in table 2 shows that students' flow experiences have increased significantly.

| | Pre-test M (SD) | Post-test M (SD) |
|---|---|---|
| Intra-mathematical problems | 3.34 (0.67) | 3.42 (0.68) |
| Word problems | 3.40 (0.69) | 3.55 (0.68) |
| Modeling problems | 3.28 (0.79) | 3.51 (0.80) |
| Number of cases | 244 | 244 |

Table 2: Students' flow experiences at pre-test and post-test

*The effect of teaching method on students' flow experiences*

Results obtained from repeated measures ANOVA type II test indicate that the effect of teaching group on students' flow experiences in intra-mathematical problems is not significant ($F(1,242)= 0.64$, $P=.425$). However, it was expected that students' flow experiences in intra-mathematical problems do not alter in any of the groups since the intervention was about solving modelling problems.

Teaching method did not have a significant effect on students' flow experiences in modelling problems (F(1,242)= 2.05, *P*=.153). Table 3 shows that the flow experiences have increased in both groups, modelling problems were new to students in both groups and that is why these problems were interesting enough and deeply involved students in solving such problems. However, it can be observed that the mean of flow in student-centered group is higher than the mean of flow in teacher-centered group.

| | Operative-strategic | | Directive | |
|---|---|---|---|---|
| | Pretest M (SD) | Posttest M (SD) | Pretest M (SD) | Posttest M (SD) |
| Intra-mathematical problems | 3.31 (0.60) | 3.34 (0.60) | 3.35 (0.70) | 3.46 (0.71) |
| Word problems | 3.17 (0.65) | 3.48 (0.53) | 3.50 (0.68) | 3.58 (0.74) |
| Modeling problems | 3.19 (0.80) | 3.53 (0.82) | 3.31 (0.79) | 3.51 (0.79) |
| Number of cases | 72 | 72 | 172 | 172 |

Table 3: Students' flow experiences in the "operative-strategic" and "directive" forms of teaching

The teaching group affected the flow experiences only in word problems. Specially, by comparing the mean of students' flow of word problems in pre-test and post-test (Table 3), we can observe that students experienced more flow in student-centered group (F(1,242)= 6.42, P=.012).

## DISCUSSION AND CONCLUSION

The main goal of the current study was to determine in which kind of mathematical problems students experience more flow. In general, students' flow experiences such as perceived balance between challenges and skills, level of interest, degree of concentration, enjoying the activity, and received control over the activity have been examined. Involvement in solving three types of mathematical problems based on their connection or lack of connection to reality is the activity which was considered for students.

The current study found that students could not differentiate between these three types of mathematical problems before the intervention in fact, students' motivation and interest were the same while solving modelling, word and intra- mathematical problems, but after it they experienced more flow while solving word problems. Statistical analysis (repeated measures ANOVA test type I) shows that there is a significant difference in students' flow score among the three types of mathematical problems (F (2,486)=3.61, *P*=.030). Table 2 shows the mean of students' flow experiences in pre-test and post-test. The mean of flow experiences in word problems are the highest in both of them. Another study comparing enjoyment related to intra-mathematical problems versus word problems (Frenzel et al., 2006; Pekrun et al., 2007) found that students enjoyed word problems more than intra-mathematical problems.

Pre-teaching modelling problems in order to improve students' performance on these problems and their familiarity with them had positive effects on increasing flow experiences in both word and modelling problems, specially, word problems among other types of mathematical problems. These results may partly be explained by the fact that word problems are both connected to the real word and easy for students to be motivated for solving them.

Another result of this study is students' flow experiences in solving intra-mathematical problems was not significantly different. Pre-teaching was related to mathematical modelling problems, hence, any increase in flow experiences in intra-mathematical problems is not expected.

This paper has compared different methods for teaching mathematical modelling problems. The teaching method affected the flow experiences only in word problems. An implication of enhancement of students' engagement while solving word problems in student-centered group is the possibility that the cooperative learning environment create positive emotions in students.

The impact of students' perceived quality of teaching on flow experiences is obvious. Student-centred method had more positive effects on flow experiences. According to the teachers' observations, most of the students in teacher-centred group tended to work in groups because of difficulty of modelling tasks. In general, therefore, it seems that cooperative learning environments have the potential for enhancing positive emotions in students (Panitz, 2000), since scholars have long debated the modelling problems are designed to solve in the small groups (Lesh and Zawojewski 2007).

This study was limited by the use of only two concepts of "Linear Functions" and "Pythagorean Theorem" from chapter 6 and 7 of Iran Math textbook, grade 9. The major limitation of this study is gender distribution of the sample. So, for generalizability carrying out more research in different grades, with males, and with other mathematical concepts is recommended. Additionally, the participants in this study were chosen from private high schools that had high math scores so it would be interesting to examine results of similar research carried out in public schools.

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# "THE OTHER TEN": AUDEN'S SENSE OF NUMBER

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*This paper presents a case study of a student (age 5 years 6 months) name Auden, who interacted with a touchscreen App called TouchCounts. This App was designed to support children's activities around counting. I use Sfard's commognitive framework to show how Auden thinks and learns about numbers. I show how Auden's exploration of numbers helps me understand the challenges children face when moving from identifying subsequent number as they appear in the natural counting sequence to identifying numbers that appears before and after each other.*

## INTRODUCTION

In this paper, I examine how the use of technology along with Sfard's (2008) commognitive framework, provides a window on how one student, Auden, thinks about what it means to count. I first explain commognition, how it is extended and applied to an episode in which Auden interacts with mathematics involving the use of *TouchCounts* (Sinclair & Jackiw 2014) and then provide a brief description of the functionalities of *TouchCounts*. Auden's engagement is not limited to a student/tool interaction but also involves exchange with peers and adults.

## BACKGROUND

In my home country (Jamaica), children are deemed "numerate" as soon as they can count from one to ten and up to twenty-five for those who are classified as "gifted". As a result, students enter the early childhood level of education as early as three years. The criteria for entry at this age are being able to count up to ten (numerate) and to recite the letters of the alphabet (literate). No doubt most, if not all, of our students can memorize and repeat number words and match them to objects at an early age—an aspect of their educational life which is left up to their parents. With this experience I often wondered how it is that this crucial aspect of children's number sense development is left up to parents to take care of. More surprisingly, I wondered about the effects this unconventional mode of building the foundation of number sense would have on children's further study of mathematics. This motivated me to conduct research on the ways children think about counting.

## LEARNING ABOUT COUNTING

Bermejo (1996) describes counting as "a complex ability [...] In fact, children need two years or more to count correctly and probably even more time to use counting competently" (p. 263). Surely, if counting is seen as "a complex ability" it is more than likely that teachers may not spend enough time to ensure that the concept is adequately developed, and highly unlikely that children will demonstrate secure foundation for further mathematics. Gelman & Gallistel (1978) outlines five counting principles they claim are essential to how children develop counting skills. These are the one – to – one corresepondence principle, the order irrelevance principle, the fixed order principle, the abstraction principle and the cardinal principle. From these five principles students are expected to count every object once and only once, recognize that there is no difference in which the order of counting is done and the number words must be produced in a constant order. In addition, children should also recognize that it is insignificant whether or not the elements counted are identical and that the last number counted represents the number of items in a set respectively. Gelman & Gallistel (1978) claim is that if a child lacks understanding of any of these principle (s) it is more than likely they will not be able to count correctly. The focus of this paper therefore, is on ordinal ability rather than cardinality because this is often times overlooked and it is also the emphasis of the "enumerating world" in which Auden was engaging. Coles (2014), in making the distinction between the two (ordinality and cardinality), refers to ordinality as "the capacity to place numbers in sequence" that is, to know which number comes before or proceed another. In demonstating this, Coles highlights Gattegno's (1994) perspective of ordinality as "primary" to number formation, where numbers are introduced as relations rather than denoting objects. As I will show, it was this relational connection that was missing from Audens' thought processes.

In combining these ideas counting, for Auden, could be seen as a complex ability which requires actions on all five counting principles in order to identify the relationship between numbers. This epitomizes Auden's experience with counting as I observed that though he is able to call the number names initially he is unable to show that ten appears after nine and he relies on the aural number names to make decisions about the relative placement of numbers. His journey through the interaction with his peers and task shows complexity in terms of how he thinks about counting and suggests that counting acts, for him, are an imitation of chanting number words. The aim of this paper, therefore, is to explore whether or not Auden's competence in counting indicates an understanding of what counting entails.

## TECHNOLOGY IN THE MATHEMATICS CLASSROOM

The widespread use of technology in the past decade has drastically changed education. As such, the challenge is for educators and researchers to create appropriate opportunities that will utilize and demonstrate effective use of them. Noss & Hoyles (1996), in their discussion on computers and educational culture, suggest that there is more power to technology in mathematics education today than its physical presence in the learning situation. Their claim is that well-designed digital technology can provide learners with a means of expressing mathematical ideas and helps to unravels ways of thinking mathematically along with fostering discursive interactions with children. My focus in this paper is to show how the affordances of *TouchCounts* helped to unravel ways in which Auden speaks about counting and numbers in general.

## TouchCounts

*TouchCounts* (Sinclair & Jackiw 2014) is an iPad application that introduces counting and adding to early childhood learners. Within *TouchCounts*, there are two microworlds called the enumerating world (for counting) and the operating world (for adding and subtracting). In the former world, learners use their fingers to tap on the screen which produces numbered yellow discs as well as aural number names. In the latter world tapping creates large circles (called herds) representing the cardinality of the set produced by the fingers. One of the main features of *TouchCounts* is the one-to-one correspondence where every touch on the screen represents only one object. In this paper the emphasis is on the enumerating world because the task demonstrated involves Auden's experience with the ordinal aspect of counting. The figures below (Figures 1 & 2) show the screen of *TouchCounts* when the children are working in the enumerating world. Figure 1 shows the horizontal bar called the shelf and a yellow disc with ten on the shelf. This was the task Auden and his peers were asked to perform. To do this, children will tap nine times on the screen below the shelf and once above the shelf. The disc of numbers representing nine taps falls from the shelf once they appear with their number sound because the program was set on "gravity on". Figure 2 shows what the screen looks like when the gravity is off after children tap ten times on the screen.

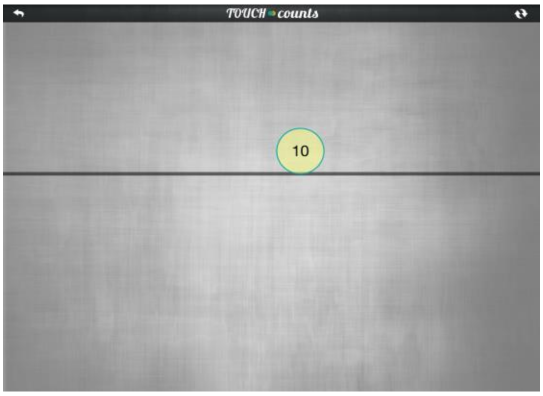


Figure 1: The tenth tap above the shelf (gravity on)

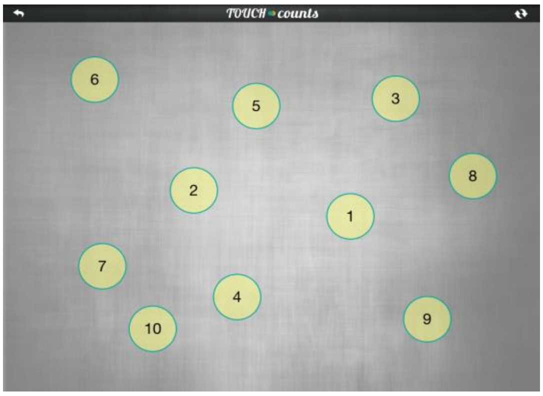


Figure 2: Ten taps (gravity off)

## THEORETICAL PERSPECTIVE

My goal in this paper is to better understand the way children think about numbers and about counting in particular. To do so, I draw on Sfard's (2008) commognition theory. Commognition is a discursive approach with its basic foundation laid on the premise that thinking is individualized communication and that mathematical activity is discursive. Sfard also believes that when learning takes place the mathematical discourse (a multimodal communicational activity) is changed. In Sfard's framework, mathematical discourse has four main characteristics: word use (what are the words used to describe mathematics objects?); visual mediators (what artefacts are used as a moderator for communication in the discourse?); routines (what *meta-discursive rules* are used to navigate the flow of communication?); and, narratives (what text are presented that can be accepted as true within the mathematics paradigm; these can be axioms, definitions and theorems?).

In the case of *TouchCounts* the visual mediators include the discs, the herds of numbers, gestures and fingers that children use to communicate within the mathematical activity and in a Sfards' framework visual *realization of a signifier* is also seen as visual mediators. Realization refers to the "perceptually accessible object that may be operated upon in the attempt to produce a description of the object" (p. 154) while a signifier refers to "words or symbols that function as a noun in an utterance" (p. 154). In other words, a signifier- realization relationship is the mediation between the signifier (in this case ten) and the object signified by its realization (such as ten counters). My interest, however, for this paper, is centred on word use, visual mediators and routines because children at this age may not be participating in 'endorsed' narratives. I also adopt Sfard's (2009) idea on gestural communication, which is used as a visual mediator. Her claim is that gestures ensure that all interlocutors "speak about the same mathematical object" (p. 197). I also draw on Châtelet's view of, "gestures as being pivotal sources of mathematical meaning" (quoted in de Freitas & Sinclair, 2014, p. 64). Though Sfard did not speak specifically about aural mediators, it was evident that the sound of number words plays an essential role in how Auden thinks about counting and numbers in general. He could hear and repeat numbers but was uncertain of what they meant. I therefore incorporated aural mediators (analogously to visual mediators) as a characteristic of discourse to analyse, with the intention to show that sounds are routine prompts whose presence increases the likelihood of enactment of a routine.

## METHOD OF RESEARCH

For 8 months I had the opportunity to interact with Auden at an after – school daycare for k-2 children with ages ranging from 5 – 8 years old in Burnaby. Auden is a 5 year 6 months old male student with high levels of interest to learn. I worked along with two other researchers, Nathalie and Sean for approximately one hour each week (30 minutes per group) with students in groups of 2 to 4. Their general custom after school is to have snack then play while they wait for parents. It was during their play time that we interacted with them using *TouchCounts* and a myriad of mathematical tasks in a small room adjacent to the common area. The episode in this paper was taken from an interaction with Auden, two students Whyles and Misha (pseudonyms) where they were asked to perform the task of putting ten only on the shelf in the enumerating world of *TouchCounts*.

Selection of children to participate in small group interactions was done in two ways. Students were randomly selected either from a list presented by one of the educators or when they walked into the research site volunteering themselves to "play". They were also purposely selected when the team of researchers recognized the need for some participants to be given more opportunities to interact with *TouchCounts.* Auden had attended more sessions than other students because he volunteered himself sometimes.

He was particularly chosen for this research because of his unusual yet stimulating way of talking about counting and numbers.

The session was videorecorded for approximately 30 minutes while students engaged in task ranging from putting 10 on the shelf, to skip counting, adding and subtracting. I viewed the video and paid keen attention to how Auden communicated about specific tasks. The video was then analysed taking into account his word use and the role of the visual and aural mediators as well as how routines were developed. In the following episode, I present a transcript in which Auden tries to put ten on the shelf (to tap below the horizontal bar nine times and then above the bar once) during his first encounter with *TouchCounts*.

## DATA PRESENTATION

The data presented here is a transcribed episode of Auden's engagement with the researchers, his peers and *TouchCounts.* It shows a portion of his journey in understanding the relational connection between numbers and ultimately counting. I outlined Auden's experience and then show how commognition was used as a lens of analysis to help me understand how he thinks about counting.

### Auden's experience with counting

In the beginning, Auden's ability to count seems secure. When asked to count to ten, he is able to call the number words from one to ten with competence and confidence. This plays an important role in his discursive fluency in two ways. Firstly, it allows him to identify the numerals as they appear on the screen and secondly, he is able to use these routines each time he tries to get ten on the shelf.

The children are asked to explore *TouchCounts* to see what happens on the screen and the episode below illustrates their first encounter.

| | | |
|---|---|---|
| 14 | A: | thirteen, fourteen, fifteen, sixteen, seventeen, eighteen,…. twenty-six [counting (listening to the sound and watching the screen) while Misha is tapping] |
| 15 | R: | How high do you want to go? |
| 16 | M: | to...to…to …to a lot. |
| 17 | A: | to a lot?...that is not even a number [opens both hands wide] |

After a few seconds had passed Nathalie ask Auden to put only 10 on the shelf.

| | | |
|---|---|---|
| 34 | A: | you said ten? |
| 35 | R: | Yeah I did say ten |
| 36 | A: | ten as in [leaving seven on the screen] |
| 37 | R: | ten comes after seven |
| 38 | A: | the one with the slanting line and one straight line …oh that ten? [Using finger to trace seven in the air] |
| 39 | W: | that ten! [Tracing ten in the air with his finger 1 and then 0] |

A few seconds after

51 A: oh! That ten

52 R: That ten! yeah

53 A: the other ten that looks like the same ten…this ten and the other ten [holding up two fingers indicating that there is another 10]

54 R: can you try to get this ten? (Pointing to 1 and 0)

Figure 3

Figure 4

Figure 5

Figure 6

**Word use and gestures**

I am cognizant that gestures can be interpreted as being visual mediators, but I have decided to incorporate gestures with word use because they usually complement a mathematical word used in an utterance. The vocabulary used by both the children and the researchers shows their thinking about numbers and constitutes words such as "a big number", "a bit less" and "a lot". The children were asked how high they wanted to go while exploring what the application can do and, interestingly, none of them gave a definitive response. Instead, they suggest a "big number" or "a bit less" as Whyles is demonstrating with the gesture shown in Figure 6. It was this gesture that helped Auden produce a number that is not big but at the same time not small either (twenty-eight). In this episode, Auden's gesture in Figure 4, which accompanies the word 'a lot', provides some insight into how he thinks about numbers, as being a range that can be of varying size. Also, while he repeats "a lot is not even a number" [17], showing that Auden went through an intrapersonal commognitive conflict; while he knows 'a lot' is not a number, he shows by his gesture, an internalization of 'a lot' as a large number. Auden continues along this path of intrapersonal commognitive conflict when he speaks about "that ten" [51] after he was asked by Nathalie to put ten on the shelf. He was unsure whether he is to put 10 on the shelf or 7.

**Visual mediators**

Auden, Whyles and Misha are able to communicate effectively about putting ten on the shelf because they are able to see the yellow disc as they appear on the iPad screen. The children are able to make decisions about when to tap above or below the shelf. Also Whyles' gesture of forming ten with his fingers (see Figure 5) helped Auden form the realization of 10 as a symbol but not as the number of times to tap on the screen: he uttered, "Oh! That ten" [51], but continued to attempt getting seven on the shelf nonetheless. Auden does not reify (interchanging discourse about a process with discourse about objects) numbers as things with properties. In other words, he sees ten as an object and not as the number that comes after nine. As a result of this he forms two different visual-verbal realizations of the signifier 'ten' and, as such, he confuses the symbol 7 for 10 [38]. He describes ten as the one with the slanting line and one straight line, as seen in Figure 3. He was in a deadlock to decide which of these symbols represent tapping ten times on the screen, and found it difficult to complete the task of putting ten on the shelf.

**Routines**

Auden cautiously uses one finger to tap the iPad screen at every attempt to put ten on the shelf. He repeats the same actions (touching the screen and listening for the number word) in the same way each time he performs the task. The sound of the number words, which is a part of the *TouchCounts* application, increases the repetitive pattern of performing the task. Auden knows the routine of tapping several times above the shelf and then once but was unclear of how many times to tap below the shelf. In addition, routines appear in word use in the logical sequence as shown below:

> # R: how high do you want to go?
>
> # M: to a lot
>
> # R: Can you get a big number?
>
> # W: A big number, but a bit less

Routines also appear in the way the children took turns in performing the task, allowing them to imitate each other.

**Aural mediators**

Auden relies heavily on the sounds of the number words in order to decide when to tap above the shelf. In addition, number words appear mainly in ritual number chanting for Auden, who acts upon the aural mediator as a routine prompt as the only means of indicating some internalization of counting [14]. When the group was asked "do you know what will come next?" Auden quickly tapped the screen to hear the number name. He is minimally reliant on what is seen on the screen rather than what is heard. As a result, he is unable to determine when to tap above the shelf and in many cases, placing just ten on the shelf was futile. He would exclaim, "I missed it!" or Nathalie would say "ten went away"

**DISCUSSION AND CONCLUSION**

The analysis provides supporting evidence that Auden utilizes gestures, visual and aural mediators along with a unique language to think and learn about counting. Though Auden is able to demonstrate a stable order in counting initially, he faces a predicament to realize that ten comes after nine, hence his inability to recognize and utilize the ordinal aspect of counting. The analysis also shows that Auden is unable to differentiate the symbols of ten and seven and is unsure about what is meant to get only ten on the shelf. The difficulty that this analysis reveals is that repetition through memorization and number chanting does not induce an understanding of counting, neither does it help children to identify the relational aspect of numbers suggested by Gattegno (1974); as in this case where Auden relies heavily on the utterances of his peers and the feedback of *TouchCounts* in the interaction. Using Sfard's framework helps me see the warning signs of ways in which non-learning may be sheltered when students are engaged in traditional ways of teaching. It brought to the fore the assumptions we make as mathematics educators of the projected and unprojected activities of students as they engage with mathematics.

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# TEACHER TENSIONS: THE CASE OF NAOMI

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*Tensions are endemic to the teaching profession. Viewed as dichotomous forces, tensions shape the experiences of mathematics teachers, affecting both their practice and professional growth. In this article, I use Berry's (2007) framework to identify and examine some of the tensions experienced by Naomi in her practice of teaching mathematics. While previous research presents the image of teachers as dilemma managers who accept and cope with continuing tensions, my research suggests that a desire to resolve these tensions may impact teaching practice and professional growth needs.*

## INTRODUCTION AND THEORETICAL BACKGROUND

Teachers are often faced with dilemmas. Lampert (1985) suggests that these dilemmas arise because the state of affairs in the classroom is not what the teacher wants it to be. Conflicts can surface when mathematics teachers encounter a disparity between what they want to do and what they are asked to do, or between what they want to do and what they know how to do. These competing influences create what Adler (2001), Berry (2007), and Lampert (1985) refer to as teacher tension, and encompass the inner turmoil teachers experience when faced with contradictory alternatives for which there are no clear answers. For both Adler (2001) and Lampert (1985), these tensions are seen as problems to be managed rather than solved. Considering that tensions are endemic to the teaching profession, mathematics education would benefit from (1) identifying these tensions, (2) understanding how teachers cope with these dichotomous forces, and (3) examining how this impacts their teaching practice and professional growth requirements. While a longer version of this paper, addressing all three areas, has been submitted to MAVI 2015, the goal of this particular paper is to explore a framework for identifying and examining these tensions, in order to better understand the wants and needs of mathematics teachers.

Whether categorized as personal, practical, or contextual, identifying tensions within the practice of mathematics is beneficial in providing a language for discourse (Adler, 1998; Ball, 1993; de Oliveria and Barbosa, 2008). Indeed, Adler (1998) refers to the language of tensions as "a powerful explanatory and analytic tool, and a source of praxis for mathematics teachers." (p. 26). Naming and exploring the tensions can present a view of teacher thinking that is broader than decision making (Lampert, 1986). To highlight this, Lampert (1985) shares an illustrative example of the tension she experienced upon choosing where to sit her students during mathematics lessons. No matter which arrangement she chose, it would be to the detriment of some of her students. Outwardly appearing as a simple 'decision', the thought process entailed in managing her tension demonstrates the complexity involved.

A framework to understand the tensions inherent in teaching emerged from the work of Berry (2007). As a former teacher who moved into the role of a teacher educator, she completed a self-study of her practice in order to improve her understanding of the process of learning to teach teachers. Building upon the work of Adler (2001) and Lampert (1985), she utilized the notion of tension as a framework for both doing and understanding her research. The result was twelve tensions expressed as dichotomous pairs that "capture the sense of conflicting purpose and ambiguity held within each" (Berry, 2007, p. 120). Noting that these tensions do not exist in isolation, she used their interconnectedness as a lens to examine her practice:

1. **Telling and growth**
   - between informing and creating opportunities to reflect and self-direct
   - between acknowledging prospective teachers' needs and concerns and challenging them to grow.
2. **Confidence and uncertainty**
   - between making explicit the complexities and messiness of teaching and helping prospective teachers feel confident to progress
   - between exposing vulnerability as a teacher educator and maintaining prospective teachers' confidence in the teacher educator as a leader.
3. **Action and intent**
   - between working towards a particular ideal and jeopardising that ideal by the approach chosen to attain it.
4. **Safety and challenge**
   - between a constructive learning experience and an uncomfortable learning experience.
5. **Valuing and reconstructing experience**
   - between helping students recognise the 'authority of their experience' and helping them to see that there is more to teaching than simply acquiring experience.
6. **Planning and being responsive**
   - between planning for learning and responding to learning opportunities as they arise in practice (Berry, 2007, p. 32-33).

Although initially applied to teacher education, it is possible the tensions that emerged from Berry's (2007) framework can be used both as a way to identify the competing conflicts experienced by mathematics teachers, and as a way to describe them. As such, my research question is aimed at identifying similar tension pairs within teachers' practice of teaching mathematics. While my eventual goal is to explore how these dichotomous forces impact teaching practice and professional growth needs, in this study my purpose is only in the applicability of using Berry's framework to emerge sets of tensions from the practice of a mathematics teacher. In what follows the methodology is addressed, one particular case is analyzed, and the results and conclusions are discussed.

## METHODOLOGY

This paper is part of an ongoing research project in which I will examine the tensions of teachers from Kindergarten to University. In the end I will have data from 25 participants (5 primary, 5 intermediate, 5 junior high school, 5 senior high school, and 5 University teachers). In what follows I present a brief analysis of one of my first participants, a teacher named Naomi.

Naomi has been teaching for eleven years: one year teaching grade five on a remote First Nation reserve, three years teaching grade four in an urban K-8 school followed by seven years teaching grade six in the same school. Her undergraduate degree required her to complete one introductory mathematics course and one mathematics for education course. A self-described 'mathematics-phobe[8]', she has had no further mathematics education other than occasional mandated professional development.

Data was collected over a one-year period during which Naomi was a participant in a District Learning Team led by myself. Notes were kept of conversations with Naomi that occurred naturally during breaks in the sessions. The opportunity was taken during these casual interactions to probe more deeply into questions Naomi had asked or about perspectives she had shared. Field notes were also taken during two classroom observations and during the lesson debrief. More formally, Naomi was engaged in two follow up interviews that were designed to illuminate tensions present in her practice of mathematics. These interviews ranged from 20 to 45 minutes in length and were transcribed in their entirety. The data collected were then scrutinized using Berry's (2007) framework as an a priori frame for identifying and coding tensions. Dichotomous tensions in Naomi's practice emerged from the data and were identified and categorized accordingly.

## ANALYSIS

In the following analysis, Berry's (2007) framework will be used to analyze data from the perspective of Naomi in her practice of teaching sixth grade children. For the purposes of brevity, only three of the tensions will be discussed here.

### Telling and Growth

For Naomi, the teaching of mathematics has been influenced by the constructivist notion of learning. She understands this to mean that the primary role of teaching is not to lecture, explain, or otherwise transmit mathematical knowledge, but to create opportunities for the student to construct their own knowledge. Naomi wants to avoid 'telling' and instead focus on growth of understanding through experience. She explains that her teaching style is "like night and day" in comparison with her own mathematical education, which she described as rote memory, drill and kill, and

[8] A 'mathematics-phobe' describes a person who has an aversion to mathematics.

algorithms. Instead, she uses "hands-on, collaborative activities where students talk and work through things together to come up with many different ways to do things". Naomi values when students come to her with a new mathematical understanding that she has not taught. Yet a tension arises when she is faced with students who are not accustomed to being taught in the "new style of teaching". She shares that she doesn't know how to deal with that, and sometimes reverts back to "the old school" despite believing that it's really not in the best interests of the child. Naomi says she attempts to help the students see the benefits of her teaching style but that there are days when she hands out mathematics worksheets. Her statement, "and you don't want to do that, you want to be egging these kids on to try and do as much as they can" reveals the tension Naomi feels. It was apparent that this tension is a driving force in Naomi's professional growth and development when she followed with "I'm always looking for better ways to teach math". This is a tension that is neither managed nor resolved, instead it has become the impetus for change.

**Confidence and uncertainty**

Teachers who feel weak in mathematics have the dilemma of whether or not to share that weakness with their students. Berry (2007) suggests there's a tension between exposing vulnerability and maintaining the respect of students, which Naomi discovered when she first began teaching. Her unexpected answer to the question, "What are you best at in teaching math?" was "Um, honestly, showing the kids my weaknesses in math". Naomi goes on to reveal that mathematics is her weakest subject and that she struggled with it throughout her life. When she began teaching, she hid this from her students because she thought they expected her to know everything. Naomi was able to accomplish this deception by a lot of traditional, direct teaching from the text and passing over questions that she couldn't answer. This eventually felt uncomfortable for her and she began telling her students when she didn't know something, stating "But then I decided, maybe it would be okay if they knew". Using the word 'maybe' in her rationale indicates that perhaps Naomi is not completely at ease with her decision but she concludes with, "and I think that the kids actually love it even more because the teacher really is struggling, and they get to teach me". Naomi has made a decision that is acceptable to her and that she can live with. Her tension is not resolved but it is managed.

**Action and intent**

Berry (2007) tells us that tension can arise when what we intend to do is in conflict with our approach in working towards that intention. In other words, what we do can inadvertently undermine our goals. Naomi provides an example of this tension when she describes her reliance on summative assessments in mathematics stating, "I would say that with math, my hardest struggle would be not to rely solely on summative assessments." She notes that her teaching style requires the use of formative assessment but she frequently forgets to document the students' learning and reverts back to her traditional summative assessment adding, "I think because I get carried away in classes and I'm not actually documenting the formative assessments as I'm going through." This causes tension for Naomi because her view of summative assessment is that it doesn't provide an accurate reflection of what the kids are doing "especially the kids that have text anxiety or they're having a bad day." Her intent appears to be to assess the students in a way that best matches the way the content was learned but her actions in relying on summative assessment interferes with that aim. And, despite acknowledging that the results from the summative assessment occasionally surprise her, Naomi manages this tension by combining the information from the summative assessment with what she has observed in class. "It (summative assessment) doesn't give a true reflection of what the kids are doing. You're looking at summative assessment marks but also trying to reflect, bring in some of your observations." Naomi shared that her decision to volunteer to be part of the District Learning Team was in part because of her desire to learn more about assessment in mathematics. Again, here is a tension in which seeking professional growth is seen as part of the outcome in resolution.

## DISCUSSION

In examining the data, it is possible to see limitations in applying Berry's (2007) framework to a teacher/student situation. Further tensions are evident that do not appear to fit within any of the categories provided by Berry (2007). Notably are the tensions the participant experiences with parents and colleagues. Naomi willingly spends time working with parents to help them understand "new math" but acknowledges that "it takes a lot of my time and effort." With colleagues, Naomi voices a tension between a latent desire to conform to "what everyone else is doing because that's just easier" and doing what she believes to be pedagogically sound. Teachers feel a great deal of pressure to conform to the norms and standards of their school, their mentors, and their grade partners. While this is especially true for beginning teachers, even experienced teachers feel tension when abiding by the norms conflicts with personal pedagogical beliefs.

It would appear that, in order to encompass Naomi's tensions surrounding parents and colleagues, new categories of tension pairs would be of possible benefit. I expect that as my research progresses and broadens I will also be capturing other tensions that are specific to the practice of teaching mathematics. This will require expanding or perhaps altering Berry's (2007) original framework to allow for their incorporation.

It may also be beneficial to consider further categorizing the tensions according to whether they are conflicts of pedagogy, conflicts of subject matter or possibly conflicts from external, systemic influences. What information might be revealed that could help in understanding how teachers experience tension? As well, on several occasions Naomi mentioned the difference between her current practice and when she first began teaching. It would be beneficial to compare the tensions felt by a new teacher with those of an experienced teacher and then examine any differences in how they manage those tensions.

## CONCLUSION

Naomi appears to fit Lampert's (1985) image of a teacher as a dilemma manager who accepts and copes with continuing tension. She initially manages the tensions that surface in her practice while never fully resolving her competing conflicts. Our results show that where Naomi may differ is in living with the consequences of her decisions, as some of her managed tensions continue to resurface. She acknowledges that she wished she "knew how to deal with that". This results in new tensions between what she wants to do and what she knows how to do. The new tensions Naomi experiences may potentially fuel a desire for change in her teaching practice. In my future research I hope to discover if this is applicable to other teachers' experiences of teaching mathematics and what impact that might have on the design and delivery of professional development. By further investigating teacher tension, I believe the field will be better informed to improve teacher education and professional development efforts related to this phenomenon. It is beneficial to mathematics education to have a fuller understanding of these tensions that drive teachers' needs, and shape both the individual and their practice.

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# A BLIND UNDERGRADUATE STUDENT'S JOURNEY TOWARD AN UNDERSTANDING OF PRE-CALCULUS CONCEPTS

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*In this paper we argue that although mathematical communication and learning are inherent multimodal and embodied subjects, sight disabled students are also able to conceptualize visuospatial information. Adapting Vygotsky's mediation theory, we show that the lack of access to visual fields in an advanced mathematics course does not obstruct a blind student in visualization, but rather modifies it. We argue proper precise tactile materials might empower blind students to better visualize mathematical functions.*

## SIGHT DISABILITY ROUSTES OTHER ABILITIES

Euler (1707–1783) and Saunderson (1682–1739) are two well-known mathematicians who had been struggling with blindness. Jackson, a contemporary blind mathematician (2002), argues that visuospatial imagination amongst people who do not see with their eyes relies on tactile and auditory activities, however mathematical instruction and communication involves gesture, gaze, posture, pointing and facial expression, while speaking (Quek, McNeill, & Oliveira, 2006). One of the central tasks for learning mathematics in general and pre-calculus in particular, is to understand, visualize and interpret mathematical concepts, graphs, and objects (Healy & Fernandes, 2011; Healy, 2012). Healy (2014) claims that visualization goes beyond "seeing" and that it can develop in the absence of vision. She explains that visualization consists of other sensorial perceptions, relationships with previous experiences and knowledge, verbalization, and more (Healy, 2014).

Mathematical and statistical graphs contain a concise, complete and *precise* summary of functions and equations such as: employing ordered pairs, axes, origin, grid lines, tick marks, intersections, labels, etc. Obviously, these components are all visual. As a result, for students with vision impairment, understanding the mathematical concepts behind them or learning the concepts themselves becomes an extremely challenging task. Teachers working with visually impaired students struggle to convey the concept of a graph to their students, specifically when the concept begins to get more advanced and complicated. This suggests that to understand the learning processes of visually impaired and blind mathematics learners, it is important to investigate how the particular ways in which they access and process information shapes their mathematical knowledge and the learning trajectories through which that knowledge is attained. In the UNESCO's World Conference on Special Needs Education (1994) in Spain, the Salamanca Statement affirmed that there is the necessity and urgency of providing education for children, youth and adults with special educational needs within the *regular education system*.

For years, most of the students with vision disabilities avoided taking mathematics courses at higher levels of education and tended to feel anxious and negative towards math. As the result, avoiding taking math courses hindered them in moving toward their desired fields of study, jobs and careers.

There are few studies on learning and teaching mathematics for learners with visual impairments from early childhood to the secondary level; in addition, there is also scant research focusing on university level students tackling understandings of advanced mathematical concepts. To help redress this situation, in this paper which is a part of larger research investigating the roles of creative haptic materials that empower blind students to visualize advanced mathematical concepts.

### A motivating example

Mathematics communication and conceptualization have strong embodied components. Embodied communication, gesticulation, gaze, pointing, and body-language in mathematics discourse has a critical role in communication between sighted individuals discussing math concepts. In "The Emperor's New Mind", mathematician and physicist Roger Penrose wrote:

> Almost all my mathematical thinking is done visually and in terms of nonverbal concepts, although the thoughts are quite often accompanied by inane and almost useless verbal commentary, such as 'that thing goes with that thing and that thing goes with that thing. (Penrose, 1989, p. 424)

While the functionality of mathematically-grounded gesticulation involved with deixis is huge, we show in the following episode that communicating in a way that is understandable to all parties in teaching and learning mathematics, could provide critical information to maintain mathematical understanding. In this paper, we are seeking how *embodiment* could be seen as the visual cue for a blind learner in mathematical communication. We also articulate that mediating resources make mathematical communication possible for blind learners in advanced mathematics courses.

## THEORETICAL FRAMEWORK

We believe mathematics to be, at least in large part, a social and cultural product. It is not an abstract, eternal and isolated subject out there, totally independent of human cognition and understanding. Therefore, construction, representation and manipulation of mathematical objects are variously invented and constructed through social and cultural activities. Our framework accords with Wittgenstein and the Participationism theory of Lave and Wenger (1991), which asserts that learning is a process when students become the participant in a certain community and engage in specific activities. So, for the blind learner many mathematical concepts that involve working with spatial representations and information, his/her hands represent the most obvious substitute for the eyes, and hence it is not surprising that research involving blind geometry learners has focused on how explorations of tactile representation of geometrical objects contribute to the particular conceptions that emerge. For this study, we adopt Vygotsky's "mediation" theory (1986) with disabled learners, and will take a qualitative perspective to explore how access to different mediating resources impacts learning mathematics (Vygotsky, 1986).

## METHODOLOGY

This study is part of a larger research project exploring "Issues And Aids for Teaching and Learning Mathematics to Undergraduate Students with Visual Impairment". Prospective participants, who are visually impaired, are identified by the Center of Students with Disabilities' (CSD) specialists. The participant focused on for this paper is a blind student named Anthony, who was taking a pre-calculus course.

Anthony, the learner described here, is a twenty-eight-year-old male, who is doing his sixth year of undergraduate studies in kinesiology and health science at a Canadian university. He was born with a profound visual disability and is completely blind. He has passed "Mathematics Foundations" successfully the prior semester as the prerequisite for enrolling in pre-calculus.

Anthony has excellent mental calculation skills, which enables him to easily do basic calculations without a calculator. He is very keen to create a pathway for other sight-disabled students to pursue their academic dreams by taking mathematics courses. What follows is a brief description of material and software use and procedures that we followed to assist Anthony through his pre-calculus journey.

### Written materials

Braille, which is well-known as a tactile writing system designed for the blind and visually impaired individuals, is a strictly linear notation and not generally useful for mathematics and its various notations (Marcone, 2013). In addition, Braille readers can only perceive what is under their fingers at the time, so it can be very difficult for them to obtain a general view of algebraic expressions and graphs as a whole. Braille extension specialized for mathematical notation known as the *Nemeth Code* has different coding than Braille itself . Considering Braille and *Nemeth code's* limitations, we chose LaTex to translate mathematical language and formulas and prepare other written materials. *LaTex* is used to prepare textbook content, class and lecture notes.

### LaTex

The most important reason that we chose to adopt *LaTex* was that we found, *LaTex* a common language that all parties (instructor, learner, tutor and Anthony's assistant) that were involved in teaching and learning the course could communicate with. A table of contents at the top of each *LaTex* and Word file was used to give insight into what is compiled in each file, so Anthony did not have to go through the whole document to figure out what is in it. Anthony also suggested that descriptive file titles are best for identification purposes rather than the date of the class. For example: \frac{numerator}{denominator} uses for fractions, % for comments and all the mathematics formulas places between $s. So, for example, `$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$` means $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.

**JAWS**

Digital technologies are facilitating conversions between written text and blind learners by offering spoken versions of written materials and mathematics equations. Furthermore, use of spoken rather than written materials suggest that the ears can also be used as alternates for the eyes. Having said the limitation and possibilities of incorporating different senses for a blind learner, for conducting this research we mostly rely on auditory and tactile materials.

JAWS[9] (Job Access With Speech), is a computer screen reader program for Microsoft Windows that allows blind and visually impaired users to read the screen either with a text-to-speech output or by a Refreshable Braille display, was used by Anthony. This program reads most of the programs and actions in Windows and *LaTex*. Anthony's laptop reads using JAWS. VoiceOver in Mac iOS also does the same action (Figure 2-A).

**Tactile graphs: A real challenge that tackled**

One of the very challenging problems here is to help a blind learner on visualizing mathematical graphs and concepts. We face this difficulty in various forms, in class and during lecture, and at home and doing exercises. To tackle this, when there was an image or figure displayed on the overhead in the class and during lecture, or when lecturer uses "this" and "that" for describing a mathematical figure, I (the first author) rapidly made graphs using Net-board and crayons (Figure 1A& 1B) or Wikki Stix (Figure 1C &2B).

Net-board is a wooden board with a metal net installed on it. As it is shown in the graph, using crayons while paper is placed on the net-board creates a texture that is tangible for Anthony. Making the graph on the net-board was fast and adequate to provide a mental image of the graph to the learner.

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[9] *JAWS is produced by the Blind and Low Vision Group of Freedom Scientific, St. Petersburg, Florida, USA.*

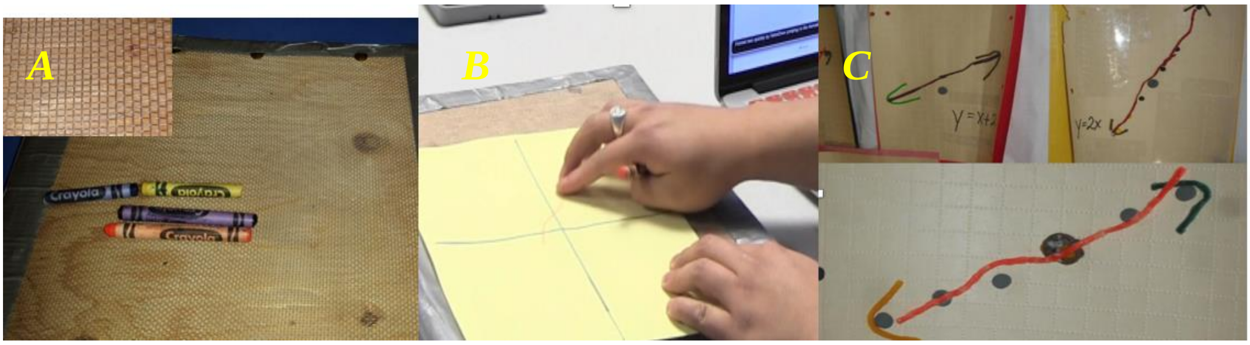



Figure 4: A&B Innovative way for drawing graphs during lectures and tutoring using "metal net-board", crayons and normal papers. C: Examples of using Wikki Stix on a raised line graph paper at Georgia Academy for the Blind

To embrace presenting textbook graphs, when there was enough time, I (first author) created graphs using various materials and textures, trying to make it as precise as possible. For this purpose, firstly axes and grids were printed on graph paper using a Braille embosser[10]. Then, I drew graphs using hot glue, a wheel tracing tool, and stickers. For labelling, I used Nemeth code on top of the graph paper, in a similar way that represented the original source (Figure 2-C).

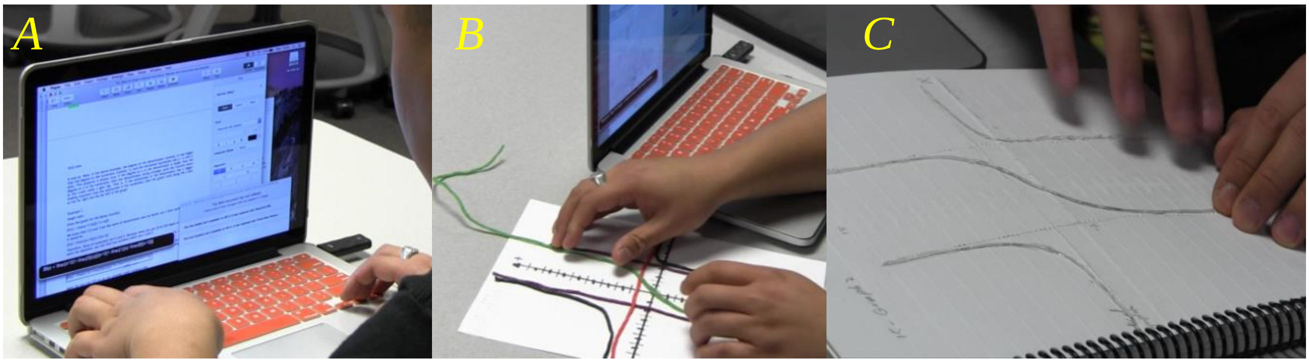



Figure 5- A: VoiceOver is reading class note. B: Anthony is creating graph of function by Wikki Stix. C: Making various textures using different materials to make graphs perceivable, and hence conceivable, as precisely as possible

**Demonstrating function's behaviour by the blind learner**

In this section, we start with describing the participant's action and communication, when he was asked to describe the behaviour of $f(x) = \frac{x^2-5x}{x^2-x-6}$ for the sighted interviewer. This episode happened after Anthony graphed the function using Wikki Stix on raised grid X-Y axes (Figure 2B), which is out of this paper's scope and we do not report on it. Then, we analyse developing gestural, embodied and discourse communication.

10 *Braille embossers are "printers" for Braille.*

Anthony taps on the quadrant two on the haptic graph created by the first author (Figure 2-C), and says: “I'm in quadrant two, the function goes like this”. He moves his left hand (illustrating function) from left to the right horizontally.

Anthony ...and hugs the vertical asymptote at y=1[ brings his right forearm parallel and right below the left forearm], and as it gets closer to [pause]…the Y-axes... [pause]...[left and right forearm moves upward] it goes up.. [pause- he seems not sure].

Then, Anthony touches the graph, and accordingly recognizes his explanation was not correct. Then, he waves his left hand to emphasize that this will demonstrate function and moves it from left to the right horizontally.

Anthony The function, goes horizontal, hugs up the horizontal asymptote at Y=1 [two forearms touch each other] (Figure 3A) and then, it goes up to the vertical asymptote at x= -2[two hands moves upward, left wrist taps right forearm] (Figure 3-C). That means that [function] doesn’t cross it [vertical asymptote] and it comes up (Figure 3-B).

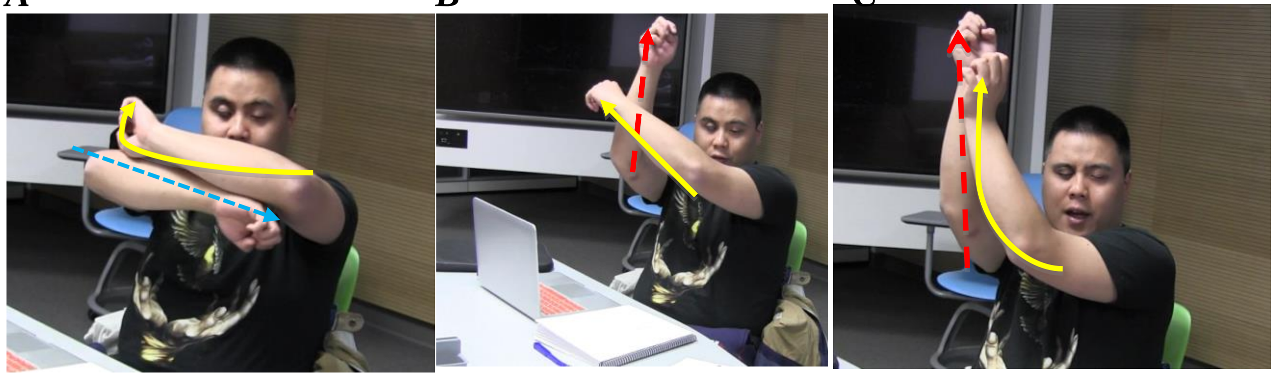
Subsequently, Anthony “*looks* at his notes” (Figure 3D) by touching haptic graph.

Anthony …And the middle [part of the] function, is where the function is between the two vertical asymptotes [-2<x<3]; doesn’t cross either ones [waves his forearms] (Figure 3E). But, it does cross upward, over the origin, and over the horizontal asymptotes at y= 1 and comes toward to the vertical asymptote at x=3 [ Taps with his right wrist(asymptote) to his left forearm (function)], and again, never crosses it (Figure 3F). And this is middle part of the function [ taps on the middle part of the graph]

Afterwards, Anthony taps in each quarter, while naming them. He explains the behaviour of right side of the function, for x>3. He points and taps on his right hand to stress that demonstrates the function and not asymptote (Figure 3G).

Anthony This is my function. It comes up along the vertical asymptote at x=3 (Figure 3L). And, as it comes up, instead of crossing over horizontal asymptote at y=1, it gets close to it [bends both forearms, while they are sticked together. Bends his body to the right], and does not cross (Figure 3M).

***A*** ***B*** ***C***



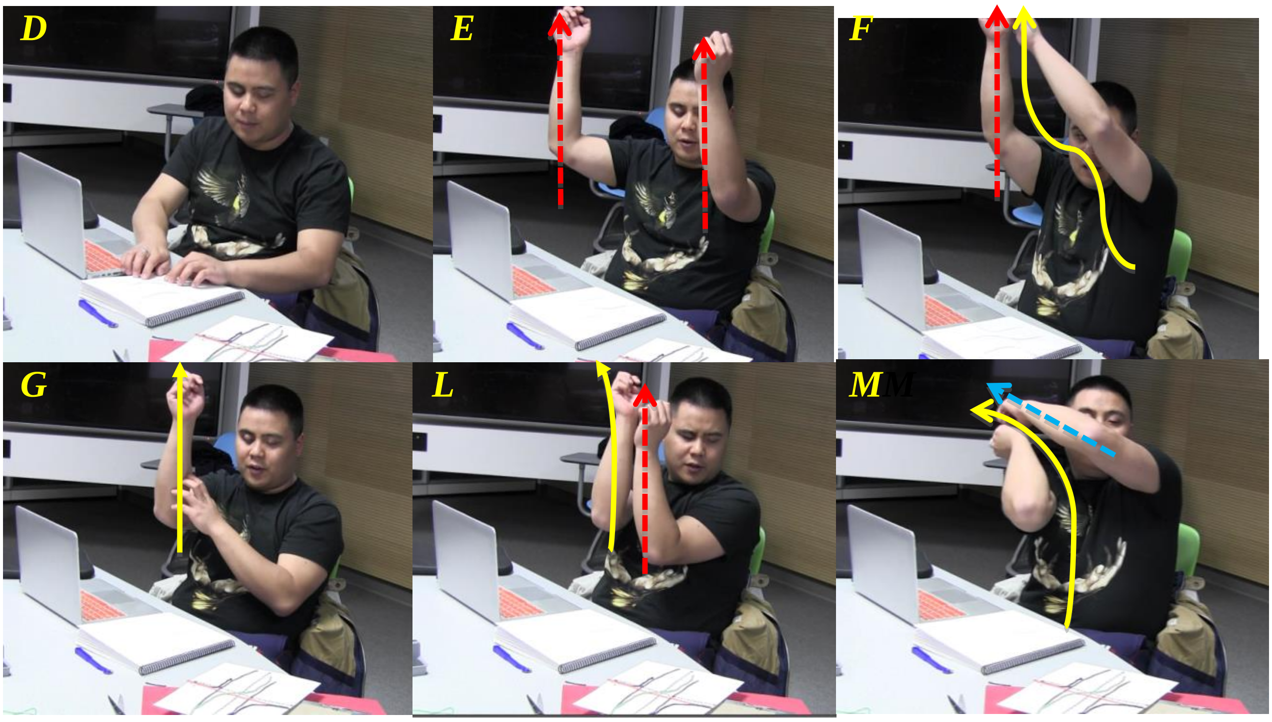



Figure 3: Anthony explains function's behaviour for $(x) = \frac{x^2-5x}{x^2-x-6}$, using his hands and forearms and body. Each arrow shows the direction and path of movement (Red dashed arrows: vertical asymptote, blue dashed line: horizontal asymptote and yellow arrow: function)

Selected episode above presented how embodiment works as *visual cue* for a blind learner. That also exemplified the *modified* multimodal communication in absence of sight. Whereas, Anthony uses his finger tips to explore diverse information on graph papers and other written resources, he does not use them, as sighted learners do. Instead, other parts of his body such as his forearms and wrist, while wave, move, touch, slide and tap demonstrates high level of body engagement and mathematical embodiment. We have shown the capacity of the blind for visualization and spatial reasoning (Quek, et al., 2008). In addition, the verbs that were used by Anthony (e.g.: moves, hugs, cross etc.) displays that he percives mathematics entities as dynamic objects.

## CONCLUSION AND REMARKS

Although teaching and learning mathematics at the secondary level seems challenging and time consuming for sight disabled learners, it is not impossible. We found a high degree of sensuous and body engagement in understanding the pre-calculus concepts by Anthony in other than visual modes. This is inline with neurological evidence to support this position suggested by Gallese and Lakoff (2005) "that circuitry across brain regions link different sensorial modalities "infusing each with the properties of others" (Healy, 2012, p.5). We also found, as Vygotsky's (1986) claims, providing proper and timely semiotic or material tools "alters its entire structure and flow", while the learner is actively engaging in processes (Healy, 2012, p.2). In other words, those activities could cover and modify visualization via other senses such as tactile and auditory rather than visual. Future researches could reveal what kind of mediators and how they could help the sight disabled learner to successfully pass other courses such as advanced biology in university level. In addition, we found that teaching and learning advance mathematics is very time consuming for visual disabled learners. So, more studies are needed to figure out how university policies should be modified for those learner.

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# MATHEMATICS LEARNING AMONG UNDERGRADUATES ON THE AUTISM SPECTRUM

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*This study examines the mathematical learning of an undergraduate student on the autism spectrum, as well as some views of autism in the Vygotskian framework. I aim to expand on previous research, which often focuses on younger students in the K-12 school system. I have conducted a series of interviews with one student, recording hour-long sessions each week. The interviews involved a combination of asking for the interviewee's views on learning mathematics, self-reports of experiences (both directly related to courses and not), and some particular mathematical tasks. I present some preliminary findings from these interviews and ideas for further research.*

## BACKGROUND ON AUTISM-RELATED RESEARCH

The Autistic Self Advocacy Network (2014) states that autism is a neurological difference with certain characteristics (which are not necessarily present in any given individual on the autism spectrum), among them differences in sensory sensitivity and experience, different ways of learning, particular focused interests (often referred to as 'special interests'), atypical movement, a need for particular routines, and difficulties in typical language use and social interaction. Over the past few decades, there have been many research studies about learning in students on the autism spectrum, such as those reviewed by Chiang and Lin (2007). A large portion of these studies focus on K-12 students, and particularly elementary students, but some of the ideas and procedures in those studies lend themselves to use in a post-secondary context.

While most mathematics-related research on people on the autism spectrum also takes place among younger children, there have been multiple reports, such as those from James (2010) and Iuculano et al. (2014) which indicate an association between autism and heightened mathematical interest and ability. Due to the young age of the students, however, it remains to be seen whether those tendencies extend to university-level mathematics content.

## THEORETICAL FRAMEWORK

I think that Vygotksian educational research provides a promising theoretical framework for further examination of the results of these comparisons, and possible reasons behind the observed differences. Vygotsky did not write about autism specifically (since the diagnosis did not exist at the time). However, Vygotsky (1993) did write about defectology (as disability research was called at the time, though it had a narrower scope than the modern concept), criticizing the focus that others took solely on the deficiencies that they saw in children. He stated that children had the potential to develop strengths to "overcompensate" for these deficiencies, in particular when those efforts were targeted toward a particular goal. He criticized other educational methods at the time which kept people with disabilities out of mainstream society, giving a sparse curriculum and low expectations for their performance in the school or later in life; many of his criticisms and more still apply to some programs today. Since Vygotsky's theory of development heavily involves cultural artifacts, and as he points out those artifacts developed for the 'typical' child, development will differ not just based on the initial differences in the child, but on how those differences affect the use of cultural tools in development. This is strongly related to current social constructionist models of disability, as outlined by Rodina (2006), as well as the social model of disability seen in contemporary activism.

The Vygotskian conception of inner speech also has interesting applications for autism-related research. According to Vygotsky (1962), speech develops from a single general function to distinct functions for oneself and for others. The 'for oneself' development is first into egocentric speech (still out loud); while this refers to the same empirical phenomenon as Piaget's egocentric speech, the conception of development here is different. Instead of simply ending, Vygotsky's egocentric speech develops into inner speech, which is no longer vocalized. However, the differences are more extensive; from Vygotsky's study of egocentric speech, he found that it gradually develops into a form which has greater distinction from social speech, more idiosyncratic and less understandable to others, before it ultimately develops into inner speech. He concluded that this form of speech relies heavily on internal conceptions and senses which could not be used in social speech, and thus it is something more than just speech without vocalization.

## AUTISM-RELATED RESEARCH IN THE VYGOTSKIAN FRAMEWORK

Some research studies have been conducted attempting to determine the level of development of inner speech in autistic children, with varying results. One study conducted by Wallace, Silvers, Martin, and Kenworthy (2009) indicated that “individuals with autism do not effectively use inner speech during the completion of cognitive tasks”, while another study from Williams, Happé, and Jarrold (2008) indicated that “individuals with ASD use inner speech to the same extent as individuals without ASD of a comparable mental age.” A third study conducted by Russell-Smith, Comerford, Maybery, & Whitehouse (2014) examined this in the particular context of executive function-related tasks (they appear to implicitly equate this to Vygotsky's 'self-regulation', and one of their citations is the Wallace et al. paper which does this explicitly), and involved a greater variety of conditions to alter the use of inner speech. They also found discrepancies in inner speech use in the same direction as Wallace et al., as well as a correlation between verbal test scores and task performance in the ASD participants. However, while the Wallace et al. study found a statistically significant worse performance in the ASD group in the control condition, the Russell-Smith et al. study did not.

It should be noted that all of these studies were mostly done on children, and only one had any participants over 18, so they were not aimed toward the same population as my own research. Also, in all three studies (like in most autism-related studies), the vast majority of study participants were boys. In this case, there is not a sufficient sample to determine if the outcomes would be different with girls (there is a widely reported disparity in diagnosis, but not necessarily in occurrence). Another unexpected finding was that of the three studies, the one where the ASD study participants had a DSM-IV diagnosis of Asperger syndrome is the Wallace study, which found a difference in both inner speech use and general performance. Given that the main difference between the autism and Asperger diagnoses was in verbal criteria, this runs counter to expectation based on those diagnostics, and provides more evidence that the developmental paths here may be more complex than these studies are able to discern.

Some reports, such as one by Patterson (2008), show cases of people on the autism spectrum who do not have much social interaction and often read a variety of 'higher level' books. Typically, they are noted for having an extensive vocabulary, but not the skills to relate to typically developing children their own age. While in Vygotsky's model, speech is most often mediated primarily by direct interaction with others, his defectology work also suggested that alternate means could be used to work toward the same goals. In this case, it seems plausible that the children involved here may have had their reading as a large component of mediation for speech development, and thus their 'unusual' vocabulary would appear to be a natural consequence of this alternative mediation. It is also worth noting that often extensive reading or other interests in children on the autism spectrum are identified in the category of “special interests”, and while some methods of therapy try to diminish those interests, Vygotsky's ideas in his defectology writing about using the child's strengths fit more with the idea of using them as part of the educational process.

Since inner speech is by its nature not directed toward others, it is reasonable to consider a process of formation could happen which does not heavily involve direct interaction with others (though cultural knowledge is still mediated from the authors through their works). Also, particularly with nonfiction, we would not expect these books to mediate elements of social interaction that are not described in their pages, giving less progress in that area. The discrepancies we have seen in studies of inner speech and task performance among children on the autism spectrum could be a result of the kinds of mediation that are available to the participants and how those work with the particular variations of the child in question (since there is a wide range of variation in the autism spectrum). Also, while Vygotsky notes that “the potential for expressive intonation, mimic, and gesture, is absent in written speech,” since it is precisely those things that are often most troublesome for autistic speakers, that could be another reason for them to be stronger with written than oral speech, and this could play into development.

## INTERVIEW PROCEDURES

I started out with the intent to find students who were on the autism spectrum and currently taking one or more mathematics courses at SFU. My advisor and I requested that the Centre for Students with Disabilities at SFU help us recruit for students who would be qualified for the study, and were put in touch with one student who was willing and able to do the interviews. It should be noted that this method will only identify those students who are both able to seek assistance from such a center and see the need to do so, and this constitutes only a portion of the fairly wide autism spectrum. The interviews were conducted with audio recording, and portions of the interview transcribed when they were examined.

The student I interviewed, Joshua (a pseudonym) was studying integral calculus and linear algebra. I conducted interviews every week for this term. These were scheduled for one hour, but were sometimes continued for a short time past the scheduled hour. I typically started by asking the student to share any particular thoughts on the week's course materials. I also asked various questions and assigned tasks related to the covered course material. Some of these were tasks that have been used with typical student populations in the literature, such as the example-generation tasks used by Bogomolny (2006) and the Magic Carpet Ride sequence used by Wawro, Rasmussen, Zandieh, Sweeney, & Larson (2012). I have also given other mathematical tasks not directly related to the material covered in the courses being taken, such as the paradoxes examined by Mamolo and Zazkis (2008); one reason for this was the interplay between visual and algebraic explanations seen in some student responses to these paradoxes. For the most part, I attempted to examine the student's thoughts without interference, but I would try to explain anything which was still incorrect or unresolved at the end of the session.

There were several reported characteristics of people on the autism spectrum which I thought could be promising for mathematics education research. In particular, I was interested in details of prototype formation, special interests, and geometric approaches. I will detail each of these with a comparison to the particular findings relevant to them in Joshua's case. One smaller finding here was that Joshua reported a heavy use of the textbook and not much gained from the lecture; this is opposite to my observations of other students and fits in with the possibilities of alternative mediations of inner speech.

## PROTOTYPE FORMATION

I started looking into prototype formation after reading a study by Klinger and Dawson (2001). It suggested that people on the autism spectrum did not form prototypes of objects when given tasks asking about group membership, instead taking an approach based on lists of rules. Although this is presented as a problem, like many other autism-related studies, I suspected that this approach could be helpful for more abstract or proof-based mathematics. In Vygotskian terms, it could be regarded as an attempt to form a scientific concept rather than building it up from an everyday concept. I have found many other students having trouble with mathematical questions that appear to result from a prototype-based approach, and this is particularly true when the course focuses on mathematical proof. In fact, I found a very similar division reported in mathematics education research by Edwards and Ward (2004), phrased as lexical or extracted definitions versus stipulative definitions. This did not appear to be the case for Joshua; he reported having this kind of thinking in the past, but was quite focused on “big picture” ideas today (this was, in fact, a recurring phrase in the interviews).

## SPECIAL INTERESTS AND LEARNING

I found the idea of 'special interests' (variously known as circumscribed interests, splinter skills, savant skills, and a variety of other names with varying connotations within the autistic and research communities) to be applicable to some of my findings. These are intense, focused interests occurring in people on the autism spectrum (often studied in children, like much autism-related research). Some of these are mathematically related, such as a focus on particular facets of arithmetic, prime numbers, or aspects of geometric shapes, as explored by Klin, Danovitch, Mars, & Voltmar (2007). The development of these interests in later life is something that does not appear to have been studied much in previous research, however. Unfortunately, research in this area (particularly from an Applied Behavioral Analysis framework) can discount these skills or even view them as detrimental to learning, as seen in Dawson, Mottron, and Gernsbacher (2008), and may even attempt to eliminate these skills. From what I have seen, however, such interests can be very helpful in imparting motivation when viewed from the right perspective. Joshua has reported a strong interest in chemistry, which he often used in analogies for mathematical concepts in our interviews. There have been several instances where he has reported more enthusiasm, better understanding, and better performance when able to see chemical applications to the topics in the courses, and has sought out additional information outside the course materials in order to make these connections.

## GEOMETRIC FOCUS AND VISUALIZATION

Particularly due to the work of Temple Grandin, one of the most famous people on the autism spectrum, there is often an association between the spectrum and visualization or spatial reasoning (Grandin, Peterson, and Shaw, 1998). There could also be a relationship between this and differences in the mediation of inner speech. While I would caution against being too broad with an association like that, I did find a strong preference for visual, spatial, or geometric reasoning in the interviews I conducted. This was particularly successful with integral calculus, where the student independently thought about what three-dimensional integrals might be like. The correct conclusion was reported for a 'flat' extension (of multiplying by a constant length), and the student did realize that this would not work for more complex three-dimensional shapes (although not to the point of developing multiple integrals). Reports of classroom progress continually reflected higher performance in areas that could be viewed in a geometric or otherwise physical way. Comparing topics across the two courses, this leads to some surprising results, such as reported satisfaction with washer and shell rotations, but issues with algebraic formulas like dot products.

I found the solution Joshua gave in one interview for the first Magic Carpet Ride task to be particularly notable. I showed the problem setup from the paper by Wawro et al. (2012), giving the two modes of travel with vectors (3,1) and (1,2), and asking for a way to get to the house at (107,64). Since I asked this relatively late, Joshua had already seen the vector material that shows the 'standard' way to do this. However, the solution he gave was instead done by drawing the vectors out on paper. He measured (30,10) and (10,20), plotted (107,64), shifted one of the vectors so that it would end at (107,64), and extended the vectors in order to find via measurement their point of intersection. (I checked this, and it was accurate enough to provide the correct solution.) When asked for an algebraic solution, he calculated the equations of the lines corresponding to those vectors and found their point of intersection, still clearly based on the same visualization.

## PARADOXES

I have also presented several paradox tasks during my interviews. I gave the Hilbert Hotel and ping-pong ball paradoxes from the paper by Mamolo and Zazkis (2008), as well as the Painter's Paradox (involving the volume and area of Gabriel's Horn obtained by rotating 1/x) and an 'infinitesimal staircase' paradox. The ping-pong ball paradox involves an infinite set of ping-pong balls (numbered 1, 2, 3, …) being inserted into and removed from a barrel over one minute. In the first 30 seconds, the first 10 balls are inserted, and the '1' ball is removed. In the next 15 seconds, 11 through 20 are inserted and the '2' ball is removed, and so on. The respondents are then asked how many ping-pong balls remain in the barrel at the end of the minute. The last one involves a staircase being divided up into finer and finer steps (at each stage having perimeter 2) to approach an incline (of length $\sqrt{2}$). Like many of the students in previous studies, the student I interviewed found these to be strange and paradoxical. However, they also appeared to inspire a great deal of enthusiasm; through having a series of interviews, I was able to see that these paradoxes had inspired independent thinking outside of the interview sessions. The response provided to the ping-pong ball paradox was something I also found particularly notable. Joshua first said that there should be infinitely many, then decided that there should be none after being asked what the numbers of the remaining balls were. However, the analogy he provided here was of slowing down molecules at extremely low temperatures in order to study them; it appeared as if this mental image was being used to accommodate the presence of the balls at times prior to the conclusion of the experiment. I also found it notable that I did not see any tendency toward rejecting the mathematical facts after they had been presented, unlike in many of the students in the prior studies. It is possible that differences in the formation of the mathematical concepts involved led to a difference in what parts of the concept are regarded as the most fundamental, so that the parts considered fundamental here may not be the ones in conflict.

## LIMITATIONS AND SUGGESTIONS

So far, I have only been able to conduct interviews with one research subject, and thus it is important to be careful not to overly generalize the results seen in the interviews. Even with more participants, there are some forms of autism which would indicate notable unwillingness or difficulty in participating in the study, so a study like this can only target a subset of the population. There are also considerations about the content that would apply to the courses that Joshua took, although I think that linear algebra is a fortunate course to have an interviewee from. Some questions that have occurred to me: Would people on the spectrum with less clearly related special interests still use them for mathematical analogies? Is there a tendency for either (more specifically) a bias toward geometric processing or (more generally) a tendency to strongly prefer one type of processing? Are people on the spectrum generally more inclined to accept conclusions that are viewed as paradoxical?

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# OVERVIEW OF RESEARCH ON STUDENTS' VIEWS OF ORAL ASSESSMENT IN MATHEMATICS

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*With the increased emphasis on closed book written examinations, there is a critical need for implementing the oral assessments in mathematics courses. Based on the overview of research on students' views of oral assessment in their mathematics courses, most of the students had positive views towards its use. In this paper, the research on oral assessment in mathematics was summarized with a specific focus on students' views.*

## RATIONALE FOR THE RESEARCH

For many years, the assessment in mathematics classroom seems to be strictly based on closed book written examinations. The USA appears to be dominated by closed book examinations (Gold, 1999; Nelson, 2011). Iannone and Simpson (2011) noted that the majority of mathematics students in the UK seem to be assessed predominately based on high stakes, closed book examinations at the end of almost every module. Joughin (1998) argued that the structure of the assessments today are either closed and formal, with little interaction between student and assessor(s), or open, with less structure and the opportunity for dialogue between student and assessor(s).

We might think that written examinations have always been used as a standard practice of assessing students' mathematical understanding, but that has not been the case. Written examination was the norm in the UK, starting at the beginning of the 20th century. Prior to the beginning of the 20th century, oral examination was a standard practice in the UK, which later on failed because of accusations of bias and the apparent efficiency of written exams (Stray, 2001). The four factors that have been identified as being crucial in this oral/written shift are: the move from group socio-moral to individual cognitive assessment in the later 18th century; the differential difficulty of oral testing in different subjects; the impact of increased student numbers; and the internal politics of Oxford and Cambridge (Stray, 2001). Today, there are some countries that still maintain an oral assessment as an important part of their assessment diet, such as Hungary, Italy and the Czech Republic (Stray, 2001).

There is a perception that an oral assessment may make students more anxious than other forms of assessment. Hounsell, Falchikov, Hounsell, Klampfleitner, Huxham, Thompson and Blair (2007) noted that "It is not clear whether oral assessments are scarier or just more novel" (p. 34). Also, Huxham, Campbell and Westwood (2012) noted that oral assessment anxiety may be primarily related to its unfamiliarity. Joughin (2007) noted, in his phenomenographic study of student experiences of oral presentations, that greater anxiety about oral compared with written assessments was associated with a richer conception of the oral task as requiring deeper understanding and the need to explain to others. On the other hand, Joughin (2007) pointed out that many students described the oral presentations as being more demanding than the written assignments, more personal, requiring deeper understanding, and leading to better learning. One of the advantages of oral assessment, noted by Joughin (1998), is that the interaction allows for probing knowledge through dialogue that is genuinely individual which "Gives assessment an inherent unpredictability in which neither party knows in advance exactly what questions will be asked or what responses will be made" (p. 371). Therefore, oral assessments might be considered fairer than some other forms of assessments. It makes plagiarizing very difficult, where students must explain their own understanding using their own words, and unlike other examinations, oral assessment with a tutor prevents one small gap in knowledge completely stalling a solution (Joughin, 1998). Also, Joughin (1998) noted that oral examinations are more authentic than most types of assessments. He described this authenticity as "The extent to which assessment replicates the context of professional practice or real life" (p. 371).

When it comes to different types of oral assessment, according to Joughin (2010), they can be categorized into three forms: presentation on a prepared topic (individual or in groups); interrogation (covering everything from short-form question-and-answer to the doctoral viva); and application (where candidates apply their knowledge live in a simulated situation, e.g. having trainee doctors undertaking live diagnoses with an actor-patient).

There is a very little literature examining the use of oral assessments. Hounsell, Falchikov, Hounsell, Klampfleitner, Huxham, Thompson and Blair (2007) noted in their comprehensive review of the literature on innovative assessment that less than 2% of the papers address the oral assessments. They reviewed the recent UK literature on 'innovative assessment' and of 317 papers considered, only 31 dealt with 'non-written assessments.' Within this category, only 13% addressed the use of oral examinations.

**PURPOSE**

Considering the fact that the literature on oral assessment in higher education is mostly dominated by the lecturer's or tutor's perspective, not much the learner's (Joughin, 1998), in this paper, I wanted to present the overview of research on students' views of oral assessment in mathematics. The purpose of this paper is to share results from research on students' views of oral assessment in their mathematics courses, with a particular emphasis on affective factors such as beliefs, attitudes, emotions, etc. The need for the study is obvious: oral assessments have a positive impact on students' learning of mathematics (Boedigheimer, Ghrist, Peterson & Kallemyn, 2015; Fan & Yeo, 2007; Iannone & Simpson, 2012; Iannone & Simpson, 2014; Nelson, 2011; Nor & Shahrill, 2014; Odafe, 2006).

## METHODS

There were several criteria used to arrive at the final pool of articles for analysis. The article had to include empirical-based work and have students enrolled in mathematics courses as participants, regardless of their age, course level, or level of learning. The research had to focus on students' views only. The main electronic database explored was ERIC, using combinations of key words: oral assessment, oral examination, oral exam, and mathematics. The electronic database search yielded a total of 26 publications. Abstracts were reviewed to see if the articles met the criteria for inclusion in this project, and the results of this search gleaned a total of 4 articles. After a review of abstracts and in-depth analysis of each paper, I was able to find 3 more articles that appeared to meet the criteria, so the final pool was comprised of a total of 7 articles.

### Participants

- 21 college algebra students (Odafe, 2006)
- ~ 200 first year, secondary students (Fan & Yeo, 2007)
- 36 Calculus I students (treatment group)/62% were considered at risk of failing Calculus I because their placement scores were less than 18/30
- 96 – 140 Calculus I students (control group)/16% were "at risk" (Nelson, 2011) - The treatment consisted of voluntary oral assessments offered before every written examination.
- 108, first-year, discrete mathematics students, graph theory (Iannone & Simpson, 2012; Iannone & Simpson, 2014)
- 28 trigonometry students, 15-16 years old (Nor & Shahrill, 2014)
- Representative comments were collected from anonymous end-of-semester student mathematics course critiques (Boedigheimer, Ghrist, Peterson & Kallemyn, 2015)

### Oral Examination

The structure of the oral examination was the following:

- Exam sessions were scheduled ahead of time
- Students were given questions before an exam (usually one week before an exam)
- Exams were conducted individually or in groups
- Each student would randomly pick a question previously assigned
- Exam questions were created to generate discussion among the students
- Exams consisted of both presentation and dialogue, had a relatively open structure, and combined oral medium with writing on a board
- The examiner would ask follow-up questions after the student's response to the initial questions.

### Tools

There were mainly two instruments designed for data collection, with a purpose to see students' perceptions about their own ability to perform, beliefs in the usefulness, and acceptance of oral assessment/presentation tasks. The two instruments were:

- *Students' questionnaire surveys* (Boedigheimer, Ghrist, Peterson & Kallemyn, 2015; Fan & Yeo, 2007; Iannone & Simpson, 2012; Nelson, 2011; Nor & Shahrill, 2014; Odafe, 2006)
- *Interview with students* (Iannone & Simpson, 2014) (19 students agreed to be interviewed)

## RESULTS: ANXIETY, ATTITUDES, AND BELIEFS

### Anxiety

Some students reported being anxious (Boedigheimer, Ghrist, Peterson & Kallemyn, 2015; Iannone & Simpson, 2014; Nor & Shahrill, 2014; Odafe, 2006). When it comes to students' anxiety, the most common students' comments were related to lack of familiarity (Iannone & Simpson, 2014; Nor & Shahrill, 2014). When students were introduced to the oral assessment, most of them were unfamiliar with this form of assessment, so it caused them to feel anxious about it.

### Attitudes

Most of the students had positive attitudes towards the use of oral assessment (Iannone & Simpson, 2012; Nor & Shahrill, 2014; Odafe, 2006). Moreover, 67 % of students preferred the oral to the written examinations, while only 9 % of students favored both forms of examination equally (Odafe, 2006). Of 85, 73 students commented, and most of them were being positive about oral assessment as an alternative to weekly coursework sheets, but negative about them as an alternative to examinations (Iannone & Simpson, 2012). Also, 65.4% of the students agreed that they 'like to do Mathematics poster and oral presentations during Mathematics lesson' in comparing with only 34.6% who disagreed (Nor & Shahrill, 2014).

### Beliefs

Taken from the provided research, students believed that oral assessments in mathematics are:

- *Reactive to students' needs* (Boedigheimer, Ghrist, Peterson & Kallemyn, 2015; Iannone & Simpson, 2012; Iannone & Simpson, 2014; Odafe, 2006)
- *Promote deep comprehension of the learned material* (Fan & Yeo, 2007; Iannone & Simpson, 2012; Iannone & Simpson, 2014; Nelson, 2011; Nor & Shahrill, 2014; Odafe, 2006)
- *Encourage students to deeply/actively engage with the course material* (Boedigheimer, Ghrist, Peterson & Kallemyn, 2015; Iannone & Simpson, 2012; Nor & Shahrill, 2014; Odafe, 2006)

**Reactive to students' needs**

When it comes to reactiveness of oral assessments to students' needs, 67 % of students claimed that they studied and learned more in the process, and that oral assessment enabled them to identify their mistakes and misconceptions and got them corrected immediately, by being provided with immediate verbal feedback during the examination (Odafe, 2006). Moreover, the assessor could respond to the performance of the student and to their needs, and this brought up the opportunity for the assessor to help the student over a gap in knowledge (Iannone & Simpson, 2014).

**Promote deep comprehension of the learned material**

In one study, 45% of the students believed that 'doing mathematics oral presentation helps me to be more aware of my understanding of mathematics,' vs. 32.5% who disagreed and 22.5% who neither disagreed nor agreed (Fan & Yeo, 2007). Also, 42.5% of the students believed that 'doing mathematics oral presentation makes me think broader and deeper about mathematics,' vs. 35% who disagreed and 22.5% who neither disagreed nor agreed (Fan & Yeo, 2007). More than 70% of the students believed that 'listening to other classmates' oral presentation is helpful for me in learning mathematics' (Fan & Yeo, 2007). Some of the students' comments, related to the use of oral assessment, were such as:

> "Let me test my knowledge, and allowed me to recognize my gaps of understanding in some areas" (Nelson, 2011, p. 51).
>
> "We went over problems in depth: it helped work out any things we didn't understand" (Nelson, 2011, p. 51).

Furthermore, the treatment students, who were considered at-risk, did significantly better than the control group on course grades, completed Calculus I and enrolled in and passed Calculus II at dramatically higher rates than at-risk students in the control group (Nelson, 2011). Many of these students reported that oral assessments were a major reason for their success. Also, 88.5% of the students believed that doing mathematics poster and oral presentations helped them to learn mathematics in comparing with only 11.5% who disagreed (Nor & Shahrill, 2014). The mean of the pre-test of the whole class were found to have improved from 29.80 to 67.05 in the post-test, which was an increase by 37.25 (Nor & Shahrill, 2014). These students were given pre-test prior to the 'project' (poster and oral presentations) and a post-test after. The questions on the pre-test and post-test were identical. On the other hand, some students saw the oral assessment as the potential value for promoting understanding of the material, while not replacing the final, high stakes, written closed book exam (Iannone & Simpson, 2012).

**Encourage students to deeply/actively engage with the course material**

Of 21 college algebra students, 19 admitted putting in more time and effort into preparing for the oral examination, compared to the time put into preparing for the traditional pen and paper examination. The other 2 students claimed to have spent just about the same amount of time and effort on both forms of assessment (Odafe, 2006). The oral assessments made students think more about the material, encouraged them to understand the material better, made them remember the material longer after the assessment, and did not allow them to get high marks without understanding (Iannone & Simpson, 2012).

## CONCLUSIONS

At the end, we can say with certainty that oral assessment can be a useful tool for assessing the students' conceptual understanding of the learned material and result in increased students' learning of mathematics (Boedigheimer, Ghrist, Peterson & Kallemyn, 2015; Fan & Yeo, 2007; Iannone & Simpson, 2012; Iannone & Simpson, 2014; Nelson, 2011; Nor & Shahrill, 2014; Odafe, 2006). During the oral examination, teachers can discover gaps in student knowledge or common misconceptions that can be corrected immediately. Moreover, it provides teachers with the opportunity to actively listen to students communicate mathematics and value the students' contributions. Therefore, when students realize that their contributions have been valued, they will start to appreciate the subject of mathematics more, and this could change completely their attitudes towards the mathematics. In addition, if we, as we do, acknowledge that each student learns differently, then having a common approach to assessment would be inadequate. Hence, educators accept the need for differentiated instruction, so in order to deal with the individuality and variability of students, they also need to accept the need for differentiated assessment to represent the learning of the fractured student collective (Liljedahl, 2010).

### Future Research

Based on these findings on students' views of oral assessment in mathematics, the following questions can be considered for future research:

- What kind of mathematical understanding can students show during the oral assessment?
- What kind of students' mathematical understanding can examiner discover during the oral assessment?
- Do the students' responses in oral and written examinations differ between different types of questions?

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