MEDS-C 2019

PROCEEDINGS OF THE 14TH ANNUAL MATHEMATICS EDUCATION DOCTORAL STUDENTS CONFERENCE

November 2, 2019

SIMON FRASER UNIVERSITY | FACULTY OF EDUCATION

TABLE OF CONTENTS

BRIEF REFLECTIONS ON THE MATHEMATICS EDUCATION DOCTORAL STUDENTS CONFERENCE

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Occurring for the first time on November 25, 2006, and marking its 14th instance with the most recent activities on November 2, 2019, the annual Mathematics Education Doctoral Students Conference (MEDS-C) continues to evolve with each passing year. Not only has its participating cohort metamorphosed over time, but, naturally, so too have aspects of its structure and facilitation seen change. In part because my time in the Mathematics Education doctoral program is gradually drawing to a close, I have inevitably begun to reflect upon the experience of engaging with MEDS-C and its related pre- and post-conference activities from one year to the next. Beyond encouraging a deeper appreciation of the conference's role in fostering both our individual and collective growth as graduate students, beyond reaffirming a great respect for the many prominent speakers and guests who have attended the event and shared their insights/expertise, beyond bringing back to mind the wealth of intriguing research themes that have been presented, doing so has also piqued an interest in (or, perhaps, curiosity about) the conference's more general history and development, not to mention the ways in which the space of the conference is gradually being reshaped.

Personal retrospection, however, is not the only reason I have opted to compose this prefacing commentary. In fact, I am also viewing this as something of an opportunity to reintroduce a formal reflective component to MEDS-C at large (if only by way of these conference proceedings). Interestingly, when looking back into past proceedings documents, I discovered that the conference activities of November 14, 2009 concluded with a 30-minute session entitled "Reflection and vision for the future of MEDS", which appears to have involved all attendees. Judging from the available conference programmes from other years, this constituted the first instance of group reflection being integrated into the conference proper and recorded as a canonical activity. More telling, though, is the *absence* of any scheduled or recorded reflective practice in subsequent years/programmes, such that this brief session in 2009 would seem to mark the sole instance of communal reflection on the past history and future potential of the conference. In addition to being curious why this is the case, I am also led to wonder if the sudden inclination to reintroduce a reflective component to MEDS-C *now* (i.e. 10 years later) might in some way point to an emerging cycle for the conference itself. If nothing else, perhaps it can be seen as a compelling coincidence.

In essence, what follows is simply an articulation of various considerations about the conference's structure, its associated activities, and a number of other factors that have

2019. In A. Hare, J. Larsen & M. Liu (Eds.). *Proceedings of the 14th Annual Mathematics Education Doctoral Students Conference* (pp. 18-26). Burnaby, Canada: MEDS-C.

entered into my thoughts. While some of these considerations might already be shared by the reader, I expect that others will be more singular/esoteric. Nevertheless, it occurs to me that there should be value in committing such thoughts to print, if even simply to foster additional reflection on the part of current and future MEDS-C participants/organizers/contributors, or to indicate a few different ways in which the overall milieu of the conference might be perceived. In keeping with this general intent, and noting that MEDS-C is a primarily student-run affair, I encourage my peers/colleagues to periodically revisit/reconsider/reaffirm their own sensibilities about the nature of the Mathematics Education Doctoral Students Conference, and to give further thought to the trajectory they might envision for it in the coming years.

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Amongst its more widely known features is the fact that MEDS-C is, by design, modelled after the annual conference in the Psychology of Mathematics Education (or PME). Inasmuch, MEDS-C mirrors/emulates aspects of PME and its accompanying formalisms, including manuscript submission, peer-review, oral presentation of research, audience Q&A, compilation/editing of proceedings, et cetera. Indeed, many MEDS-C participants parlay their work into subsequent PME papers and presentations, and the adoption of the PME template for MEDS-C manuscripts illustrates one particular manner in which MEDS-C is especially well-suited as a sort of staging ground for PME submissions.

While its purposeful similarities to PME are part of MEDS-C's structural DNA, I would suggest that it is also important to acknowledge that MEDS-C need not be perceived as existing within the shadow of PME, or as being subordinate to that conference in any appreciable way. By stating this, I do not intend to minimize the obvious significance of the two conferences' connections, or to suggest that anyone overlook/ignore the good sense of utilizing MEDS-C as a space for refining work in progress. Rather, I mean only to point out that MEDS-C carries much innate value of its own, outside the context of its links to PME. Should one choose to perceive it in such a light, MEDS-C can be seen as occupying quite a unique position insofar as academic conferences in our field are concerned.

As I have come to know it, MEDS-C realizes a number of different, yet overlapping, spaces for our growth as doctoral students, teachers/practitioners, and researchers in mathematics education. Although the notion of *developing and sharing graduate research* might broadly encapsulate many of these, it would not be all-encompassing. Thus, I briefly outline a few notable spaces in the pages to follow, whilst acknowledging that my perspectives may not necessarily reflect or fully represent those of my peers/colleagues.

MEDS-C AS A SPACE FOR COMMUNITY BUILDING AND NETWORKING

With the overall expansion of the mathematics education doctoral cohort, the extent to which MEDS-C instantiates a community-oriented programme has become increasingly apparent. Particularly for more senior graduate students who are no longer engaged in course work or attending seminars, the annual conference comes as a regular opportunity to reconvene with colleagues and faculty, to engage with visiting scholars, and to forge new relationships with beginning graduate students as well as potential inductees to the Mathematics Education program. Albeit less frequently than the semesterly book club gatherings, MEDS-C facilitates both professional and casual socialization, each of which contribute to its communal atmosphere and discursive relevance.

As all those who have been involved with the conference are aware, MEDS-C also makes possible a wide range of collaborative activity. Whether at the level of organizing the event itself, facilitating peer review, compiling/editing conference proceedings, curating digital spaces, troubleshooting technical issues, or ensuring that conference-goers are well-fed and comfortable, the impact and importance of these paired or small-group partnerships cannot be overstated. It is true that many aspects of the more general graduate student endeavour can involve spates of largely solitary activity; however, the various collaborations afforded by our annual conference have the potential to be greatly rewarding and enlightening for those who seek them out.

MEDS-C AS A SPACE FOR ENCULTURATION INTO ACADEMIA

Despite being a single conference internal to a doctoral program within a specific university, MEDS-C's influence extends well beyond our Mathematics Education doctoral program and its parent institution. By virtue of the tasks/activities it entails and the community it draws together, the conference plays a significant role in helping to enculturate its participants into the world of academia. With the members of our cohort coming from many different backgrounds and walks of life, and pursuing diverse interests and goals, MEDS-C reflects not only the work being done within the doctoral program itself, but also the many ways in which fledgling members of the mathematics education research community are choosing to engage with the field at large.

In keeping with this theme of enculturation, as a largely student-driven event MEDS-C is simultaneously a representation of and a precursor to the established academic culture into which we are being inducted. As its participants, then, we are shaping aspects of that culture at the very same time as we are being shaped by it. By virtue of this mutual interplay, for some, MEDS-C can be instrumental in determining how they might wish to begin navigating the academic terrain and negotiating its myriad streets. Subsequently, much can be said of the relief/comfort that comes in knowing that it is possible for more tentative steps into this complex environment to take place in the company of empathetic peers, under the guidance of supportive elders.

MEDS-C AS A SPACE FOR EXPERIMENTATION AND EXPLORATION

While the sentiment may not be as commonplace as those already expressed, I will venture here to suggest that MEDS-C can also be seen as realizing a powerful space for experimentation and exploration, possibly even self-discovery. Invariably, facets of the conference do pose constraints; yet, with those constraints also come other affordances that may not be immediately evident. By this, I mean to emphasize that the conference sets forth various opportunities to experiment with modes of expression and the ways in which we, as participants/contributors, ultimately choose to articulate scholarly ideas in both written and verbal form.

Although it may not always seem to be the case, it is entirely possible to experiment with voice and tonality, to play with structural conventions, to explore alternative presentation styles and engage with one's audience/readers differently, and even to subvert expectations if one might be so inclined. In spite of certain restrictive structures inherent to MEDS-C, I am of the impression that the more permissive setting of our conference does allow for such experimentation to be highly fruitful/generative, and that it enables (or at least empowers) us to actualize/employ more tentative approaches than might be possible in other conference settings. At the time of drafting this reflective piece, it is unclear to me how common it is to be afforded opportunities to play around the borders of a given academic field, and possibly even to push at them a little. In that sense, I suggest that MEDS-C instantiates quite an interesting and liberating set of circumstances, for it necessitates adherence to certain common academic constraints whilst being entirely *free* of a great many others. As a result, for those graduate students who might be looking to broaden the space of possibility that is examined within the field of mathematics education, it can be rather fertile ground.

A SPECIAL NOTE ABOUT THE PROCEEDINGS

At the expense of a more elaborate discussion, but in favour of keeping things concise, I return to the underlying motivation of this prefacing commentary, namely the desire to reintroduce a reflective component to MEDS-C at large. Although it is not my place to specify what form such reflection might take, the *process* of engaging in meaningful reflection is certainly something I hope to promote. That said, it seems clear that the ongoing evolution of our annual graduate student conference will continue to be informed by its history, and it is for this reason that I briefly speak to another curious discovery gleaned from the examination of existing conference materials, which has prompted me to include additional comments about the MEDS-C proceedings themselves. As with the dearth of formal reflective practice following the 2009 conference, this particular discovery is also marked by an interesting absence.

Noted earlier, MEDS-C has now reached its 14^{th} instance, yet this is not true of the accompanying proceedings. While a conference did indeed occur on November 29, 2014, it would appear that a proceedings document was never actually generated; therefore, an unusual gap has manifested in the lineage of the conference records. As an unfortunate result, certain traces of the 2014 conference can now only be accessed/revisited anecdotally, which is to say that these traces exist primarily within the memories of the various individuals who attended the conference or were otherwise involved in its planning/execution. To date, MEDS-C 2014 is the only instance of the conference characterized by such circumstances. Albeit an isolated case, it does raise the interesting question of how else we might point to, or index, this occurrence of the conference beyond the remembrances of those involved, or any fragmentary allusions to it that may exist elsewhere.

I cannot offer much insight into the conference's history before my own induction into the Mathematics Education doctoral program; however, exploring the traces of past conferences embodied by their proceedings has given some indication of the overall "flavour" of those events. Even for an academic conference as young as MEDS-C, it is clear that the proceedings not only serve as a reflection of the various research contributions from one year to the next, but also satisfy the function of marking/recording the *existence* of the conference as an actual series of events that transpire in a specific place and time. Thus, they are tangible, historical artefacts as much as they are condensed overviews of research in progress.

Via a loose metaphor, the reader might even envision the conference proceedings as a sequence of figurative snapshots/photographs of the mathematics education community (or a small part thereof), and as indicators of the diverse topics and themes that motivate its newest members. In that sense, the proceedings resemble a cross-sectional image of our evolving field, captured mid-growth. I articulate this point in order to stress that these documents extend beyond the anecdotal, allowing even previous conferences to persist/endure in such a way that they might be re-entered anew at any time (by anyone). Granted, engaging with the written content of the proceedings undoubtedly differs from the lived experience of the conference itself; nevertheless, the former affords all interested parties (be they attendees of the original conferences or not), a kind of asynchronous and extended/ongoing access to the latter. As a cumulative whole, the MEDS-C proceedings help to tell the story of how a portion of our field is developing over time. By collectively preserving that history, illustrating growth, and pointing toward future trends, they can be considered a deeply informative and powerful tool.

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Having very likely said too much at this point, I close out this prefacing commentary by once again encouraging my peers/colleagues to revisit/reconsider/reaffirm their own sensibilities about the nature of the Mathematics Education Doctoral Students Conference, and to reflect upon the constant evolutionary change it is undergoing. I encourage future participants/organizers/contributors to consider how MEDS-C also exists as a distinct entity, and how they might *already* perceive its many overlapping spaces (including those not mentioned in the current preface). While there is certainly value in working within the existing structures and constraints of MEDS-C, much potentially stands to be gained by periodically re-evaluating those same structures and constraints, and thoughtfully/judiciously modifying them in response to the changing goals of the graduate community that MEDS-C represents. Finally, I encourage my peers/colleagues to explore the invaluable traces of past conferences that are housed within the available MEDS-C proceedings. At the time of writing this prefacing commentary, those documents may be readily accessed via the following URL:

https://www.sfu.ca/education/mathphd/program-expectations/meds-c/past-meds-conferences.html

It is my sincere hope that current and future members of the Mathematics Education doctoral cohort will deem it worthwhile to dip into these documents on occasion, so as to engage with the multifaceted works within and to develop a greater appreciation of the vastly different perspectives being articulated in our field. On behalf of the Mathematics Education doctoral cohort and our supervising faculty, I also humbly extend additional thanks to the array of exceptional plenary speakers who have, to date, greatly enhanced the MEDS-C activities with their diverse contributions and insights. For ease of reference, a comprehensive list of these speakers and the titles of their respective talks has been included below:

- November 25, 2006: Brent Davis
  Complexity Thinking as a Pragmatic Pedagogical and Investigative Tool

- December 8, 2007: Nicholas Jackiw
  Perspectives on Dynamic Geometry Software

- November 22, 2008: David Pimm
  Mathematics Education as an Interdisciplinary Endeavour: Over Thirty Years of Looking Elsewhere

- November 14, 2009: Thomas O’Shea
  The Accidental Professor

- September 25, 2010: Robin Barrow
  Some Fundamental Questions and Egregious Errors in Educational Thought

- November 19, 2011: David Robitaille
  International Studies in Mathematics Education: TIMSS and PISA

- December 8, 2012: Lulu Healy

Mathematical Cognition and Embodied Experience: Learning from Students with Disabilities

- November 23, 2013: Beth Herbel-Eisenmann
  From Talking the Talk to Walking the Walk: An Exploration of Data Using Tools from SFL

- November 29, 2014: John Mason
  A Few of My Favourite Tasks

- October 17, 2015: Cynthia Nicol
  Slow Pedagogy, Research and Relations: Building Relationships for Research that Matter

- December 3, 2016: Alf Coles
  An Enactivist Story of Researching the Teaching and Learning of Mathematics

- November 4, 2017: Ofer Marmur
  Undergraduate Student Learning During Large-Group Calculus Tutorials: Key Memorable Events

- November 10, 2018: Xiaoheng Kitty Yan
  What's the Story? Identifying Key Idea(s) in Proof in Undergraduate Mathematics Classrooms

- November 2, 2019: Egan Chernoff
  The Canadian Math Wars: A Disagreement over School Mathematics

Respectfully,

Jason T. Forde

MEDS-C 2019 PROGRAMME – NOVEMBER 2, 2019

8:30 – 9:00	Welcome and Coffee – Learning Hub EDUC 8620		
	EDB 8620.1	EDB 8620.2	EDB 8625
9:00 – 9:35	Andrew Hare Getting New Writing on the Board in an Undergraduate Mathematics Lecture	Peter Lee Dyscalculia in the Media: A Critical Discourse Analysis of Two News Articles	Annette Rouleau “I Don’t Want to Be *That* Teacher”: Anti-Goals in Teacher Change
9:40 – 10:15	Sandy Bakos Teacher-as-Learner or Teacher-as-Expert?	Pauline Tiong Talking in Mathematics – Do We Know How?	
10:15 – 10:30	Break		
10:30 – 11:05	Srividhya Balaji Understanding Creativity Through Emergence: A Process Perspective	Victoria Guyevskey Learning Geometry Through Drawing and Dragging in a Primary Mathematics Classroom	
11:10 – 11:45	Rob Sidley Changing Assessment Practice and Teacher Conceptions of Student Improvement in Mathematics	Canan Güneş Reciprocal Influences in a Duo of Artefacts	
11:50 – 12:25	Mike Pruner Thinking Classrooms and Complexity Theory	Sam Riley Logarithms through Textbooks: From Calculating Tool to Mathematical Object	
12:25 – 1:25	Lunch		
1:30 – 2:15	Plenary Speaker: Egan Chernoff The Canadian Math Wars: A Disagreement over School Mathematics		
2:15 – 2:30	Plenary Q & A		

2:35 – 3:10	Niusha Modabbernia Not Choosing is Also a Choice	Max Sterelyukhin Exploring the Bentwood Box: Collaboration in Lesson Design and Implementation	
3:10 – 3:25	Lunch		
3:25 – 4:00	Yumi Clark Dot Product - It's so Easy?	Wai Keung Lau Mathematical Problems that have No Known Solution in the Secondary Curriculum	
4:05 – 4:40	Sheree Rodney Embodied Curiosity, Geometer's Sketchpad and Mathematical Meanings	Jason Forde Engaging with Metaphors: Borderless Puzzles and Defragmentation	
4:40 – 5:00	Wrap up – EDUC 8620		

CONTRIBUTIONS

MEDS-C 2019 was organized by members of the Mathematics Education Doctoral Program. The conference would not have been possible without the following contributions:

Conference Coordinators:	Sandy Bakos and Robert Sidley
Final Survey:	Pauline Tiong
Lunch Coordinator:	Max Sterelyukhin
Photographer:	Victoria Guyevskey and Sam Riley
Proceedings Editors:	Andrew Hare, Judy Larsen, and Minnie Liu
Program Coordinator:	Srividhya Balaji and Annette Rouleau
Review Coordinators:	Jason Forde and Sheree Rodney
Snack Coordinators:	Canan Günes and Niusha Modabbernia
Technology Support:	Michael Pruner
Timers:	Yumi Clark and Peter Lee

PLENARY SPEAKER

Egan Chernoff

THE CANADIAN MATH WARS: A DISAGREEMENT OVER SCHOOL MATHEMATICS

The Math Wars, Eh? Believe it or not, the teaching and learning of mathematics has become a staple of local, provincial and national media coverage over the last five years. The purpose of this talk is to provide an abridged version (5 years presented in 1 hour) of the recent heated debate over the teaching and learning of mathematics.

ABSTRACTS

Sandy Bakos

TEACHER-AS-LEARNER OR TEACHER-AS-EXPERT?

This paper examines two primary teachers' explorations with TouchTimes *(*TT*), an iPad touchscreen application designed to provide a visual and kinaesthetic way to engage with multiplication directly through touch. Using the Theory of Semiotic Mediation, I will focus on the key notion of semiotic potential by analysing each teacher's first interactions with* TT. *While seeking to gain insight into the use of the app from the perspectives of these teachers, there were interesting shifts observed between the teacher-as-expert and the teacher-as-learner. The semiotic potential, however, did not always arise from the artefact and lead to the mathematics, as expected. Rather, there were instances where the teachers' prior knowledge of the mathematics actually led to understanding the artefact.*

Srividhya Balaji

UNDERSTANDING CREATIVITY THROUGH EMERGENCE: A PROCESS PERSPECTIVE

Creativity has always been assessed from two important viewpoints: novelty and usefulness. However, these perspectives enrich our understanding only of the creative product and not of the underlying process. To gather a holistic appreciation of what creativity entails, both the creative process and the product need to be studied hand in hand. To accomplish that, it would be worthwhile for us to understand the nature of the creative process and its unfolding. In this paper, I primarily argue that creativity is an emergent phenomenon with the help of a modern theoretical framework called enactivism and I examine the creative processes involved in a mathematical proof generation. Interview excerpts from two expert mathematicians are analysed and the idea of emergence involved in their creative endeavours are discussed.

Yumi Clark

DOT PRODUCT - IT'S SO EASY?

In many college level math classes, concepts are often presented in such a way that overlooks the remarkable mathematical achievements culminating in those concepts. How the dot product of two vectors is introduced in the 2- or 3-dimensional space is a prime example. Given two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\vec{v} = \langle v_1, v_2, v_3 \rangle$*, students often find it very trivial to compute* $\vec{u} \cdot \vec{v} = u_1 v_{1_1} + u_2 v_2 + u_3 v_3$*. This trivial computation often gives them false impressions about their understanding of the concept. How do* *textbooks facilitate students' understanding of dot product? With this question in mind, a textbook analysis involving 13 textbooks in mathematics, physics, and engineering was conducted. This paper discusses its results and findings.*

Jason Forde

ENGAGING WITH METAPHORS: BORDERLESS PUZZLES AND DEFRAGMENTATION

In a deliberate move away from the typical MEDS-C submission, this paper focuses less on the articulation of specific research findings, and more on the communication of insights gleaned from the process of writing about my primary research themes (namely the nature of mathematics and the notion of material assemblage). Specifically, two complementary metaphors which have proven useful in formulating my view of mathematics as the science of material assemblage are discussed at a metacognitive level, using the writing process itself as a framing device. I also briefly indicate how the metaphors of interest (i.e. borderless puzzles and defragmentation) can be linked to underlying mathematical considerations.

Canan Günes

RECIPROCAL INFLUENCES IN A DUO OF ARTEFACTS

The combined use of a physical pedagogical artefact and its digital counterpart is described as a duo of artefacts. In the literature duos of artefacts are presented with a certain order assuming that the digital counterpart enhances mathematical knowledge by adding affordances to the physical artefact. This study examines the effect of reciprocal use of artefacts in a duo on a 5-year-old child's identification of relationships between the objects. Data is created through the video record of two clinical interviews with the child. The results show that unless they are used reciprocally, none of the artefacts were enough to mediate relationships between the objects of artefacts which are important for multiplicative thinking.

Victoria Guyevskey

LEARNING GEOMETRY THROUGH DRAWING AND DRAGGING IN A PRIMARY MATHEMATICS CLASSROOM

This project was carried out in a mathematics classroom in an affluent and culturally diverse urban elementary school in North America. We conducted a month-long classroom intervention with Grade 2/3 students, experimenting with geometric tasks within physical environment of paper-and-pencil, and virtual multitouch environment of dynamic geometry. In our experiments, we were interested in specific ways these two contexts give rise to mathematical concepts, and how learning affordances of digital and tangible tools are complementary and different. We wanted to see (1) what *the students would learn, and (2) what the constraints and liberations of those environments would be. We found that there was no binary distinction between the two.*

Andrew Hare

GETTING NEW WRITING ON THE BOARD IN AN UNDERGRADUATE MATHEMATICS LECTURE

In this paper I take seriously the task of the lecturer in undergraduate mathematics: to write on the board a selection of true results and precise definitions while providing convincing argumentation justifying these inscriptions. Using a microethnographic approach that emphasizes contexts and the role of the hands, I analyze a few moments to highlight some common writing/speaking/gesturing actions: construal of a piece of writing in order to make a contrast, construal of a piece of writing in order to make a specific choice from a general type, nonlinear writing, grasping and circling to indicate "many", and moving the hands from one place to another while keeping the shape constant in order to indicate equality.

Wai Keung Lau

MATHEMATICAL PROBLEMS THAT HAVE NO KNOWN SOLUTION IN THE SECONDARY CURRICULUM

It is possible to find a connection between high school mathematics and mathematics beyond the curriculum. In this paper, I offer two well-known examples, namely, Euclid's fifth postulate (parallel postulate) and Ping-pong ball conundrum (Littlewood-Ross paradox). The former is equivalent to say that "sum of the angles of a triangle is 180°", and the latter involves considerable cognitive conflict in different sizes of infinity. I argue that good examples not only can reveal the beauty of mathematics but also can inspire students' interest in mathematics. I also say that despite intuitive ideas without rigours proof may not be valid for formal mathematics, students may gain extra benefit rather than just delivering the conventional lessons. However, how to choose good representative examples for high school students is one of the crucial points for high school teachers and the researchers.

Peter Lee

DYSCALCULIA IN THE MEDIA: A CRITICAL DISCOURSE ANALYSIS OF TWO NEWS ARTICLES

Dyscalculia has received relatively little attention in the popular media compared to other disabilities such as dyslexia. This paper applies the tools of critical discourse analysis to examine two rare articles on the disability. Discursive representations of dyscalculics, cognitive neuroscientists and their research on the brain and the roots of

number sense are examined for how such representations are influenced by ideology and the media genre.

Niusha Modabbernia

NOT CHOOSING IS ALSO A CHOICE

Although counting problems are easy to state there is much evidence that students struggle with solving counting problems correctly. As this topic became part of K-12 and undergraduate curricula, there is a necessity to study factors that might have affected students' success. Detecting all the choices in solving a counting problem is one of the factors of students' success. The option of not choosing which may not often be considered as a choice is the core of this research. A pair of prospective high school teachers participated in this research. Their combinatorial thinking was examined in term of Lockwood's model (2013) with the focus of detecting the option of not choosing.

Mike Pruner

THINKING CLASSROOMS AND COMPLEXITY THEORY

In this article I will look at how Thinking Classrooms can be described and studied through the lens of Complexity theory. A Thinking Classroom is a teaching framework developed by Peter Liljedahl to occasion greater supports for student activity and engagement through the extensive use of randomized groupings, problem solving on vertical whiteboards, and sequenced tasks to maintain flow. The public nature of the whiteboard surface and the close and fluid interactions of the students affords the potential for ideas, hunches, queries and representations to move freely through the room. In this article, I describe the connection between Thinking Classrooms and Complexity theory and how emergent events may be observed in this environment.

Sam Riley

LOGARITHMS THROUGH TEXTBOOKS: FROM CALCULATING TOOL TO MATHEMATICAL OBJECT

Throughout history, logarithms have been understood and therefore presented in many different ways. How they were introduced in a textbook affected the work done with logarithms throughout the text, as well as affecting what sort of understanding readers could take from the text. By looking at three texts used in the same area over 150 years, through the lens of Anna Sfard's Operational/Structural conception duality, I will analyse how these texts built understanding in their readers.

Sheree Rodney

EMBODIED CURIOSITY, GEOMETER'S SKETCHPAD AND MATHEMATICAL MEANINGS

This paper is a corollary to a larger research study. It examines how two grade nine students, both 15 years old, from a secondary school in Jamaica, interacted with circle geometry theorems in a Dynamic Geometry Environment (DGE) called The Geometer's Sketchpad (GSP). I utilize the notion of Embodied Curiosity, as well as, Andrew Pickering's idea of agency (the influence of human and non-human actions against each other), to analyse the ways in which Embodied Curiosity emerge when students interact with their peers on geometric tasks. In addition, I adopt parts of Berlyne's curiosity dimension model as a methodological tool to identify physical markers of when and how students become curious. I argue that curiosity along with digital technology, body movements and mathematical meanings work hand-in-hand for learning to take place. I also suggest that curiosity; the main ingredient, plays an important role in shaping the body and the mind.

Annette Rouleau

"I DON'T WANT TO BE *THAT* TEACHER": ANTI-GOALS IN TEACHER CHANGE

This paper uses the theory of goal-directed learning to examine anti-goals that arise as teachers implement change in their mathematics practice. Findings suggest that anti-goals develop as teachers begin to recognize who they ***do not*** *want to be as a mathematics teacher. Accompanying anti-goals are emotions that can be useful in measuring progress towards anti-goals (fear and anxiety), and away from anti-goals (relief and security). Furthermore, acknowledging anti-goals allows mathematics teachers to focus on the cognitive source of their difficulties rather than be overwhelmed by the emotional symptoms.*

Max Sterelyukhin

EXPLORING THE BENTWOOD BOX: COLLABORATION IN LESSON DESIGN AND IMPLEMENTATION

This work resulted from an attempt to collaboratively design and implement a lesson with the lens of First Peoples Principles of Learning in a high school Mathematics 8 class in 2018-2019 academic year. We describe the process, outline the key objectives and challenges in both design and implementation stages. We also discuss the reflections and the learning observed by teachers as designers as well as learners along with the students. The analysis of noted teachers' experiences and observations showed the complexity of the challenge to incorporate the indigenous ways of learning into teaching practice is substantial among mathematics teachers and the lack of *knowledge in the subject matter remains large. We discuss possible approaches of bridging the gap between the current state of First Peoples Principles of Learning to what it is mandated to be by the Ministry of Education in British Columbia mathematics classrooms.*

Pauline Tiong

TALKING IN MATHEMATICS – DO WE KNOW HOW?

The notion of talking in mathematics or what is more commonly referred to as spoken communication in mathematics classrooms has been an increasingly important yet demanding task for both students and teachers. Specifically teachers face the challenge of orchestrating and facilitating meaningful mathematical talks with and for their students. As an in-depth literature review of the notion of spoken communication in mathematics classrooms, this paper serves as a preliminary exploration to address what teachers need to know or do to help students develop their mathematical spoken communicative competence. A possible framework which may explicate why and how spoken communication (or mathematical talk) can contribute to mathematics teaching (and learning) is proposed as a result of this exploration.

TEACHER-AS-LEARNER OR TEACHER-AS-EXPERT?

Sandy Bakos

Simon Fraser University

This paper examines two primary teachers' explorations with TouchTimes *(*TT*), an iPad touchscreen application designed to provide a visual and kinaesthetic way to engage with multiplication directly through touch. Using the Theory of Semiotic Mediation, I will focus on the key notion of semiotic potential by analysing each teacher's first interactions with* TT. *While seeking to gain insight into the use of the app from the perspectives of these teachers, there were interesting shifts observed between the teacher-as-expert and the teacher-as-learner. The semiotic potential, however, did not always arise from the artefact and lead to the mathematics, as expected. Rather, there were instances where the teachers' prior knowledge of the mathematics actually led to understanding the artefact.*

INTRODUCTION

Multiplication is commonly introduced in primary schools through counting in groups or repeated addition. Supported by curriculum documents and deeply rooted in teaching practice, multiplication as repeated addition has become the dominant perception of multiplicative situations for both primary students *and* primary teachers (Askew, 2018). This is problematic because although multiplicative structures do rely partly on additive structures, Vergnaud (1983) points out that they also possess "their own intrinsic organization which is not reducible to additive aspects" (p. 128). Davydov (1992) argues that "multiplication does not receive any special advantage" (p. 9) when thought of as repeated addition. Thinking about multiplication only in this way is limiting and later creates difficulties for students when they progress to more complex mathematical concepts requiring a direct capacity to think multiplicatively (Siemon, Breed & Virgona, 2005).

In an effort to engage primary school children with multiplicative relationships in an easily accessible, visual and kinaesthetic way, an interactive touchscreen iPad application called *TouchTimes* (hereafter TT) has been developed. Inspired by Vergnaud's (1983) relational and functional aspects of multiplication, and Davydov's (1992) double change-in-unit, TT provides students with a model for multiplication that does not rely on repeated addition.

Brief description of TouchTimes

Prior to reading the ensuing description of TT, it may be helpful to view this short video demonstration of the app (m.youtube.com/watch?v=L3BRXZfBbZo). When first opened, the screen is split in half by a vertical line. Different coloured discs (called 'pips') appear beneath the user's fingers upon contact with whichever side of the screen is touched first (the app is designed to be symmetric) and will remain so long as the user's fingers maintain continuous screen contact. The numeral that corresponds to the number of pips created is displayed at the top of the screen (Figure 1a) and adjusts instantly as pip-creating fingers are added or removed.

Once the pips are established, the user is then able to create bundles of pips (termed 'pods') by tapping on the side of the screen that is opposite the vertical line. The pods reflect a duplicate of the colours and configuration of the original pips (Figure 1b), and as the pods are created, a multiplication (×) sign, and a second numeral appear as part of the mathematical expression. The pips represent the multiplicand, and the pods, the multiplier. Both pips and pods may be produced by sequential or simultaneous screen contact, but unlike the pips, the pods do not require continuous finger contact in order to remain visible on the screen. TT encircles all of the pods in a single unit with a white 'net' while simultaneously completing the mathematical expression by displaying the product in white (Figure 1c). The content of the pods and the subsequent product are directly affected by the addition and removal of pips, and if all of the pip-creating fingers are lifted from the screen, then all of the pips and pods will disappear.

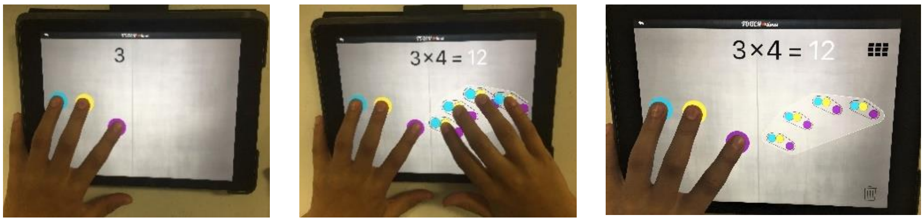

Figure 1: (a) Creating pips (b) Creating pods (c) Finished expression

Modeled upon the double change-in-unit process, the first unitising in TT takes place when the multiplicand (represented by the pips) is determined, and the second unitising occurs when the number of units (or pods) is established. Consistent with the Davydovian approach to multiplication where the *unit quantity* is identified prior to the number of units, the multiplicand precedes the multiplier in TT. One way to conceptualise this action is to see the pips unitised into pods (first unitising), and then the pods unitised into the product. This order is not commonly found in Alberta textbooks where the multiplier always precedes the multiplicand and is the opposite of how many teachers think about and teach multiplication to their students.

THEORETICAL FRAMEWORK

Built upon the Vygotskian perspective of signs as semiotic mediators, the Theory of Semiotic Mediation (hereafter, TSM) (Bartolini Bussi & Mariotti, 2008) further extends Vygotsky's ideas into the context of the mathematics classroom. From an educational perspective, semiotic mediation refers to the study, use and interpretation of signs and symbols (semiotics) and the potential use of these signs and symbols as mediators between learners and knowledge. The TSM provides a framework for describing the teaching-learning sequence which begins with the integration of an artefact for use in completing a task, and proceeds with the teacher guiding students towards mathematical understanding. Knowledge of the *semiotic potential of the artefact* in use and thoughtful engagement with the *didactic cycle* are key components that underlie the semiotic and educational perspectives of the TSM. Though more commonly used in situations involving students learning mathematics, the TSM is also well suited for the purposes of this paper, with its focus on the semiotic potential of TT and two teachers' initial exposure to it.

An essential component within the TSM, an *artefact* is viewed as a material or symbolic object, which has been created to answer a specific need (Rabardel, 1985), while *signs* are psychological tools that support and develop mental activities. The relationship between an artefact used to accomplish a task and the specific mathematical knowledge that emerges varies depending on the experience of the user. Whereas a novice will construct personal knowledge while completing a task using an artefact, an expert will recognize the mathematical knowledge that can potentially emerge by using the artefact to accomplish a task (Mariotti, 2012). Therefore, the semiotic potential of an artefact involves the double semiotic link which may be established between an artefact, the learner's personal meanings evoked by its use to accomplish a task and the mathematical meanings that emerge from its use that can be recognized by an expert as mathematics.

Adopting a semiotic perspective involves "recognizing the central role of signs in the construction of knowledge" (Vygotsky, 1978, p. 60). When first using an artefact to complete a task, the initial situated signs evoke the artefact (*artefact signs*) and emerge from the learner's personal experiences (Bartolini Bussi & Mariotti, 2008). The primary objective of semiotic mediation is student acquisition of mathematical content which requires that the teacher carefully orchestrate the transition from artefact signs that express the relationship between the artefact and the task towards culturally determined signs connected to the mathematical knowledge evoked by the artefact (*mathematical signs*). During this process, *pivot signs*, hinting at both the artefact and mathematical knowledge, appear and must be recognized and exploited by the teacher in order to facilitate this transition for the learner.

In the context of a mathematics classroom, this evolution will be facilitated through task design and guidance by the teacher through the didactic cycle, which consists of activities with the artefact, individual and/or small group production of signs as well as the collective production of signs during mathematical discussions. However, for the purposes of this paper, which consists of individual interviews with two teachers in a research context, the unfolding of the semiotic potential (Mariotti, 2012) of TT will be examined. Consequently, the focus will remain on the first phase of the semiotic mediation process and the emergence of the teachers' (rather than students') personal signs when interacting with the TT artefact.

There is additional complexity to be recognized in using the TSM in a research context with teachers interacting with a digital app not previously experienced (teacher-as-learner), that presents an already known mathematical concept (teacher-as-expert) (Bakos & Sinclair, 2019). The emergence of subtle shifts between teacher-as-learner and teacher-as-expert occurred throughout the interviews and must be considered in relation to their use of TT as an artefact, their previous knowledge of multiplication itself, and their prior teaching practices related to multiplication.

METHODS

The data for this paper comes from a small-scale qualitative study involving two experienced elementary school teachers, both of whom have spent the majority of their teaching careers in Alberta classrooms. The two teachers, who are referred to as Hannah and Katie, were chosen to be part of this study due to their extensive experience teaching grades in which multiplication is first introduced and practiced with Alberta students. Single session exploratory conversations were conducted with each participant separately, lasting between 10 and 20 minutes in length. Audio and video-recordings of each session were created and later transcribed.

Each teacher was given time to become familiar with the TT app through independent exploration prior to any specific requests or questions from the interviewer. After this short period of free discovery, the participants were provided with some open-ended tasks to complete using TT. These exploratory situations provided an opportunity to observe how each teacher made sense of TT using their pre-existing personal knowledge of multiplication as well as a chance to learn their thoughts about the app in relation to their experiences teaching children multiplication.

The semiotic potential of *TouchTimes* in relation to multiplication

A full analysis of the semiotic potential of TT is beyond the scope of this paper, therefore the key affordances of TT will be linked to meanings emerging from its use and how these meanings may elicit specific mathematical meanings of multiplication. The unfolding of the semiotic potential of TT, according to the TSM, will result in the emergence of different personal signs related to the use of the artefact to complete a task. These personal signs have the potential to evolve into mathematical signs.

RESULTS

From the video transcripts of both interviews, episodes have been chosen that illuminate aspects of the semiotic potential of TT in relation to multiplication. Prior to their first task, Hannah and Katie were given a short time to freely explore the app. It was not intuitive for them that continuous finger contact with the screen is required for the creation and maintenance of pips, but *only a tap* is required to create pods. This proved to be problematic throughout the initial exploration of the task and persisted even after reminders that continuous contact is not required to keep the pods visible.

Katie: Teacher-as-learner of the app

Once comfortable with the app, each teacher was given the task: “*The product is 24, how many pips are there?*” Katie immediately began sharing her thoughts while exploring TT. Her first comment was, "The only thing is, it's hard to see the groups when you're touching them." An artefact sign which prompted me to instruct Katie to lift her right hand (the one creating the pods) so that she could see that the pods would remain after removing her hand. Later, while placing both hands on the screen and then removing one or other first, Katie noted, "Oh the groups stay. That's a good thing because I'd not want my fingers to stay there. But they don't stay on the other side." Even after this Katie still used both hands to create 6×4=24, saying "I can make 24 but I have to use both hands, which is a little difficult".

Mid-way through the interview, while discussing the mathematical expression on the screen, Katie maintained continuous pip-contact, but removed her hand from the pods, which remained. Then, wanting to modify the equation, Katie removed some of her fingers so that there were fewer pips. When doing this, she appeared not to notice the changing configuration of pips within the pods, and once finished removing pips, she began tapping the right screen (RS). Seeing the addition of more pods, she stated in surprise, "Oh, you just have to tap them!" and then clarified, "So I can tap them, and they stay?" This statement and question, each in succession, are both artefact signs that signify a transition in Katie's understanding of how the app operates.

The artefact sign in TT relating to the maintenance or removal of finger contact and its subsequent effect on the pips or the pods, demonstrates how Katie was a teacher-as-learner when interacting with the digital tool. This would become a pivot sign, prompting her to pay closer attention to the effects that finger placement or removal had on the pips and the pods already on the screen.

Hannah and Katie: Teachers-as-experts of multiplication

Hannah and Katie's engagement with the TT app during the first task seemed to activate what I shall term a *semiotic sifting and imposition of the mathematics*. The signs emerged not from the artefact itself, but instead from the mathematics prompted by the task. Each teacher had pre-existing knowledge of multiplication and methods of teaching multiplication, and both endeavoured to impose their personal meanings which were intertwined with the mathematics, upon their use of this artefact.

When asked how many pips would be required for a product of 24, both teachers gave the answer first and then created it using TT. Katie stated, "Well I think of 6×4, right now." Her emphasis on *right now* indicates how quickly she knew the answer (teacher-as-expert). Later, when asked to make 24 another way, she said: "24, yeah. 24×1, 12×2" before engaging with the app. These are both examples of mathematical signs that emerged from the task given, but *prior to* the use of the artefact.

When Hannah was first given the task, she questioned: "The product is 24, there's only going to be 12?" and used TT to create 12 pips and a single pod for a product of 12. After being prompted, "So the product is 24", Hannah responded, "So we didn't do enough? [*Creates another pod*] Oh, I see, I have to put two fingers down over here". This scenario began with Hannah providing a correct answer to the task (a mathematical sign) and then working to explain the mathematics using TT to build 12 pips and two pods (an artefact sign). She goes further by stating, "So 12 pips, because they're all going to go in groups of two", which is a pivot sign between the mathematics and the personal meanings developed through Hannah's use of the artefact.

Hannah and Katie: Teachers-as-experts *and* teachers-as-learners

When asked to create a product of 24 in another way, Hannah made 24=4×6 but had difficulty describing how the changes made were reflected in the pips and pods. Hoping to encourage the use of TT's affordances in "building" multiplication, I had Hannah remove two pip-fingers (Figure 2a) and asked, "Can you adjust that now to get 24 again?" She added two pods to create 24=6x4 (Figure 2b) and explained, "There's more of these things. I mean four in each one [*indicates the pods*], there's six with four in each one [*gestures over the pods]*". Wanting her to notice the spread effect of pips in the pods, I asked her to recall what she created earlier. Hannah placed two pip-fingers down, noted 36=6×6 at the top of the screen, dragged two pods to the trash (24=4×6) and said, "So I could take two of those away, so there's six in each one" (Figure 2c).

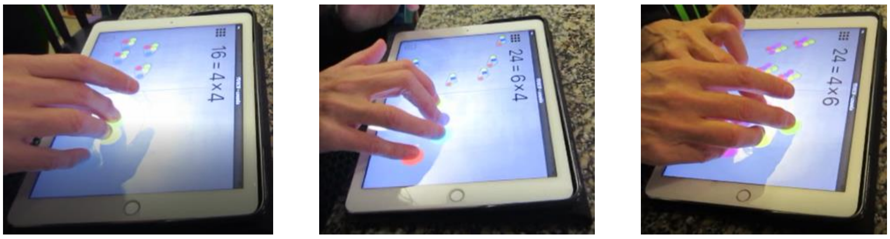

Figure 2: (a) 16=4×4 to start (b) 24=6×4 (c) 24 =4×6

In this episode, Hannah was using her pre-existing knowledge of the commutative property in concert with the multiplication expression displayed at the top of the iPad screen (mathematical sign) in order to create a product of 24 that reflects six pods made up of four pips *and* four pods composed of six. Throughout this process, she was working towards a more solid understanding of how the creation of pips and pods in the app (teacher-as-learner) reflects what she already knows about the commutative property (teacher-as-expert). Her reference to the expression at the top of the iPad screen through gestures, indicates that this mathematical sign continued to work as a pivot sign between the mathematics that she already possessed and the personal meanings that she was developing through the use of TT to solve the task given.

Katie also seemed to shift back and forth between teacher-as-expert and teacher-as-learner in regards to the mathematics and the use of the artefact. After Katie created a configuration of 6×4=24 using TT I asked, "Can you remove more fingers perhaps and create more pods?" To which she replied, "Probably yeah…but I'd have to really think about this because then I'd have to make the other side…" She then removed a pip-finger to create 5×4=20, pointed at the pods and explained, "I could make six here", lifted another pip-finger (4×4=16), "I could make four here" and finished by tapping one, two more pods to make 4×6=24. When asked how the pods were different, Katie stated, "They only have four in them. […] Okay that's weird because now they're six of them [*makes a circling gesture over the six pods*] with four in them [*points at a pod*]. I guess that's how it changed. So, how did that work?"

As Katie interacted with TT to address the task, she was creating artefact signs. Through physically changing 6×4=24 to 4×6=24, and her accompanying explanation about what she was doing, Katie appeared to recognize the relationship between the pips and pods in TT (mathematics signs) until she questioned, "So, how did that work?" A pivot sign that she was not understanding how the pips comprised the pods. At this point I intervened by continuously adding and removing two pips while questioning Katie about what was happening. She pointed back and forth between the pips and pods, and explained that, "You've just added two more and then they've added two more on both sides." Wanting to know specifically *where* the two pips were going, I prompted further. Katie responded by pointing at my pip-creating fingers and said, "There's two here and they've added two there [*indicating a single pod*]". But she then questioned, "Haven't they? Or are they always six there? No, they always were six. Okay. They've added two more here [*points back at my two pip-fingers*]. Two more pips, I don't know where they're going". Trying to direct her attention away from the unchanging number of pods and towards the changing composition of pips *within* the pods, I suggested that Katie pay attention to the colours. Although she noted that my pips "go into there [*the pods*]. They're the same colour as what you're doing", it wasn't until I said, "Yes, but there are *two more in each pod*. So it's not just two, *it's two more in each group*" that Katie verbalised, "Yeah, so it's like 12 more? Yeah, yeah, okay."

The artefact sign that I was paying particular attention to was Katie's difficulty in describing what was happening within each of the pods. Although she noticed the addition of pips within a pod, it was only within a *single pod*, rather than in *all of the pods*. Her question, "so it's like 12 more?" was a pivot sign that indicated her transition towards mathematical meaning in relation to the artefact used (teacher-as-learner). After this, she experimented by adding and removing a single pip while watching the effects of her actions on the pods. She then explained, "So really, what I thought is they're [*points back and forth between the pips and pods]* working together. They're really not working together. That's [*indicates the pips*] building this [*points to a pod*]. Is that what we're saying? […] it's building the groups".

Hannah also elaborated further about her observations regarding the impact of the changes she had been making to the pips and pods in TT. "It can either make less groups, more groups, it changes how many. […] It has to take however many fingers you have over here [*circles over pips*], has to be in your group over there [*points to pods*]. So, that's [*points to pips*] going to change what's in over here [*points to pods*]". Throughout both explanations, the intertwined nature of gesture and speech is readily apparent while also demonstrating the double semiotic link between the task and the artefact, and the resulting development of personal meanings that were linked with mathematical understanding as it was related to the use of TT specifically.

DISCUSSION AND CONCLUSION

Besides shifts between teacher-as-learner and teacher-as-expert, there were also unexpected shifts that occurred in relation to semiotic potential throughout Hannah and Katie's explorations with the artefact. Initially, both appeared to draw upon their knowledge of mathematics prior to engaging with TT, which influenced their interactions with the app in unexpected ways. The teachers first engaged in a *semiotic potential of the mathematics* itself, rather than exploiting the semiotic potential of the artefact. Then, after becoming more familiar with TT there was a transition towards using the app to make sense of the mathematics in a way that was more consistent with the *semiotic potential of the artefact* within the TSM. However, further research related to the role that semiotic potential plays when introducing TT to those who already possess pre-existing ideas about the process of multiplication is necessary. If primary teachers are to effectively implement this digital technology as a teaching tool, then it is necessary that they possess a clear understanding of the multiplicative principles that underly it. An understanding that is not as straight forward as previously presumed prior to these interactions with two very experienced educators.

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UNDERSTANDING CREATIVITY THROUGH EMERGENCE: A PROCESS PERSPECTIVE

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Creativity has always been assessed from two important viewpoints: novelty and usefulness. However, these perspectives enrich our understanding only of the creative product and not of the underlying process. To gather a holistic appreciation of what creativity entails, both the creative process and the product need to be studied hand in hand. To accomplish that, it would be worthwhile for us to understand the nature of the creative process and its unfolding. In this paper, I primarily argue that creativity is an emergent phenomenon with the help of a modern theoretical framework called enactivism and I examine the creative processes involved in a mathematical proof generation. Interview excerpts from two expert mathematicians are analysed and the idea of emergence involved in their creative endeavours are discussed.

INTRODUCTION

Creativity plays an important role in the development of new knowledge in any human endeavour. The definition of creativity, what it is or how to measure it, however, is elusive. One distinction however is that there seems to be an over reliance on creativity associated with the human mind. (Sawyer, 2012). That is, creativity is often seen as a personal trait or an attribute that one is born with. We usually find studies about the 'person' behind a creative act, acknowledged as being 'talented' or 'gifted' (Leikin et. al., 2010; Sriraman, 2005). Less frequently, we find creativity framed in terms of a person's social milieu, like being raised in a certain family environment, or something similar. For example, socio-culturalists argue that creativity ('imagination' in Vygotsky's terms) develops through the interiorization of cultural tools available to an individual (Gajdamaschko, 2005). There are many other challenges in describing creativity or coming to terms with what it could mean in different situations. For example, creativity is often times judged and measured using a finished creative product rather than the creative process underlying it (Sawyer, 2000). Is there a way to conceptualize creativity without having to think about the creative product and if so, what might that look like?

A possible solution for this question would be to turn our attention towards the 'process' aspect of creativity, where the primary focus of analysis would be on the unfolding of events in a creative act, in an unpredictable fashion. In other words, I suggest that in order to understand creativity in its totality, 'emergence' might be an important lens to use. Emergence, according to philosopher George Henry Lewes

2019. In A. Hare, J. Larsen & M. Liu (Eds.). *Proceedings of the 14th Annual Mathematics Education Doctoral Students Conference* (pp. 26-32). Burnaby, Canada: MEDS-C.

(1877), is an effect that is unpredictable and arises out of the combined agencies of its interactants, but in a form that does not display these interactants (in Sawyer 1999). In other words, an emergent effect is neither predictable nor decomposable into its components. This idea could be nicely tied into creativity, which has mostly been treated as a static trait that only belongs to the individual involved in a creative act, whilst neglecting influences of the multiple interacting factors involved like tools, social situations, mood of the individual etc., to a large extent. Thus, situating creativity in the idea of emergence helps in decentering the human agency involved, thereby providing equal emphasis on other influential factors involved in the creative process. Also, thinking about creativity in terms of the creative process, instead of the finished creative product sheds light on the development trajectory of the product, which is otherwise often hidden if only looking at the product. Put differently, the improvisational aspect of creativity, where the creative process *is* the product, brings forth a perspective that is otherwise concealed when conceptualizing creativity from the product perspective alone (Sawyer, 2000).

In this paper, I analyse data collected from newspaper/magazine interview excerpts of some expert mathematicians' reflections on their own creative process while engaging with proofs/problem solving and I look for evidence that bring forth the idea of emergence and unpredictability involved in the unfolding of their creative process. I then argue that creativity is primarily an emergent phenomenon, following which I delve into exploring the possible factors this emergence depends on and furnish some pointers as to what this might inform us about creativity in general. I use enactivism, a modern theoretical framework on interaction as a way to look at creativity as emergence. I begin with a brief exposition of enactivism and its main tenets, to give the reader a theoretical lens to engage with the paper.

ENACTIVISM AND EMERGENCE

Moving on from the constructivist approaches where the primary emphasis is on knowledge rather than knowing, enactivism was developed to bring forth the idea of the inseparability of the individual and the world (Begg,1999; Lozano, 2005). The term 'enactive' was coined in the 1992 seminal book *Tree of Knowledge* written by Francisco Varela and Humberto Maturana, to bring forth the view of knowledge not as existing "in any place or form, but enacted in particular situations" (Kieran & Simmt, 2015). Enactivism is a theory of cognition that comes from a biological viewpoint and posits that 'knowing' neither represents grasping knowledge from the outside world nor constructing knowledge inside our minds, but is a result of interactions of an individual with the environment. It further explains that since each individual's personal history and experiences are different, the outcomes of their interactions with the environment will be quite different. In other words, the outcomes of the mutual interaction an organism and its environment are not predetermined; rather, they emerge real-time as a product of the interaction, directly and continually influenced by both (Proulx, 2013). This sits perfectly well with the description of emergence, as proposed earlier, as an effect that is not additive in nature, but a phenomenon that surfaces due to the combined agencies of its interactants. Although enactivism is not explicitly a theory of emergence, its essence is rooted in the unfolding of unpredictable outcomes, the cause of which cannot be traced back to the properties of either the organism or the environment. Enactivism is grounded in Darwin's theory of biological evolution and some associated concepts like 'structural coupling' and 'structural determinism' are discussed below.

Structural Coupling

The concept of 'structural coupling' is rooted in Darwin's conception of how organisms and environment co-evolve to adapt and become compatible to each other (Maturana & Varela, 1992). Enactivism builds on this idea to explain how an organism and its environment co-adapt and mutually influence each other during the process of their interaction. This then means that, both the organism and its environment go through constant transformations in their structures, as a result of their interaction with each other. From this notion of structural coupling, we can say that the outcomes of the interaction are highly dependent on both the environment and the organism. This leads us to another important concept of enactivism, namely structural determinism.

Structural Determinism

The phenomenon called structural determinism addresses the importance of the structural makeup of an organism that allows itself for transformations and changes (Maturana & Varela, 1992). Here, structural makeup not only refers to the physical structure of an organism, but a collective of the physical, psychological, neurological and sociological make-up of the being. Thus, two people trying to solve a problem, which can be thought of as a 'disturbance' from the environment, would engage with it differently based on their structural makeup and their interaction with the problem. This constraint and limitation possessed by each organism is precisely called structural determinism, in enactivist terms. Thus, the interaction between each structurally determined organism and its environment, is guided very much by the organism's way of making sense and understanding, which Varela (1996) refers to as 'problem posing'.

As an emergent effect arises from the combined agencies of its interacting components, structural coupling and structural determinism help us in making sense of what those agencies of the components are. Specifically, though emergence is not predictable from the knowledge of its components and their agencies, we can at least understand what each component brings with itself and how each of these components is transforming, as they all interact.

METHODS

The data for this study was collected from newsletter/magazine interview excerpts of two expert mathematicians and Field medallists, Sir Andrew Wiles and Maryam Mirzakhani. A 10-page interview of Sir Wiles for the European Mathematical Society Newsletter and a 3-page interview of Mirzakhani for a Clay Mathematical Institute

Annual Report were taken for analysis. The gathered data was then examined to find traces of the notion of emergence in creativity and subsequently discussed using enactivism. Interview excerpts where the mathematicians precisely spoke of their creative process were chosen for this paper.

EMERGENCE IN MATHEMATICAL PROOF GENERATION

The words 'proof' and 'proving' are arguably two of the most important words associated with the discipline of mathematics. A mathematical proof entails a systematic series of successive mathematical statements, each of which follow logically from what has gone before (Bell, 1976). The evolution of the finished creative product named 'proof' is not just a mechanical activity based on axiomatic deductive reasoning starting with an infallible truth, rather, it begins with the emergence of a conjecture which is then continuously transformed through the process of structural coupling by its interaction with the mathematician, and through the emergence of supporting lemmas, axioms and counter examples. Coming up with a proof from the same mathematical assumptions by different mathematicians seem to highly depend on their knowledge base and various other factors that come into play during this process. Maryam Mirzakhani, the late 2014 Field's medallist, mentions about this idea of looking at a problem from different perspectives.

> I find it fascinating that you can look at the same problem from different perspectives, and approach it using different methods. …I would prefer to follow the problems I start with wherever they lead me. (Clay Mathematics Institute Annual Report, 2008, p. 12)

Sir Andrew Wiles, the famous mathematician who proved Fermat's Last Theorem, in a detailed interview, describes how many different factors along the course of many years influenced the way he started writing his seminal proof.

> I started off really in the dark. I had no prior insights how the Modularity Conjecture might work or how you might approach it. …To start with, there are three ways of formulating the problem, one is geometric, one is arithmetic and one is analytic. …I think I was a little lucky because my natural instinct was with the arithmetic approach and I went straight for the arithmetic route, but I could have been wrong. …Partly out of necessity, I suppose, and partly because that's what I knew, I went straight for an arithmetic approach. I found it very useful to think about it in a way that I had been studying in Iwasawa theory. With John Coates I had applied Iwasawa theory to elliptic curves. When I went to Harvard I learned about Barry Mazur's work, where he had been studying the geometry of modular curves using a lot of the modern machinery. There were certain ideas and techniques I could draw on from that. I realized after a while I could actually use that to make a beginning – to find some kind of entry into the problem. (EMS Newsletter, September 2016, p. 34)

Even though we might think that Wiles proved Fermat's last theorem in isolation, in reality, it emerged from the constant interactions between him and the environment that he was situated in. From his interview, we see that his interactions with peers like John Coates and his situations in Harvard that led him to Barry Mazur's work, were influential in helping him to formulate his proof pathway. The process involved in the generation of a proof is quite emergent and lively, as opposed to the mechanical activity of lining up of pre-determined logical statements. Thus we could say that the generation of a mathematical proof depends on a lot of factors like the mathematician's knowledge base, personality, mood at any given time and the situation that they are situated in, making it as a result of their mutual interaction, as suggested by the enactivist theory.

DISCUSSION

From our data in the previous sections, we see that the process involved in the generation of a mathematical proof is quite unpredictable in nature, even to the individual who is creating it, emerging at each step from the constant mutual interaction between them and their surroundings. As true to our conception of the notion of emergence, the causes behind the outcomes of a creative process cannot be completely mapped to either the individual or their surroundings; a one-to-one mapping between cause and effect ceases to exist during any kind of improvisation. Nevertheless, I find that these unpredictable outcomes do depend highly on the characteristics of the interacting entities, thus accounting for the uniqueness that accompanies each interaction, as suggested by the enactivist lens. In the following paragraphs, I will discuss how the emergence of these outcomes depends on the interacting individual followed by how the surroundings contribute to the emerging outcomes of a creative process.

In Sir Andrew Wiles's interview, when he says, " *Partly out of necessity, I suppose, and partly because that's what I knew, I went straight for an arithmetic approach",* we see how the personal preferences and limitations of the individual has a great impact on how a problem or a conjecture is being approached. This is consistent with Varela's notion of *structural determinism,* where the organism's structural make-up, with its physical/psychological features allow for a certain kind of action. Thus, the knowledge base of a person in a certain domain seems to be directly influential in bringing forth the optimal situations for them to explore. Also, when he says, *'I started off really in the dark. I had no prior insights how the Modularity Conjecture might work or how you might approach it',* it gives a glimpse of the internal feelings of doubt and uncertainty, that he was going through at that moment, which might have influenced his work in a certain way.

In terms of the influence of the environment, when Wiles says, '*When I went to Harvard I learned about Barry Mazur's work, where he had been studying the geometry of modular curves using a lot of the modern machinery… There were certain ideas and techniques I could draw on from that',* it is evident that the exchange of ideas from fellow researchers from the domain play a significant role in constituting the surroundings of a mathematician engaged in the generation of a proof. It also informs us as to how a mathematician depends on the knowledge and work of others, to build their own work. In a non-collaborative setting, where an individual is generating a proof in solitude, there is still a collaboration between them and their respective discipline, which exerts an agency in the emergent creative process. In Andrew Pickering's (1995) terms, this is called disciplinary agency. He claims that the concepts or materials belonging to a discipline, exert agency during practice, that is quite significant. When the expert mathematician Maryam Mirzakhani says, "*I would prefer to follow the problems I start with wherever they lead me",* we see the exploratory nature of her pursuit and her reliance on the agency of the problems. Here, we see how Mirzakhani gives an equal emphasis on her co-evolving creative entity, the mathematics that she works with, which constitutes the environment that she is in. This co-evolution of the individual and their environment is what I identify as *structural coupling*. There is also evidence of the individual's momentary de-centering of themselves from the creative act, as they give away their agency to the problem or the collaborative setting that they are in, to lead them into some unknown territory as Mirzakhani does when she says she would follow her problems wherever they lead her to.

Thus, we see that creative processes are zig-zag in nature and seldom linear. The unpredictable emergence of ideas and thoughts in a creative process plays an important role in stimulating the enthusiasm of the creator, for if the outcomes were known beforehand, the joy of creation is lost. Thus, I argue that emergence plays a vital role in keeping the individual engaged in their creative activity, or in other words, creativity is primarily an emergent phenomenon. Also, we could possibly say that the emergence of creativity is a product of multiple interacting factors, for which optimal conditions might be created, but can seldom be reduced or dismantled to the properties of the component parts. This is quite consistent with the notion of emergence mentioned in a previous section as something that is not additive in nature.

CONCLUSION

Once a product is formed, we tend to overlook the dynamic emergent processes that were involved in the creation of the product and look at it as a static entity. However, creative processes are seldom linear, because the product thus formed is due to the result of constant emergent deformations, caused by the interaction between the individual and their environment. In this paper, interview excerpts of two expert mathematicians regarding their creative processes were analyzed to find evidence for the notion of emergence in creativity, to advocate for the idea that, creativity, in general, is an emergent phenomenon that unfolds due to the constant mutual interactions between a person and their environment. Several statements from the mathematicians emphasized the role of their collaborators, mood, personality and their disciplines in the emergence of their creative process.

The descriptions from the expert mathematicians about the unpredictability and suspense experienced during a proof generation sheds light on the significance of emergence imbedded in the creative process. On the surface, mathematical proofs appear exceptionally precise and logical, however, the inner core of their improvisational aspect is very much imprecise, zig-zag and uncertain, much like a freeflowing musical improvisation. I conclude that the unpredictable emergence of the outcomes is imperative for an individual to feel the liveliness in their creative process, for which de-centering the human is essential. In other words, creativity does not reside in any one entity, but emerges in the interaction of multiple interacting factors.

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DOT PRODUCT - IT'S SO EASY?

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In many college level math classes, concepts are often presented in such a way that overlooks the remarkable mathematical achievements culminating in those concepts. How the dot product of two vectors is introduced in the 2- or 3-dimensional space is a prime example. Given two vectors $\vec{u} = \langle u_1, u_2, u_3 \rangle$ *and* $\vec{v} = \langle v_1, v_2, v_3 \rangle$, *students often find it very trivial to compute* $\vec{u} \cdot \vec{v} = u_1 v_{1_1} + u_2 v_2 + u_3 v_3$. *This trivial computation often gives them false impressions about their understanding of the concept. How do textbooks facilitate students' understanding of dot product? With this question in mind, a textbook analysis involving 13 textbooks in mathematics, physics, and engineering was conducted. This paper discusses its results and findings.*

INTRODUCTION

Achieving deep understanding, or what Richard Skemp calls relational understanding, of mathematical concepts is challenging. In particular, "concepts of higher order than those which people already have cannot be communicated to them by a definition, but only by arranging for them to encounter a suitable collection of examples" (Skemp, 1987, p.18). APOS theory and Anna Sfard's process vs. object conceptions also suggest that there are stages an individual learner must goes through in order for an unfamiliar concept to be encapsulated into a more static, structural object in one's mind (Dubinsky & McDonald, 2001; Sfard, 1991). In terms of developing object conception of any mathematical ideas in learners, how the concept is communicated to them can have significant impact. It is particularly the case for more advanced mathematical concept, as the level of abstraction is so far removed from our immediate experiences with the surrounding physical environment. It is; therefore, paramount to introduce a mathematical concept in such a way that can facilitate this learning process. In this paper, we focus on the concept of dot product. How do textbooks communicate this concept to the students? 13 textbooks, including calculus, linear algebra, physics, and engineering, were studied in order to analyze how they introduce the concept of dot product. These books were studied with the following questions in mind:

1 Is the motivation for the definition given?

2 How is it defined; algebraically or geometrically? Does the book make connection between the two ways? And if so, how?

3 Does it relate to physical/concrete examples of applications outside of geometry?

4 Any notable differences between math and physics/engineering? And if so, what implications do they have on teaching and learning of this concept?

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Following the example in Park (2016), in analyzing the findings, attention was given to how these textbooks communicate the concept of dot product to the learners. Park's study relied on Anna Sfard's interpretive framework stemming from the idea that thinking is a form of communication (Sfard, 2007).

COLLECTED DATA

In the U.S. college curricula, students typically encounter the notion of dot product in one of three ways, depending on which course they take first; 1) multivariable calculus course, 2) calculus-based physics course, or 3) linear algebra course. To answer the above listed questions, 13 textbooks were studied consisting of 7 calculus books, 2 linear algebra books, 1 integrated physics and calculus book, 1 calculus-based physics book, 1 engineering statics book, and 1 upper division physics (electrodynamics). Table 1 on page 5 summarizes the findings. The detailed explanations of the column headings and the data in this table follows.

Textbooks

7 Calculus Books: Riddle's text (1984) can be considered a book from the pre-calculus reform era. Apostol's text (1964) is a textbook chosen frequently for honors level calculus courses in the U.S. It is much more theoretical and abstract than typical 1st year calculus textbooks. Stewart's text (2008) and *Thomas' Calculus* by Haas and Weir (2008) are books that are popular for calculus sequence. McCallum and Iovita (1998) can be considered one of the most popular reform calculus textbooks. Herman and Strang (2017) and Schlicker (2017) are books born out of the opensource movement to provide low cost options to students. The book by Herman and Strang is gaining popularity because of the credibility of the authors.

2 Linear Algebra Books: Anton and Rorres (2014) and Lay (2000) are both introductory linear algebra textbooks. Lay's book emphasizes more applications.

1 Integrated Physics & Calculus Book: Rex and Jackson (2000) provides a rare example of a textbook that presents calculus with physics content focus.

3 Physics & Engineering Books: Serway (2000) represents a typical textbook for calculus-based physics courses that most students majoring in STEM fields would take. Hibbeler (2001) is one of the popular engineering statics textbooks. Griffiths (1999) is an example of a textbook used typically by physics majors in upper division who take a course such as electricity and magnetism, which is essentially an applied vector calculus course.

Motivation

"Motivation" here refers to an attempt made by each textbook to motivate the definition of dot product. The data in this column indicate if each textbook introduces the notion of dot product by making a connection to students' prior knowledge or by making a reference to any other associated concepts as a segue into the notion of dot product.

Definitions

Generally, there are two ways to define the notion of dot product; algebraic and geometric. The letter "A" in this column indicates the algebraic definition of dot product. More specifically, this indicates that the dot product of $\vec{u} = \langle u_1,\ u_2,\ u_3 \rangle$ and $\vec{v} = \langle v_1,\ v_2,\ v_3 \rangle$ being given by: $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$. In contrast, the letter "G" indicates the geometric definition of dot product; namely, $\vec{u} \cdot \vec{v} = \lfloor \vec{u} \rfloor \lfloor \vec{v} \rfloor \cos\theta$, where θ is the smaller angle between the two vectors. As summarized in Table 1, there were 4 possible outcomes for this column. A-G indicates that the book defined the dot product algebraically, then derived the geometric definition as a theorem. G-A indicates the opposite approach. A+G means that both definitions were provided simultaneously. "A" indicates that only "A" was given as the definition without stating the geometric definition (or as theorem). "A"; however, is very similar to G-A approach as the motivation of the dot product for this approach was the angle formula. So, in a sense the geometric definition was given as part of the motivation for the notion of dot product.

Connection

"Connection" refers to each textbook's attempt to relate the two definitions. It answers the question, "how does the textbook connect the two definitions?" We can see that A-G definition is followed by an application of the law of cosine to establish the geometric definition. It is interesting to note that all physics and engineering books relied on algebraic computations to show the equivalence of the geometric and algebraic definitions even though using the law of cosine is a viable approach.

Applications

"Applications" column indicates any examples provided for applications of dot product beyond the geometric ones, such as vector projection and orthogonality. As the table shows, the most common application came in the form of work as it relates to force and displacement vectors.

DISCUSSION

We will now analyze the findings from the perspective of possible implications each approach may have on the formation of more complete understanding of dot product.

Motivation

On the notions of process vs. object conceptions, Sfard (1991) views processes as operations performed on previously established objects. In this light, a motivation justifies the need for the new process represented by dot product being performed on vectors as objects. The findings indicate the common ways to motivate dot product are to appeal to the usefulness of dot product in computing work or to the usefulness in computing angles between two vectors. Regardless of which type of motivation is given, providing one facilitates the learners' understanding of the "need" to define such notion, aiding in their attainment of the process conception of dot product.

Definitions

We can see in Table 1 that there is a notable difference in the tendencies between mathematics textbooks vs. physics & engineering textbooks in ways they define dot product. Whether a motivation for the notion of dot product is provided, mathematics textbooks tend to give the algebraic definition first before the geometric. In contrast, physics and engineering textbooks tend to define dot product geometrically first. Perhaps, this is not surprising since the common example of dot product in physics and engineering tend to be that of work, which involves the projection of force vector along the displacement vector. This lends itself to the geometric definition of dot product. In mathematics textbooks, we see some variations in the way they define dot product; however, predominantly, “A-G” is the most common approach. Further investigation is needed before we can conclude whether the A-G tendency for mathematics and the G-A tendency for physics and engineering are due to common belief within each discipline as an effective approach or merely a default position due to tradition. McCallum’s book’s approach of A+G is certainly unique. This can be viewed as the impact of calculus reform, in an attempt, to provide students with various representations of concepts.

Connection

Regardless of which approach of definition one takes initially, if we were to instill in learners more complete understanding of the notion of dot product, it is essential to provide evidence of equivalence for the two definitions and how they are related. If no connection is made by the learner, it is more likely that learners’ conception of dot product would remain at the process level conception, at least longer than otherwise. Particularly, if they solely rely on the algebraic definition to compute the dot product, it seems as though “dot product” is merely a name you give to the process of computing such product. They may also be regarded as pseudo-object by Sfard (1991). Pseudo-objects are those that are operated on by other processes, yet not fully reified into objects. It may be possible that the notion of dot product remains at this stage for a long time for many students.

For this aspect of dot product, we also see a notable difference between mathematics and physics/engineering textbooks. Again, it is difficult to ascertain whether it is due to the pedagogical belief held within each discipline or if it is due to each discipline’s tradition in practice.

Applications

Applications of dot product beyond those of geometric ones extend the scope of learners’ understanding. These also supplement the motivation for the definition by providing the evidence of the need for such conception. By and large, the application is limited to that of work as you see in Table 1. Griffiths provides an example of application in terms of E-field flux. By the time students reach the upper division undergraduate courses, the discussion of dot product is limited to a brief review of how one computes the dot product; however, it provides further applications that the lower division undergraduate courses do not provide. One may argue that being able to apply dot product in these contexts may be an evidence of object-conception of dot product. Making such claim may be premature, as we do not know whether students are performing such operations with pseudo-object conception of dot product alone.

CONCLUSION

As we tread through our mathematical journey, the more advanced the content becomes, the more detached concepts encountered are from our immediate experiences with the physical world. Definitions often seem to come from nowhere in particular. In this paper, various textbooks' approaches to the notion of dot product were studied. This was done in an attempt to understand how the textbook discourse may or may not facilitate more complete understanding of dot product. There is a strong tendency in mathematics textbooks to state the definition algebraically whereas physics and engineering tend to define it geometrically. There is also a notable difference in the way mathematics textbooks tend to connect the two definitions as opposed to how physics and engineering textbooks tend to do so. It is not clear if either approach is better than the other with regards to the effectiveness of communicating the concept. Perhaps, being exposed to both approaches greatly enhance the likelihood of attaining structural conception of dot product. It however, warrants caution on either side to be cognizant of potential confusion these approaches may cause in learners.

How textbooks provide a motivation for dot product is not consistent, and often no motivation is given at all. Providing some motivation can be argued as crucial in order to facilitate learners' understanding for the necessity of such conception. Providing application examples is essential according to Skemp (1987), however, the findings indicate that often examples provided are very limited.

Further study may shed more light on the existing notable differences between math and other disciplines in their approaches with regards to definition and connection. It may turn out that these differences are, indeed, more beneficial than harmful for learners as they highlight the multifaceted nature of the concept of dot product.

Discipline	Textbook (by Author)	Motivation	Definition	Connection	Application
Calculus	Riddle	Angle formula	A		N/A
	Hass (Thomas')	Angle formula	A		Work
	Stewart		A - G	Law of cosine	Work
	McCallum	*	A + G	Law of cosine	Work

	Apostol		A - G		
	Schlicker	N/A	A - G	Law of cosine	Work
	Herman	Work	A - G	Law of cosine	Econ, Velocity, Work
Linear Algebra	Anton	N/A	G - A	Law of cosine	N/A
	Lay	N/A	A - G	Law of cosine	Least Squares
Integrated	Rex	**	A - G	Law of cosine	Work
Physics & Engineering	Serway (Physics)	Work	G - A	Algebraic derivation	Work
	Hibbeler (Statics)	Angle	G - A	Algebraic derivation	Work
	Griffiths (Electrodynamics)	N/A	G - A	Algebraic derivation	Flux of E-field

Table 1: Summary of findings

* This book tried to motivate the definition as a useful tool to connect algebra and geometry.

****** This book was exploratory in nature. Asked the reader to go through computation to see any patterns emerged.

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ENGAGING WITH METAPHORS: BORDERLESS PUZZLES AND DEFRAGMENTATION

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In a deliberate move away from the typical MEDS-C submission, this paper focuses less on the articulation of specific research findings, and more on the communication of insights gleaned from the process of writing about my primary research themes (namely the nature of mathematics and the notion of material assemblage). Specifically, two complementary metaphors which have proven useful in formulating my view of mathematics as the science of material assemblage are discussed at a metacognitive level, using the writing process itself as a framing device. I also briefly indicate how the metaphors of interest (i.e. borderless puzzles and defragmentation) can be linked to underlying mathematical considerations.

AN UNUSUAL ACKNOWLEDGMENT

There is a sense in which I must immediately offer an apology to the reader, for I have intentionally avoided the kind of contribution customarily found within the proceedings of our Mathematics Education Doctoral Students Conference (MEDS-C). Though still connected to the overall themes of my dissertation work, and while still grounded in a perspective that draws upon the new materialist discourse, this particular piece is not focused on the reporting of additional research findings (see related background discussions in Forde, 2018). Instead, it is intended to foreground a number of reflections that have informed my dissertation writing at key intervals. In part, because I have come to view the evolving doctoral dissertation as a *kind* of material assemblage (i.e. a dynamic configuring and reconfiguring of entangled ideas), and as a living document whose features continue to develop in response to my interpretations of the literature with which I engage, I have also opted to extend aspects of its internal discussion so as to reflexively comment on the writing process itself.

While I shall make reference to broader theoretical considerations arising from the new materialist discourse, the forthcoming discussion is more directly rooted in the phenomenology of my own experience. As a result, the current paper constitutes a conceptually-motivated exploration more than it does a theoretically- or methodologically-motivated one. Moreover, insofar as written MEDS-C contributions are concerned, it also constitutes something of an experimental piece, in that it adheres to certain MEDS-C guidelines while openly eschewing others. Though I will address singular aspects of the dissertation writing process, I also hope to make clear the fundamental mathematical themes that prompted the writing of this paper.

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OVERARCHING CONCEPTUAL CONSIDERATIONS

As I have come to know it, the new materialist discourse does not currently have well-defined disciplinary boundaries. This is largely to be expected, as the majority of new materialist scholars actively engage in the deconstruction and reconsideration of commonly-accepted boundaries within and between their respective fields of interest. Consequently, the themes explored (and the works produced) by new materialist scholars are highly varied. Consider, for instance, Barad (2007), de Freitas and Sinclair (2013, 2014), and Meillassoux (2008), which delve into the physical sciences, mathematics education, general philosophy, and socio-politics as well. Amongst the wide-ranging cross-disciplinary discussions within these works is the generative topic of *assemblage theory*, which I find particularly compelling by virtue of its parallels with modern physics. As with entangled quantum systems, *assemblages* involve/necessitate scenarios wherein contributing components co-evolve with and within the larger systems in which they are embedded, and it is by so doing that they "form emergent unities that nonetheless respect the heterogeneity of their components" (Smith, 2013). The features of these unified systems are neither static nor predetermined, such that contributing objects or processes can play different roles and exhibit greater or lesser prominence at different times.

Of notable importance to my own work is the assemblage theory of French philosopher Gilles Deleuze (1925-1995), which emphasizes the importance of structural interplay and reconfiguration within assemblages. As John Macgregor Wise writes in editor Charles Stivale's *Gilles Deleuze: Key Concepts* (2011):

> Assemblage, as it is used in Deleuze and Guattari's work, is a concept dealing with the play of contingency and structure, organization and change […] The term in French is *agencement* […] It is important that *agencement* is not a static term; it is not the *arrangement* or *organization* but the *process* of arranging, organizing, fitting together. The term […] is commonly translated as *assemblage*: that which is being assembled […] An assemblage is not a set of predetermined parts […] Nor is an assemblage a random collection of things […] An assemblage is a becoming that brings elements together.
>
> (Wise, 2011, p. 91)

At its heart, the sense in which I see mathematics as *the science of material assemblage* is also deeply concerned with ongoing processes of reorganization/structural change and the bringing together of elements. One of the key ways I articulate this is by working to substantiate the claim that "mathematics embodies the very principles according to which matter organizes and reorganizes itself" (Forde, 2018, p. 24). In keeping with this motif, my broader mathematical worldview leverages a portion of the new materialist discourse that skews less toward *object-oriented* ontology and more toward *relational* ontology (with a specific interest in entanglements stemming from the quantum theoretical discourse).

Following forward from these explorations into material assemblage and the associated notions of structural change and reconfiguration, I have inevitably begun to think about the *dissertation writing process* in similar terms. More specifically, I have come to envision my dissertation as an evolving assemblage of entangled ideas, and I am increasingly concerned with the structuring principles that are at play within it, and through which the document's discussions are being realized. It is for this reason that I have more recently reconceived of *metaphors* in a manner analogous to my overall mathematical worldview. Just as *mathematics* might be seen as embodying the principles according to which *matter* might be reorganized, I here entertain an extended notion in which *metaphors* embody the principles according to which *meaning* might be reorganized. Granted, this may strike the reader as being quite far removed from a mathematically-grounded discussion; but, I shall shortly revisit this analogy in terms that should allow the underlying mathematical significance to become more evident.

METAPHORICALLY SPEAKING

The excerpt from Wise provided on the previous page is rather emblematic of my thoughts about the activity of scholarly writing. Indeed, I believe that the writing process is wonderfully illustrative of the sensibilities Deleuze and Guattari express about the nature of *assemblage* (or *agencement*); for the composition of coherent, engaging, and accessible text with sensible structure, agreeable flow, and rigorous discussion clearly has much to do with the processes of *arranging*, *organizing*, and *fitting together* authors' ideas. There also exists a corresponding and ever-present need to be cognizant of structural interplay and possibilities for reconfiguration at multiple levels. Whether literally (at the level of word choice and grammar), or figuratively (at the level of connotation and intentionality), the conveyance of meaning through written text is very much an act of *bringing together* or *entangling* ideas in such a way that a larger, unified whole emerges from more singular elements. Syntactic/technical and aesthetic/stylist considerations are all deeply implicated in the negotiation of meaning common to written communication, and it is by facilitating the negotiation of meaning that *metaphors* are of immense value to authors and readers alike. The following *two* metaphors have been particularly useful to me throughout the endeavour to craft my own cohesive and (hopefully) well-structured dissertation document.

Borderless Puzzles

It has occurred to me that the earlier stages of my dissertation work were largely analogous to the construction of a jigsaw puzzle that had no border pieces (or rather, one whose borders were not apparent at the onset). The puzzle pieces also seemed to vary not only in shape but in size. The initial absence of an outer bound, compounded by this variation in granularity essentially meant that neither the scale of the puzzle image nor its resolution were entirely obvious, and the required scope and depth of the research task did not become evident until much later on. It was only when smaller clusters of related ideations began to accrete and connective tissues emerged from the interstices *between* neighbouring clusters that the general features of the puzzle image (and their orientations with respect to one another) started to reveal themselves. In addition to indicating how the complexity of a given research topic might impact the features of the puzzle being constructed, these factors also made clear the necessity of oscillating back and forth between single puzzle pieces (i.e. individual words/ideas) and the larger puzzle image (i.e. the overarching vision for the dissertation). The need for this oscillation would not always be immediately apparent at the time of writing; however, in retrospect, it does seem to be an essential activity, as both the *part* and the *whole* must be given similar (if not equal) consideration.

With this in mind, the "dissertation as borderless puzzle" metaphor can be seen as thematically derived from (perhaps even fundamentally rooted in) mereological underpinnings, and as marking the importance of attending to part-whole relationships. Of course, while the contexts of scholarly writing do involve markedly different moment-to-moment concerns when compared to explorations in number theory, this metaphor nonetheless helps to draw attention to the mathematically-significant notions of *the discrete* and *the continuous*, and indicates how they may actually be implicated in activities that are not typically associated with mathematics.

In fairness, while the notion of writing as akin to puzzle construction is likely very common amongst writers, I have a sense that a direct association with mereological themes, specifically, might be less so. That said, just as fragmentary elements at a very small scale support more cohesive, unified structures at a larger scale, I would suggest that the corresponding reverse relationship is equally important. Often, the role of the solitary part in shaping the intended whole is given the greater priority, yet the reciprocal contribution of the intended whole in shaping the solitary part should not be overlooked. Both facets of the relationship are mutually informative, and neither should be dismissed/neglected.

Mechanically, structural concerns at the level of individual characters, words, phrases, and so on inevitably preface/precede more expansive concerns at the level of sentences, paragraphs, chapters, et cetera, and all of these ultimately contribute to a single unified document whose subsections have the potential to cohere structurally and thematically. At the same time, ideationally/conceptually, the composite features of the document at large invariably shape the singular choices that are made as writing is in progress. Thus, rather than viewing these as two opposed perspectives on (or approaches to) writing, I instead prefer to think of them as complementary facets of one unified process of *becoming*, and I believe that such an interpretation would be aligned with the concept of *assemblage*/*agencement* as it has already been expressed by Deleuze and Guattari.

Defragmentation

Though the turn may be abrupt, at this juncture I ask the reader to move with me into the informational realm of binary data and digital file management. In particular, it is the processes of data storage and retrieval which will be highlighted here. When dealing with digital storage space under specific file management systems, the effects of data fragmentation can be an occasional concern. In brief, fragmentation arises when a file system is unable to reserve enough contiguous file space for data to be written as a unified "block", and must instead reallocate data fragments elsewhere on the storage medium. While not all file management systems operate in this manner, computer systems that do experience substantial file fragmentation can suffer a number of deleterious effects. The most apparent of these will manifest as general performance issues resulting from the scattered distribution of data and less efficient use of the available storage space. Essentially, the more fragmented a file system, the greater the time/effort necessary for a processor to read/access stored data and to reconstitute the information pertaining to any given file.

As a preventative measure, *defragmentation* can be performed so as to combat these effects. Defragmentation is a redistribution of the binary data, a reallocation of information into more contiguous "chunks". Inasmuch, it is also an extremely powerful/highly illustrative example of material assemblage in action, and another instance that nicely embodies the *play of contingency and structure, organization and change* spoken to by Deleuze and Guattari. It is a literal *bringing together of elements*, and the second metaphor to which I shall speak.

This notion of redistributing and "tightening up" information has been integral to the latter stages of my dissertation writing. In a way, the defragmentation metaphor is simply a variation on the puzzle metaphor offered earlier; however, the contexts that motivate it are quite different and I have found it useful in characterizing another aspect of the writing process. Whereas the "dissertation as borderless puzzle" metaphor could be seen as relating more to the resolution of a coherent image with clear boundaries (i.e. a unified document with well-defined scope), this "defragmentation" metaphor might be comparable to *sharpening* the image (i.e. of restructuring/refining the document in order to further consolidate the content within). It could even be said that the former is concerned with the overall *construction* of the dissertation, while the latter is concerned with the *efficiency* of that construction. I grant that these analogies might be a little loose, but it has been no less helpful to think of my evolving dissertation work in these terms.

To be clear, by drawing upon the notion of defragmentation, I am in no way meaning to invoke the long-standing and potentially problematic "brain as a computer" metaphor that continues to be debated. This is to say that I am not suggesting that human brains *actually* reorganize written information in the same way that the operating systems of digital computers defragment file space. Rather, I am simply articulating a figurative connection that has informed my own writing efforts. In fact, I would also suggest that the defragmentation metaphor I have put forth *does not* carry any of the problematic baggage associated with the aforementioned debate; for it makes no real claims about brain *function* and merely speaks to an underlying *mathematical* process. Indeed, if defragmentation is approached through the sense in which I see mathematics as *the science of material assemblage* (i.e. where mathematics embodies the principles according to which matter might be organized/reorganized), then defragmentation is an *inherently* mathematical process, albeit not necessarily a digital one. Alternately put, one can think of defragmentation as solving a specific kind of *optimization problem*, where the optimization constraints are not only associated with grammatical syntax and the economy of word use, but also stylistic, semantic, and various other social factors as well. In this way, the notion of defragmentation (applied to the scholarly writing process) retains a mathematical basis. I will even go so far as to note that this is in keeping with the manner in which I see mathematics as being encoded in matter.

> As the science of material assemblage, mathematics embodies the essential framework upon which material reality is built, from which its objects emerge, and according to which they (co-)evolve […] In this view, mathematics not only underlies the material structure of reality, but also encapsulates the conditions and constraints through which the dynamic processes of material assemblage are manifested.
>
> (Forde, 2018, p. 24)

AN INTERESTING EXTENSION

Thus far, I have framed this discussion in terms of my overall dissertation writing, because that is the materially-grounded process that originally turned my attention in this direction. However, since mulling over the two metaphors addressed in the previous pages, I have also begun to wonder if these metaphors, or others like them, might be more generally applicable in terms of characterizing how we organize and reorganize, configure and reconfigure, our mathematical knowledge as well. For clarity, by using the pronoun ‘we’, I do not necessarily mean members of the mathematics education community alone, but ‘we’ in the more inclusive sense of human beings as a species.

As suggested earlier, there are other aspects of my dissertation research that draw upon the quantum theoretical discourse. In light of these, I believe that both of the metaphors discussed herein might fall under the more encompassing thematic of *granularity* (to which I have alluded on p. 3). By employing *this* term, I mean to emphasize the fundamental tension between the discrete and the continuous that resurfaces in both metaphors, and which seems to speak to a key aspect of our sense-making activities (again, by ‘our’, I refer to the greater human collective).

Interestingly enough, in his recent publication *The Order of Time*, theoretical physicist Carlo Rovelli (2018) puts forth the view that continuity “is only a mathematical technique for approximating very finely grained things. The world is subtly discrete, not continuous” (p. 75). Also noting that granularity “is ubiquitous in nature” (ibid.), he remarks that “The good Lord has not drawn the world with continuous lines: with a light hand, he has sketched it in dots, like Seurat” (ibid.). Beyond this artful allusion to Seurat’s pointillist style as a proxy for the quantum paradigm, it is Rovelli’s reference to continuity as a *mathematical technique* that piques my curiosity. While I cannot personally commit to the stance that continuity is simply a shared illusion, I am intrigued by the notion that the *perception* of continuity might arise from (or be grounded in) some sort of mathematical process. In a manner of speaking, this would suggest that the perception of continuity is not only a means of making sense of the lived experience of time, but also a powerful form of *mathematization* that is integral to the ways in which human beings make sense of the material world as a whole.

It is conceivable that a metaphor which characterizes "continuity as mathematization" might actually be closely aligned with the two metaphors already discussed in this piece; however, I hesitate to delve deeper into that possibility here. Instead, I encourage the reader to consider the sense in which written articulations depend upon some amount of continuity emerging from the combination of discreet elements. Individual units of text (i.e. characters, punctuation, other symbols, et cetera), which are largely devoid of meaning when removed from the systems they comprise, can be assembled according to various technical and stylistic constraints so as to produce words and passages designed to communicate intentionality and meaning. Via a surface reading of Rovelli's remarks, any semblance of continuity that accompanies a given text *could* be interpreted as a simple trick of perception; but, I find it far more compelling to interpret this as mathematically-driven activity. The case of speaking could also be treated in a similar manner, as longer and more significant utterances are built up from individual phonemes. Thus, I am left wondering if it might be appropriate/fitting/reasonable to discuss both writing and speaking as activities that are also enacted, in some sense, through a capacity to *mathematize*.

While I am tempted to continue extending the metaphors already presented, I have a suspicion that the connective threads I am drawing may now be wearing a little thin, and that I could be stretching them to their limits. In any event, I acknowledge once again that this document may, by design, not conform to the expectations of the reader. Nor is it likely to align with the structure of the other contributions to these proceedings. Nevertheless, I do hope that it has offered some insight into how the two metaphors of interest (borderless puzzles and defragmentation) have informed different aspects of my ongoing dissertation writing.

I have chosen to describe the use of literary metaphors as a process consistent with the *assemblage*/*agencement* described by Deleuze and Guattari, and as one that facilitates the negotiation of meaning common to written (and possibly spoken) communication. By extension, I have also attempted to make a case for treating two specific metaphors as being innately mathematical. In the case of the "dissertation as borderless puzzle" metaphor, the construction of coherent and unified written work from singular characters/words/ideas has been reframed as a process that is deeply concerned with mereology and the complementary relationship between the part and the whole. Via the "defragmentation" metaphor, the redistribution of fragmented writing/ideas into more contiguous passages has been likened to an optimization problem whose constraints involve a range of technical as well as social considerations. Both instances entail the *rearranging*, *reorganizing*, and *fitting together* of text and/or ideas, and both appear to be fundamentally rooted in considerations falling under the greater thematic of *granularity*. In itself, this evokes the fundamental tension between the discrete and the continuous; yet some scholars, like Rovelli, question whether the discrete and the continuous are as distinct as traditionally believed. Through my reading of Rovelli, I have tentatively proposed that the perception of continuity might actually involve a form of *mathematization*.

As a closing remark, I openly acknowledge that the perspectives I have shared here are, themselves, largely built up from metaphor, and the *rearranging*, *reorganizing*, and *fitting together* of ideas that I have been mulling over for some time. As more pieces of the larger dissertation puzzle fall into place, and as I continue to defragment the ideational content that emerges, I am confident that the deeper significance of these perspectives will become clearer.

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RECIPROCAL INFLUENCES IN A DUO OF ARTEFACTS

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The combined use of a physical pedagogical artefact and its digital counterpart is described as a duo of artefacts. In the literature duos of artefacts are presented with a certain order assuming that the digital counterpart enhances mathematical knowledge by adding affordances to the physical artefact. This study examines the effect of reciprocal use of artefacts in a duo on a 5-year-old child's identification of relationships between the objects. Data is created through the video record of two clinical interviews with the child. The results show that unless they are used reciprocally, none of the artefacts were enough to mediate relationships between the objects of artefacts which are important for multiplicative thinking.

INTRODUCTION

Learning by doing is encouraged in classrooms as the embodied theories gain weight in the mathematics education. Studies show that mathematical tasks which require students to manipulate physical artefacts enhance mathematical knowledge. However, the rigid structure of artefacts might prevent teacher from modifying them in a way to increase their mathematical potentials. At this point, their digital counterparts add value to the use of physical objects as classroom teaching equipment because different artefacts trigger different signs (e.g. natural language, gestures, and mathematical semiotic systems), and different signs lead to different cognitions. Digital counterpart can achieve this through "offering students a new opportunity to identify the mathematical properties embedded in the artefact behavior and more abstract and conventional representation of mathematical objects" (Soury-Lavergne, 2017, p.1). This combined use of a physical pedagogical artefact and its digital counterpart is described as *duo of artefacts* (Maschietto & Soury-Lavergne, 2013).

Integrating duo of artefacts in mathematics classes is a recent practice, but it has already demonstrated some positive outcomes. Young children who learn combinatorics via a duo of artefacts were found to keep more systematic records of the situations and to enhance their understanding of what a combinatorial problem encompasses (van Bommel & Palmér, 2018). Duo of artefacts pushed six-year old French students to connect the separate conceptualisation processes related to the numbers (Soury-Lavergne & Maschietto, 2015). In addition to young children, duo of artefacts is also beneficial for older children. Using a duo of artefacts to prove Pythagorean theorem enhanced 7th grade students' visualization and added a dynamic dimension with to the students' personal drawings in their pencil-and-paper-based proofs (Maschietto, 2018).

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In all these studies, the duo of artefacts is presented with a certain order: first, students are introduced the physical artefact and then they are given the digital counterpart. This restrictive order suggests that the duo of artefacts enhances mathematical ideas through the added value of digital counterpart only. However, this one-directional approach might hinder the potential of physical artefact to enrich the affordances of the digital counterpart. I will study how reciprocal influence of a duo of artefacts enhances mathematical ideas. In this study, the digital artefact is a tablet application called TouchTimes. It is designed to develop multiplicative thinking through creating numbers in specific ways. The physical artefact is the pencil and paper, through which students draw the target numbers they created with Zaplify – one of the TouchTimes "worlds".

ZAPLIFY

TouchTimes is an iPad application designed to enhance multiplicative thinking. It consists of two models or "worlds" – The Zaplify and the Grasplify. This paper will focus only on the array model in the Zaplify. It starts with an empty screen. When the tablet is placed horizontally on a surface, seven fingerprints and a diagonal line appear respectively in order to guide users to place their fingers both horizontally and vertically in the designated areas separated by the diagonal (see Figure 1a &1b).

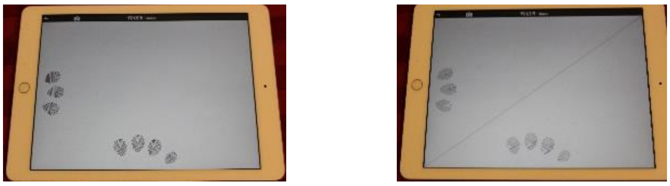


Figure 1: (a) Fingerprints, (b) Fingerprints and the diagonal.

When a user places and holds any finger on the screen, a "lightening rod" (I will call them "lines" from now on), which passes through the point of touch, appears on the screen either horizontally or vertically according to the position of the touch with respect to the diagonal. The upper triangular area formed by the diagonal allows horizontal lines (HL), while the lower triangular area allows vertical lines (VL). Screen contact can be made with one finger at a time or with multiple fingers simultaneously. Multiple fingers that maintain continuous contact can create either only HL, only VL or both VLs and HLs (see Figure 2 a-c).

Whenever two perpendicular lines intersect, an orange disc gradually appears on the intersection points. The numerical value of the total number of intersections, which is the product of the two factors, appears in the upper right corner of the screen (see Figure 2c). If there is no intersection, only the number of factors appear (see Figure 2 a,b).

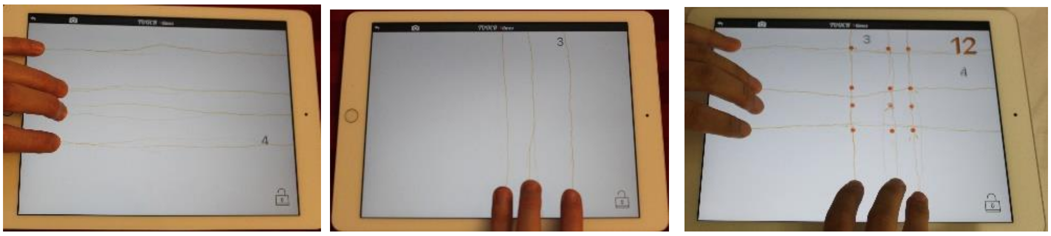


Figure 2: (a)HLs, (b) VLs, (c) VLs and HLs.

There are two modes of manipulation of the app: locked and unlocked. In the unlocked mode, the lines disappear as the fingers separate from the screen, whereas in the locked mode, lines remain on the screen even when the user's finger is lifted. This allows a user to create products that involve more than ten fingers.

THEORETICAL FRAMEWORK

This study draws on Bartolini Bussi and Mariotti's Theory of Semiotic Mediation (TSM). This theory focuses on the relationship between the representation systems and the human cognition. Human beings create representations through using artefacts and this has two consequences: the modification of the environment and the cognitive development. TSM is based on this double nature of artefacts.

Given an artefact, it does not guarantee a specific use for the subject. At this point Rabardel (as cited in Bartolini Bussi & Mariotti, 2008) distinguishes artefacts from instruments. An artefact is a concrete or a symbolic object itself. It becomes an instrument by the subject through its particular use. For example, a glass is an object which is designed to carry liquid. If a cook uses it to crash some walnuts into smaller pieces by pressing the walnuts between the bottom of the glass and a cutting plate, the glass becomes an instrument.

Instrumental approach to artefacts can be informative in analyzing the cognitive processes related to the use of a specific artefact and its semiotic potential. However, it is not adequate to analyze the more complex process of teaching and learning mathematics through artefact use. At this point, Bartolini Bussi and Mariotti (2008) resort to Vygotsky's approach to artefacts.

Vygotsky talks about the difference between an individual's developmental levels in two different situations: (1) when an individual is able to accomplish a task him/herself, and (2) when an individual can accomplish a task with the guidance of a more knowledgeable individual (as cited in Bartolini Bussi & Mariotti, 2008). This difference is called *zone of proximal development*. Within this zone, the communication between the individual and the more knowledgeable one leads to the cognitive development of the learner. Theory of semiotic mediation elaborates more on the relationship between tasks, signs and mathematical meaning making within this process and distinguishes semiotic mediation of artefacts from teachers' cultural mediation.

Using an artefact in a social context, learners produce certain signs which are essential for semiotic mediation. These signs have a dual role: expressing the relationship between the task and the artefact on the one hand, and the relationship between the artefact and mathematical meaning on the other hand. The former is called *artefact sign* and their meaning is associated with the operations conducted to achieve the task. The latter is called *mathematical sign* and it is aligned with the existing mathematical culture. On the way to the evolution of artefact signs into mathematical signs, pivot signs are important. The pivot signs “may refer both to the activity with the artefact [...] and to the mathematical domain” and they are distinguished from the other signs based on the extent of generalization they carry (Bartolini Bussi & Mariotti, 2008, p.757).

METHOD

Data is created through the video recording of a 5-year-old child’s interaction with both TouchTimes’ Zaplify world and pencil-and-paper. The participant is recruited through convenience sampling. I (denoted as R in the below transcripts) conducted two interviews with the participant, Zach (denoted as Z in the below transcripts) at his home. The interviews consisted of number-making tasks, drawing tasks, and what-happens task in which I asked Zach to anticipate how the number would change, if I added more fingers. Zach’s father (denoted as F in the below transcripts)) was present during the first interview, and he participated in the interview by asking questions to Zach when Zach seemed hesitant to respond. Both interviews lasted for approximately half an hour.

In this analysis, I focused on the signs Zach created via duo of artefacts, drawing from Arzarello, Paola, Robutti, and Sabena's (2009) concept of *semiotic bundle*. There are two ways to analyze a semiotic bundle: synchronic and diachronic analysis. The former focuses on a specific moment where the subject produces different signs spontaneously. The latter focuses on the evolution of the signs produced by the subject in successive moments. In my analysis, I also analyzed different signs created by different artefacts at different time points in a synchronic manner in order to examine the relationship between the artefact signs.

FINDINGS

At the beginning of the first interview session, Zach randomly made one orange disc on Zaplify. Zach described the orange disc as a dot. When I asked him to make one more, he could not make it. During the following 18 minutes, while Zach was holding HLs, I was adding VLs one by one, making 2, 4, 6, 8, 10, and 3, 6, 9, 12, 15 respectively. Then I asked Zach to make “one” again, assuming that creating numbers repeatedly on Zaplify might have helped Zach to identify the relationship between the lines and the discs. As I pointed to the upper right corner of the screen, I said: “I want to see the [numeral] one here and one orange ball”. After a few attempts, he could not make any disc. Then I asked him to draw one disc:

1 R: In order to get one dot, what we should see? How does one dot appear? Can you draw one dot? How was it on the screen when we see one dot?

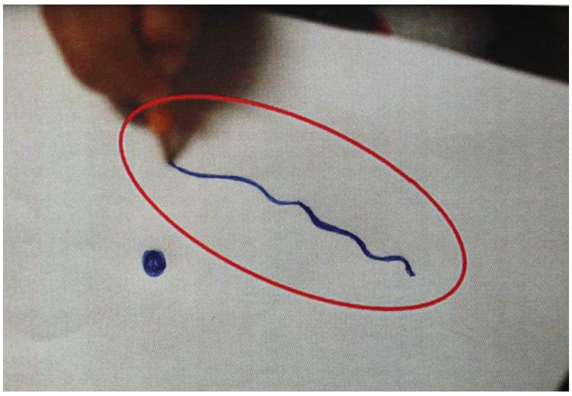
2 Z: It was small and red [*drawing a circle*]

3 R: Were there anything else other than the dot?

4 Z: A yellow line

5 R: Where was it?

6 Z: … [*drawing a curvy line which looks like a wave just below the circle*]



Note. The author retraced the pencil marks in the pictures to improve visibility.

Figure 3: Horizontal curly line.

Zach used the words "small" and "red" in order to describe the dot. These artefact signs refer to physical features of the ball unlike its position, which might suggest a relationship between the other artefact signs such as lines and the intersection point. When I drew Zach's attention to the other artefact signs (line 3), Zach uttered the word "yellow line". This artefact sign includes a mathematical sign, which is a "line", yet it also refers to the color of the line in order to describe it. Again Zach created signs related to the physical features of the objects rather than their orientation, which is important in terms of multiplicative relationships. When I hinted the orientation by asking where it was (line 5), Zach created a sign in another modality. Rather than describing it with verbal signs, he created a visual sign with his drawing (see Figure 3). This sign illustrates the line in horizontal orientation as in the Zaplify, yet separate from the disc. So it seems that Zach did not relate the disc with the HL. They were two independent entities for him.

The relationship between the signs appeared in our second trial. After Zach and I together made a disc the second time on Zaplify, I asked him to draw a disc on the paper.

7 R: How did we do one dot? Can you draw it?

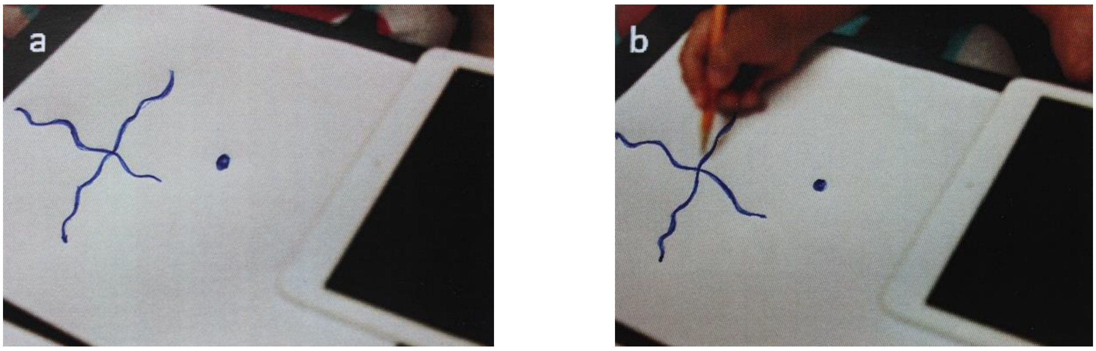
8 Z: … [*drawing a circle*]

9 F: Draw what you saw on the screen. Where were the yellow lines?

10 Z: Where were the yellow lines? One is here and one is here.

11 R: Why don't you draw it here [*pointing to the paper*]

12	Z:	… [*drawing one vertical curly line from top to the bottom of the paper, then another one from left to right of the paper crossing over the VL*]
13	F:	[*pointing to the dot on the paper*] Is this dot on the same spot compared to the screen?
14	Z:	No.
15	F:	Draw the dot. Where should it be?
16	Z:	It should be in the middle of here [*pointing the intersection of the lines*]



Note. The author retraced the pencil marks in the pictures to improve visibility.
Figure 1: (a) Dots and the intersecting lines, (b) Pointing to the intersection of the lines.

Compared to the first drawing, Zach produced more signs in this episode. First, he drew one disc and then two lines next to the disc, which intersected each other. So this physical separation between the lines and the disc in Zach's drawing indicates partial relationship between the artefact signs. The lines are related to each other, but they are not related to the disc.

Zach transferred the orientation of the lines from Zaplify to the paper directly. He drew two perpendicular lines as in Zaplify (see Figure 4a). When we made one disc together, Zach first held his finger and made a VL, and then I put my finger and made a HL. Similarly, first he drew the VL in this episode. While Zach transfered the order of the lines from Zaplify to his drawing, the order of the disc was not transfered. In Zaplify, Zach created two lines on the screen and then the disc appeared out of the lines, but on the paper, he first drew the disc and then the lines. Thus, he did not transfer the location of the disc in relation to the lines. Zach connected the disc with the lines (see Figure 4b) only after he was asked to compare his drawing of the disc with the diagram in the Zaplify (no. 13-16).

Zach started to create the intersecting lines on the screen after he used his second drawing as a reference to make one disc in Zaplify. However, the relationship between the intersecting points and the discs became solid after we discussed the relationship between the lines at the second interview. Until this episode, Zach answered few "what happens" tasks correctly. After our discussion, he started to demonstrate a consistent strategy to answer these tasks correctly. The following episode presents these discussions.

After Zach made one disc on the screen, I asked him: “What happens here?” as I pointed to the intersection of the lines.

17 Z: One dot.

18 R: What is happening to the lines here where the dot stays [*pointing to the intersection*]?

19 Z: The dot stays in the middle [*pointing to the dot*] of these [*tracing the VLs and the HLs*] lines

20 R: How did you make this [*pointing to the dot*] in the middle?

21 Z: I put my finger here [*pointing to the bottom of the VL*] and make the line, and then I put my finger here [*pointing the HL*] and make the line, and then I make the dot with this line [*tracing the HL back and forth*]

22 R: You made this line [*pointing the VL*] first, and this one [*pointing the HL*] second, right?

23 Z: Yes.

24 R: What did the second line do to the first line? What happened here [*pointing the intersection*]?

25 Z: Second line crossed [*tracing the HL*] the first line [*tracing the VL*]. The dot is with the second line.

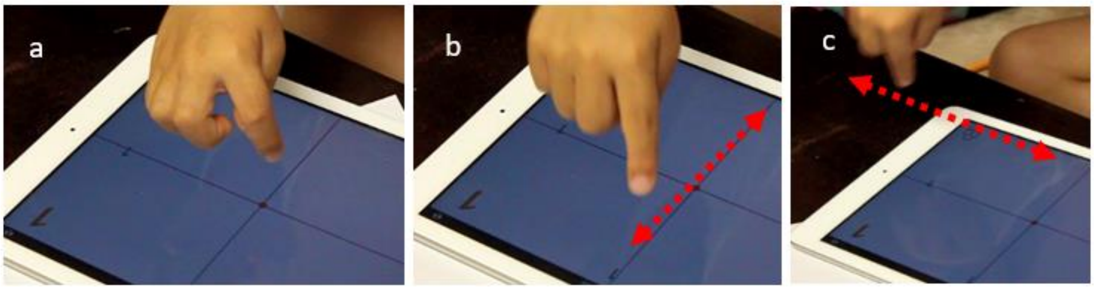


Figure 5: (a) Pointing to the dot, (b) Tracing the VL, (c) Tracing the HL.

Zach referred to the intersection point via a sign “the middle”, which he created during a drawing task in the previous interview (line 16). The verbal sign “the middle” and “these lines” are used together with the gestures (line 19). They all together suggest that the orientation and the intersection point of the lines are both related to the location of the disc. The pointing gesture (see Figure 5a) and the word “middle” refer to the intersection point, and the tracing of the lines (see Figure 5b & 5c) refers to the perpendicular lines. According to Zach’s verbal accounts, the intersection seems to be necessary for the disc to appear. He stated that he made the disc with the second line, which crossed the first line (line 25). Thus, the sign “cross” constitutes the relationship between the lines and it is an important sign to create the disc.

CONCLUSION AND DISCUSSION

Multiplicative thinking differs from additive thinking when a child identifies the factors of multiplication as two distinct referents and conceptualizes the multiplication as the coordination of these units (Clark and Kamii, 2006). Therefore, distinguishing

HL's and VL's of Zaplify which represent two factors of multiplication and identifying the coordination between the lines and the product are important for multiplicative thinking.

The findings show that continuous interaction with Zaplify was not enough for the child to identify the relationships between the objects which are important for multiplicative thinking. Pencil and paper provided an environment for the child to think in another modality, which was transferred back to the digital artefact via Zach's gestures. Zach first expressed the location of the lines by drawing on the paper and then he expressed it by tracing the lines on the screen. However, drawing through pencil and paper was also not effective itself to make the relationships between the objects salient. Zach created several pivot signs in different modalities via reciprocal use of this duo of artefacts. Moreover, Zach's discussions with adults through these signs seems to play a role in mediating the identification of the relationship between one disc and the intersection point of the two lines. Even though it is problematic to equate this relationship with multiplicative relationships, it might mediate identifying the two separate units and the relationship between them in a multiplication. The future research will be conducted to examine this potential mediation through the reciprocal use of a duo of artefact.

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LEARNING GEOMETRY THROUGH DRAWING AND DRAGGING IN A PRIMARY MATHEMATICS CLASSROOM

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This project was carried out in a mathematics classroom in an affluent and culturally diverse urban elementary school in North America. I conducted a month-long classroom intervention with Grade 2/3 students, experimenting with geometric tasks within physical environment of paper and pencil, and virtual multitouch environment of dynamic geometry. In my experiments, I was interested in specific ways these two contexts give rise to mathematical concepts, and how learning affordances of digital and tangible tools are complementary and different. I wanted to see (1) what the students would learn, and (2) what the constraints and liberations of those environments would be.

INTRODUCTION

In recent years, much research has focused on the use of manipulatives in mathematics classroom (e.g. Murray, 2010; Clements & Sarama, 2014). Physical manipulatives were reported to increase student engagement, gains in students' mathematical ability and communication. Virtual manipulatives were found to improve progress, promote problem-solving skills, facilitate changes in mental representations, create complex spatial patterns, and assist in understanding abstract concepts, among other benefits (Spencer, 2017). More recently, researchers focused on combining virtual and physical manipulatives (e.g. Soury-Lavergne & Maschietto, 2015). Soury-Lavergne (2016) suggested that concrete manipulatives, when used in isolation, limit knowledge transfer between situations, and proposed to design *duos of artefacts*, associating a concrete manipulative tool to a technological tool in order to combine the advantages of both types.

In specific case of geometry, Perrin-Glorian et al. (2013) distinguished three kinds of spaces: *physical* (the world of physical objects), *graphical* (diagrams, drawings, and artifacts) and *geometrical* (Euclidian theory, deductions, and tools to solve problems). According to Perrin-Glorian, it is the graphical space that acts as a 'bridge' between the other two. Indeed, Soury-Lavergne & Maschietto (2015) found that DGE acted as such graphical space since one could produce different graphical representations of objects by referring either to physical or geometrical space: "Actions in dynamic geometry carry some of the properties of physical space, since objects can be moved, grabbed, suppressed, hidden, folded, etc. Dynamic geometry provides pupils with continuous feedback as they drag the points of a diagram. This is something that cannot

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be produced using a paper-and-pencil diagram" (p. 6). In addition, Thom and McGarvey (2015) argued that the act of drawing served as a means by which children became aware of geometric concepts and relationships and should be viewed as a mode of thinking rather that a product of that awareness.

This paper reports on a qualitative, empirical study, conducted in a primary mathematics classroom. The purpose of this study was to explore how this combination of paper-and-pencil drawings and DGE would give rise to geometric concepts.

THEORETICAL FRAMEWORK

Since my interest lies primarily in geometry, I will use Raymond Duval's geometric constructs to guide the analysis of students' learning in each of the two environments.

Dimensional Decomposition

Duval emphasized "ways of seeing" and considered visualization to be fundamental in geometric thinking. He argued that the cognitive power of visualization lied in its tendency to fuse units of inferior order into one unit of superior order (Sinclair, Cirillo, & de Villiers, 2016), and that it was important to achieve synergy between the visual and discursive registers of geometry. This could be done through engaging a "constructor" way of seeing, which was supported by the process of dimensional decomposition. Dimensional decomposition involves two aspects: seeing the basic shapes as constructed from lines and points and seeing that many two-dimensional shapes could emerge from networks of lines.

Duval proposed that the learning of geometry could begin with exploration of the different configurations that could be formed with lines. To support dimensional decomposition, he recommended construction as the point of entry:

> The shape is no longer a stable object but one that evolves over time, capable of being decomposed and reconfigured. This non-iconic visualization is characteristic of geometric thinking… It is a way of seeing that eventually enables all the discursive procedures in geometry (pp.459–460).

Duval distinguished four apprehensions of a geometric figure: (1) perceptual (recognition), (2) sequential (construction), (3) discursive (property-based recognition), and (4) operative (processing, as in transforming and reconfiguring). He viewed geometry as joining two representation registers: the visualization of shapes and the language for stating and deducing properties - seeing and saying. Since our primary concern was development of geometric thinking, Duval's vision was found helpful in analysing progression of students' learning.

Dynamic Geometry

As was mentioned above, spatial thinking was found fundamental to mathematics learning and a predictor of future mathematics achievements. It has been recently acknowledged that spatial reasoning skills are not genetically predetermined like once thought, but can be developed with practice, and that students with well-developed spatial reasoning skills succeed in STEM subjects. Davis et al. (2015) articulated the need to bring a stronger spatial reasoning emphasis into school mathematics, while Sinclair & Bruce (2015) explored the role of tablet technology as a mediator of spatial reasoning for young children, and concluded that compared to handling physical tools, engaging with virtual tools provided students with different kinaesthetic experiences. Research has also demonstrated that the use of Dynamic Geometry Software (DGS) can significantly enhance the learning process, combating prototypical thinking. DGS can lead users to think about geometric objects and relations in different ways, thereby changing the fixed, linear development, proposed by the van Hiele's model (Sinclair & Moss, 2012). It is important that learners understand dragging as a manipulation that preserves the critical attributes of the shape.

METHOD

Interested in ways different contexts give rise to mathematical concepts, I conducted a classroom intervention, experimenting with geometric tasks within two environments: (1) physical environment of paper-and-pencil and hand-held manipulatives, and (2) virtual multitouch environment of DGE. I wanted to see what the students would learn in each of the environments, and how learning affordances of digital and tangible tools are complementary and different. In this paper, the physical environment is understood to encompass both tools like pencils, and manipulatives like pattern blocks. Virtual environment encompasses virtual tools (e.g. geometric primitives provided by the widgets) and virtual manipulatives (e.g. a constructed shape that can be manipulated on the screen).

I conducted this research in a Grade 2/3 classroom in an affluent and culturally diverse urban elementary school in North America. While students were exposed to tablet and Smartboard technology since Kindergarten, they never worked with DGS before. It was a month-long classroom intervention, which consisted of four weekly sessions. Each session was an hour-and-a-half long. In my lessons, I opted for a combination of whole-class Smartboard discussions, followed by students working on iPads or using paper and pencil. I used web sketches based off The Geometer's Sketchpad (Jackiw, 2012) Specifically, I used "Triangle Shapes" and "Point, Segment, Circle" web sketches designed by Sinclair (sfu.ca/geometry4yl.html):

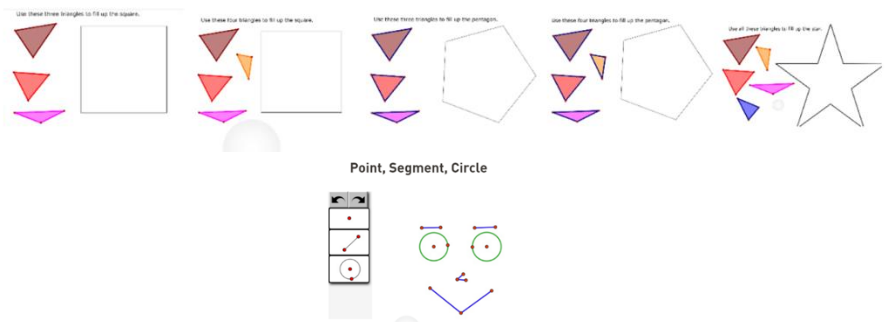

Figure 1: Web sketches

In my tasks, I focused on properties of a triangle. Each session was thoroughly documented through notetaking, photographs, and examples of students' work, both in static and dynamic format. Records of classroom activities were analysed for the evolution of students' conceptual understanding of a triangle.

FINDINGS

I will now describe some of the findings according to the environment that housed the activities, while attempting to interpret them through the lens of dimensional decomposition and its role in the development of geometric thinking.

Pencil-and-paper environment

Most students drew their first triangles in a prototypical way – with bases parallel to the bottom of the page, which indicated perceptual apprehension. When requested to not have a side on one of the lines, most still had a side parallel to the edge, flipping the shape upside down, or rotating it just slightly. Some children drew triangular prisms, positioned like pyramids. These initial drawings provided an important window into students' conception of a triangle: an iconic equilateral triangle with one of its sides parallel to the bottom of the page.

Drawing solutions to DGE problems turned out to be challenging - students frequently ended up with drawings of pentagons and trapezoids, even though there were no such shapes on the screen. Having to construct a shape from scratch needed not only well-developed perceptual apprehension, but sequential as well. This drawing experience provided an important insight: more DGE experiences were needed to help consolidate the ability to think sequentially.

The situation has changed when the students were offered to make free drawings, with the constraint to use triangles only. The difficulty to decompose and recompose quickly resurfaced to reveal accurate level of geometric thinking, which was still perceptual. No one used two triangles to form a square, but many were comfortable adding smaller triangles inside a larger one for eyes or mouth.

In the paper-and-pencil environment, the unsatisfactory construction has to be erased and re-done, and that could involve multiple attempts. It did not have the affordance to fluidly move from one construction to another, allowing for observation of an infinite number of transient stages, one of which could represent the solution. The benefits of this pencil-and-paper activity were in the construction process itself - it engaged the "constructor's way of seeing": being able to see that a triangle can be constructed from lines and points, and that many various triangles could emerge from networks of lines. This emphasis on one-dimensional prior to two-dimensional is a precursor for development of geometric thinking (Duval, 1998).

Dynamic Geometry Environment

Judging by the silent focus during the first web sketch demonstration, the students were stunned to see a dynamic triangle, whose behaviour disrupted the expected equilateralness of the green wooden pattern blocks. One of the students, who volunteered to come to the board, first made his triangle "dance" by dragging its vertex – an impossible skill to teach a triangle within a paper-and-pencil environment. As the boy was playing, his body was mimicking the trajectory in a subtle way. I did not observe anything like this during the paper-and-pencil session. Now, this virtual experience paradoxically seemed more embodied than the physical, which means it could help mediate mathematical cognition (Radford, 2008). Previous research showed that virtual things could be even more concrete than physical because students come to be in close relation with them through interaction. This instance precipitated entry into operative apprehension of geometric diagram.

The first DGE task was to fill in a square with two dynamic triangles (Fig. 5), which did not feature much variation. There would usually be an isosceles triangle nicely centered, with its base taking up the entire lower side of the square. With three triangles, the design featured a perfectly centred equilateral triangle, embraced by a symmetrical "curtain" (Fig. 2).

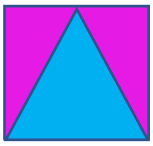

Figure 2: Filling up a square with three dynamic triangles

Even when the challenge included four triangles, this design persevered, leaving the fourth triangle unused. One student solved the problem creatively: he shrank the leftover triangle into a near-line and added it to one of the sides. Others protested: "You can't do that! You should have left more space!" This difficulty to see this "thorn-like" triangle as a triangle indicated that discursive apprehension was not quite developed yet, as the collaborative definition now included the length of the sides.

Once the class moved to the template of a star, the difficulty students faced has increased significantly (Fig. 3).

Figure 3: The Star challenge

The break-through happened as one student saw the possibility to drag one of the vertices into the pentagonal area. After that, students seemingly saw that any rectilinear shape could be made of triangles.

When I introduced construction, as their first task the students were assigned to construct anything using a segment, a point, and a circle widget at least once. Everybody without exception made use of the segment tool. Second task was to construct a house only with triangles built from scratch, but most houses came out crooked (Fig. 12a). However, an instance of breakthrough thinking was observed, when a student applied prior knowledge and used two almost-right triangles to make a door (Fig.4).

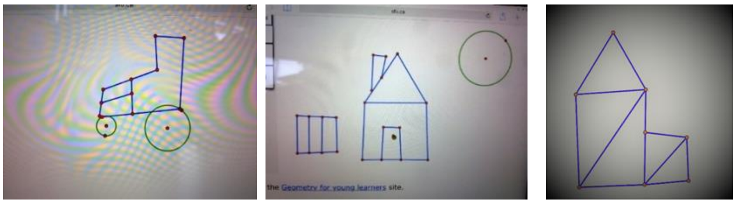

Figure 4: First constructions

DISCUSSION AND CONCLUSION

Now I will discuss the results as they pertain to the literature, and answer the two research questions: (1) what the students would learn in each of the environments, and (2) what the constraints and liberations of those environments would be.

Constructor's way of seeing

Using Duval's construct of visualisation as a lens, I concluded that the majority of students learned to decompose a triangle: they were able to move from 1D to 2D when drawing and constructing shapes, and move from 2D to 1D when they needed to re-draw a side or use the "go back" function in the sketch pad. Also, the vast majority of students could recognise a triangle - even in disguise of a "thorn" (perceptual apprehension), construct it in both environments (sequential apprehension), talk about some of its properties and prove that it was indeed a triangle (discursive apprehension), and finally process and transform both prefabricated triangles and triangles constructed from scratch (operative apprehension).

Our experiments showed that one level of visualisation was not necessarily a precursor to the next, but the two or more were developing concurrently, and some students exhibited signs of operative level long before they had a chance to master sequential level: I could see triangles being constructed and manipulated in sketchpad, when students still had difficulty recognizing shapes based on properties. It is this non-linear experience that allowed the students to frequently revisit and reflect on various activities, while building on prior knowledge: one apprehension level was supporting another, while still being under development itself. Ultimately, such spiral combination of tools and environments supported the development of geometric thinking and evoked all four apprehensions of a triangle in primary students, albeit to a varying degree.

I carefully observed how these two contexts of physical and virtual gave rise to mathematical concept of a triangle, but it is not possible to single out one task or one environment that could be considered more important or more effective: it was the combination of the two, and frequent switching between them, which helped students improve their spatial reasoning skills and ultimately develop geometric thinking.

No binary distinction

Thom and McGarvey (2015) argued that the act of drawing served as a means by which children became aware of geometric concepts and relationships and should be viewed as a mode of thinking rather that a product of that awareness. This was evident during our process of drawing: children experienced dimensional decomposition first hand, planning to produce a 2D shape they knew well, but then realizing they needed to produce 1D object first, and that would be a significant step in understanding that a triangle was made of segments.

DGE is complimentary to drawing in that it invites the learner to construct a 2D shape from a number of 1D objects as well. Dimensional decomposition can also be performed effectively, since the first step of engagement with DGE is often a construction task. However, the complexity that this environment brings into the picture is impossible to achieve with paper-and-pencil drawings. Once the initial construction is complete, the fun is just beginning. Learners come to understand that dragging preserves the critical attributes of the shape, if the construction is robust. They must eventually understand dragging as an action that can be used to generate a family of shapes, with a certain set of properties, determined by the way the shape was constructed, which involves a significant discursive shift (Sinclair & Yurita, 2008). Being able to see all the transitional variations of a shape, whose vertex is being dragged from point A to point B, is a one-of-a-kind experience, opening up the familiar constraints and leading a student towards abstraction.

Soury-Lavergne & Maschietto (2015) argued that the construction of geometrical knowledge was based on spatial knowledge and implied building relationships between the three spaces: physical, graphical and geometrical. They proposed that the use of digital technologies supported the connections between spatial and geometrical fields. I witnessed in my experiments how today's activity seemed difficult, but tomorrow it was appropriated and applied in a new context, as students were making meaningful connections (e.g. learning how to fill up a square with two triangles in a web sketch and then transferring that skill to the construction of a door, or learning the concept of a right angle, which came to fruition in DGE play).

I found that each context boasts a unique set of features, benefiting the learner in a particular way, but that there is no strict binary distinction, and multitouch environment often makes experience more physical due to whole body involvement. The physicality of digital environment was ever-present as students were developing intimate relations with the triangles they were constructing and instantly appropriating as their own, making DGE more concrete than abstract, or indeed a "bridge" between the two.

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GETTING NEW WRITING ON THE BOARD IN AN UNDERGRADUATE MATHEMATICS LECTURE

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In this paper I take seriously the task of the lecturer in undergraduate mathematics: to write on the board a selection of true results and precise definitions while providing convincing argumentation justifying these inscriptions. Using a microethnographic approach that emphasizes contexts and the role of the hands, I analyze a few moments to highlight some common writing/speaking/gesturing actions: construal of a piece of writing in order to make a contrast, construal of a piece of writing in order to make a specific choice from a general type, nonlinear writing, grasping and circling to indicate "many", and moving the hands from one place to another while keeping the shape constant in order to indicate equality.

INTRODUCTION

Many undergraduate mathematics classes are taught using a lecture approach. The professor stands in the front of the room, usually at a blackboard or whiteboard, perhaps with a transparency or other means of projecting text onto a large visible screen, writes out theorems and their proofs and their corollaries, writes out examples and exercises and their solutions, draws diagrams of all kinds, points repeatedly at elements of these things, gestures, tells stories and jokes, gives historical background and motivation, fields questions, asks questions of the class and engages in dialogue with those who answer, and more. The class sits in their seats, usually do not talk except when whispering to the person next to them or answering a question or asking a question or pointing out an error they think the professor has made. Their seats face the front of the room where the professor and the board are, and there is usually an unbroken spatial boundary between them (Barany & McKenzie, 2014; Greiffenhagen, 2014). Such an arrangement for university mathematics teaching and learning has been very common for a very long time – Hilbert and Klein would find nothing unusual about it.

Mathematics was already a deep subject in the 1890s, with an incredible amount of mathematical truths already proven in the literature. Already professors had to be extraordinarily selective in what they chose to lecture on to their undergraduates. Every generation of mathematicians since has inherited the previous generations' literature, and has attempted to compress it, to rewrite it using newer mathematical concepts and results. The subject known now as group theory was still very young then; by now it has a very large literature indeed. Famously, the theorem which classifies all finite simple groups has a proof that extends to thousands of pages, is the collective work of

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many many mathematicians, spread over hundreds of research articles. Even the attempt at a “second-generation” proof runs now to many volumes (8 of an expected 11 have been published thus far). No one individual in the world understands the full proof. I will take from this example two lessons (neither of them I hope surprising). First, that those who teach group theory to undergraduates have an enormous wealth of material to draw on – there are always far more theorems and definitions than are possible to include in a first course, even when you restrict to material accessible to an undergraduate (even group theory textbooks designed for first courses in group theory are much longer and denser than can be reasonably covered in a course). Second, that mathematics is supremely a collective human endeavor, the result of a community of people sharing a common mathematical language and heritage. I combine these lessons with a third, also hopefully unsurprising observation, that the results in mathematics must be proven – with arguments that satisfy this community, using definitions that are precise and unambiguous, relying on previous results that are trusted and can be verified. Taking these assertions seriously, this paper seeks to find some partial answers to the question: how does a lecturer manage to successfully write new pieces of mathematical material on the board. In the next section I describe work by two mathematics education researchers, Elena Nardi and Keith Weber, that impacted this study.

MATHEMATICAL LECTURING, WRITING, AND REASONING

In 2002-2003 Nardi conducted eleven half-day focused group interviews with twenty mathematicians from different parts of the UK. Their ages ranged from early thirties to late fifties and the length of their teaching career varied from a few years to more than thirty. A week before the interviews she distributed data samples consisting of students’ written work and interview transcripts. This data had been collected from students attending first year courses in calculus, linear algebra, and group theory. When the participants arrived for the four-hour interview they usually came with comments and questions based on their careful consideration of the samples. She took a narrative approach in presenting the results of her analysis of these interviews (Nardi, 2007), structuring her book as a dialogue between two characters, M (the mathematician, a composite of all the participants) and RME (the researcher in mathematics education). Nardi offers mathematicians revealing their pedagogical beliefs and strategies insofar as they are conscious of them and can articulate them.

Concerns about their students’ mathematical writing expressed by M are a dominant theme:

> And words, sentences, those creatures ever absent from students’ writing exist exactly for this purpose: of emphasis, of clarification, of explanation, of unpacking the information within the symbols. (151)

Nardi’s mathematicians reveal over and over that they are keenly interested in the details of precise and careful use of language, including syntactic and paragraphic structure. Another common theme is that of mathematical viewpoint and perspective:

> I see this business of sharing landscapes as my main business as a lecturer. We are not just communicating facts, we are saying that this is one way you can view it and that is another way you can view it, let's put these together somehow. And it's not an easy job, believe me! (218)

Weber performs an analysis conducted largely at the level of the (75 minute) lecture, and concludes that they can be sorted into three categories that reveal three different teaching styles: logico-structural, procedural, semantic (Weber, 2004). In his discussion of the procedural style, Weber notes that it appeared to him that in lectures delivered in this style, the professor aimed to explicitly teach his students strategic knowledge, which Weber defined as "heuristic guidelines that they can use to recall actions that are likely to be useful or to choose which action to apply among several alternatives" (Weber, 2001). In this paper I conduct a case study of a lecturer teaching a course, just as Weber did: I hope to analyze the actions of the professor in the classroom that they in practice use when faced with alternatives or when faced with the question of what to do next and how.

CONTEXTS AND MICROETHNOGRAPHY

Chafe (1994) has emphasized the importance, in spoken discourse, of local contexts where a specific topic is being discussed. He notes that beginnings and endings of these contexts are communicated by speakers in a variety of ways: intonation, volume, pitch, tempo, pauses, body movements, eye gaze, gestures. In the context of mathematics education research, Staats (2008) defended the division of transcripts of classrooms into poetic lines, citing workers in linguistic anthropology such as Dell Hymes and Dennis Tedlock who helped develop the field of study known as ethnography of communication.

This paper adopts a microethnographic approach (Streeck, 2017). This framework does not posit thoughts inside people's heads that are not visible. Instead, close attention is paid to visible actions, occurring inside local contexts, where the participants visibly adapt their movements and their gestures and language to meet their own answer to what they understand the communicative event they are involved in to mean.

METHODOLOGY

35 50 minute lectures in group theory were videotaped. A 240 000 word corpus was constructed from this data. The transcript was divided into 3004 stanzas, using the tools described by Chafe.

Some notes on the transcripts below. In the transcripts of speech, when mathematical objects are referred to, they will be put within dollar signs delimiters, with no space between the variable and the dollar signs. In the transcripts of writing, when mathematical objects or equations are being used, a dollar sign and a space will be used as delimiters. Any other mathematical relationship in the transcripts of writing will be written using a backslash and then an abbreviation of the relationship (in fact, we use the format mathematicians use when producing their published texts – LaTeX). When speech is referred to, single quotation marks will be used; when writing on the board is referred to, double quotation marks will be used.

DATA ANALYSIS

Construing to contrast; grasping/circling to show ‘many’; equating.

This section analyzes the following stanza (the 42nd in the 23rd lecture):

1 ok so here it is as a definition

2 a subgroup of a group is normal

3 if every left coset equals the right coset

4 that doesn’t mean that a little h equals little h a for every h

5 for every little h in the subgroup ok?

6 it just means that this collection of objects

7 that’s a coset

8 that’s order H elements in there

9 that collection of objects is the same as that collection of objects

10 no matter which coset rep we take. (23.42)

The writing that exists on the board as he begins this stanza (he has just written it to begin the new chapter of the textbook they are using) is as follows:

A subgroup $ H $ of $ G $ is _normal_, written $ H \nsubgp G $, if

$ a H = H a $ for all $ a \in G $.

J begins the stanza by taking four steps to the board in order to arrive at the equation on the board just as he says ‘here’, using his index and pinky finger of his left hand to simultaneously point to either side of the equation. The previous stanza had seen him make a spoken comment from a distance. Keeping his fingers in their configuration he pulls his hand back and pushes his hand to the board twice in emphasis as he says ‘definition’. In his next spoken line he reads the first written line while touching “$ H $” when he says ‘subgroup’, “$ G $” when he says ‘group’, and “normal” when he says normal. While the last is a direct transposition of writing to speech, the first two are small modifications: by now in the course he has stopped writing “a group $ G $”, trusting that “$ G $” alone will carry the meaning. This is one of numerous compressions of writing that occur on the micro-scale.

The reading continues in the next line, where he touches the left and right sides of the equation while saying ‘left coset’ and ‘right coset’ respectively. As soon as he concludes this line he turns his gaze for the first time in the stanza to the students, marking a transition in the stanza. He maintains the same body position, orienting his upper body and head in order to face the students, but shifting his body weight to the leg closer to the class. There is little to no pause and no change in volume or tone as he continues, so it is clear that the same work is continuing in the same stanza. In line

4 he turns again to the board and begins touching the equation again, this time touching each of the 4 letters in it as he says 'a', 'little h', 'little h' and 'a' respectively, ending with holding the last letter and turning his gaze back to the class as he says line 5. Here although he is touching the capital letter $ H $, which denotes a subgroup, he is deliberately construing this mathematical object to be an arbitrary element in the subgroup ("little h"). He is contrasting the actual written truth with what it might easily be mistaken with, cautioning them against this interpretational error.

Line 6 begins with him turning his head back to the board, and as he speaks the first few words he forms a grasping shape with his left hand, and touches the board with all fingers and thumb while surrounding the left hand side of the equation. He appears to grab "$ a H $" in a clutching move as he says "collection". His hand looks like he is trying to contain a bunch of objects, rather than the single element that he had been touching with a single finger in line 4. In line 7 he shifts to a different approach to indicate that there are many objects living in the set that is the left hand side – he uses his left index finger to circle around and around the "$ a H $" term as though to draw a physical boundary around these elements and corral them safely together as a single grouping. Such movements of the shape of his hands while he distinguishes the one from the many recur frequently in the course – both versions, the grasping gesture, and the circling/corralling gesture.

Now that his hand has placed the notion of many elements on the (vertical) table, he moves to revealing how many elements there are. His left hand briefly imitates the vertical line that would be drawn on the board on either side of "$ H $" in order to write down the order of this subgroup (and hence also the order of the left coset he is talking about, tacitly appealing to an earlier proved theorem). This writing-in-the-air lasts for only a half-second (not even as long as he needs to say the words 'order H'). This is an instance of an occasion where perhaps the communicative feature of the gesture is of less importance (because if you blink you miss it) than the function of the gesture to help J himself think through what he wants to say. On many other occasions the writing-in-the-air (treating the air in front of him as if it were a surface he was writing on) lasts longer, and this is a regular feature of his hand movements in the course. By the time he says 'there' his hand is shaped like an open palm and he is pressing on the board with all fingers and thumb, entirely covering the left coset. Now when he turns his gaze to the class again, he is ready for the punchline (lines 9 and 10) of his stanza: he holds his hand where it is as he says 'that collection of objects', lifts it up from the board without altering the shape of his hand at all, moves it over the few centimetres needed (while saying 'is the same as'), and places his hand over the right coset (the right hand side of the equation) as he says 'that collection of objects'. Transferring the hand in this way in order to indicate equality is a regularly recurring gesture. Line 10 finishes the observation – saying 'no matter which' he leans over to tap underneath the phrase "for all $ a \in G $".

J marks the conclusion of his 29-second-long stanza by turning his gaze from the students on the word 'which' to the page of notes that he has been carrying in his right hand throughout the stanza, and looking at it as he finishes ‘coset rep we take’. In addition, he swivels again to the board, switching the notes to his left hand so that his right hand is free to write, and the next stanza is begun with the words ‘ok so’ and his hand moving up to the board to write. The first line of this stanza continues ‘actually I’d better write that’, and he proceeds to write on the board a parenthetical comment that comes after the written definition that captures the warning he has spent the last stanza explicating with the help of his hands and the previous written material on the board. Very often his spoken comments remain only spoken, but often too J follows his spoken comments with a written capture (and frequently such comments are marked on the board by being contained within parentheses).

Nonlinear writing, construing to specify.

I turn now to a stanza that begins about three and a half minutes later:

1 and going in the other direction

2 what do we want

3 we know that this is a subset of this for all x

4 and what we want is that this holds for all a

5 so let’s pick an a

6 and what we want to end up with

7 is aH equals Ha

8 that’s where we’re going

9 now do we know immediately where to go

10 well we’re trying to prove that these things are equal

11 and what we’re given is a condition on inclusion of subsets

12 so that is a clue as to how we are gonna prove

13 that those two sets are equal is it not? (23.50)

This stanza occurs as he begins the backwards implication portion of the proof of the theorem that he labels as the “Normal Subgroup Test”, the statement of which is:

Suppose $ H \subgp G $. Then

$ H \nsubgp G $ \iff $ x H x^{-1} \subseteq H $ for all $ x \in G $.

In the previous stanza, having completed his justification of the forwards implication, he concludes the stanza by turning his gaze to his notes, and saying the line “so that’s one direction”. As he says this he picks up an eraser, says the word “and” from the new stanza, erases the next board (that had writing on it already from the previous chapter), and when he’s finished erasing, completes line 1. He then writes a backwards implication symbol as a heading, almost begins writing, then changes what he does at the moment the pen would have hit the board and says line 2 as he walks back over to the neighboring previous board that contains the writing of the definition of normal subgroup as well as the normal subgroup test theorem. Again, stanza transitions are marked in multiple ways, here with body movement, erasing a whole board, and also the structure of the proof signalling a clear break between one context and the next.

On the first ‘this’ of line 3 he touches the “$ x H x^{-1} $ expression; on the second ‘this’ he touches the $ H $ on the other side of the subset inclusion relation; on ‘for all x’ he touches the quantifier clause. This is much like the reading-with-touching that occurred in the last section. The touching in the line 4 has a different character. On ‘want’ he touches the $ H \nsubgp G $ condition, whereupon he rapidly moves his index finger up a few lines (and out of the theorem environment and into the definition environment) in order to touch both sides of the $ aH = Ha $ condition while saying ‘this holds for all a). Walking back to the new board with purpose, he says line 5 and writes “Let $ a \in H $” right next to the backwards implication symbol heading. With no delay he then says line 6 while staring at his notes, then leaves a space of two lines on the board, and then writes “$ aH = Ha $” while saying line 7. This last bit of writing will end up being the concluding part of the proof of this backwards implication. He has written only the beginning and end of a short paragraph of argumentation. This is an example of nonlinear writing, of which there are many instances in the course. This is a beginning-and-end form of nonlinear writing; two other popular kinds are headings-first nonlinear writing, and leaving-small-gaps nonlinear writing.

He takes three steps back from the board as he says line 8. This moment, like a similar moment analyzed in the last section, is a little transition inside this stanza, but does not rise to the level of a stanza transition. As he says line 9 his gaze is back on the board, and he has begun to walk back to touch it, which he accomplishes on the word ‘equal’ in line 10, holding the $ aH = Ha $ condition with a right handed index and pinky finger double hold while he stares at the class for a short pause. Then he walks back over to the previous board as he says ‘and what we’re given’, arriving at the subset inclusion relation just as he says the word ‘condition’ in line 11, again using his right hand to touch both sides of the inclusion with his index and pinky. By the beginning of line 12 he has removed his hand, turned to face the class with his back to the board, and the final 2 lines are delivered looking at the students. There is one final pointing in the last line, in the general direction of the new writing on the new board as he says ‘those two sets are equal’.

There is next a stanza where he elaborates a bit on this clue (showing two sets are equal can be achieved by showing that each set contains the other). The next moment I wish to highlight occurs in lines 3 to 7 of stanza 52:

3 we know- let’s just try-

4 oh what value do you think I should-

5 this holds for all x

6 what value for- for x do you think would be a good one to pick

7 [a] a would be a splendid idea

On ‘we know’ he touches the “$ x $” of the subset inclusion; during ‘let’s just try’ he touches the “$ H $” and “$ x^{-1} $” in succession. Some halting phrases here, atypical for this lecturer. In line 4 he switches to asking the class, and he touches the condition twice again. In line 5 he touches the “$ x $” term again as he says it, and by line 6, as he gets his full question out, he is looking at the class, and a student provides the answer. Here he has been touching a term but wanting to construe it as a specific value of that term (picking the specific value of $ a $ for $ x $). Touching a general term and construing it as a specific instance of that generality is a common move in the course. He is now ready to write the first line of his missing argumentation on the new board.

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MATHEMATICAL PROBLEMS THAT HAVE NO KNOWN EXPLANATION IN THE SECONDARY CURRICULUM

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It is possible to find a connection between high school mathematics and mathematics beyond the curriculum. In this paper, I offer two well-known examples, namely, Euclid's fifth postulate (parallel postulate) and Ping-pong ball conundrum (Littlewood-Ross paradox). The former is equivalent to say that "sum of the angles of a triangle is 180°", and the latter involves considerable cognitive conflict in different sizes of infinity. I argue that good examples not only can reveal the beauty of mathematics but also can inspire students' interest in mathematics. I also say that despite intuitive ideas without rigorous proof may not be valid for formal mathematics, students may gain extra benefit rather than just delivering the conventional lessons. However, how to choose good representative examples for high school students is one of the crucial points for high school teachers and the researchers.

TERMINOLOGIES

(1) Parallel postulate: there is at most one line that can be drawn parallel to another given one through an external point. One of the equivalent statements which has often been used in high school geometry is "the sum of the angles in every triangle is 180°".

(2) Great circle (of a sphere): is the intersection of the sphere and a plane that passes through the centre point of the sphere. Note that the shortest path between two points on a sphere lies on the segment of a great circle (see fig. 5).

(3) Ping-pong ball conundrum:

> You have an infinite set of numbered ping-pong balls and a very large barrel and you are about to embark on an experiment, which lasts 60 seconds. In 30 seconds, you place the first 10 balls into the barrel and remove the ball numbered 1. In half of the remaining time, you place the next 10 balls into the barrel and remove ball number 2. Again, in half the remaining time (and working more and more quickly), you place balls numbered 21 to 30 in the barrel and remove ball number 3 and so on. After the experiment is over, at the end of the 60 seconds, how many ping-pong balls remain in the barrel?
>
> (Mamolo & Bogart, 2011, p. 615)

(4) Cardinality: Two sets have the same cardinality if and only if they can be put in one-to-one correspondence (Mamolo & Bogart, 2011, p. 616).

Noting for finite sets *A* and *B*, the one-to-one correspondence implies they contain the same number of elements; but for infinite sets, the one-to-one correspondence implies

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they have the same cardinality only, but not "the same number of elements," because 'infinity' is not a number. Conventionally, $A \approx B$ denote sets A and B have the same cardinality. Besides, we use the Hebrew alphabet $\aleph_0$ (aleph null) to denote the cardinality of the set of all natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$. Since $\mathbb{N}$ is countable, hence, if $A \approx \mathbb{N}$, then set A is also countable with the cardinality of $\aleph_0$.

INTRODUCTION

Euclid's fifth postulate and Ping-Pong ball conundrum are famous mathematical puzzles related to the properties of parallel lines and the "lens of 'measuring infinity'" (Mamolo & Bogart, 2011), parallel lines and infinity are linked with high school curriculum but have no deeper in explanation. The purpose that I highlight these two examples is to advocate students thinking, visualize, classify geometric objects, and compare different sizes of infinite set through in-class discussion. To do so, I offer examples and tasks to develop creativity in students. Noting a good curriculum is more than a collection of examples and activities, as stated in *Principles and Standards for School Mathematics (2000),* one of the significant components of curriculum is "[t]o ensure that students will have a wide range of career and educational choices, the secondary school mathematics program must be both broad and deep" (p. 287). Therefore, in the broad sense, examples and tasks in this paper are appropriate for secondary or even elementary students. They are, in some sense, visualizable, tangible, entertaining, and situated in the real-world. In the deep sense, they all seem easy to formulate at first sight, but in fact, those examples have been attacked by many mathematicians and philosophers for a long time; hence, good examples are representative, motivative, and worth learning, by deliberating these examples, secondary school students can augment extra insight in mathematics.

THEORETICAL ORIENTATION

Scholars agree that learning-from-examples plays an essential role in learning mathematics, accordingly, how to select a meaningful and representative example for the learners, and how to promote learner create their own examples become an issue for teachers and researchers. In this regard, we can reference the research of Zhu & Simon (1987) and Zazkis & Leikin (2007). Zhu & Simon state that through carefully designed worked-out-examples, "students were at least as successful as, and sometimes more successful than, students learning by conventional methods, and in most cases, they learned in a shorter time" (p. 160). Zazkis & Leikin indicate that "[w]e all learn from examples, …, we believe that learners' example spaces, and their relationship to the conventional ones, provide a window into their understanding of mathematics" (p. 15, 21). Another pedagogical strategy that appears practicable to motivate students toward the idea of non-Euclidean geometry and the notion of infinity is demonstrating tangible geometric solids in class or presenting famous paradox regarding infinity. It might facilitate students' initiatives for learning. For example, the ping-pong ball conundrum contains not only immediately the idea of the *infinite* but also the interplay of cognitive conflicts between *potential infinity* and *actual infinity* (Mamolo & Zazkis, 2008). Despite numerous research articles regarding Euclid's fifth postulate and Ping-pong ball conundrum are highly technical in philosophy and mathematics, this paper releases a unique message that highly technical thoughts still be able to understand by high school students.

Also, proof by contradiction has commonly been used in mathematics, which means something either is, or is not, but cannot be both. At least in most of the formal logic, we must obey the law of excluded middle; that is, paradoxes or contradictions are not allowed. However, students could interest in whether paradoxes exist. For example, according to Parmenidean logic, contradictions are allowed to exist, and indeed, usually do exist in the realm of senses (Campbell, 2001). Nevertheless, I like the term 'incommensurability' or lack of a common unit of measure (ibid.). In fact, in high school textbook, $\sqrt{2}$ is an irrational number using the property of incommensurability; moreover, Cantor's one-to-one correspondence approach provides a unique way to measure the sizes of infinite sets, namely by its cardinality. It should also worth knowing for high school students.

RESEARCH METHODOLOGY

Although this paper does not involve the explicit contents in non-Euclidean geometry and the debates between potential and actual infinity, it does engage with the students' interests and matches the high school curriculum. I purposefully emphasize the value of *visualization*. According to van Hiele's model, visual perception (level 1) is the most basic level of thinking and understanding in geometry to achieve the next sequential levels, namely analysis, abstraction, deduction, and rigour. However, Mason (1997) shows that some mathematically talented students appear to skip the van Hiele levels which are supposed to be hierarchical, Mason further suggests that, even for the talented students, geometry course "should be taught in a much less abstract manner in its initial stages than the current traditional geometry course" (p. 51).

For the Euclid's fifth postulate

The goal of this paper is not guiding the students to write a formal proof. Instead, I intend to demonstrate a big picture regarding the difference between Euclidean geometry and non-Euclidean geometry. I begin by showing the students a world map and world globe, obviously, the map is a plane while the globe is spherical, they are tangible, visualizable, and real-life related. Then I will guide them with tasks to find the shortest distance on the surface of a sphere as well as on a cone (a type of hyperbolic geometry). I do prepare six questions with various figures to lead them into the exciting part of non-Euclidean geometry step-by-step.

For the Ping-pong ball conundrum

This conundrum is a thought experiment involving a finite interval of time (1 minute only) and infinitely many steps. That is, we assume that one minute can be divided into infinitely many parts, namely, $\frac{1}{2}$ minute, $\frac{1}{4}$ minute, $\frac{1}{8}$ minute, and so on. Hence, its summation involves an infinite series. Indeed, high school students could have no trouble to find the sum of the geometric series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots=1$. In this paper, I skip the philosophical debates regarding infinite divisibility of time, potential infinity, and actual infinity; I even skip the '*time*' consideration. Instead, I emphasize the Cantor's one-to-one correspondence which states, as mentioned before, that "[t]wo sets to have the same cardinality if and only if they can be put in one-to-one correspondence" (ibid., p. 616), I further assume that by "working more and more quickly" (Mamolo & Bogart, 2011), the supertask (with infinite steps) can be completed. I will provide two simple tasks in class to encourage them to discuss the different ways of 'measuring infinity,' and thereof, understand more about the mystery of infinity.

ANALYSIS & DISCUSSION

The following diagrams could be very beneficial for a better understanding of the questions and the discussions.

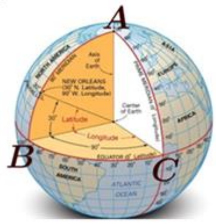

fig. 1


fig. 2


fig. 3


fig. 4

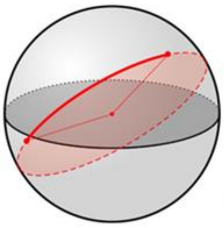
fig. 5

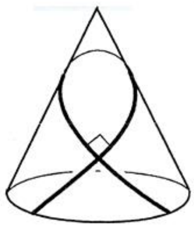
fig. 6

fig. 7

fig. 8

fig. 9

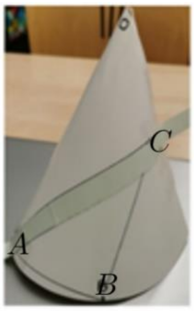

fig. 10

fig. 11

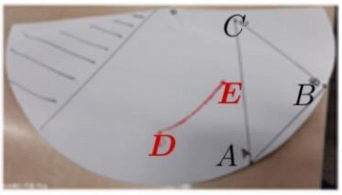

fig. 12

Guiding questions and discussions about the Euclid's fifth postulate

1) Do you think the angles of any triangle on the map add up to 180˚?

Discussion: students may notice that the sum of the interior angles of any triangle is sum up to 180˚. Good to be mentioned that, all the reasonings and 'proofs' within the high school textbook are using Euclid's parallel postulate, which is taken to be true without proof.

2) Suppose there are three non-collinear points on the globe, they undoubtedly form a triangle inscribed on the surface of a sphere (ΔABC in fig. 1). Do you think the angles of this spherical triangle (spherical geometry is non-Euclidean) also add up to 180˚?

Discussion: This question is not easy for many students, but the teacher could lead them to imagine a virtual tour starting at the north pole, following any longitude flying to the equator, then make a 90˚ turn along a quarter of the equator, then make another 90˚ turn flying back to the north pole. In this virtual scenario, a classroom world-globe could be a useful teaching material because students can draw a visible path of the tour on the globe (see figure 1); amazingly, the angles of this triangle add up to 270˚ which is equal to 3π/2 when measured in radians. Furthermore, the teacher could have their students check whether in this particular case (see figure 1), the equality below holds,

$$A + B + C = \pi + \frac{area\ of\ the\ spherical\ triangle}{R^2}$$

where *R* is the radius of the sphere. Without so much effort, many high school students could be able to prove the above equality algebraically by using the surface area of a sphere $SA_{sphere} = 4\pi R^2$. However, students may ask why, for example, line *AB* is the shortest distance from *A* to *B* but not otherwise? I will give an intuitive paper-task regarding '*shortest distance*' in the next question #3.

3) How can we find the shortest distance between two points on the sphere? And on a cone?

Discussion: as mentioned, this paper is not focusing on the formal proof, therefore, despite a line (shortest distance) between two points on a sphere indeed lies on the great circle (e.g., longitude and equator on a globe), and it can be proved by solid geometry at the high school level. A teacher could use a paper strip to surround the globe, as shown in figures 2 and 3. Obviously, the paper strip is 'flat' while globe is spherical, students may notice that it is not possible to embed a paper strip on the globe entirely (see fig. 3), they also notice that the nearest embed is putting the strip, for example, on the equator (fig. 4). In fact, using a cone instead of a sphere could be better, because paper strip can be embedded perfectly on a cone, and one can expand a cone to become flat by cutting its slant. To do the task, step 1: make a paper cone (see figure 7), and a paper strip with the width about 2cm; step 2: use this paper strip to draw any triangle on the cone, noting paper strip must be fully embedded on the surface of the cone as shown in figures 8, 9, and 10; step 3: expend this cone by cutting its slant (see figure 12, do not cut any slant passing through the triangle *ABC*). Then students can see the triangle on the cone really form three straight lines. Hence, it must be the '*shortest distance.'* Besides, the teacher could show them if the strip is not entirely embedded on the surface of the cone (e.g., red line *DE* on figure 11), then the red line appears to be a curve in a plane (see red line *DE* on figure 12). Therefore, it cannot be a 'straight-line' because it is not the 'shortest distance.'

4) Can anyone find two parallel lines on the globe?

Discussion: parallel lines do not exist in spherical geometry. Intuitively, students will notice that any two great circles always intersect with each other at two points. Hence, they are not parallel. In other words, any two different lines on a sphere, if we extend them, must intersect each other.

5) Do two points determine exactly one line on the surface of the globe?

Discussion: for example, there are infinitely many lines joining the north pole and the south pole (all longitudes work). In fact, except two points are the endpoint of the diameter, any two points on the sphere determine exactly two lines, which correctly form a great circle. Students may verify it by the red disc in figure 5. Interestingly, it is also not valid for a cone (hyperbolic geometry). However, it is not easy to find such two lines; figure 6 could be a good counterexample for students to think with.

6) Can a line on a sphere be extended to infinitely long?

Discussion: this is an interesting question. Intuitively, students might think that any line on a sphere can only be extended to a great circle at most, because this line segment, when extends, will connect with itself at some point. I think this conclusion should be acceptable at the high school level. However, it is no harm to mention that in spherical geometry, a line segment can go around the sphere an infinite amount of times; hence it can be extended indefinitely long theoretically.

Tasks and discussions about the Ping-pong ball conundrum

The ping-pong ball conundrum is a thought experiment involving a finite interval of time (lasts exactly 60 seconds) through infinitely many steps. This conundrum has been studied intensively by Mamolo and others. As said, I skip the philosophical debates on the infinite divisibility of time and potential/actual infinity. Instead, I borrow the ideas from Manolo & Bogart (2001), give two new variants (scenarios A and B) to encourage high school students to open their thinking about infinite quantities. I also give credence to both the *intuitive* and *normative* approaches. Roughly speaking, the intuitive approach means employing part-whole consideration (the whole is greater than those parts). At the same time, the normative approach is relying on Cantor's notion of *cardinality* (use one-to-one correspondence to compare different sizes of an infinite set). Two scenarios below should be intelligible to high school students.

Scenario A: Imagine we are in a virtual classroom: there is an infinite number of ping-pong balls in the right corner, and we have a barrel, we are going to put the balls in the barrel. So, the teacher goes to the pile and picks up 10 of these, comes over the barrel, dumps 9 balls in the barrel and throws one in the left corner. Then the teacher goes over again, picks up the other 10 balls, dumps 9 in the barrel, throws one in the left corner, and so on. Now let go to infinity, we take the limit, the teacher goes super-fast, and there are no more balls left in the right corner (assume that this supertask has been completed). Now, the problems are: How many balls remain in the barrel? How many balls are in the left corner? Are we all agree that there are infinitely many balls both in the barrel and in the left corner? Below are some common responses for scenario A:

A1) There is no end for this task. Putting infinite balls in the right corner cannot be completed no matter how fast the teacher could move. Thus, the hypothesis "assumes that this supertask has been completed" is wrong.

A2) There are infinitely many balls both in the barrel and in the left corner because the in-going balls in the barrel are more than the throwing-out balls in the left corner, its ratio is always 9:1 (Rates of infinity).

A3) Initially, there are ℕ = {1, 2, 3, …} many ping-pong balls in the right corner, according to Cantor's one-to-one correspondence criterion, despite the ratio of in-going (countable) and throwing-out (countable as well) seem to be 9:1; however, they have the same cardinality. It raises a question whether 'having the same number of elements' is equivalent to 'having the same cardinality'?

Scenario B: take the same setting as in scenario A, but now all balls are *labelled* as 1, 2, 3, … and so on. At the first step, the teacher dumps ten balls (#1 to 10) in the barrel, then he/she picks one ball from the barrel, throws it in the left corner; at the second step, he/she dumps other ten balls (#11 to 20) in the barrel, then picks one ball from the barrel, throws it in the left corner; same manner for third step for balls #21 to 30 and so on until no more balls left in the right corner. In this task, how many balls are in the barrel, and the left corner? Noting scenario B is different from the Ping-pong ball conundrum (Mamolo & Bogart, 2011, p. 615) and scenario A. What differences can you find?

Nevertheless, a normative resolution of ping-pong ball conundrum is constantly meeting Cantor's one-to-one correspondence, which claims that the barrel will be *empty*:

> […] In order for the barrel to be empty at the end of the experiment, the ping-pong balls must be removed consecutively, beginning from ball numbered 1. Consequently, there will be a specific time at which each in-going ball is removed: ball numbered *n* is removed at time $\left(\frac{1}{2}\right)^n$, for all $n \in \mathbb{N}$. Thus, at the end of the experiment the barrel will indeed be *empty*.
>
> (Mamolo & Bogart, 2011, p. 616)

Below are some common responses and questions regarding scenario B:

B1) At first sight, scenario B is the same as the ping-pong ball conundrum and scenarios A, for in each step, only one ball was thrown, and 9 balls were stayed in the barrel. Question: are all responses in scenario A is reasonably valid in scenario B?

B2) Scenarios A and B are different. Students will recognize that the balls in scenario B are *labelled* by 1, 2, 3, … while scenario A does not. Thus, whether balls are numbered might play an important role. In general, the mystery is still hard to perceive for most of the people, not only for high school students.

B3) It is good to know that Cantorian one-to-one correspondence is considerably normative because many mathematicians acknowledge it, but it is also good to know that it sounds not 'more correct' than the intuitive 'part-whole relation' because one-to-one correspondence often leads inconsistency in many thought experiments, especially in the real world. Mamolo & Bogart indeed did list many interesting 'riffs' that lead paradox in logic. I will give an alternative resolution in B4. Noting Mamolo and Bogart also highlight that 'one-to-one correspondence' may necessary but not sufficient to solve the paradox (ibid.).

B4) Here is an alternative resolution of scenario B. Pick any natural number, say 3, then after this supertask is completed, there remain *exactly 3 balls* in the barrel.

How could that be? Here is the clarification: in step one, the teacher picks ball #4 from the barrel (instead of ball #1), then throws it in the left corner; in step two, the teacher picks ball #5 from the barrel (instead of ball #2), then throws it in the left corner and so on. That is, in step *n*, the teacher picks ball #(*n*+3) from the barrel. Thus, at the end of this supertask, the left corner has a set of balls numbered by {4, 5, 6, …}, which is very close to the initial set of balls in the right corner $\mathbb{N} = \{1, 2, 3, 4, \ldots\}$. However, if so, the barrel remains exactly three balls, no more, no less, namely ball #1, #2, and #3. Are there any mistakes, or mystery, or trick in this clarification? I think it is beneficial for the students to think with even though they might not resolve the puzzle right by the way because paradoxes "might influence students' understanding of infinity, as well as the persuasive factors in students' reasoning" (Mamolo & Zazkis, 2008, p. 167).

CLOSING REMARKS

Discussion is one of the common ways to get students to open their imagination and intuitions, defend their reasoning by relevant facts, learn how to cooperate with others, and increase participant engagement, especially through discussion on the open-ended questions among the peer group. I hope this paper can help secondary students gain extra mathematical insight into non-Euclidean geometry and the essence of infinity.

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DYSCALCULIA IN THE MEDIA: A CRITICAL DISCOURSE ANALYSIS OF TWO NEWS ARTICLES

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Dyscalculia has received relatively little attention in the popular media compared to other disabilities such as dyslexia. This paper applies the tools of critical discourse analysis to examine two rare articles on the disability. Discursive representations of dyscalculics, cognitive neuroscientists and their research on the brain and the roots of number sense are examined for how such representations are influenced by ideology and the media genre.

INTRODUCTION

The purpose of this paper is to analyse some of the discursive practices at play within two media texts on dyscalculia. Relatively little is known about dyscalculia (compared to dyslexia for instance) and research on it crosses over into many different fields such as medicine, cognitive psychology, neuroscience and mathematics education. Debates over its origins, causes and core deficits continue in the literature (Chinn, 2015). While much of the reported research on dyscalculia occurs in the academic literature, the popular media, on occasion, picks up on the topic. Such was the case with *Discover Magazine* (Flora, 2013) and the journal *Nature* (Callaway, 2013) (in a news feature). It is of interest to analyse media representations of dyscalculia as researchers begin to learn more about the disability to see how those labelled as dyscalculic are positioned in relation to others and how different conceptions of the roots of number sense are debated over. This paper is a brief exploration of the discursive terrain on which dyscalculia is written about in the news and due to space limitations, I only analyse the headline, kicker, image and image captions for each article. The rationale for such an exploration is to denaturalize the "taken-for-granted" language and identify the ideological underpinnings around disability and number sense in relation to dyscalculia.

THEORETICAL AND METHODOLOGICAL FRAMEWORK

Norman Fairclough's (1993) critical discourse analysis (CDA) is a useful framework for analysing media discourses. For Fairclough, language is a socially and historically situated mode of action that is both "socially shaped and socially constitutive" (p. 134). His CDA applies Halliday's multifunctional view of language as constitutive of social identities, social relations and systems of knowledge and belief (systemic functional linguistics). Language use is socially shaped by societies and institutions through often competing discursive practices. These discursive practices (e.g. those exemplified

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through the media), often adhere to certain conventions of language use, but may not always follow them as straightforwardly as one may think. Fairclough (1993) categorizes discursive practices between *discourses* "ways of signifying areas of experience from a particular perspective (e.g. patriarchal versus feminist discourses of sexuality)" (p. 135) and *genres*, "uses of language associated with particular socially ratified activity types such as job interview or scientific papers" (p. 135). I use both terms in these senses throughout this paper. Fairclough's framework is also critical in the sense that it aims to "investigate how [discursive] practices, events and texts arise out of and are ideologically shaped by relations of power and struggles over power; and to explore how the opacity of these relationships between discourse and society is itself a factor securing power and hegemony" (p. 135). For Fairclough (1989), ideology is, loosely, common sense in the service of power.

I follow Fairclough's (1993) three-dimensional framework for analysing any communicative event in terms of three facets: "it is a spoken or written language *text*, it is an instance of *discourse practice* involving the production and interpretation of text, and it is a piece of *social practice*" (p. 136). Due to space limitations, I only apply the first two dimensions of Fairclough's framework. I apply the techniques of systemic functional linguistics to analyse the articles as text and I apply the concept of interdiscursivity for the second dimension. *Interdiscursivity* highlights "the normal heterogeneity of texts in being constituted by combinations of diverse genres and discourses" (p. 137) and "a historical view of texts as transforming the past—existing conventions or prior texts—into the present" (p. 137). This analysis asks: "How are dyscalculia and dyscalculics represented in the articles?", "How is the concept of number sense represented?" and "How do the two articles differ in their representations?".

ANALYSIS AND DISCUSSION

Starting with the *Nature* news feature, the headline and kicker (introductory paragraph) dominate the first page, together taking up almost half the page. The headline itself ("NUMBER GAMES") is capitalized and in large font, an allusion to the dystopian novel *The Hunger Games*. The movie adaptation of the book was released the year before the article was published to much box-office success and this pun capitalizes on the movie's popularity. Indeed, the headline has a similar font style as the title in the theatrical release poster. It is unclear, however, what the connection is, if any, between the themes of the novel and the topic of the article. The Hunger Games is a televised fight to the death set in a post-apocalyptic future, and the novel touches on themes such as power, poverty and oppression. "Number Games", on the other hand, refers to one of the main topics of the news feature: the research of cognitive neuroscientists in the development of computer games in order to improve the number sense of children with dyscalculia. Some readers may draw their own conclusions regarding the connection between the use of computer games as a treatment and intervention for dyscalculia, and the Hunger Games as a cruel punishment imposed by the "Capitol" of the nation Panem on neighbouring districts for a past rebellion, although such a connection could not be made until after the article is read—it is not until the second section of the article before it becomes clear what “Number Games” has to do with the rest of the article. (It is also notable that the strength of the allusion will be lost as time passes on as interest in the movie slowly fades—as the power of the pun diminishes, the heading will be less effective at drawing the reader’s attention.) Nevertheless, the author’s decision to use “Number Games” for sensationalistic reasons has the ideological effect of placing greater focus on the interventions for dyscalculia rather than other equally important aspects of the article such as discussion about the origins, symptoms and causes. Furthermore, this focus hides the individuals to whom the treatments are for—those labeled as dyscalculic. The heading makes clear that the article is more about the treatments themselves rather than the individuals effected.

The kicker of the *Nature* article (“Brian Butterworth is on a crusade to understand the number deficit called dyscalculia—and to help those who have it”) is placed before the heading, and Brian Butterworth’s name is thematically prominent at the beginning of the sentence (and the entire article itself). The use of the word “crusade” (sometimes used sensationalistically in newspapers) is suggestive of a religious war but, in this context, is suggestive of a campaign for a type of social change and Brian Butterworth is set up as the leader of this campaign. Note how the dash creates an emphasis on the parenthetical statement “and to help those who have it.” Brian Butterworth is undeniably the agent and those with dyscalculia (who are nameless) are the patients of his “help.” Also note the modality of the sentence suggested by the phrase “to understand.” Butterworth is not on a crusade *against* the number deficit dyscalculia in the same way other campaigns crusade *against* gambling or *against* crime where it is abundantly clear what “gambling” or “crime” is. Instead, he is on a crusade *to understand* the number deficit called dyscalculia. That is, dyscalculia is not yet fully understood by scientists despite the fervent desire for change suggested by the word “crusade.” There is also the presupposition of the existence of dyscalculia and the reality of it is brought to life by the definite article “the” used to identify a particular “number deficit.” A further assumption is that readers of the news feature are unfamiliar with dyscalculia but may have some understanding of what is meant by a “number deficit.” (I unpack the use of “number deficit” in more detail below.)

At this point it is worth analyzing the image of Brian Butterworth that takes up half of the third page of the article, as there are parallels between the caption for the image and the kicker. In the image, Brian Butterworth sits in front of a shelf full of scholarly books and various artifacts while holding what appears to be a three-dimensional glass model of a brain. The caption reads: “Brian Butterworth hopes that his number games will help dyscalculic children—and open a window on how the brain processes numbers.” Note the linguistic similarities with the kicker. “Brian Butterworth” is once again thematically prominent at the beginning of the sentence and he is the central and only figure in the image itself. There is again the “hope” that his number games will “help” dyscalculic children. Note the truth modality inherent in the author’s use of the word "hope" which parallels the modality in the kicker—there is the possibility that his number games do NOT in fact help dyscalculic children. There is also the informal use of a dash to emphasize the parenthetical statement "and open a window on how the brain processes numbers." There are two key metaphors at play here that are suggestive of the way cognitive neuroscientists understand how the brain functions. Firstly, the "open a window" metaphor is a metaphor of seeing and alludes to the research of neuroscientists who observe how certain neurons can literally be "seen" firing in specific regions of the brain in response to some numerical task. Secondly, the word "processes" suggests how the brain performs numerical tasks as a series of steps or as a mechanical procedure similar to how operations are carried out by a computer. These metaphors tend to support a particular perspective on numerical thinking, particularly that of number sense originating from specific regions of the brain (such as the intraparietal sulci as noted on page 3 of the article) which can be "exercised" through "number games" in order to hone certain numerical abilities.

This idea of the brain as being transparent and amenable is literally depicted in the image of Brian Butterworth holding a glass model of a brain between his fingers. This image also places him in a powerful position as a knowledge keeper, and the article itself emphasizes his role as a leading scholar in the research on dyscalculia (the voices of researchers with competing views on number sense such as Stanislas Dehaene are not as dominant—his views are buried in the middle of the article and he certainly does not get a half page image of himself). Thus, image, image caption, and kicker, combine to establish Butterworth's particular research as the primary view on number sense even before a paragraph needs to be read. Yet, in contrast to this construction of him as a leading academic scholar is an ethical dimension to his character. He is depicted by the author as a type of saviour (suggested by the use of the word "crusade" and the repeated use of "help") to those identified with dyscalculia. In the only pull quote of the article (shown prominently in the middle of the last page used both to entice readers and highlight a point), we get a further sense of his caring for children and the seriousness of his conviction. In this pull quote, he describes how he is struck by the distress that children feel by being bad at maths: "Every day they go to school. Every day there's a maths class. Every day they're shown up to be incompetent." Note the shock value in this quote created by the short, snappy sentences and the repetition of "Every day" at the beginning of each one. There is a narrative created by the three sentences and readers themselves must draw the conclusion implied by last two sentences: Maths makes some children feel incompetent. In this brief analysis so far, it is becoming apparent that characteristics of two traditional print genres combine in this news feature to create an interdiscursive mix: the magazine and the scholarly journal. The sensationalism of the header, kicker and pull quote are characteristic of a magazine (note that these are characteristics of newspapers too), while the circumspection of the claims made, the academic jargon ("number deficit", "dyscalculia") and references at the end of the article not commonly found in magazines are typical of the scholarly journal.

I now turn to an analysis of the headline, kicker and image in the *Discover* article. Firstly, it should be noted (as indeed it is at the end of the online edition of the article) that the headline for the original print edition is “No Head for Numbers”. The headline for the online edition is “How Can a Smart Kid Be So Bad at Math?” Interestingly, this headline is the first subheading in the print edition. “No Head for Numbers” appears nowhere in the online edition other than in the aforementioned note at end of it. I cannot account for this authorial or editorial change other than perhaps to speculate that the online headline is catchier (although wordier) and more suggestive of the topic of the article. (“No Head for Numbers” may well be suited as a headline for the *Nature* article as there is a focus on number sense and the connection with the brain.) Whatever the case may be for this difference between the two mediums, “No Head for Numbers” is still an effective headline when one considers what the characteristics of an effective headline may be. There are two uses of metonymy within it: “Head” serves as adjunct for “cognitive ability” and “Numbers” for “mathematics”, “number sense”, or possibly “arithmetic”. The headline draws readers into the topic of the article in an informal yet concise manner taking a rather complex topic and encapsulating it into a mere four words. The secondary effect of this conciseness is that there remain hidden meanings within the headline, creating a bit of mystery for the reader and enticing him or her to read further. A possible critique of such a headline is that it oversimplifies complex ideas for rhetorical (even ideological) effect and (perhaps falsely) makes assumptions about where “number sense” may come from or how we learn mathematics. Nevertheless, a headline’s purpose is not to tell the entire story, but to encourage further reading in search of answers or further debate, and I believe this headline attempts to do so.

The first subheading for the print edition “How Can a Smart Kid Be So Bad at Math?” is “promoted” to the headline of the online edition (there is no first subheading for the online edition although many subheadings appear later on in the article). This serves as an appropriate subheading for the print edition as the author addresses the answer to the question by the third paragraph of the article: “There are many reasons for a bright student to be bad at math, including poor learning environments, attention disorders and anxiety. But Steph’s struggles typify a specific math learning disability known as developmental dyscalculia.” This subheading and the introductory paragraphs serve well to introduce Steph, a college bound student prominently featured in the article, and the general topic of the article. By moving this subheading to the more prominent position of headline in the online edition, Steph’s story about having dyscalculia and her struggles become front and centre—she is indeed the “Smart Kid” who is “Bad at Math” mentioned. The headline “No Head for Numbers,” in comparison, may well refer to a general disorder and is less explicit about any particular individual, and it hints more at the cognitive neuroscience aspects of the article. Moreover, “No Head for Numbers” is less colloquial and engaging than the rhetorical question “How Can a Smart Kid Be So Bad at Math?” (Consider the usage of the colloquialisms “Smart Kid” and “Bad at Math.”) The informal nature of both headlines is consistent with *Discover Magazine*’s appeal to a more general audience. (Note also how the latter headline has a higher word count, but is less prominent with a much smaller font size than the *Nature* article headline.) By deciding to use this as the headline for the online edition rather than as a subheading creates a different framing effect (more provocative and relatable) than if it were left as a subheading.

A closer analysis of the headline “How Can a Smart Kid Be So Bad at Math?” reveals a number of presuppositions inherent in the rhetorical question that make a general appeal to readers’ “background knowledge” and hence give it a “dialogic” property (Fairclough, 1989). There is the presupposition of the notion of a “Smart Kid” (many widespread discourses connect “smart kid” with “high I.Q.,” “high emotional intelligence,” “high test scores,” “well read,” “well spoken,” or “street smarts”). There is the presupposition of what it means to be “Bad at Math” (often associated with many widespread discourses such as “can’t do arithmetic,” “did poorly in math at school,” “math anxiety,” or “math learning disability”). Most importantly, there is the presupposition that readers consider these two ideas to be mutually exclusive (“Smart Kids” are often associated with “good at math”) for the headline to be impactful and newsworthy—it is the unexpectedness of the event that gives it value as a news item (Richardson, 2007). The headline “How Can a *Stupid* Kid Be So Bad at Math?” is not newsworthy at all. The ideological effect of the headline “How Can a Smart Kid Be So Bad at Math?” is to elide the heterogeneity of how MLD is experienced and the co-occurrence of MLD with things such as dyslexia, anxiety, slow processing or low socioeconomic status. Thus, the headline presumes that the dominant interest (those of “smart” people) is the interest of society as a whole and the interests of kids who may suffer more global disadvantages co-incident with math difficulties are less notable. Indeed, the article introduces us to the story of Steph who excels in many areas including chemistry, writing and literature but struggles with math, and her diagnosis of “dyscalculia” is the explanation for such a discrepancy in her learning.

The kicker makes this explanation explicit: “Scientists search for the cause, and treatment, of a mathematical learning disability called dyscalculia.” Note the shift in tone between the headline and kicker from informal to formal as the kicker responds to the question posed in the headline. The use of the words “scientists,” “search,” “cause,” “treatment,” “disability,” and “dyscalculia” all suggest a medical condition and gives the sentence a feeling of scholarly importance. It is more understated and less informal than the kicker used in the *Nature* article—contrast the use of the word “search” in the *Discover* article with “crusade” in *Nature*, and the use of “treat” versus “help.” Moreover, the *Discover* article uses the general term “scientists” rather than a specific name as in the *Nature* article (“Brian Butterworth”). Indeed, the *Discover* article is less bias towards the research of one particular scientist and more about the story of an individual who suffers from dyscalculia—about Steph and her experiences with dyscalculia as opposed to Brian Butterworth and his “crusade” to help those with dyscalculia. While both articles contain voices from various scientists, the *Discover* article maintains a balance between those voices whereas in the *Nature* article Butterworth’s research is more prominent.

Another notable point of contrast between the two kickers is how "dyscalculia" is introduced to readers—as a "number deficit" in the *Nature* article versus a "mathematical learning disability" in the *Discover* article. "Number deficit" suggests that dyscalculia is characterized by a difficulty in dealing with numbers (or poor number sense) and reflects the *Nature* article's focus on Butterworth's research on the cognitive basis of numeracy (e.g. approximate number sense) and how an understanding of dyscalculia can shed light on it. "Mathematical learning disability," on the other hand, is more suggestive of a disability in learning *mathematics* in its broadest sense rather than a deficit in mere numbers or arithmetic (although like the *Nature* article it also explores the core deficits of number sense). This is reflected in the *Discover* article's description of Steph's struggles to learn mathematics by using "flashcards, computer games, videos, math songs, summer tutors" and how she was eventually able to overcome her challenges: "Steph persevered through multiplication tables and ratios, fractions and decimals. It was never fun—and geometry in particular is an adventure she'd prefer to forget." Moreover, the phrase "learning disability" suggests a deficiency in cognitive processing that might affect mathematical learning (such as poor working memory or executive functioning) in a way that "number deficit" does not. Thus, the use of MLD is more suggestive of the different ways people may experience dyscalculia, how it affects learning school mathematics, and the co-occurrence of it with other conditions that might affect learning (such as dyslexia).

I will end this section by analyzing the image and image caption in the *Discover* article and comparing them to the ones in the *Nature* article. The image in the *Discover* article follows immediately after the kicker and shows Steph smiling toward the reader with one hand on a sign with the words "Coe College" on it. She is standing in front of Nassif House that the sign indicates to be the office of admission and financial aid. The caption reads: "Steph Zech, who has developmental dyscalculia, will attend Coe College in Iowa this fall." Note the non-essential clause "who has developmental dyscalculia" set off by commas. While non-essential from a meaning perspective (the caption will still make sense without it), it is essential from a news value sense in that it is a "happy news story" that someone with a disability is able to "overcome" it to be admitted into college—the image itself suggests a happy ending to the story despite the negative sounding headline. This common narrative found in popular media representations frames impairment as a problem that is solvable through willpower and possibly the help of technology. Critiques of such frames note how they limit the access the disabled have to the media and such stories may not be relevant to the everyday lives of those living with disability (Grue, 2015). Notice how both the stories in *Nature* and *Discover* follow particular scripts in order to be deemed newsworthy. The *Nature* article is about a leading cognitive neuroscientist on a "crusade" to help dyscalculic children through the use of computer games (hero helping the disabled script). The *Discover* article, on the other hand, is about a "smart" girl diagnosed with dyscalculia who, despite this, is accepted into a high-ranking college through hard work and support (the disabled lifting themselves up script). Both types of narratives focus on well-educated, middle-to-upper class individuals that may overlook the lives of lower class individuals who suffer from dyscalculia and co-morbid disorders. For example, in the *Nature* article, a young student with dyscalculia named Christopher serves as the beneficiary of Brian Butterworth's computer games research. He is described as struggling at first, but slowly improves to the point of optimism and serves no more than a foil to Butterworth's research. Unfortunately, a more nuanced view of those who struggle with dyscalculia outside of these scripts (such as a depiction of the everyday life of a student with both dyscalculia and dyslexia and his or her family and teachers) may not be "appropriate" or even worth a magazine or journal's attention.

CONCLUSION

By applying the tools of critical discourse analysis, this brief analysis has demonstrated the heterogeneous nature of the news articles in terms of genre (the conventions of news, magazines, and journals), the discourses employed (e.g. discourses on disability), and the formal and informal nature of the writing. An analysis of just a few elements of the articles already begins to reveal much about whose voices are heard and how the media frames what is newsworthy or not. For instance, the *Nature* article presents a very distinct perspective on the neurocognitive roots of number sense that may not be shared by the wider academic literature. The choices that the authors made or did not make in their representations of individuals labelled with dyscalculia and the nature of the research being reported is often ideologically influenced depending on the politics and/or economics of the discursive practices involved. A detailed explanation of such choices is beyond this paper and would require an analysis of the articles as a form of social practice (this would be the third dimension of Fairclough's framework).

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NOT CHOOSING IS ALSO A CHOICE

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Although counting problems are easy to state there is much evidence that students struggle with solving counting problems correctly. As this topic became part of K-12 and undergraduate curricula, there is a necessity to study factors that might have affected students' success. Detecting all the choices in solving a counting problem is one of the factors of students' success. The option of not choosing which may not often be considered as a choice is the core of this research. A pair of prospective high school teachers participated in this research. Their combinatorial thinking was examined in term of Lockwood's model (2013) with the focus of detecting the option of not choosing.

INTRODUCTION

Combinatorics topics have become part of K-12 and undergraduate curricula. Over the last two decades, the number of research articles about students' combinatorial reasoning has increased. Examining common errors, strategies, and ways of thinking related to students' solving of counting problems were the core of those research articles (e.g., Annin & Lai, 2010; Eizenberg & Zaslavsky, 2004; English, 1991; Halani, 2012; Lockwood, 2013, 2014; Lockwood, Swinyard, & Caughman, 2015; Tillema, 2013). In addition, as counting problems play an important role in computer science and probability some researchers were interested in studying these types of problems (e.g., Abrahamson, Janusz, & Wilensky, 2006). Furthermore, counting problems are accessible and easy to state but need critical mathematical thinking to solve (e.g., Martin, 2013; Tucker, 2002). Hence, this topic has absorbed educators' attention.

Several researchers reported low overall success rates among postsecondary students in solving counting problems (e.g., Eizenberg & Zaslavsky, 2004; Godino, Batanero, & Roa, 2005, Batanero et al.1997). They mentioned several error types they found in students' work, including errors of order, errors of repetition and using incorrect arithmetic operations.

A counting problem points out some options/choices to present a group of objects. The number of these objects would be the answer of the problem. In solving counting problems, errors may come from not correctly detecting options/choices which the problem specifies. Options can be presented in different ways, including the possibility of not choosing. In this paper, I narrow the focus to students' understanding of a particular option that is the possibility of not choosing as a choice and how students may consider it in solving counting problems.

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THEORETICAL FRAMEWORK

To be able to understand how students conceptualize counting problems, a model of students' combinatorial thinking was developed (Lockwood, 2013). This model facilitates a conceptual analysis of students' thinking in facing counting problems and it provides a language to address and describe different aspects of students' activities related to combinatorial enumeration (counting). In fact, it has equipped educators to elaborate on ways that students might think in solving counting problems, and it has provided a powerful lens to analyse different facets of students' counting.

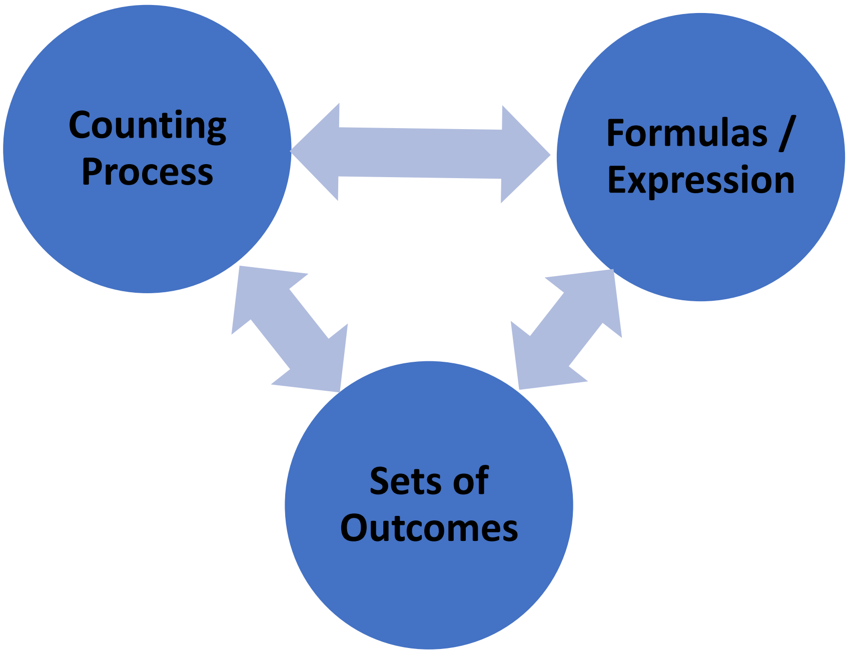


Figure1. Lockwood's model of students' combinatorial thinking

This model has three components, *formulas/expressions*, *counting processes*, and *sets of outcomes*. Any mathematical expressions that turnout some numerical values can be considered as a formula/expression. While solving a counting problem, a counter may engage in an enumeration process or series of processes. The counter either does some steps or procedures or even imagines doing them. Both cases are considered as enumeration processes which describe counting processes as a component in this model. Sets of outcomes refer to the sets of elements which a counter wants to count. Any collection of objects that the counter tries to generate or calculate by a counting process, is considered as a set of outcomes. This model is a helpful lens to analyse students' combinatorial thinking which is proper for this research. Considering students' combinatorial thinking with the focus on students' detection the option of not choosing as a choice is the main focus of this research.

METHOD

Participants and data collection

The participants were two prospective secondary teachers who attended a mathematics content course. Although combinatorics was not in the content of this course, they both had been taught the multiplication and addition principles with the focus on solving counting problems. They both participated in individual semi-structured interviews. The interview instrument consists of 3 problems that aimed at probing students' understanding in how they consider the possibility of not choosing as an option and comparing students' understanding of "*choosing*" to "*not choosing*". The option of not choosing presents in different formations in each problem. Problem 1 was designed to consider if the option of not choosing would be detected by students. Problem 2 was designed with the focus on two purposes, 1) evaluating students' understanding on using the multiplication principle in solving counting problems, 2) considering students' combinatorial thinking on the option of not choosing which is started by expressions component. Investigating students' understanding of the option of not choosing without recalling it is the purpose of designing the third problem.

Tasks

Problem 1: There are 9 international students and 5 domestic students to fulfil English language requirement. International students have to take one of three English courses (ENGL 100A, ENGL 100B, ENGL 100C). There is no requirement for domestic students to take any of these three courses. Taking one of these three courses is an option for them. How many possibilities are there for students' choices (for example, one possibility is 3 international students take ENGL 100A, 3 international students take ENGL 100B, 3 international students take ENGL 100C and all domestic students take ENGL 100B).

Problem 2: A group of 8 people had a meeting. Three of them had a presentation in this meeting. After their presentations they desperately needed a cup of coffee. The people who did not present can choose to have coffee. There were 11 different types of coffee to choose from. How many possibilities are there for these people's choices?

Amy wrote the following expression as her answer:

$$\binom{5}{0}\times 11^3+\binom{5}{1}\times 11\times 11^3+\binom{5}{2}\times 11^2\times 11^3+\binom{5}{3}\times 11^3\times 11^3 + \binom{5}{4}\times 11^4\times 11^3+\binom{5}{5}\times 11^5\times 11^3$$

Do you agree with her? What was she thinking about?

Nikki's answer was $11^3 \times 12^5$. What do you think about Nikki's answer?

Problem 3: How many factors of $2^5 \times 3^2 \times 7^4$ are divisible by 7?

DATA ANALYSIS AND RESULT

Most of the focus of this research is on Alex, an interviewee who didn't consider the possibility of not choosing as an option while the other one initially did. The two interviewees' works show both of them were consistent in considering the option of not choosing in the three problems, either consider it as an option or not. Although different formations of presenting this option did not have an effect their consideration of the option of not choosing as an option or not, it affected on their combinatorial thinking. I decided to organize the data by considering the option of not choosing through each component of Lockwood's model of students' combinatorial thinking.

The option of not choosing and

> "If the domestic students have three options for either taking or not taking an English course, so it should have been 2×3. But it has supposed to have three choices for the don't as well which is not our case. It's not 2×3 as for the don't you just don't take any course. For the options within the take, I would like to say $\frac{1}{2}$. $\frac{1}{2}$ for take and $\frac{1}{2}$ for don't. So, for the $\frac{1}{2}$ they have three options. I just don't know how to write it down."

Alex clearly showed by slightly changing the problem he could get a correct answer but dealing with the extra option for domestic students while it was not considered as an option brought some confusion. Although he could solve the changed version of the problem, eliminating options to be able to reach a solution for the original problem was challenging. Thinking about $\frac{1}{2}$ may come from him calculating the probability of taking a course in the changed version of the problem. Lack of understanding of the addition principle and the way it can affect expressions may be the reasons behind these two confusions.

The two numerical expressions in the second problem pushed the interviewees to start their combinatorial thinking by formulas/expressions component. To be able to evaluate $11^3 \times 12^5$ as an answer, the extra option of not having a coffee needed to be numerically explained.

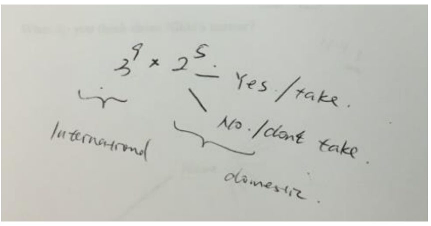
> "I agree with 11^3, but the 12 probably is 11 types of coffee plus 1 which is represents not having a cup of coffee. I don't think they can add that. It's like you are adding a different type of coffee but you can't add a choice by not having a coffee."

Although $11^3 \times 12^5$ brought up the idea of considering the option of not choosing as an option, supportive counting processes was needed to convince Alex. In addition, numerical expressions and sets of outcomes may have one to one relationship in his understanding which I will discuss it in the section which focuses on the component of set of outcomes.

The option of not choosing and counting processes

Although multiplication principle which is one of the foundations of counting processes presented in Alex's counting process, detecting options correctly was absent in his combinatorial thinking.

His answer to the first problem was $3^9 \times 2^5$.



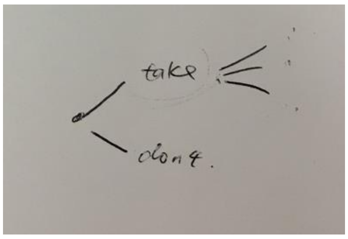
Considering 2^5 as the total possibilities for domestic students shows 1) his attention on the two options of taking a course or not taking a course which were explained by "yes" or "no", 2) missing the three options in the case of taking a course. He was asked what the difference is between the way he counted all the possibilities for international students compare to domestic students. His answer was:

> "The 3 is three different courses, but the 2 is yes or no. **They don't lie together**. Domestic students they either take an English course or they don't take it. If they do want to take an English course, it doesn't matter which one they still have 3 choices."

By not detecting the option of not choosing as an option, counting all the possibilities for domestic students needs different counting processes compare to counting all the possibilities for international students. Hence, not considering the option of not choosing as a choice may increase the level of difficulty in solving counting problems.

The option of not choosing and set of outcomes

Although Alex could correctly get the tree diagrams for the group of people who had an extra option in solving the first two problems, the correct counting processes and correct numerical expressions were missing.



Drawing a tree diagram can show students awareness of the overall conditions of the problem. It can be considered as an activity to generate organized list of outcomes. Hence, based on the model, it can be described as the relationship between counting processes and sets of outcomes. In addition, getting an expression based on the diagram can be illustrated as the relationship between formulas/expressions and sets of outcomes.

While he could count all the possibilities for international students, not considering the option of not choosing as an option for domestic students, became an obstacle in his combinatorial thinking to move smoothly from the component of set of outcomes to the other two components.

While Alex was working on the second problem, he rejected $11^3 \times 12^5$ as 11 types of coffee can not be added to the option of not having a coffee. His reason was:

> “Another explanation is these 5 people not only all choice cup of coffee, they chose from 12 flavours instead of 11 flavours which is not the case. They had 11 flavours plus not having a cup of coffee. This means they had it for sure, just from 12 different flavours.”

After being reminded that the number of two different counting can be equal, he continued by:

> “But to me the thing is you have to think this **linearly**. The first you need to make a choice of having a coffee or not. And then within the choice of having a coffee you have 11 options. But for this 12 is like you only have one option of having a coffee, it just you have 12 options, you have a coffee for sure but from 12 flavours. I don’t consider not having a coffee equivalent to the 12th flavour.”

As was mentioned before, he might incorrectly assume there is a one to one relationship between sets of outcomes and expressions.

DISCUSSION AND CONCLUSION

Having awareness of options/choices is critical in solving counting problems. Although detecting the option of not choosing is not a counting process, it definitely affects the processes of counting. Not considering it as an option may cause a detour in students’ paths of solving counting problems by increasing the number of processes. In addition, not having it as an option may force counters to go through different structures of counting processes. Both scenarios make counting processes become more complicated. In addition, while detecting the option of not choosing cannot be considered as any combinatorial thought related to the other two components, not detecting it as an option may affect both. As detecting options should be part of combinatorial thoughts in solving counting problems, and it cannot be considered as any of the three components in Lockwood's model, refining this model could be a future research study.

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THINKING CLASSROOMS AND COMPLEXITY THEORY

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In this article I will look at how Thinking Classrooms can be described and studied through the lens of Complexity theory. A Thinking Classroom is a teaching framework developed by Peter Liljedahl to occasion greater supports for student activity and engagement through the extensive use of randomized groupings, problem solving on vertical whiteboards, and sequenced tasks to maintain flow. The public nature of the whiteboard surface and the close and fluid interactions of the students affords the potential for ideas, hunches, queries and representations to move freely through the room. In this article, I describe the connection between Thinking Classrooms and Complexity theory and how emergent events may be observed in this environment.

INTRODUCTION

Thinking Classrooms and Complexity theory are both relatively recent ideas in the field of mathematics education and not often discussed together. Thinking Classrooms are a model of instruction developed by Peter Liljedahl (2016) to occasion student activity, autonomy and collaboration towards solving rich mathematical tasks and creating a problem-solving culture in a classroom. Complexity theory describes events of emergence, "those instances where coherent collectives arise through the co-specifying activities of individuals" (Davis & Simmt, 2003, p. 140). In Thinking Classrooms, the "coherent collectives" can be thought of as creative moments in problem-solving, the genesis of ideas as groups work to solve common problems or mutual understanding of new concepts. Davis and Simmt describe the mathematics classroom as having the potential to benefit from Complexity Science if certain conditions are present or nurtured. A Thinking Classroom has most of these conditions in place by its very structure, and so a Thinking Classroom is predisposed to exhibit or occasion emergent events such as creative ideas or collective understandings. In this article I will describe the deep connection between Thinking Classrooms and Complexity Theory and outline some possible avenues for study or observation of the emergent events within a Thinking Classroom.

THINKING CLASSROOMS

The Thinking Classroom framework was developed to occasion greater supports for student activity and engagement through the extensive use of randomized groupings, problem solving on vertical non-permanent surfaces (whiteboards), and sequenced tasks to maintain flow. The framework has fourteen elements for teachers to consider when structuring their lessons; three of these elements are the most impactful on

2019. In A. Hare, J. Larsen & M. Liu (Eds.). *Proceedings of the 14th Annual Mathematics Education Doctoral Students Conference* (pp. 95-98). Burnaby, Canada: MEDS-C.

changing classroom norms and providing spaces that are conducive to problem solving (Liljedahl, 2016). These are visibly random groupings, vertical non-permanent surfaces, and rich thinking tasks. I have been teaching using the Thinking Classroom model for six years now and have found it to be quite successful in creating environments where students willingly engage in collaborative problem-solving tasks and classroom cultures that are inquisitive and mutually supportive. Over the past six years of implementation, I have been impressed with how my students attack problems and work together to find creative and at times innovative solutions. I have been struck with how the problem-solving process in a Thinking Classroom is so different from that found in traditional mathematics classrooms. The public nature of the whiteboard vertical surface and the close and fluid interactions of the students affords the potential for ideas, hunches, queries and representations to move freely through the room. In a Thinking Classroom, students are noisily exchanging ideas within groups and between groups, conjectures and diagrams evolve on the whiteboards around the room and shouts of joy can be heard as progress is made. It is only recently where I have begun to see Complexity Theory as an instrument to describe student behaviour and learning in this environment.

COMPLEXITY THEORY

The field of Complexity science or Complexity theory bears some resemblance to enactivism, radical constructivism, situated learning, and some versions of social constructivism and has only come together over the last 45 years. Like enactivism, complexity theory arose from cybernetics and deals with themes of emergence, interaction between object and environment and adaptation.

Two key qualities are used to identify a complex system: adaptivity and emergence. Adaptivity is the change in the object and the change in the environment as the system interacts and evolves. Emergence is the self-organization of the individual agents into a collective with a clear purpose. Weaver (1948) described complexity by contrasting it with not-complex. Not-complex can be simple systems such as trajectories, orbits or billiard balls where actions and interactions can be characterized and even predicted in detail. As the number of variables increase, they become exponentially more difficult to predict and scientists move to new analytical methods such as probability and statistics to interpret, and the systems move from simple to disorganized complex. These two systems do not cover the range of possibility. It is when we recognize that objects within systems may not operate based on a set of known inputs/outputs (not deterministic), but rather their operations emerge in the interaction of the agents that we begin to think of it as a complex system. It is this key quality of emergence that is the focus of discussion in this paper.

Complexity science is interested in events of emergence that come from these complex systems. Illustrative examples of emergence are flocking of sandpipers, the spread of ideas, or the unfolding of cultural perspectives. These are collective possibilities not represented in individual agents (Davis & Simmt, 2003, p. 140).

In terms of mathematics education, mathematics classes are adaptive and self-organizing. Behaviours and norms of the group emerge from the collective. Complexity theory represents a move toward understanding the collective as a cognizing agent (as opposed to a collection of cognizing agents). It is this self-organization that is a key quality in a complex system and may be present within a mathematics classroom.

Simmt and Davis (2003, p. 145) outline five conditions of complexity as necessary but insufficient conditions for systems to arise and to learn: (a) Internal Diversity – enables novel actions and possibilities. (b) Redundancy – sameness among agents is "essential in triggering a transition from a collection of *me's* to a collective of *us*." (c) Decentralized Control – locus of learning is not always the individual. (d) Organized randomness – emergent behaviours are about living within boundaries defined by rules, but also using that space to create. Liberating constraints draw a distinction between proscription and prescription in tasks. (e) Neighbour Interactions – there needs to be collaboration… not necessarily people to people but more for ideas to bump up against one another. These five conditions are stated as *necessary but insufficient*, because a complex system cannot be forced or coerced into existence. Its very nature requires a randomness and a freedom amongst the individual agents for self-organization or emergence to occur.

THINKING CLASSROOMS AS COMPLEX SPACES

A Thinking Classroom as a complex space can be justified through the five conditions of complexity described above.

Internal Diversity: The daily randomizing of the groups produces diversity within the groups.

Redundancy: All students are at the same age and have similar prior mathematical experiences, they are members within random groups, they are all working at vertical spaces and all working towards completing a common task. This sameness helps to trigger the transition from individual learners to a learning community.

Decentralized control: Thinking Classrooms are neither teacher-centred nor student-centred. Rather, learning is shared and emergent and control is distributed amongst the groups. Knowledge and ideas are not coming from the teacher alone; they ebb and flow through the classroom as progress is made or seen or heard from others.

Organized Randomness: The tasks in a Thinking Classroom may have structure or provide constraints (proscriptive), but the ways that groups progress through the tasks is completely random and unstructured. The natural randomness of human cognition thrives in a Thinking Classroom and is what allows creative ideas to develop, evolve and migrate throughout the room.

Neighbour Interactions: The very nature of the groups working side-by-side in a vertical and public medium facilitates the interactions within groups and among groups.

Because a Thinking Classroom has all five of the conditions for complexity, it is a learning space that is ripe for the collective emergence of understanding or creativity within a mathematical solution.

EMERGENCE IN THINKING CLASSROOMS

What does emergence look like in a Thinking Classroom? Emergence in any classroom is when intellectual movements arise spontaneously and may quickly exceed the possibilities of any of the individuals – the knowledge, idea or understanding is a property of the collective. This is not only present in a Thinking Classroom, but it is amplified. Due to a Thinking Classroom satisfying all the conditions for complexity, it is a fertile space for observing emergence. Emergence can be observed by noting student representations, diagrams and solutions, and the evolution of these, on the public space of the vertical whiteboards. Observing conversations and student interactions is another method for noticing emergence in a Thinking Classroom. Neighbour interactions in the form of inter and intra-group communications as well as the interactions between ideas should be observable, and then emergence may also be noticed in these observations. Emergence in a mathematics classroom is a key solution strategy, a unique diagram or method, an A-Ha moment of inspiration or a new understanding of a concept; in a Thinking Classroom, these moments can all be observed, and their emergence can be mapped because of students working in the public space of vertical whiteboards.

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LOGARITHMS: FROM A CALCULATING TOOL TO A MATHEMATICAL OBJECT

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Throughout history, logarithms have been understood and therefore presented in many different ways. How they were introduced in a textbook affected the work done with logarithms throughout the text, as well as affecting what sort of understanding readers could take from the text. By looking at three texts used in the same area over 150 years, through the lens of Anna Sfard's Operational/Structural conception duality, I will analyse how these texts built understanding in their readers.

BACKGROUND

While relationships between geometric sequences and arithmetic processes had been known for centuries, they were not formalized until the 16th century under the work of John Napier, Joost Bürgi, and Henry Briggs (Cajori, 1913a). Their work was foundational in creating a process to evaluate logarithms, their operations, and how these can revolutionize arithmetic calculations. It was not until 1685 that an English mathematician named John Wallis first made the connection between exponents and logarithms in his text *Algebra*, though he does not fully define them through exponents. That first happened when William Gardiner published *Tables of Logarithms* in 1742, but it reached the wider mathematics world in Leonhard Euler's publication *Introductio*, in 1748 (Cajori, 1913b). In there, modern readers would recognize the connections he makes, defining logarithms as an exponent and as the inverse of the exponential function (Dunham, 1999).

In the following 250 years, logarithms continue to be introduced through exponents, but not always in the same way. Their introduction, along with how they use the process surrounding a logarithm can set the tone in how student understand this new concept.

RESEARCH QUESTION

My goal with this study was to look at three textbooks from different points in history to see if they introduced and worked through logarithms differently. Further, I wanted to see how their presentation guided the readers understanding of the concept.

The textbooks were all written for University or College students by professors from the American Midwest.

2019. In A. Hare, J. Larsen & M. Liu (Eds.). *Proceedings of the 14th Annual Mathematics Education Doctoral Students Conference* (pp. 99-106). Burnaby, Canada: MEDS-C.

- *A School Algebra: Designed for Use in High Schools and Academies,* by Emerson White, 1896
- *Introduction to College Algebra,* by William Hart, 1947
- *Intermediate Algebra (11^{th} ed),* by Mark Bittinger, 2011

THEORETICAL FRAMEWORK

To examine the effect that the presentation of logarithms would have on the reader, I turned to Anna Sfard's theory around the duality of operational and structural understanding. She comes from the school of ideas around Piaget, and others in that field, she is interested in the interiorization and encapsulation of an idea. Operational understanding is the process that created the idea and the calculations or computations that come from or use that process. Structural understanding looks to the idea beyond its process, becoming more than just calculations, but taking shape and becoming an object. The process is still inherent in understanding though, so mathematical concepts are both operational and structural at all times (Sfard, 1991).

Mathematical concepts are cyclic, (Figure 1), with 3 phases, first on a process level with concepts that are already objects, then an emergent idea of the product of the process being an object, and finally the ability to see the new object and all that entails.

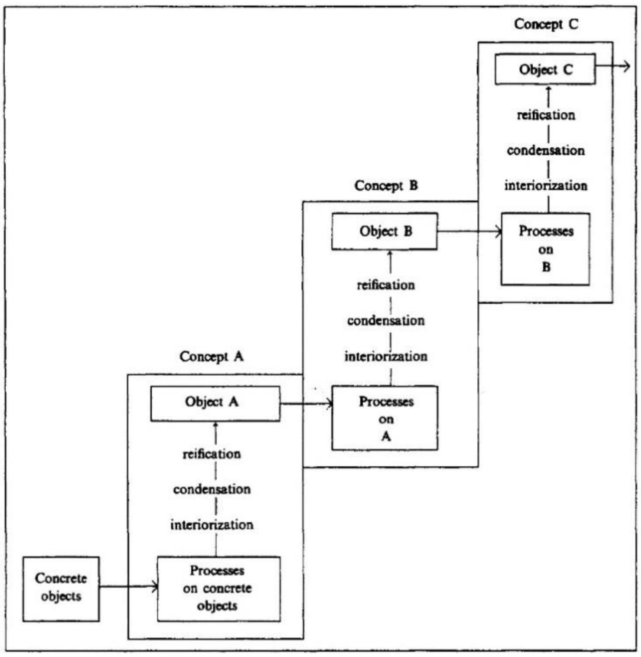

Figure 1: Sfard's model of concept formation (Sfard, 1991, p. 22)

The first phase involves interiorization, being able to do the process that leads to the new object internally, so with a mental representation. Condensation is ability to alter between different forms of the concept and being able to see some parts of the concept as a mathematical object and not just an operation. "Reification is defined as an ontological shift […] detached from the process which produced it and begins to draw its meaning from the fact of its being a member of a certain category" (Sfard, 1991, p.

19–20). This last one does not build from continuous practice, but is more an ‘aha’ moment, where you change the way you understand and relate to the concept.

While she does argue that there is a hierarchy apparent in these understandings, she does feel it is possible to introduce things outside of procedure as long as the student comes back to that important step of interiorization. This alternate pathway is often true in textbooks where they will introduce a concept by definition as an object, so structural conception may come before operational (Sfard, 1991). Separately, structural understanding cannot come without a reason for the concept to be an object, without a higher level process that the concept leads into, then that ontological shift will not happen (Sfard, 1995).

METHOD

As I wanted to focus on how the presentation of logarithms would affect how the reader viewed them as a concept, I conducted an analysis of the specific chapters and categorized them according to the theoretical framework expounded upon above. As all of the chapters did use exponents in some way to introduce logarithms, I paid particular focus to how they built that introduction into the basis of a separate operation (or function). Overall, I had five themes: Introduced through a repeatable process, Used for calculations, Interiorization, Condensation, Reification.

ANALYSIS

School Algebra: Designed for Use in High Schools and Academies

This first text was published in 1896 by Emerson Elbridge White, a math instructor and a school administrator, who rose to become the third president of Purdue University in Indiana. As Purdue University was an agricultural school at this time, it follows that he had a strong belief in applied learning (Norberg, 2019). His text on algebra follows that pattern, pushing for learning through applications over theory.

White begins by introducing logarithms through a definition relating it to exponents. He follows that by working through base-10 logarithms and how that relates to the formation of logarithmic tables (pp. 309–311). White’s work stays mainly in the world of calculations. While he does use the idea of exponents to build up the operations with logarithms (pp. 311–312), those operations are practiced only with the use of the tables of logarithms (pp. 313–323). Even when he does get into using logarithms within an equation (pp. 323–328), so as an operation, he does the work and then sets that up as a formula that the reader can use to plug in numbers as seen in Figure 2. So a reader would see logarithms used as an operation, and some operations with logarithms, so have taken a small step toward condensation. But while the idea of logarithms is interiorized, logarithms themselves stay a calculation and never become much of an object.

613. Exponential equations (§ 569) are most readily solved by the aid of logarithms. When, for example, the a, l, r, of a geometrical progression, are given to find n, the value of n must be found from the equation $l = ar^{n-1}$; whence $r^{n-1} = l \div a$.

Taking the logarithms of both members of this equation,

$$(n-1)\log r = \log l - \log a\cdot$$

whence $$n = 1 + \frac{\log l - \log a}{\log r}.$$

By taking from the table the logarithms of the given quantities a, l, r, and performing the indicated operations, we obtain the value of n. If in the foregoing formula we suppose $a = 3$, $l = .00019683$, $r = .3$, we obtain

$$n = 1 + \frac{\log .00019683 - \log 3}{\log .3},$$

or $$= 1 + \frac{\bar{4}.29409 - 0.47712}{\bar{1}.47712};$$

whence $$\mathbf{n} = 1 + \frac{\bar{5}.81697}{\bar{1}.47712} = 1 + 8 = \mathbf{9}.$$

Figure 2: Using logarithms to create a formula (White, 1896, p. 323)

Introduction to College Algebra

William Hart was a mathematician turned mathematics educator who lectured at the University of Minnesota. He was very influential in broadening the idea of applicable mathematics, writing textbooks on statistics and the mathematics of investments. He used the above skills, along with assistance from his brother, Walter Hart, a mathematics teacher at the secondary level, to write a series of algebra books for students to use from secondary into their first University classes (Price, 1986).

In *Introduction to College Algebra,* Hart also introduces logarithms through an exponential definition and spends a few pages evaluating logarithms through this means (pp. 212–214). He then combines this evaluation with the operations of logarithms, derived through exponents, so the reader gets practice with two views of logarithms (pp. 215–226), starting the work from interiorization to condensation. He also spends time working on the operations outside of evaluations, in both logarithmic and exponential equations (pp. 227–229). His equations combine ideas behind both exponents and logarithms so they would be hard to solve without understanding the logarithm as an exponent and the logarithm as its own thing. That he has his readers working with the logarithm as a value (something in a table), a process (which is exponential in nature), and an operation which one can use within an equation, helps the reader condense the logarithm into an object.

Lastly, he introduces the logarithmic graph (Figure 3) and uses that to make some conclusions about its boundaries and limits. While reification would be hard to get from a text, as it is a change in how a person views the concept, it is something that a text can encourage. By including a graph as well as putting in some other limitations on logarithms (Figure 4), Hart pushes his reader to see logarithms in multiple ways, to think more about their structure, to question them limitations of a logarithm. He puts them on the path toward reification.

★190. Graphs of logarithmic and exponential functions. We recall that $y = \log_a x$ and $x = a^y$ are equivalent relations. We call $\log_a x$ a *logarithmic function* of x and a^y an *exponential function* of y.

ILLUSTRATION 1. In Figure 28, we have the graph of $y = \log_e x$. For any base $a > 1$, the graph of $\log_a x$ would be similar. This graph assists us in remembering the following facts.

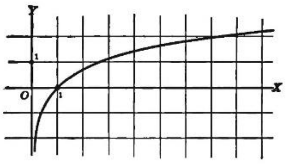

FIG. 28

I. *If x is negative,* $\log_a x$ *is not defined.*

II. *If* $0 < x < 1$, $\log_a x$ *is negative, and* $\log_a 1 = 0$.

III. *If x increases without limit,* $\log_a x$ *increases without limit; if x approaches zero,* $\log_a x$ *decreases without limit.*

Since $y = \log_a x$ is equivalent to $x = a^y$, *these equations have the same graph.* Thus, in Figure 28 we have a graph of $x = e^y$.

Figure 3: Hart, 1947, p. 229

Note 1. We do not use $b = 1$ as a base for logarithms because every power of 1 is 1 and hence no number except 1 could have a logarithm to the base 1.

In the definition of $\log_b N$, we stated that N was a *positive* number. That is, in this book, if we speak of the logarithm of a *number* N we shall mean a *positive* number N. Also, we stated that the base b is *positive*. These agreements were made to avoid meeting imaginary numbers as logarithms. In advanced mathematics it is proved that, if N and b are positive, there exists just *one* real logarithm of N to the base b.

Figure 4: Hart, 1947, p. 213

Intermediate Algebra (11th ed)

Marvin Bittinger was a professor at Indiana University who has written over 250 textbooks, mostly focusing on algebra (Bittinger, n.d.).

Unlike the other textbooks, logarithms here are not given their own chapter. They are introduced after he has covered exponential functions and inverses, so he introduces the logarithmic function graphically as the inverse of an exponential function (Figure 5). The initial work in the section is to relate a graph and a table of values, and use those to evaluate logarithms (pp. 703–705). The setup of a new concept should lead to interiorization as the reader works to practice the process that gave way to that concept, but by introducing logarithms through an exponential graph and table of values, Bittinger creates something that would be difficult for a student to recreate mentally. It is not until he defines logarithms as the inverse of exponential functions through an equation and practices conversions between the two forms (pp. 705–708) that readers have something they could start to interiorize.

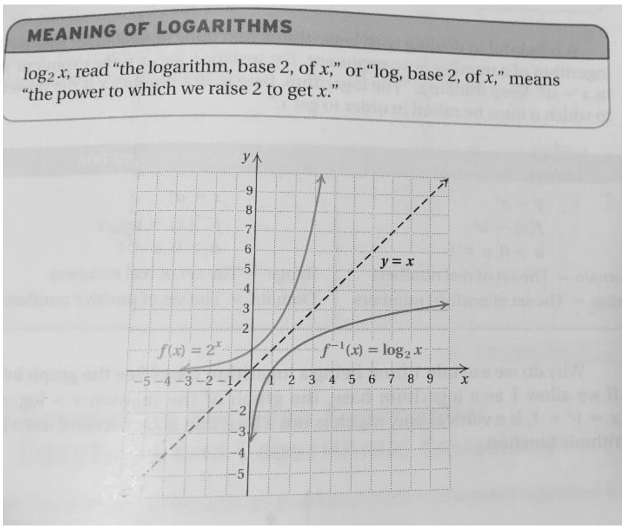

Figure 5: Logarithmic graph as inverse of exponential (Bittinger, 2011, p. 703)

That interiorization may not always take hold in his presentation though, the readers are not given much practice in evaluating logarithms without either rewriting it as an exponential equation, or using a calculator (pp. 708–709). The few times the text does discuss evaluating logarithms without those methods, it is either a sidenote or a one-time example that is followed by a litany of examples that use one of the above methods. A clear preference is present in the text, a preference that highlights the use of exponential equations or calculators.

The operations of logarithms are presented as a property with an optional proof below. The proof leans into the conversion between logarithmic and exponential form, bypassing the procedure that introduced the concept (pp. 714–715). The rest of the chapter is spent on manipulating logarithms, logarithmic graphs, and solving logarithmic and exponential equations. Even though logarithms may not be interiorized in Bittinger's text, as they are presented in so many different forms, they begin to become condensed into different parts of an object. They are given a shape, and some structure, perhaps leading a reader toward reification.

DISCUSSION

While Anna Sfard argues for the hierarchy of her model, she acknowledges that it is not a requirement, as long as you have operational and structural understanding than you can have full understanding of the concept. If we are going to look at how her model applies to logarithms as presented in these textbooks, then it will appear as it does in Figure 6.

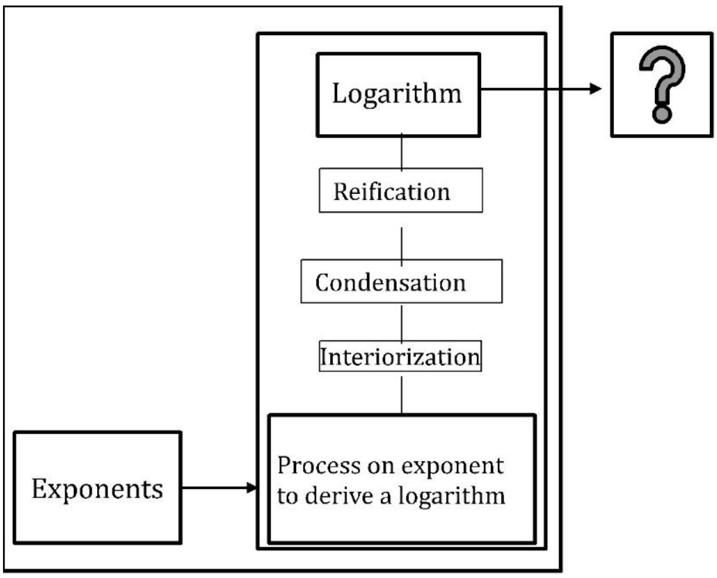

Figure 6: Concept formation of logarithm

All three texts introduced logarithms through exponents, assuming that the readers had an operational and structural understanding of exponents. From there, the three steps of interiorization, condensation, and reification need to happen before there is a complete understanding of logarithms. While interiorization and condensation can be worked toward through practice, reification is more of a flash of insight. Furthermore, Sfard (1995) believes that reification cannot happen without leading into a new process. In each of these books, logarithms are an end goal, so reification is an impossible task. Given that, it is still possible to move readers toward an operational and structural understanding by building up the processes and structure, but full understanding in any of these texts cannot happen.

Tying this back into my initial themes, where aside from interiorization, condensation, and reification, I had looked at whether logarithms were introduced through a repeatable process and whether they were employed for calculations, we make some inferences.

Both White and Hart introduced logarithms through exponents, and practiced evaluating logarithms in a consistent way with that introduction. They used that process to establish operations with logarithms, and then used those to evaluate logarithms. By having the reader recreate the process of a logarithm repeatedly, and by combining that process with others, the reader should be able to interiorize the concept. By showing logarithms in different forms, as a value, as an exponent, as an operation, they begin to work toward condensation. Hart even takes that initial introduction to start showing some of the structure of logarithms, exploring the graph and its boundaries, and in doing that he begins to build structure.

Bittinger, in introducing logarithms graphically, puts his readers at a disadvantage. They may immediately begin to see the structure of logarithms, but they will not be able to interiorize the process of a logarithm. To move into any operations, he has to introduce logarithms in a different way, which in his text is to convert between logarithmic and exponential form. In any of the following examples logarithms are quickly disposed of in favour of the more familiar exponential form. They do come back in different forms over the rest of the chapter, but not in a way that builds off of the initial introduction, or even from the secondary introduction. So while readers will see logarithms in many ways, leading to condensation, they have yet to interiorize it.

CONCLUSION

It seems that even though Anna Sfard argued that there did not have to be a strict hierarchy in understanding a new mathematical concept, for logarithms that hierarchy is useful. An introduction that students can recreate, as well as practice in evaluating and using logarithms, make a great difference in the interiorization of the concept. That bridge that builds to a structural understanding should come out of the same procedure. From there, new forms of the concept will help fill out the structure that can lead to reification.

Starting with different forms would seem to lead to a shallow understanding of the concept. While readers may have an idea of some rules and boundaries surrounding the concept, they may still be missing what exactly is the operation of a logarithm. Trying to build on a scattered structure, without that operational foundation underneath could lead to future problems with understanding logarithms.

I feel the next steps to this study are to do a wider exploration of textbook introductions of logarithms and how they build up the structure looking at whether different definitions lead to a different understanding of this concept.

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EMBODIED CURIOSITY – A RECIPE FOR THE MATHEMATICS CLASSROOM

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This paper is a corollary to a larger research study. It examines how two grade nine students, both 15 years old, from a secondary school in Jamaica, interacted with circle geometry theorems in a Dynamic Geometry Environment (DGE) called The Geometer's Sketchpad (GSP). I utilize the notion of Embodied Curiosity, as well as, Andrew Pickering's idea of agency (the influence of human and non-human actions against each other), to analyse the ways in which Embodied Curiosity emerge when students interact with their peers on geometric tasks. In addition, I adopt parts of Berlyne's curiosity dimension model as a methodological tool to identify physical markers of when and how students become curious. I argue that curiosity along with digital technology, body movements and mathematical meanings work hand-in-hand for learning to take place. I also suggest that curiosity; the main ingredient, plays an important role in shaping the body and the mind.

INTRODUCTION

Over the years, mathematics education research has concentrated on the nature of mathematics (Begg, 1994), and on issues relating to the teaching and learning of the subject (Ellerton and Clarkson, 1996). Furthermore, through engaging with literature, as well as my own experiences, I became aware that although much emphasis is placed on the cognitive aspects of mathematics learning, insufficient consideration is given to the affective in research on digital technology. A research conducted by Sinclair and Heyd-Metzuyanim (2014), which focused on the role of the body and emotions in mathematical communication sought to shed light on this. However, in this research consideration is given to how the body (specifically the hands and fingers), as well as emotions, influence how young children communicate mathematically in a touch-screen environment. My interest is in pushing the boundaries a bit further to examine how the body (as a whole) and emotions contribute to mathematical learning in Secondary School children, particularly in a digital technology environment such as *The Geometer's Sketchpad*. As a result, this paper seeks to address how children's emotions (curiosity) and physical beings (their body) contribute to construction of 'new' mathematical meanings. I refer to 'new' in the sense that the knowledge which may emerge might not be readily accessible in students' mathematical register, but can be implicitly or explicitly identified when children interact with each other.

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Additionally, in a technology-rich era such as the 21st century, I believe that advancement of modern technology is closely linked to humans' constant need to explore the world. Seymour Papert (1980) had envisioned an abundance of technology use in society at a time when sceptics were unable to articulate the many positive benefits of computers. According to him, the invention of computers has influenced "the way people think and learn" (p. 3). His particular interest was on the way computers transmit ideas and promote cultural change. Papert also saw computers not solely as a tool that programs children, but also as children programming them. Pickering (1995) describes this two-way relationship between the children and the computer, as "the dance of agency" (p. 78), implying that both depend on each other in the process of learning. It is within this two-way relationship that I believe human curiosity is situated. As a consequence, I have dedicated this paper to the use of *The Geometer's Sketchpad* as the digital technology tool of interest and examine its role in triggering curiosity via geometry.

Curiosity

As far back as the 14th century, the word "curiosity" has taken on several meanings. For example, *curiositatem* (Latin) meant "a desire of knowledge, inquisitiveness" which implies something good, while in Middle English "curiosity" meant "to pry, idle or a vain interest in worldly things", which positioned curiosity as something bad. A more contemporary, scholarly view of curiosity emerged from George Loewenstein (1994) as a term evolving through the lens of psychology and philosophy. He defined curiosity as "a form of cognitively induced deprivation that arises from the perception of a gap in knowledge or understanding" (p. 75). He also suggested that one of the reasons for the constant change of the meaning is that there is insufficient account of why people voluntarily seek out curiosity. As a result, Loewenstein argued that curiosity is positioned at the intersection of both cognition and motivation. In this paper, curiosity is operationalized within Loewenstein's views but, in addition, bears a connection to embodiment.

Embodiment

Like curiosity, the term 'embodiment' (the noun), emerged from the disciplines of psychology and philosophy and shared a similar unstable pathway in terms of its meaning. According to Kiverstein (2012), embodiment first became popular in the work of Varela, Thompson and Rosch (1991), whose intention was to discredit traditional ways of thinking about cognition as solely mind-based, following the Caretsian approach in which body and mind are seen as separate entities. Kiverstein suggests that the term has now transitioned in meaning through three fundamental perspectives. The body-functionalism (the body is understood to play an important role in implementing computational machinery), body-conservativism (the body's contribution to cognition is solely as an input–output channel for the brain) and body-enactivism (the body is seen as the primary source of meaning). In keeping with the body-enactivism perspective, on which the theoretical framework of this paper rests, the body is seen not solely in a sensorimotor realm, but also as placing emphasis on the affective factors as well.

THEORETICAL FRAMEWORK

In order to pursue a more enactivist approach to curiosity, I have coined the term Embodied Curiosity (EC), which stresses not only the body as a primary source of meaning, but also the distributed and material nature of curiosity. As a result, I argue that, within an Embodied Curiosity framework, curiosity, digital technology, body movement and mathematical meanings work together in a unified and harmonious way to enhance mathematical learning. I also suggest that Pickering's (1995) idea of human and non-human agency acts as an adhesive that keeps the elements of Embodied Curiosity framework connected. Pickering's notion of "the mangle of practice" of individual, material and discipline was also useful in helping me understand mathematics in particular. In this "mangle of practice" agency is shifted away from the individual and as a result the claim is that curiosity no longer arises from the individual. According to Pickering, there is a line drawn between the actions of human and the actions of nature (other material things), which assigns agency both to people and to things; that is, agency has to do with the influence of one thing onto another. Two important aspects of his work (which form the basis of agency in this paper) is the emphasis that he placed on temporality and posthumanism. In these two ideas Pickering claimed that human and non-human agencies are not known in advance but rather they emerge during an experience and, that human agency is not given precedence over any other form of agency (material or disciplinary). This means that, within an Embodied Curiosity framework, the interactions among curiosity, digital technology, body movement and mathematical meanings are equally important.

METHODS

The students (age 15 years) who participated in the episode used in this paper, were part of a larger pool of Grade 9 students from two secondary schools in Jamaica. However, the boys were from one of the schools, which I called School X. The data was collected over a six-week period. The interaction represented in this research was randomly selected from approximately 24 video recordings (approximately 12 hours) of classroom observations. The purpose for a random selection of this interaction was to determine, at the initial stage of data analysis (for the larger research project) ways in which the elements of Embodied Curiosity were present in the data. Randomization was also important in giving me that first-hand look on any relationship which might be present between the components of EC. The episode was taken from an interaction between the two boys, which I called Dani and Mio while they were engaged in a task given by their teacher (Sammy). The task involves investigating what happens to the angle formed in a semi-circle – one of the circle geometry theorems.

Research design and methodology

Since this paper is concerned with body movement and the ways in which curiosity influences these movement, I utilized Berlyne's (1954) curiosity dimension model as an analytical instrument for Embodied Curiosity. Within this model Berlyne suggested that curiosity can emerge in two ways, that is, by perceptual (sensation-seeking) experiences or by epistemic (knowledge-seeking) experiences. He further argued that these two ways require a need to actively seek depth (specific) or variety (diversive) when an activity is performed. As a result, his model comprises four categories of curiosity. On one hand, there is *epistemic specific curiosity,* which focuses on a desire for a specific piece of knowledge, and *epistemic diversive curiosity* which requires an exploration and focuses on a desire for multiple ideas. On the other hand, there is *perceptual specific curiosity* with a desire for a particular sensation, and *perceptual diversive curiosity* which requires the use of multiple senses to explore the environment. In this paper, although his model has been a useful tool in helping me to identify curiosity, I deviated from the polarization he suggested and instead propose that in order for curiosity to be identified, perceptual and epistemic curiosity should work hand-in-hand. This is in keeping with my enactivst commitment. Therefore, I see curiosity as sensation-seeking coupled with a desire for knowledge.

DATA ANALYSIS AND RESULTS

In order to provide an account for the co-ordination of curiosity, body movement, digital technology and mathematical meaning, I used an interaction between two boys, whom I call Mio and Dani along with their teacher (Sammy). All names are pseudonyms and the episode which was used in this data analysis, was documented using a transcript, pictures, diagrams and anecdotes to express how the boys' interaction played out. Upon analysing the data, I recognized that utterances were frequently accompanied by body movement. Therefore, I also describe the action of the participants within the transcript by placing them in curly brackets at the turns where they occurred simultaneously with speech.

Mio and Dani

Mio (in green shirt) and Dani (in white shirt) were working on a task together using separate computers and *The Geometer's Sketchpad* software. The task for the day was for the students to explore what happens when an angle is formed in a semi-circle. Sammy (the classroom teacher) was in charge of the session and she asked the students first, to use the circle tool to construct a circle on the computer screen. All the students were able to produce the circle. Then next instruction was to construct a diameter and Mio quickly constructed his but noticed that Dani had constructed a radius instead of a diameter, as seen in Figures 1 [a] and 1[b].

[a] Dani’s radius instead of the diameter.	[b] Mio observed Dani’s radius.	[c] Mio explains about the diameter.	[d] Mio fingers moved inwards.

Figure 1: The boys collaborate to construct the diameter

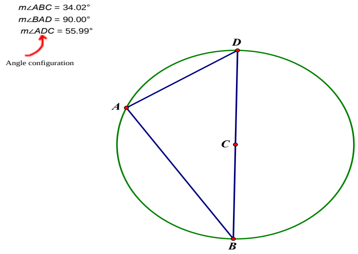
Having observed that Dani had drawn the radius (Figure 1[b]), Mio in a surprised manner exclaimed to Dani that the diagram he drew was in fact the radius (Figure 1[a]), Dani then erased the circle with the radius and started the task again. With a second circle on the screen, Mio recognized that Dani faced a challenge when constructing the diameter because he was reluctant to use the line segment tool in his second attempt. Instead of telling Dani what to do, Mio used his index finger to show that the diameter could have a possible starting point on the circumference of the circle as seen in Figure 1[c] despite drawing his in a symmetrical manner shown in 1[b]. Perhaps this was because he was surprised that Dani’s knowledge about the diameter did not seem secure and he wanted to reinforce the idea of the diameter ‘running through the centre and touching at both ends from any point on the circumference’. He further moved his index finger inward towards the centre of the circle indicating that the diameter, while having a starting point on the circumference should also pass through the centre of the circle (Figure 1[d]). Dani was visually fixated on the screen, while Mio negotiated a possible diameter as shown in Figures 1 [c] and [d].

One can argue that when Mio looked across at Dani’s screen, as seen in Figure 1 [b], he also demonstrates curiosity in a similar manner, but Mio has always been an assertive student throughout the interactions. This limits my judgment in being able to decide whether or not he was curious or merely demonstrating an interest in Dani’s work. Furthermore, Dani’s fixation was followed by him leaning closer to his screen as he demonstrated an interest in knowing what the diameter looks like. After a few minutes had passed, and Sammy walked around the room to check on other students performing the task, both boys managed to inscribe the triangle within the semi-circle based on the instructions given by their teacher. Dani’s diagram in Figure 2 [a] below shows the orientation and labelling of each vertex of the triangle. Sammy noticed that the boys had their full construction on the computer screen and stopped by Dani to probe his understanding of the task. The transcript which follows represents the dialogue which ensued between Sammy and Dani.

1. S: Move this (points to angle A in table 2 [a] below).
2. D: (Drags point A along a small portion of the circumference between angles B and D repeatedly in a back-and-forth manner).

3.	S:	What do you notice with angle A? Move it again, which angle is changing?
4.	D:	(Drags point A further). The first one Miss?
5.	S:	No, it depends on which A you are taking about. The angle you are measuring should be in the middle. So, it's BÂD. (points to the angle notation shown in Figure 2 [a] and then points to the vertices of the triangle, B, then A, then D (Figure 2 [b]).
6.	D:	But Miss, the size is not changing.
7.	S:	It is not changing.
8.	D:	Miss bring it over that side to see?

While Dani dragged point A (turn 2) along the circumference between angles B and D both angles were constantly changing while the invariant angle, A, remained unchanged at ninety degrees (turns 6 &7). Sammy drew his attention to the angle when she asked "what do you noticed with angle A?" at turn 3. Dani wondered what would happen if he dragged point A over the opposite side of the diameter (turn 8) which turned his focus from the measurement of the angle as Sammy implied in turn 5 to the angle itself as he asked, "miss, bring it over that to see?" portraying the geometric angle as a moveable object which can move from side to side.



[a] Dani's diagram after construction

[b]Sammy pointing to the vertices of the triangle

[c] Dani position his hand to show how the angle should move

[d] Dani sways his hand from left to right

[e]Dani's hand movement continued from[d]

Figure 2: Dani's hand movement illustrating a flip.

His wondering accompanied by the swaying of his hand from left to right of the screen (Figure 2 [c] to [e]) while using the diameter as a reference for the action to be carried out, as well as the repetition of performing the dragging motion (turn 2) suggested that there is evidence of curiosity. His hand movement implied that if a reflection was done and the angle remained unchanged then there is something to be said about this angle in particular. Furthermore, while Dani performed the swaying hand movement, he simultaneously dragged the point A along the circumference of the circle which meant that while his hand (left hand) represented a reflection of the angle, dragging the point along the circumference (using his right hand) in a circular motion represented a rotation. I found this intriguing because the functionality of *Sketchpad* played an important role in allowing both actions to be performed instantaneously, implying that, by using *Sketchpad*, Dani was able to experience implicitly a relationship between the geometric transformations – reflection and rotation.

DISCUSSION: DANI'S EXPERIENCE OF EMBODIED CURIOSITY

Dani began to experience Embodied Curiosity when his visual fixation (a sensory experience), his leaning forward coupled with his uncertainty about the diameter (the knowledge he was seeking) prompted him to construct the chord which runs through the centre of the circle. He had initially represented this concept as the radius but Mio's hand movement from the circumference of the circle inwards through the centre, helped him to formulate the concept and construct it on his second attempt. This knowledge appeared to be 'new' for Dani. Additionally, in performing the task Dani seemed curious again when he wondered about what would happen to the angle which is formed in the semi-circle, if it was reflected through the diameter. His wondering was evident when he asked, "Miss bring it over that side to see?" He was uncertain whether or not the size of the angle would change or remain the same if the point was dragged over the diameter. Also, his repeated dragging action in a back-and-forth manner suggested that he had a desire to know what was happening each time the angle moves. The draggability of *Sketchpad,* coupled with the aesthetic appeal of the coloured triangle and the symmetrical manner in which the diameter was constructed (Figure 2 [a]) indirectly led to the representation of a composite geometrical transformation involving a reflection and a rotation. In this sense, Dani's hand movement and the dragging function of the *Sketchpad* co-ordinated with each other to produce the composite geometrical transformation. I also believed, that the computer stimulated the boys' interest in performing the task not only because it was efficient in constructing the various parts of the circle, but also allowed them to see how each part related to each other. For example, by dragging one point of the diameter along the circumference, it maintained its property as long as it passes through the centre of the circle. This knowledge would not be readily accessible to the students if the task was performed in a static environment.

CONCLUSION

The analysis provides supporting evidence that in understanding mathematics, students utilize their senses (in this case, sight), their bodies (hand movement, leaning forward and backward) and their uncertainties together to generate ideas and perform an action such as constructing a diameter. The data also revealed that in using *Sketchpad* to perform these actions, new mathematical meanings may emerge. That is, students may become aware of a piece of knowledge which was not previously known to them, such as, that there is a relationship between a geometric reflection and rotation. The data also revealed that curiosity plays a significant role in triggering students' ability to act and that the functionality of *Sketchpad* was instrumental in attaching meanings to the student's hand motion. The hand motions in this case represented a reflection and a rotation occurring at the same time. Finally, the implication of this research finding suggest that curiosity is a central aspect of learning and should be exploited in the mathematics classroom.

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"I DON'T WANT TO BE *THAT* TEACHER": ANTI-GOALS IN TEACHER CHANGE

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This paper uses the theory of goal-directed learning to examine anti-goals that arise as teachers implement change in their mathematics practice. Findings suggest that anti-goals develop as teachers begin to recognize who they ***do not*** *want to be as a mathematics teacher. Accompanying anti-goals are emotions that can be useful in measuring progress towards anti-goals (fear and anxiety), and away from anti-goals (relief and security). Furthermore, acknowledging anti-goals allows mathematics teachers to focus on the cognitive source of their difficulties rather than be overwhelmed by the emotional symptoms.*

INTRODUCTION

In *Intelligence, Learning, and Action*, Skemp (1979) describes a thought experiment in which we are to imagine two events. In the first, we strike a billiard ball causing it to move across the table into a pocket. In the second, we are in a room with a child who, upon our command, moves across the room to sit in a chair. Superficially, these are two similar events: we have 'caused' the child to cross the room and we have 'caused' the billiard ball to roll into the pocket. This is basic stimulus and response in which an object remains in a state of rest, or uniform motion in a straight line, unless acted upon by an external force. Now imagine inserting an obstacle into the pathways of both the ball and the child. What a strange billiard ball it would be if it could detour around the obstacle and continue on its path. But a child will do this with no change in stimulus — perhaps by going around the obstacle, hopping over it, or even moving it. Skemp suggests that, unlike those of a billiard ball, the child's actions are goal-directed, and necessary to reach her goal state (sit in the chair).

Recognizing that many human activities are goal-directed is essential if we want to understand their actions. In other words, we need to attend to their goals with the same importance as we do their outwardly observable actions. Skemp offers the example of someone crawling around on their hands and knees on the office floor. There is no point in asking them what they are doing — we can see that for ourselves. A better question might be "Why are you doing that?", which might elicit a reasonable answer such as "I'm looking for the cap for my pen".

Let us try another thought experiment. Imagine observing a secondary mathematics teacher in her classroom. She is standing by an overhead projector demonstrating how to solve a problem while the students sit quietly at their desks and take notes. Another

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adult sits with a notepad at the back of the room. Students who talk are met with a polite reminder to raise their hand if they wish to speak. What might the actions of the teacher suggest? An immediate response might be the teacher is teaching, albeit in a somewhat traditional manner. Now imagine talking with the teacher after the lesson as she describes her frustration with having to conform with school norms regarding effective teaching of mathematics in order to successfully pass a probationary evaluation. To observe that the teacher was teaching traditionally is accurate, but incomplete. To make sense of her actions also requires consideration of her goal — she was teaching traditionally in order to achieve her goal of maintaining employment.

The aim of this study is to make sense of teachers' actions through consideration of their goals. As the pursuit of a goal is an emotive experience (Skemp, 1979), I begin by describing some of the literature regarding emotions in teaching. To connect emotions with goals, I then outline Skemp's theory of goal-directed learning.

EMOTIONS, ACTIONS, AND GOALS

Liljedahl (2015) describes emotion as the "unstable cousin" of beliefs yet there is much to be learned from their study; it is the erratic cousin at the family dinner who is most likely to blurt out uncomfortable truths. Fortunately, while pursuit of these uncomfortable truths was once the least researched aspect of teaching practice, over the past two decades there has been an increased focus on what emotion reveals to us about teaching and its implications for teacher change (Zembylas, 2005). As Kletcherman, Ballet, and Piot (2009) suggest, "A careful analysis of emotions constitutes a powerful vehicle to understand teachers' experience of changes in their work lives" (p. 216). The unstable cousin is being heard.

Prior to this, teacher cognition had been the primary focus of research on teachers (e.g., Richardson, 1996) and its underlying assumption was that teacher actions and behaviour were strongly influenced by cognition. More recently has come the recognition that emotion and cognition are inseparable and that emotions may provide insight into the relationship between a teacher and the socio-cultural forces that surround her (Van Veen & Sleegers, 2009). According to Hannula (2006), emotions (along with attitudes and values) encode important information about needs and may even be considered representations of them. While cognition is related to information, Hannula understands emotions as affecting motivation and therefore as directing behaviour by affecting both a person's goals and choices. Emotions constitute a feedback system for goal-directed behaviour, and thus shape a person's choices. Hence, emotions, cognitions, and actions turn out to be strongly intertwined and inseparable; it is necessary to consider these factors and their relationship in order to understand teachers' actions.

Skemp's (1979) theory of goal-directed learning takes into account these connections between emotions and actions. His framework was built on fundamental ideas in psychology and links emotions to goals which a learner may wish to achieve, and also to anti-goals which a learner wishes to avoid. Goals can be short-term, such as the desire to learn a procedure for solving a routine problem or long-term, as in the desire to be successful in mathematics. A short-term anti-goal may be to avoid failing a test, while a long-term anti-goal may be to avoid future mathematics studies altogether. Skemp emphasizes that goals and anti-goals are not simply opposite states, rather, a goal is something that increases the likelihood of success, while an anti-goal is something to be avoided along the way.

For Skemp, emotions come into play as they provide information about progress towards either goal state in two distinct ways (see Figure 1). First are the emotions experienced as one moves towards, or away from, a goal or anti-goal (pleasure, unpleasure; fear, relief). For example, moving towards a goal brings pleasure, while moving towards an anti-goal results in unpleasure. Although the four emotions bear similarities, there are subtle differences. Consider relief and pleasure; the relief one feels upon not failing a test is a different feeling from the pleasure one experiences upon learning the correct procedure for a problem. The second aspect of emotions concerns one's sense of being able to achieve a goal or, conversely, avoid an anti-goal (confidence, frustration; security, anxiety). For example, believing one is able to achieve a goal is accompanied by confidence while believing that one is unable to move away from an anti-goal state induces anxiety.

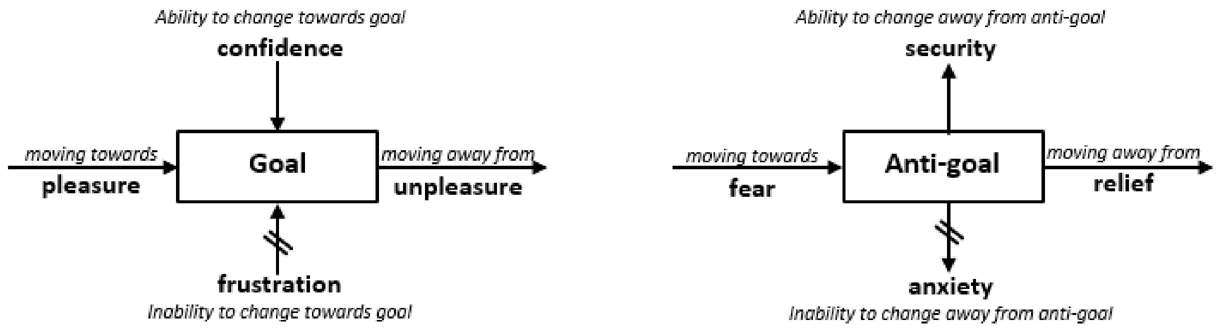

Figure 1. Emotions associated with goal states (adapted from Skemp, 1979).

Although initially designed to examine goals related to the learning of mathematics, in this study I will be using Skemp's theory to examine goals related to the teaching of mathematics, particularly those of teachers who are trying to change elements of their practice. In doing so, I follow Jenkins (2003) who suggests that change is not just to make different, but, like learning, it is also to continually improve in skill or knowledge. Specifically, I use Skemp's notion of goals and anti-goals to better understand the actions of teachers involved in changing their mathematics practice.

METHODOLOGY

McLeod (1992) suggests that detailed, qualitative studies of a small number of subjects allow for an awareness of the relationship between emotions, cognitions, and actions that large-scale studies of affective factors overlook. Accordingly, in this study, I adopt an exploratory and qualitative approach that focuses on documenting that relationship. Data for analysis was taken from a larger study involving 15 teachers whose teaching experience ranged from 1 to 16 years. The data was created during semi-structured interviews that ranged from 40 to 60 minutes. The interviews were audio-recorded and then fully transcribed. The structure of the interview aimed at letting emotions emerge organically through a narrative rather than by direct questioning. For example, the teachers were asked to describe the changes they had implemented in their mathematics practice without explicitly asking them to describe the emotions they felt. This allows for richer descriptive data of personal experiences that leading questions may inhibit. With DeBellis and Goldin (2006), I am aware that emotional meanings are often unconscious and difficult to verbalise but, with Evans, Morgan, and Tsatsaroni (2006), I believe that textual analysis of teachers' narratives allows the identification of emotional expressions that function in teachers' positioning. As such, the transcripts were scrutinized for utterances with emotional components such as "I was worried…" and then re-examined for their potential connections to goals. Due to space limitations, I report only on those findings related to anti-goals.

THE DEVELOPMENT OF AN ANTI-GOAL

For the teachers in this study, the decision to implement change in their mathematics classrooms stemmed from dissatisfaction with their current practice. Most had learned mathematics as learners in traditional mathematics classrooms and had simply gone on to replicate that for their own students. As Kelly recalled, *"There was nothing during my journey to becoming a mathematics teacher that made me think of another way to teach math."* Their collective desire to move away from the notion of teaching as telling and learning as listening (and remembering) so permeated their interviews that I originally coded these excerpts as 'That Teacher'. However, it was this tension between who they were and who they wanted to be that led to change in their mathematics practice, as who they wanted to be as a teacher became their goal, while who they had been, or wanted to avoid becoming, became an anti-goal.

Many of the teachers described similar situations where tension with their teaching practice drove them to seek out professional development. For example, the development of Amy's anti-goal began with the feeling that *"I was boring, like they just weren't getting from me what they needed."* It coalesced into an anti-goal as she realized her practice was harming, rather than helping, her students:

> My practices resulted in increased anxiety and frustration amongst my students; damaged their mathematical confidence; removed their desire to think deeper and search for understanding; as well as robbed my students of conceptual experiences. Valuing speed and accuracy comes at a great cost for students and gives them little mathematical benefit.

To alleviate the anxiety this caused her, Amy sought out professional development *"for some new ideas".* Instead she experienced a student-centred teaching style that *"completely transformed my pedagogy."* No longer content with her product-oriented mathematics classroom where students worked individually to develop fluency with procedural skills, she turned to a process-oriented model that valued conceptual understanding and collaboration. In searching for relief from her anti-goal, Amy found security in the new practices she implemented.

For other teachers, it was attending professional development that caused the development of an anti-goal. Kelly described the same sort of experiential learning from professional development as Amy but added, *"I never questioned it* [her practice] *until my eyes were opened — when I saw another way. Since then, I have felt my teaching pedagogy do a complete 180° shift."* Although she had willingly attended the professional development session, it was not due to tension with her own practice; it was more a matter of convenience and opportunity: *"It was our district ProD and it was a math topic. I was there because I was a math teacher."* Describing herself as a typical, traditional mathematics teacher, the experience provoked a desire to implement changes in her teaching as she noted:

> It was confounding to learn that something I was doing in my class was actually taking away from students' learning. It really makes you think about and reflect on what you are doing as a teacher.

Like Amy, the traditional teacher she once was became her anti-goal as she emphasized, *"I knew I never wanted to be that teacher."*

For both Amy and Kelly, their use of figurative language like *"transformed"* and *"eyes opened"* suggests the core of who they were as a teacher had been unexpectedly altered and the result was the development of an anti-goal. They may have set out to change some things about their practice but ended up changing themselves. For other teachers, this alteration appeared to be a more purposeful decision. Sam spoke of being at a *"crossroads"* where anxiety with his teaching style caused him to ponder two choices: seek out professional development or quit teaching. In the end he chose the former as he explained, *"I'm going to try out for one more year and I'm going to become better."* No mention of transformational experiences, this was a deliberate response to relieve the pressure of an anti-goal: he was not happy with who he was as a teacher and he set out to change that. This sense of deliberation appears again in David, a new teacher assigned to teach mathematics. Having never planned to be a mathematics teacher, he first turned to colleagues for advice on what to do:

> I asked them, how do you teach math? How can I make this fun? And they're...like, I hate to say it, but they're older teachers, and they have very traditional views on math, and they kind of do it how I was taught math. They just work on the problem on the board, show them how it's done, and get them to practice, practice, practice until they get it. And I knew that's not how I wanted to do it. That's not who I wanted to be.

Although David had not yet developed a mathematics pedagogy, he knew who he did not want to be as a teacher. This anxiety led him to sign up for a series of professional development sessions that focused on progressive teaching practices in mathematics.

And, over time he implemented the strategies he learned in his classroom. Again, there is less a sense of an unexpected transformation and more of a determined decision to avoid an anti-goal.

Like the others, Corey had implemented new practices in her classroom that required changes not only in the physical movements of her students but in her own as well. She mentioned, *"Physically the vertical learning can be challenging for me. I struggle to stand for the whole day, so I have to make sure I'm doing a mix of things throughout the day."* During the interview, she let this thought be and then came back to it unexpectedly about 10 minutes later as she further explained, *"I just don't want to be that teacher."* When asked to clarify, she added:

> Because I struggle to stand. I don't ever want to be that teacher that sits at the desk all day, because that's not effective at all. I think if it's this bad, I'm 43, what am I going to do five years from now? Six years from now? How's it going to look? That's something that keeps me up at night. How am I going to best serve these kids when I can't move around the room? So, yeah, it's a concern. That's one of the reasons I might not always be a classroom teacher; it might not be an option for me physically, to do a really good job of it.

There are two things to note here. Like Kelly and David, Corey's use of the adjective/noun combination '*that* teacher' suggests she has developed a schema of what a teacher is and is not. This sets up an anti-goal as she knows what kind of teacher she does not want to be, and despite the tension that results from worries over her physical limitations, she does not veer from that. Second, it is interesting that while Corey does later mention solutions such as a *"motorized scooter"* or *"mixing things up",* moving away from the new practices that are taxing her physically is not mentioned. Like Sam, it seems she would rather leave the profession than move towards her anti-goal state.

RECONNECTING WITH AN ANTI-GOAL

Traditional mathematics practices comprise universally accepted norms such as teacher-led examples, individual seat work, and silent practice that are especially difficult to displace. Such a strictly controlled environment offers the illusory appeal that serious learning is taking place. This notion is embedded in the mathematical backgrounds of the teachers in my study for whom the pull of traditional practices lingered. This created anxiety and fear for those attempting to suppress these desires and for those who succumbed. Lily recalled that in her early teaching career she believed that, *"The quieter the class the more I thought learning was happening."* She had come to recognize that this is not the case, yet acknowledged:

> I do on occasion go back to this method because of a bad day or I am not prepared. When I do go back to this traditional method, I am aware that it was not a good teaching day for me or the students.

This created anxiety as she realized that her decision, while satisfying her immediate needs, had unintended consequences for both her and her students. Interestingly, this notion of being unprepared appeared to be the impetus for several others who also return to traditional practices to satisfy their own needs. As Kelly recounted:

> So today I sort of reverted. I have not been feeling great and I needed something quick and easy to put together for a lesson. I started the class with a review/notes of all the topics we have been doing. We did some examples together on the board then I gave them a worksheet. This class has rarely come into the room to see desks and chairs set out that are available to sit in. But today I caved. I was hoping for some quiet time while they worked.

This backfired for Kelly as she later admitted, *"For the most part I spent the rest of class going from one student to another with hands up helping them with problems."* Like Lily, her anxiety lay in knowing that her decision to 'revert' had had unintended consequences for both herself and her students. It appears that the challenge of implementing change can occasionally nudge teachers towards that which was once familiar and therefore seen as easier. Hoping for a respite, they instead experience the emotion that accompanies a move towards an anti-goal.

This return to the familiar also occurred for several teachers not because it was easier but rather, they missed the reassurance of traditional teaching. This created anxiety for them as they struggled to suppress this need. Linda mentioned wanting to be sure she was covering the content since she implemented the changes in her classroom:

> I still occasionally like to start by demonstrating something new and then having students do similar problems or problems connected to what was demonstrated. This comforts the 'conventional' teacher in me, but I do feel like it is cheating or missing the point.

This need for reassurance is also apparent in Diane who mentioned occasionally returning to her previous teaching practices:

> I really want to make sure that everybody's learning. When they're quiet and they're all looking at me I know I have their attention. I'm not sure if everybody is paying 100% attention when they're working in the problem-solving groups.

When speaking later of year-end assessments she added, *"I know I don't need to do it* [teach traditionally]. *I know I shouldn't. They all did so well that it solidified for me that the way I was doing it was already working."* This suggests that anti-goals serve another purpose. Teachers might purposefully move towards an anti-goal in order to experience the relief it brings when they move away. In essence, they are reconnecting with their anti-goal in order to affirm the changes they are making in their practice.

CONCLUSION

Anti-goals develop during teacher change as teachers come to recognize and articulate who, and how, they *do not* want to be in the mathematics classroom. For some, this process occurs during change, for others this recognition propels them to seek out ways to change. In either instance, I suggest anti-goals are useful in three ways. First, having teachers reflect on the emotions they feel may be useful in reasoning why they felt this way and how they might use this knowledge to their advantage. Doing so allows teachers to focus on the cognitive source of their difficulties rather than being overwhelmed by the emotional symptoms. For example, a teacher who can connect the anxiety she experiences to the action she is undertaking, can take steps to alter the action. Second, I suggest that recognizing what one *does not want* to be brings into sharp relief what one *does want*. Having that clarity might enable teachers to seek out the actions and changes that will help them reach that goal. For example, a teacher who realizes she does not want to be *that* teacher who only uses unit tests for assessment may look for learning opportunities that broaden her assessment practice. Finally, anti-goals also prove useful in keeping change alive. Teachers who find themselves pulling back from the changes they have implemented, find in the emotional reconnection with their anti-goal the encouragement or reinforcement needed to continue with the change. As Zembylas (2005) suggests, "Teaching practice is necessarily affective and involves an incredible amount of emotional labor" (p. 14). Harnessing that emotion during teacher change may prove valuable for teacher educators.

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EXPLORING THE BENTWOOD BOX: COLLABORATION IN LESSON DESIGN AND IMPLEMENTATION

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This work resulted from an attempt to collaboratively design and implement a lesson with the lens of First Peoples Principles of Learning in a high school Mathematics 8 class in 2018-2019 academic year. We describe the process, outline the key objectives and challenges in both design and implementation stages. We also discuss the reflections and the learning observed by teachers as designers as well as learners along with the students. The analysis of noted teachers' experiences and observations showed the complexity of the challenge to incorporate the indigenous ways of learning into teaching practice is substantial among mathematics teachers and the lack of knowledge in the subject matter remains large. We discuss possible approaches of bridging the gap between the current state of First Peoples Principles of Learning to what it is mandated to be by the Ministry of Education in British Columbia mathematics classrooms.

INTRODUCTION

In 2006/2007, the BC Ministry of Education partnered with the First Nations Education Steering Committee (FNESC) to create the English 12 First Peoples course. FNESC was founded in 1992 and works at the provincial level to provide services in the areas of research, communications, information dissemination, advocacy, program administration and networking (FNESC, 2019). The development of this curriculum included a significant input from Indigenous knowledge-keepers and educators from BC and was unique in a number of ways. The Indigenous Elders, scholars, and knowledge-keepers helped to ensure that the course was able to authentically embody aspects of First Peoples' values around teaching and learning. This meant that the course had to take into account not only authentic First Peoples content, but also reflect First Peoples' epistemology and pedagogy. Second, it included the development of the First Peoples Principles of Learning (FPPL), which will be elaborated on further in the paper (FNESC, 2019). The development of the FPPL has had a major influence in the writing of the new BC K-12 Curriculum, as the necessary curricular competencies now include elements of indigenous ways of knowing and learning (BC Ministry of Education, 2019). This created a new avenue for explorations for teachers and potentials opportunities for learning. Thus, another avenue of interest in this paper is teacher collaboration in lesson design and implementation. There is a growing body of research confirming that participation in more collaborative professional communities impacts teaching practices and improves student learning (Vescio, Ross, & Adams,

2019. In A. Hare, J. Larsen & M. Liu (Eds.). *Proceedings of the 14th Annual Mathematics Education Doctoral Students Conference* (pp. 123-130). Burnaby, Canada: MEDS-C.

2008). Among many positive results, one stands out that more cohesive professional community correlates with an observable increase in the use of more desirable kinds of pedagogy (Louis and Marks, 1998). Such positive outcomes are suggested to occur because more collaborative working conditions can lead whole departments or even whole schools to develop shared norms, values, practices, and orientations towards colleagues and students (Achinstein, 2002; McLaughlin, 1993; McLaughlin & Talbert, 2001; Westheimer, 1998). Teachers working in cohesive and highly collegial professional communities also report high levels of commitment to teaching all students, high levels of energy and enthusiasm, and high levels of innovation (McLaughlin, 1993). Now with these two orientations in mind, we will present a project where three teachers engaged in the design and implementation of a lesson in a Mathematics 8 classroom where the focus was on the following Curricular Competencies (BC Ministry of Education, 2019):

- Incorporate First Peoples worldviews, perspectives, knowledge, and practices to make connections with mathematics concepts.
- Engage in problem-solving experiences connected with place, story, cultural practices, and perspectives relevant to local First Peoples communities, the local community, and other cultures.

THE PROJECT

As part of a BC Ministry of Education Open School project, a group of 14 teachers were selected to create lessons that would showcase the new mathematics curriculum. There were a number of stages to the project: initial planning, writing, editing, implementing and videoing, editing of the footage, and finally releasing the videos as resources for other teachers in the province. Initial planning stage took place in the BC Ministry of Education office in Victoria, BC. Teachers self-selected the topics of interest for the lessons and the planning and design process began. Teachers were collaborating in the planning of the lessons, with ongoing check-ins of sharing between groups. In what follows, we will outline the framework, discuss the observations and teacher interviews, as well as offer analysis with possible implications to lesson design, particularly dealing with the FPPL orientation.

THE STUDY

This study positions itself at an intriguing crossroads of teacher collaboration and new curriculum implementation efforts with the FPPL at the centre. The collaborative nature of lesson design has been a well-researched topic in the mathematics education community. At the same time, First Peoples Principles is a much less explored avenue with the mathematics education lens. The new British Columbia mathematics curriculum calls for “Engaging in problem-solving experiences connected with place, story, cultural practices, and perspectives relevant to local First Peoples communities, the local community, and other cultures”, as well as “Incorporate First Peoples worldviews, perspectives, knowledge, and practices to make connections with mathematical concepts" (BC Ministry of Education). Furthermore, the following are the First Peoples Principles of Learning as put forward by FNESC, taken from their website and frequently found on the walls of many BC schools and classrooms:

- Learning ultimately supports the well-being of the self, the family, the community, the land, the spirits, and the ancestors.
- Learning is holistic, reflexive, reflective, experiential, and relational (focused on connectedness, on reciprocal relationships, and a sense of place).
- Learning involves recognizing the consequences of one's actions.
- Learning involves generational roles and responsibilities.
- Learning recognizes the role of indigenous knowledge.
- Learning is embedded in memory, history, and story.
- Learning involves patience and time.
- Learning requires exploration of one's identity.
- Learning involves recognizing that some knowledge is sacred and only shared with permission and/or in certain situations.

We have decided to take both frameworks and look at the data we collected from the participating teachers with the above-mentioned findings and principles.

ENVIRONMENT

As mentioned above, there were several stages in this project. The first stage took place in Victoria, BC at the office of Ministry of Education. The teachers who volunteered their time to participate were instructed to design their lessons with the new curriculum in mind, in particular the curricular competencies. The mandate of the Ministry of Education was to step away from the content-centred lesson planning to a more curricular competencies-based orientation. A group of three teachers came together to plan, design and implement a mathematics lesson with the focus on the curricular competencies that deal with the above-mentioned First Peoples Principles of Learning. During the planning stage of the lesson design, the teachers brainstormed ideas, narrowed down topics and potential plans of implementation to agree to focus on the bentwood boxes as an indigenous artefact and mathematics that can be discussed pertaining to the making, the stories, and places connected to the bentwood boxes. After the initial planning stage was complete, the teachers summarized it in the cleaner version of the above whiteboard snapshot. This was done on paper, with a noticeable structure of a flow chart. The idea of recording the summary of the discussion as a flowchart had emerged organically and continued to be present in the subsequent stages of the project. At the next stage of refining and writing up of the lesson, one teacher was responsible for producing an initial activity plan followed up by the feedback of the other two members of the team. This involved mainly back and forth email correspondence with an occasional skype meeting session when needed. Once the edits of the above-mentioned document were complete, one teacher volunteered to teach this class. First component of this class was done at the school: students were shown a video where a wood bender is making a bentwood box from the piece of wood with every stage of the process explicitly demonstrated. One thing to note here is that this video appears to be a test of patience and resilience from the audience as it does not offer instant gratification: it takes time to observe the progress. The process of making of the bentwood box is a tedious and meticulous one and the video demonstrates every stage in nearly exact time. This aspect aligns well with one of the FPPL, namely: "Learning involves patience and time." Furthermore, it was decided that the connections with the story and place should play a central role in the lesson implementation and thus it was agreed that one portion of the lesson will be delivered through the experience of visiting a museum with the artefacts of actual bentwood boxes. The students were given the tour of the museum with the emphasis on the stories of bentwood boxes. Not only the boxes were discussed from the mathematical point of view, such as the process of making and practical usage, but also from the perspective of story and place, such as the artists and owners and their families, particular anecdotes and stories that are inevitably a part of the uniqueness of every box presented to the students. A whole class discussion followed upon the return from the museum, showing evidence of students' authentic interest and appreciation for the complexity, ingenuity, and creativity of the box makers and artists.

PARTICIPANTS

Three teachers involved in the project and the design and implementation of this lesson were interviewed about the process. In what follows, we give a brief summary of each individual and their involvement in the project.

(1) Mike. Mike teaches mathematics in the rural area of British Columbia. He is very involved in the mathematics education community and often takes part and organizes professional development for the teachers in his district. Mike is very well-versed in the areas of Numeracy teaching and learning as well as computational thinking.

(2) Nathan. Nathan teaches mathematics at a Lower Mainland independent school. He is also an education coach at his school and closely works with faculty on improving instruction, collaboration, communication and research in mathematics education. Nathan is a graduate student in the mathematics education doctoral program.

(3) Sally. Sally is an assistant professor of teacher education at a BC university. She has been a very active member of mathematics education community, FNESC, professional development around the province and beyond. Sally is a pioneer in many initiatives in mathematics education, particularly the ones dealing with the FPPL ideas and notions.

METHOD AND DATA

As outlined earlier, our interest was situated in teacher collaboration and FPPL integration into lesson planning and implementation. The participant teachers were interviewed over email with the following questions, adapted from the Ministry of Education post-filming questions:

1. Please provide a brief description of the activity including curricular competency and course(s).

2. What were some considerations when designing the activity?

3. What did you notice about the shift in designing with curricular competencies in mind in addition to content?

4. Did you make it explicit with students about how to progress with respect to the curricular competencies? If so, how did you accomplish this?

5. Any advice for colleagues on ways to implement the new curriculum in their math classes?

6. Are there things you'd like to change the next time you use the activity?

7. Are there any extensions or adaptations you can think of to do with this activity?

8. Any other reflections?

All interview data was collected and after reading and coding all the responses, three themes were identified that surfaced from looking at the data. The emerged themes are as follows: (1) **Importance of Story and Place**; (2) **Teacher Collaborative Experience**; (3) **Evidence of Learning**.

Due to the limits of this paper, we will only present two of the three themes. Below is the data created with the first two themes heading each set of participants' responses:

Theme 1: Importance of Story and Place

(1) Mike:

> … permission to discuss, explore and use another cultures property such as art, crafts.
>
> … seeking the expertise of an outside expert I felt during this activity I was just like my students and I needed to listen, reflect, question, and decide where I wanted to go next.

(2) Nathan:

> …and, the focus was on the place and the setting, which we thought, well, we can talk about those things in the classroom, but we [teachers] thought it would be a great idea to actually step away from there and actually be in the place where we'd be surrounded by the artefacts, and having people who have knowledge in the subject matter be leading guides for this.
>
> …where's the mathematics in this? I purposely left it for later because I wanted them [students] to have the experience with the actual, the history and the components that have to do with, not just the object itself, but everything that surrounds it, the stories. And, today was a wonderful experience of that, they really got to see and got to learn from the expert about that.

…it was sort of a collective decision to move outside of the classroom because the place and setting play such an important role in those curricular competencies we're looking at, and we thought about maybe we'll bring someone in into the classroom, but we thought it's actually probably better if we can find a place where we could go, where it'd be more authentic, in a sense, that we're surrounded by things we're looking at.

(3) Sally:

… being authentic to the First Peoples, the culture, and context of the bentwood box; focus on the bentwood box first, then mathematics

…learning in place... although we could gather videos and links to the bentwood box... being on the land was critical to the learning experience

… attaching story and history to the bentwood box provided more meaning and purpose to the learning activity that extended beyond the calculation of surface area and volume, or maximizing or minimizing

… students were learning in place, through story, and experientially with and about the bentwood box.

Theme 2: Teacher Collaborative Experience

(2) Nathan:

… I think the whole experience of, it taught me a whole lot. From the first day when we gathered in Victoria to make the lesson, and then the communication that was happening in between when we were done with that. And the topic we particularly tackled, I didn't have any expertise, not to say that I have expertise now, but at least I know a little bit more because I was able to engage with it, it's such an invaluable experience by working together. So, working with other people has been an immense, we don't get to do that very often in our practice because everybody's busy and doing their own thing, but it was extremely valuable to talk to people I normally don't ever get to talk to because Mike is up north and Sally is as well, and we don't ever get to really meet in person. But, that was, three of us bringing in our experience and our knowledge together, made really a huge difference because I don't think any one of us would be able to accomplish what some of the ideas that we were able to come up with.

(3) Sally:

… we were operating as a collaborative team (i.e. Sally, Nathan, Mike) to co-design the learning activity; being humble in the process

… BE THE LEARNER. Collaborate with others. Invite community members in. Go find local knowledge keepers and elders to help facilitate the learning. Co-teach

… This was an incredible learning experience and collaboration working with Mike and Nathan. I loved working with colleagues/mathematics teachers who were equally excited, curious, and scared to learn more about embedding Indigenous Education, First Peoples curricular competencies, and First Peoples Principles of Learning into secondary mathematics. We asked tough questions of ourselves and our practices.

THEMES AND ANALYSIS

In this section we will elaborate on the observations from the data on the three themes. It was very satisfying that these three themes emerged so clearly from all the teachers, as the interviews were done over email and the responders never got to see their colleagues' answers. Furthermore, the themes did not directly follow from the questions that were asked.

First, we turn our attention to the Importance of Story and Place consideration and the opinions about it. Clearly, the three teachers decided to highlight it, thus allowing to conclude that they found it important and central to the activity. Mike, Nathan and Sally all comment on the crucialness of having the activity outside the classroom and "in the land", "surrounded by the artefacts" were what made it successful. From these positive reflections we are confident that our original idea to make the place and story central to the activity was a worthwhile endeavour. This showcases the ability of teachers to implement the prescribed curricular competencies and echoes the FPPL, namely: "Learning is embedded in memory, history, and story" (FNESC, 2019). For the second theme of Teacher Collaborative Experience, one can easily identify the elements of positive experience in lesson design and implementations. Teachers are telling us that they learned as they engaged with the process, thus improving their teaching repertoire and teacher knowledge, which corresponds well with improving teaching practices with collaborative approaches to instruction (Vescio, Ross, & Adams, 2008). Furthermore, we can say that our data shows that the 'unlikely' collaboration from the teachers who normally do not work together on a frequent basis goes hand in hand with the cross-department, cross-school working together to produce positive outcomes in teaching and learning (Achinstein, 2002; McLaughlin, 1993; McLaughlin & Talbert, 2001; Westheimer, 1998).

CONCLUSION

As we gather from all three participants of this study, this experience was very rewarding and positive. All three participants spoke highly of usefulness and how ground-breaking and valuable the collaboration was for them. There is also an element of teacher education and professional development embedded in some of the potential extensions experiences such as this could offer. We particularly were pleased with the positive outcomes of this study as the topic and integration of First Peoples Principles of Learning is still a challenge for many educators in BC. This study clearly showcases one possible approach to this challenge and hopefully is an inspiring example for other educators to try to engage with the Indigenous ways of learning and teaching. We would like to conclude with the quote from Sally, one of our teacher-participants:

> "I feel that we had a shared mindset on how we could approach this learning activity. We were willing to compromise and listen. And, there were a few moments where we caught ourselves in our cultural bias, shifted our thinking, and made amendments accordingly. We took risks and were willing to extend ourselves to do something different. This is a

friendship and collegial relationship that will be career long and it's been one of my most memorable in my 25-years of teaching. Thank you for this opportunity. I believe that serendipity brought us together and I hope that other math educators can do the same to move forward in our practices together".

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TALKING IN MATHEMATICS – DO WE KNOW HOW?

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The notion of talking in mathematics or what is more commonly referred to as spoken communication in mathematics classrooms has been an increasingly important yet demanding task for both students and teachers. Specifically teachers face the challenge of orchestrating and facilitating meaningful mathematical talks with and for their students. As an in-depth literature review of the notion of spoken communication in mathematics classrooms, this paper serves as a preliminary exploration to address what teachers need to know or do to help students develop their mathematical spoken communicative competence. A possible framework which may explicate why and how spoken communication (or mathematical talk) can contribute to mathematics teaching (and learning) is proposed as a result of this exploration.

MOTIVATION FOR THIS EXPLORATION

The notion of spoken communication, especially in the form of discussions between two or more people, seems awkward in mathematics education, as mathematics has always been considered culturally as an impersonal subject with universal mathematical truths where there is no need for any discussions to make any personal or collective meaning on what is being learned or taught (Bishop, 1991). Consequently, most people, including teachers, are more familiar with the communication of mathematical ideas in the written form of a solution to a problem for the purpose of teaching and assessing if one has learned and understood the mathematics ideas taught.

Hence, with communication being highlighted as a process standard in The Curriculum and Evaluation Standards for School Mathematics by the National Council of Teachers of Mathematics (NCTM) in 1989 (NCTM, n.d.), the notion of communicating mathematically (Pimm, 1991) has since become an even more demanding task for both students and teachers. It seemed to have brought both clarity and confusion to how the notion of communication (from written to spoken, or even drawings or gestures) should be considered as a process in the teaching and learning of mathematics. On the one hand, NCTM defines mathematical communication as “a way of sharing ideas and clarifying understanding” (p.4) in both spoken and written forms which is important as students are given opportunities to reflect, refine and share ideas clearly and precisely using mathematical language, thus emphasizing the value of communication in developing mathematical thinking. On the other, their deliberate focus on the use of conversations or discussions in problem situations where “mathematical ideas are explored from multiple perspectives” so as to “sharpen thinking” (p.4) may have been misunderstood by many that mathematics can be learned simply through conversations,

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i.e. communication in the form of conversations develops mathematical thinking - an idea which was contested by Sfard, Nesher, Streefland, Cobb and Mason (1998).

While some teachers resist this notion of communication as they are not prepared to teach mathematics differently from the drill-and-practice way, many have swarmed towards it by changing their classrooms to embrace some form of spoken communication or talk of mathematical understanding, believing it will develop mathematical thinking as NCTM (1989) suggested or on the premise of promoting student-centric learning. However, the role of spoken communication and how it should look like to effectively help in the development of mathematical thinking did not seem clear. The biggest assumption for the belief that spoken communication can help to develop mathematical thinking is the ability of students to know how and what to communicate in the mathematics classroom, i.e. mathematical communicative competence was assumed to be a given (Adler, 2002; Pimm, 1987; Sfard et al., 1998) when it is the exact opposite.

As Sfard et al. (1998) further argued, it is “an extremely demanding and intricate task” (p.51) for conversations (either orchestrated or unintentional) to be meaningful or productive in the mathematics classroom when students, and probably even teachers, need to be taught how to tap on communication to learn mathematics. Thus, there is a need to further understand the value and process of spoken communication in the mathematics classroom; and its corresponding implications on the teaching (and learning) of mathematics. With the limited scope for this paper, the focus here explores what teachers need to know or do to help students develop their *mathematical spoken communicative competence* - ability to communicate ideas or make meaning verbally in the appropriate language, i.e. mathematics register, in the context of mathematics discussions (Pimm, 1987) - which “cannot be taken-for-granted” (Adler, 2002, p. 10).

SPOKEN COMMUNICATION IN THE MATHEMATICS CLASSROOM

Perhaps due to the NCTM’s standards (1989), the level of research interest in the area of spoken communication in mathematics classrooms has increased over the years (e.g. Durkin & Shire, 1991; Morgan, Craig, Schuette, & Wagner, 2014; Moschkovich et al., 2018). Studies have looked into students’ spoken communication, such as mathematics discussions, conversations or discourse, as part of research on language and communication in mathematics education. While this may implicitly lend support to the value of spoken communication in the teaching and learning of mathematics as suggested by NCTM, more clarity needs to be provided on how spoken communication actually develops mathematical thinking (Sfard et al., 1998) and how this can look like in the classroom.

Talking in mathematics - The value and form

Based on earlier literature, Sfard et al. (1998) described three common arguments in support of the value of spoken communication in mathematics teaching and learning -

namely the cognitivist, interactionist and neo-pragmatist perspectives. Yet they challenged the mathematics education field to reconsider whether or not these three perspectives indeed explain why and how spoken communication adds value to mathematics education. Some questions included thinking about the mechanism behind communication which develops mathematical thinking (cognitivist perspective); how a community of practice in the classroom supports mathematics learning through conversation when it does not really reflect how mathematicians work (interactionist perspective); rethinking if knowledge can be simply equated with conversation (neo-pragmatist perspective). Seemingly, the basis of each perspective may just be its own pitfall in trying to advocate the role or value of spoken communication in the learning of mathematics.

In addition, what constitutes spoken communication in the mathematics classroom continues to be ambiguous. While researchers tend to use words such as mathematical *conversation*, *discussion* (as used by NCTM, 1989; n.d.), *talk* or even *discourse* (which has grown to be one of the favourite terms in recent years although Ryve (2011) has found the concept to be unclear in its use in mathematics education), are they simply terms that can be used interchangeably to refer to the same process? Are teachers or researchers clear about what and how they should look like in the mathematics classroom? Do teachers suppose that the process of putting a group of students together in the classroom to talk or discuss about a mathematics problem constitutes the kind of spoken communication leading to the development of mathematical understanding, as envisioned by NCTM (1989; n.d.)? Are these the only forms of spoken communication in the mathematics classroom which will aid mathematical learning? These are some questions which need to be further explored, particularly if spoken communication is to be effectively used as a teaching strategy to learning mathematics.

Talking in mathematics - The "language"

Beyond the value and form of spoken communication in the mathematics classroom, talking in mathematics, on its own, is an intriguing idea. It implies that mathematics is a language, analogous to natural languages such as English, Mandarin, a claim that many mathematics education researchers (e.g. Wheeler, 1983; Pimm, 1987) may disagree with. This confusion may have arisen due to the need for mathematics to be communicated in or through a natural language (e.g. English in many current curriculums), in order to express mathematical ideas. Instead, a more apt representation of the so-called "mathematical language" should be the *mathematics register* which involves a unique use of words and structures (both written or oral) in a natural language, e.g. English, to express "the set of meanings that is appropriate to" the mathematics discipline, i.e. "the mathematical use of natural language" (Halliday, 1975, p. 65). However, it is not simply a collection of mathematics-related words or terms - which seems to reside in the surface understanding of the "mathematical language" (Halliday, 1975). Specifically, the mathematics register determines how these words or terms are used or structured, in conjunction with the everyday language to form unique phrases or clauses that can precisely represent both explicit and implicit mathematical relationships or ideas (Schleppegrell, 2007; Wilkinson, 2015).

Notably, the mathematics register has been developed as a rich language resource to support students' acquiring and learning of ways of mathematical thinking (Wilkinson, 2015), which is crucial in the teaching and learning of mathematics. Yet, it is a pity that students and even most teachers do not seem (or do not have the opportunity) to fully grasp or develop fluency in the register as part of the learning and teaching process due to gaps in their awareness and understanding of the mathematics register.

AN IDEA TO TALKING IN MATHEMATICS

With such ambiguity in the purpose and form of spoken communication in the mathematics classroom, it is necessary to revisit the intent of spoken communication in the mathematics classroom to better clarify what and how it may look like. With regard to the notion of *communicating mathematically*, Pimm (1991) suggested how spoken communication can be considered as the pathway to written communication if used purposefully with the intent of acquiring the mathematics register. In the process, students can be guided to move from informal mathematics talk, using everyday language, to formal ones, using the mathematics register, before progressing to formal mathematical writing which is deemed difficult but valued in school mathematics (Adler, 2002). As there is great independency between the development of mathematical understanding and the use of the mathematics register, Pimm's suggestion helps to illuminate the purpose of spoken communication in the mathematics classroom.

As for the form of spoken communication in the mathematics classroom, Barnes' (1976) studies on classroom talk are a possible source of reference in providing a frame to understand classroom talk which contributes to learning. Particularly he identified two types of talk, namely exploratory talk and final-draft talk (c.f. expository talk coined by Crespo, 2006), which can be structured to bring about different learning outcomes. In his research, Barnes noted that exploratory talk happens when students are in the process of surfacing and refining ideas (usually with informal language) while final-draft talk happens when students are presenting or sharing ideas which have been thought through in advance (usually with more formal language).

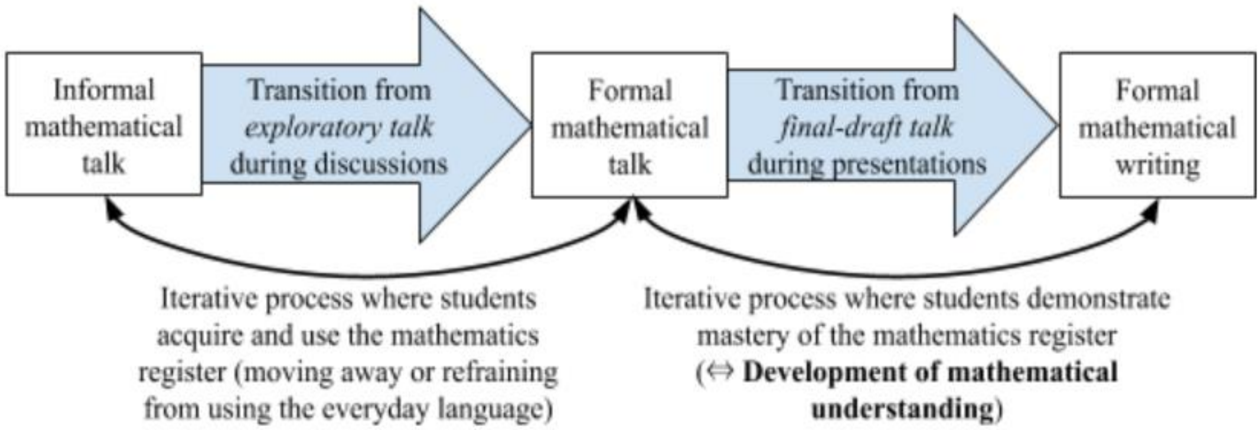

Figure 1: Spoken communication as a process in mathematics classrooms.

Based on the ideas from both Pimm (1991) and Barnes (1976), Figure 1 is my preliminary attempt to propose a framework which integrates both sets of ideas, with the intent of explaining why and how spoken communication (or mathematical talk) can contribute to mathematics learning. While it may not fully explicate the value and process of spoken communication in the mathematics classroom, I am hopeful that this idea can be further explored and refined through future research with teachers for a start, with the intent of answering the questions raised in the earlier section of this paper.

POSSIBLE IMPLICATIONS FOR TEACHERS

With the framework (Figure 1) illustrating the possible potential and value that spoken communication may bring to mathematics learning, the next step will be to think about integrating it meaningfully into the teaching and learning process. Particularly, it is important to note that for the process of spoken communication to achieve learning outcomes (as in Figure 1), it very much depends on students' mathematical spoken communicative competence (Adler, 2002; Pimm, 1987; Sfard et al., 1998), an ability which should not be assumed, but rather explicitly taught or developed with the help of teachers. Putting students in groups to talk or discuss does not necessarily equate to any mathematical thinking or learning simply because the context or task is mathematics related. Teachers need to play a very crucial role in ensuring that mathematics learning takes place during the process of spoken communication in the classroom. However, this is probably what challenges or deters teachers from tapping into communication as a strategy in the mathematics classroom. It requires a different set of mathematical knowledge for teaching (Ball, Thames & Phelps, 2008) to orchestrate and facilitate mathematics talk in the classroom (Sfard et al., 1998), unlike what is required in a traditional or direct teaching classroom.

Firstly, to ensure that spoken communication is effective in developing mathematical understanding, teachers need to purposefully design learning tasks. Students need to have the chance not only to explore ideas through informal mathematics talk, but also to present final-draft ideas through formal mathematics talk. Secondly, throughout the learning process, teachers will need to mediate at appropriate times, such as to provide support on the use of the mathematics register; to probe or prompt students with questions which will help them review and refine their ideas or; even to tap into students' final-draft talk to bring about richer discussions, etc. As such, this entire process of orchestrating and facilitating mathematics talk in the classroom is certainly not easy as there are many times when teachers need to review their lesson plans or make impromptu decisions to best support students' learning, based on what they notice during or even after the teaching process.

In particular, Adler (2002) identified three possible teaching dilemmas which require teachers' decisions in this process, namely:

- *code-switching* where teachers need to decide whether or when to change the language of teaching to ensure mathematical understanding (without compromising learning of the mathematics register);
- *mediation* where teachers need to decide when to intervene to validate students' meanings during the process of mathematical talk (without compromising the opportunities for them to develop mathematical communicative competence);
- *transparency* where teachers need to decide the extent to explicitly teach the mathematics register (without compromising the focus on the development of mathematical understanding).

Certainly, knowing or understanding these dilemmas may inform the practice, but it is not enough to help teachers make decisions as they facilitate the process of spoken communication in the mathematics classrooms. Moreover, these are probably not the only situations where teachers need to make decisions to ensure students' learning. For example, the rationale and choice of students' ideas from their informal or formal mathematical talk to consolidate learning is also a difficult decision many teachers have to know or learn how to make. If teachers do not use any of the students' ideas in the consolidation of learning, students may feel that their ideas are not of value and gradually not be bothered with any exploratory talk. Conversely, if teachers attempt to use all the ideas, they may be overwhelmed and lose focus of the learning outcomes.

Ultimately, the ability to make appropriate decisions when faced with teaching dilemmas as teachers orchestrate and facilitate mathematics talk in the classroom is highly dependent on their mathematical knowledge for teaching. Beyond the necessary mathematical content knowledge (including the mathematics register), teachers need to be equipped with or acquire pedagogical content knowledge (Shulman, 1986) which will inform the decisions to be made in mathematics classrooms which tap into spoken communication as a teaching and learning strategy. It may be necessary for teachers to have more opportunities to experience mathematics talks as learners themselves (e.g.

Crespo, 2006) before bringing it to the classroom to impact student learning in mathematics.

Specifically, the framework (Figure 1) can be used to frame teacher professional development activities so as to provide such experiences for teachers. Through these experiences, teachers can better understand and also reflect upon the dynamics of such processes before they orchestrate and facilitate students' mathematics talk. Mathematics talks for teachers may not always need to just replicate what students will discuss. Teachers can also come together in study groups to discuss the mathematics register (e.g. Herbel-Eisenmann, Johnson, Otten, Cirillo & Steele, 2015) or even to discuss strategies which may overcome a certain teaching dilemma they are facing, i.e. turning a dilemma or difficulty teachers face into a professional learning strategy or resource (e.g. Zazkis, 2000), etc.

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