MEDS-C 2016

PROCEEDINGS OF THE 11th ANNUAL MATHEMATICS EDUCATION DOCTORAL STUDENTS CONFERENCE

December 3, 2016

SIMON FRASER UNIVERSITY | FACULTY OF EDUCATION

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MATHEMATICS EDUCATION DOCTORAL STUDENTS CONFERENCE 2016 PROGRAMME – DECEMBER 3, 2016

08:30 – 09:00	**Welcome and Coffee**	
	HUB 8620.1	**HUB 8620.2**
09:00 – 09:35	**Sheree Rodney** Starting from scratch: An investigation of polygons	**Lyla Alsalim** Using patterns-of-participation approach to understand a high school mathematics teacher's practice
09:40 – 10:15	**Andrew Hare** Words in contexts: 'proof' and 'prove' in a course of mathematics lectures	**Darien Allan** Understanding and getting a good grade: An examination of two mathematics motives
10:15 – 10:30	**Break**	
10:30 – 11:05	**Jeffrey Truman** Algebraic and geometric reasoning preferences in adults on the autism spectrum	**Tanya Noble** Identification of surface markers for positioning of mathematics in student written discourse
11:10 – 12:15	**Plenary Speaker: Alf Coles** An enactivist story of researching the teaching and learning of mathematics	
12:20 – 13:20	**Lunch**	
13:30 – 14:05	**Minnie Liu** Students' modelling process – A case study	**Jason Forde** Maxim adherence in secondary school mathematics textbooks
14:10 – 14:45	**Annette Rouleau** Creating tension between action and intent	**Leslie Glenn** I wonder as I wander
14:50 – 15:05	**Break**	
15:05 – 15:40	**Masomeh Jamshid Nejad** Undergraduate students' perception of transformation of sinusoidal functions	**Judy Larsen** Discursive patterns in the mathematics teacher blogosphere
15:45 – 16:20	**Peter Lee** Representing mathematical learning disabilities: An analysis of a CBC radio interview on developmental discalculia	**Milica Videnovic** Oral vs. written exams: What are we assessing in mathematics?
16:20 – 16:30	**Break**	
16:30 – 17:05	**Robert Sidley** Are they getting any better at math? Reflections on student evaluation and *mathing*	**Harpreet Kaur** Young children's understanding of benchmark angles in a dynamic geometry environment
17:05 – 17:30	**Wrap up**	

CONTRIBUTIONS

MEDS-C 2016 was organized by members of the Mathematics Education Doctoral Program. The conference would not have been possible without the following contributions:

Conference Coordinators: Annette Rouleau and Sheree Rodney

Evaluation: Judy Larsen and Andrew Hare

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Snack Coordinators: Sandy Bakos, Minnie Liu, and Robert Sidley

Technology Support: Jeffrey Truman

Timers: Tanya Noble and Masomeh Nejad

PLENARY SPEAKER

Alf Coles

AN ENACTIVIST STORY OF RESEARCHING THE TEACHING AND LEARNING OF MATHEMATICS

In this talk, Alf will reflect on the methodological principles he has used in his research, drawing on examples from work in early number, teacher learning and the links between mathematics education and environmental sustainability. These principles include: the use of 'story'; the significance of 'detail' and mechanisms for seeing what is 'not' seen.

ABSTRACTS

Darien Allan

UNDERSTANDING AND GETTING A GOOD GRADE: AN EXAMINATION OF TWO MATHEMATICS MOTIVES

From the teacher's perspective, student actions run the gamut from compliant and expected to irrational and unpredictable. Yet, from an activity theory perspective, all student activity is driven by motive. This research, conducted in three secondary mathematics classrooms over one semester, explores the actions that students perform in the mathematics classroom with a goal of identifying student motive and analyzing relationships between the actions and motives. In this paper I focus on two of these identified motives: understanding, and getting a good grade. Analysis of student actions suggests two key findings: first, that in any single activity setting, student actions are a poor indicator of motive; and second, that understanding is a continuous motive, whereas getting a good grade is discrete.

Lyla Alsalim

USING PATTERNS-OF-PARTICIPATION APPROACH TO UNDERSTAND A HIGH SCHOOL MATHEMATICS TEACHER'S PRACTICE

In this paper, patterns-of-participation theory serves as a lens to interpret and understand Saudi high school mathematics teachers' practices. This framework focuses mainly on understanding what practices and figured worlds are significant for the teacher and how the teacher engages in those figured worlds. The data presented is about Noha, a high school mathematics teacher in Saudi Arabia. The data generated suggests that there are five significant practices or figured worlds to Noha's sense of her practice as a mathematics teacher. The paper discusses and explains these figured worlds.

Jason Forde

MAXIM ADHERENCE IN SECONDARY SCHOOL MATHEMATICS TEXTBOOKS

This paper extends H.P. Grice's four conversational maxims of Quality, Quantity, Relation, and Manner into the genre of secondary school mathematics textbooks. In order to determine if and how the supervening Cooperative Principle may be applicable to the author-reader relationship, the general notion of division by zero is explored across Grade 9, 10, and 12 mathematics textbooks from a single publishing group. Examining a formal definition of the rational numbers, considerations associated with simplifying rational expressions, and an introductory discussion of the tangent function in these respective textbooks reveals likely authorial assumptions *regarding students' prior mathematical knowledge, as well as certain inconsistencies in representing undefined cases connected to potential divisions by zero.*

Leslie Glen

I WONDER AS I WANDER

In this paper, I describe an informal self-study experiment during which I use an unfamiliar DGE (Geometer's Sketchpad). The use of previously identified dragging modalities in predictable ways is obvious, but I wanted to explore the possible need for an additional modality and examine how it might fit into this familiar landscape. I use the framework of instrumental genesis to explore the differences between the modality known as wandering dragging and a potential new modality I call "wondering dragging". The results of the experiment are discussed, not in terms of squeezing the new term in to the old lineup, but of creating a new "dimension" in which to use this term and making room for others.

Andrew Hare

WORDS IN CONTEXTS: 'PROOF' AND 'PROVE' IN A COURSE OF MATHEMATICS LECTURES

This paper addresses the question: "what do the lexemes PROOF and PROVE mean when they are uttered in a course of undergraduate mathematics lectures". 35 lectures of a third-year abstract algebra course were videotaped and transcribed. The transcript was broken into units called stanzas, and the stanzas into units called lines. A corpus linguistic approach to the transcript is taken, and we use the surrounding stanza as the context for our lexemes. We find that: 1. 'Proof' gets explicitly defined by the professor. 2. Two written proofs are often explicitly compared and contrasted. 3. Whether or not some argument constitutes a proof is contested on a few occasions by the students and the professor. 4. 'Proof' gets contrasted with conceptually close but distinct notions, including illustration, model for a proof, outline, and main idea.

Harpreet Kaur

YOUNG CHILDREN'S UNDERSTANDING OF BENCHMARK ANGLES IN A DYNAMIC GEOMETRY ENVIRONMENT

This paper examines young children's thinking about benchmark angles in a dynamic geometry environment. Using the dynamic sketches in Sketchpad, kindergarten children were able to develop an understanding of angle as "turn," that is, of angle as describing an amount of turn. Children experienced different realizations about the benchmark angles and showed a shift from context specific descriptions to more general descriptions. Children's gestures, motion and environment played an important role in their thinking.

Judy Larsen

DISCURSIVE PATTERNS IN THE MATHEMATICS TEACHER BLOGOSPHERE

Teacher collaboration is essential for improving teaching, but is often difficult to establish and sustain in a productive manner. Despite this, an unprompted, unfunded, unmandated, and largely unstudied mathematics teacher community has emerged where mathematics teachers use social media to communicate about the teaching and learning of mathematics. This paper presents an analysis of one episode where teachers engage in a prolonged exchange about responding to a common mathematical error. Analytical tools drawn from variation theory are used to explain generative moments of interaction. Results indicate that discursive patterns signal taken-as-shared pedagogical approaches, which can extend the space of possible variation while establishing a range of permissible change.

Peter Lee

REPRESENTING MATHEMATICAL LEARNING DISABILITIES: AN ANALYSIS OF A CBC RADIO INTERVIEW ON DEVELOPMENTAL DYSCALCULIA

On October 23, 2015, CBC radio host Rick Cluff conducted an interview with cognitive neuroscientist Daniel Ansari on developmental dyscalculia, discussing what it is, its causes and its treatments. The purpose of this paper is to apply the methods of Critical Discourse Analysis to examine the interaction between host and interviewee to see what lines of inquiry emerge. The intent is to demonstrate how the nature of the medium positions the host and interviewee in relation to dyscalculia, and how the medium represents developmental dyscalculia, those who have it, and its treatments to the CBC audience. Analysis suggests that the radio interview enables certain traditional storylines regarding developmental dyscalculia to be told while also allowing some alternative ones to emerge.

Minnie Liu

STUDENTS’ MODELING PROCESS – A CASE STUDY

In this paper I discuss a case study where two grade 8 students worked collaboratively to solve a modelling problem. Their modelling process shows that rather than closely following the modelling cycle suggested by modelling literature closely, where they develop a real model, a mathematical model, a mathematical solution, and a real solution for the entire situation and repeat the modelling cycle to improve their solution, these students broke down the modelling problem into smaller pieces and went through the modelling cycle multiple times in order to generate a realistic solution to the modelling problem.

Masomeh Jamshid Nejad

UNDERGRADUATE STUDENTS' PERCEPTION OF TRANSFORMATION OF SINUSOIDAL FUNCTIONS

Trigonometry is one of the fundamental topics taught in high school and university curricula, but it is considered as one of the most challenging subjects for teaching and learning. In the current study mason's theory of attention has been used to examine undergraduate student's perception of the transformation of sinusoidal functions. Two types of tasks – (a) recognizing sinusoidal functions and (b) assigning coordinates – were used in this study. The results show that undergraduate students participating in this study experienced difficulties in identifying a period of a sinusoid, especially when it was a fraction of π radians.

Tanya Noble

IDENTIFICATION OF SURFACE MARKERS FOR POSITIONING OF MATHEMATICS IN STUDENT WRITTEN DISCOURSE

The study explores the movement of mathematics from the classroom into the lived experience of students. The Positioning theory from Wagner & Herbel-Eisenmann (2013, 2014) is used to identify different authoritative structures within student mathematical discourse. British Columbian students enrolled in Workplace Mathematics were given the task to pose a math question of interest. The task anticipated the creation of personally relevant questions instead student responses mimicked the dominant curriculum resource. Understanding a student's authority within the discipline of mathematics is critical for the mathematical applications beyond the classroom.

Sheree Rodney

STARTING FROM SCRATCH: AN INVESTIGATION OF POLYGONS

In this paper, I report on the work of one student in a computerized environment - scratch programming-and consider ways in which children learn to assign and internalize meanings to geometrical ideas, specifically polygons. I use the Vygotsky inspired theory of semiotic mediation as an analytical lens, to show how technology tools (as mediators) enable the invention and use of signs as auxiliary means of constructing mathematical meanings. I argue that scratch programs and the potentialities provide a deeper understanding and educe creative innovations, which may not be possible in paper-and-pen environments.

Annette Rouleau

CREATING TENSION BETWEEN ACTION AND INTENT

Pre-service teachers come to mathematics methods courses with well-established conceptions of what it means to teach and learn mathematics. Images of teaching reinforced by their own lived experiences shape their pedagogy. This can be problematic for a teacher educator for whom it may be necessary to offer a way of reframing traditional notions of teaching and learning. The research presented here examines that process of reframing. In this study we deliberately introduce a tension in pre-service teachers' conception of timed drills and examine the resulting process of transition they undergo. Using a tension pairing from Berry's (2007) framework, our findings suggest that the introduced tension provided the means for reflection on intent and resulted in a subsequent change in action.

Robert Sidley

ARE THEY GETTING ANY BETTER AT MATH? REFLECTIONS ON STUDENT EVALUATION AND MATHING

Conversations with stakeholders about students' improvements in mathematics invariably focus on student grades and work habits. Further, research into improvements in mathematical performance focus almost exclusively on the acquisition of mathematical content and improvement in test scores. This narrow focus makes assumptions about what it means to know and do mathematics. By analyzing traditional evaluation data gathered from a year-long grade 10 mathematics class, I evaluate the usefulness of this data in determining student improvement and, by exploring the micro and macroculture of mathematics classrooms, reflect on the role traditional evaluation and pedagogy have in shaping how students "math".

Jeffrey Truman

ALGEBRAIC AND GEOMETRIC REASONING PREFERENCES IN ADULTS ON THE AUTISM SPECTRUM

This study examines the mathematical reasoning of college-educated adults on the autism spectrum. I aim to expand on previous research, which often focuses on younger students in the K-12 school system. In this report I focus on a case study of Joshua, an undergraduate student in science, and Cyrus, a college graduate in mathematics. The interviews involved a combination of asking for the interviewee's views on learning mathematics, self-reports of experiences (both directly related to courses and not), and some particular mathematical tasks. In particular, I examine observations related to preferences in types of reasoning that I have encountered.

Milica Videnovic

ORAL VS. WRITTEN EXAMS: WHAT ARE WE ASSESSING IN MATHEMATICS?

One of the most striking differences between Canadian educational system and most of the other European educational systems is the importance given to oral examinations, particularly in mathematics courses. In this paper, seven mathematics professors share their views on mathematics assessment, and types of knowledge and understanding in mathematics that can be assessed on written and oral exams. With the increased emphasis on closed book written examinations, there is a critical need for implementing the oral assessments in mathematics courses.

UNDERSTANDING AND GETTING A GOOD GRADE: AN EXAMINATION OF TWO MATHEMATICS MOTIVES

Darien Allan

Simon Fraser University

From the teacher's perspective, student actions run the gamut from compliant and expected to irrational and unpredictable. Yet, from an activity theory perspective, all student activity is driven by motive. This research, conducted in three secondary mathematics classrooms over one semester, explores the actions that students perform in the mathematics classroom with a goal of identifying student motive and analyzing relationships between the actions and motives. In this paper I focus on two of these identified motives: understanding, and getting a good grade. Analysis of student actions suggests two key findings: first, that in any single activity setting, student actions are a poor indicator of motive; and second, that understanding is a continuous motive, whereas getting a good grade is discrete.

INTRODUCTION

Students' actions within the mathematics class often appear to occur with little reason. In any given setting, an individual student may perform one action one day, and a different action the next, or exhibit two seemingly contradictory behaviours in two separate settings. This seemingly random or unpredictable behaviour actually has a common source, according to activity theory: motive. In fact, according to activity theory, motive is the driver of all human activity. Thus, establishing the nature of this driving force plays a role in explaining the actions that students perform, and reconciling behaviours that appear to be irrational.

What students do in the classroom is called *studenting.* The term 'studenting' was coined by Gary Fenstermacher in 1986. Initially, he describes this concept in terms of a cohort of student behaviours including "*getting along with one's teachers, coping with one's peers, dealing with one's parents about being a student, and handling the non-academic aspects of school life*" (p. 39). In essence, Fenstermacher describes studenting as what students do to help themselves learn. A later definition encompasses other behaviours such as "*'psyching out' teachers, figuring out how to get certain grades, 'beating the system', dealing with boredom so that it is not obvious to teachers, negotiating the best deals on reading and writing assignments*" (Fenstermacher, 1994, p. 1) and other similar practices.

Analyses of student behaviour support Fenstermacher's expanded definition. The process of schooling produces a number of unintended consequences, some desirable, but also many that are patently objectionable (Engeström, 1991) and counterproductive to the goal of student learning. Preliminary studies in mathematics classrooms have shown that students often act in ways that subvert the expectations of the teacher or in ways that comply with expectations[1] but are unlikely to result in learning (Liljedahl & Allan, 2013a; 2013b).

The research data that founds this particular analysis is part of a larger project, aimed at investigating two key questions: *What are the behaviours that students exhibit in different activity settings in the mathematics classroom*; and, *What are the motives that drive their behaviours?* The basis for this paper lies in the answers to the latter question. However, to put this in context I provide an overview of the entire research project in Figure 1, below.

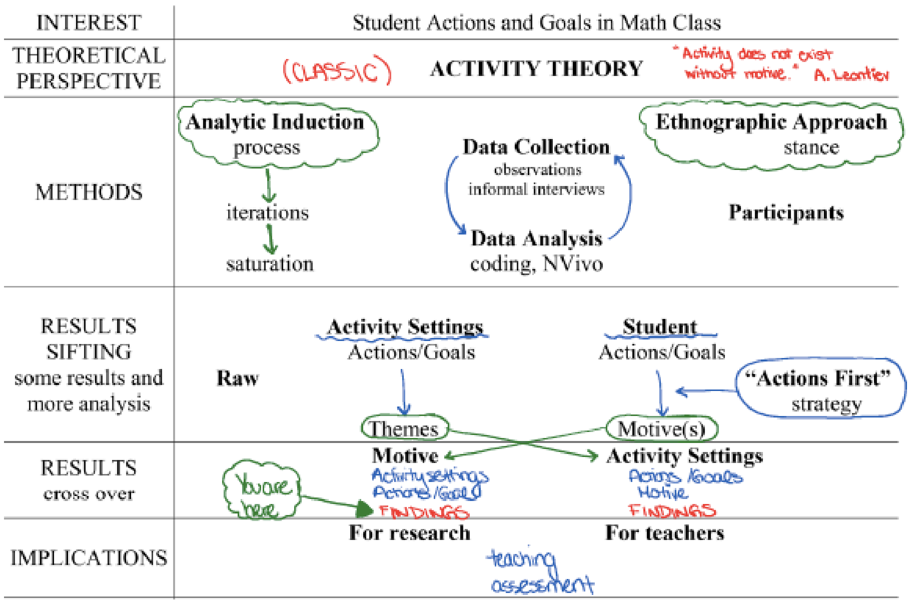

Figure 1: Overview of research.

Student motives have significant impact on their actions in mathematics class. Overall, five distinct motives emerged: *understanding*; *get a good grade*; *get credit for the course*; *pass the course/get through it*; and *avoid work and/or attention*. In this paper I now compare and contrast the actions of students holding two particular motives with respect to mathematics: *understanding* and *getting a good grade*. First, however, I provide a brief description of activity theory and the methodology I used to investigate and analyse student behaviour.

THEORETICAL FRAMEWORK

As mentioned earlier, the force that drives student activity is motive and it is the key component of Leontiev's Activity Theory (1978). As a theoretical lens, activity theory permits a description of what students do and say without overlaying pre-existing assumptions or judgments. These observations, taken together, can then be used to

[1] For the purposes of this paper the term 'expectations' refers to the actual expectations of the teacher, and/or the students' understanding of the teacher's expectations.

develop a hypothesis for the student's motive, which may be something other than a desire to learn.

For Leontiev, "*[a]ctivity does not exist without a motive; 'non-motivated' activity is not activity without a motive but activity with a subjectively and objectively hidden motive*" (1978, p. 99). The object of an activity is its motive, and is something that can meet a need of the subject. Motives arise from needs, which are the ultimate cause of human activity. Figure 2, below, illustrates the relationship between the elements in Leontiev's development of activity theory.

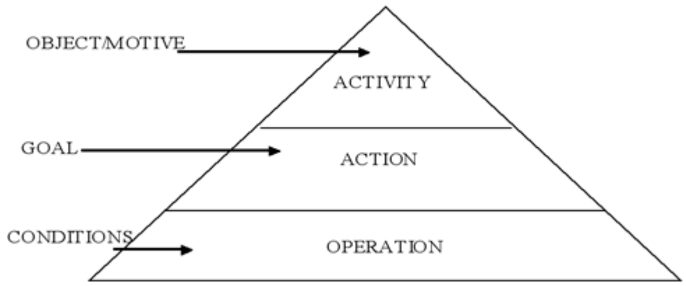

Figure 2: Leontiev's Three-Level Model of Activity (1974)

Motives sit at the apex of the triangle and drive *activity*, and activities are directed at goals. People have many goals, which shift in importance and in content on the basis of both contextual and intrapersonal factors. At any time an individual has a hierarchy of these motives, the order of which is determined through and as a result of one's activity.

Actions are the many steps that comprise an activity, although not all are immediately related to the motive (Kaptelinin & Nardi, 2012). Actions are directed towards specific targets, called *goals*. Goals are conscious, in contrast to motives, of which a subject is not usually aware.

The fact that motive is often hidden from the subject suggests difficulty in determining the ultimate motive. This obstacle can be overcome by utilizing an "actions first" strategy (Kaptelinin & Nardi, 2012). The strategy begins at the level of goals, which people are generally aware of and can express, and the analysis is subsequently expanded up to higher goals and ultimately the motive. Given the primary interest of this study is the students' actions/goals and activity/motive, only the top two levels of Leontiev's pyramid will be considered.

Student motives have significant impact on their actions in mathematics class, and hence their learning. Cataloguing the observable aspects of studenting (actions) and analysing these together with students' goals through the lens of activity theory can offer new insights into student motives and provide researchers and educators with evidence to better understand student behaviour.

METHODOLOGY

The nature of the research questions necessitates an ethnographic approach, and in this spirit, I spent significant time immersed in the classes under study observing and interacting with students, taking fieldnotes, and asking questions. Analysis occurred throughout the process whereby data collected in one lesson provoked questions and shifted focus for subsequent observations and interviews.

Participants

The data for analysis are taken from a larger study conducted in three secondary school mathematics classes in British Columbia. Three classes were observed: two at the grade 11 level (PreCalculus 11 and Foundations 11) and one at the grade 10 level (PreCalculus 10). All teachers had at least ten years teaching experience.

Data Collection and Analysis

Data were collected during the 2013-2014 school year. Throughout the fall semester the class was observed for twelve periods, each period ranging from 60 to 75 minutes. Classroom lessons and informal interviews were audio recorded and transcribed for later analysis and comparison with field notes taken during the class. Activity settings, discussed below, are used as a unit of analysis by which student actions can be organized and analysed.

Activity Settings in Mathematics Class

Activity settings serve as a unit for analysing students actions in mathematics class. Defined as units of "*contextualized human activity*" (O'Donnell & Tharp, 1990), they are the specific settings that provide the context in which activities take place and that influence the types of activities subjects are likely to encounter. According to Tharp and Gallimore (1988), activity settings are the *who*, *what*, *when*, *where*, and *why* of everyday events that take place in what Mariane Hedegaard (2012) calls *institutions*.

Activity settings within the secondary mathematics classroom include (but are not limited to): *now you try one*; *taking notes*; *problem solving*; *doing homework*; and *doing review*. Given that students behave differently in different activity settings it is of interest to investigate the nature of these particular behaviours, in what settings they occur, and what force drives students' actions.

The data discussed here has been subjected to an analysis using Leontiev's activity theory in order to determine the likely primary motive underlying the students' behaviour. Then the actions are re-examined through the lens of the driving motive.

RESULTS AND ANALYSIS

The following investigation draws from the actions of students whose motives were determined using the "actions first" strategy. Though students' motives were determined by considering their actions and goals in all settings, not all settings are represented in the diagrams or discussion.

Of the aforementioned five identified motives, the scope of this paper is limited to examining two: *understanding* and *get a good grade*. Two diagrams represent the range of actions performed by students, together with students' rationales. Activity settings and significant action choices (such as cheating and volunteering) form the main structure of the diagram; student actions branch from these nodes. The interior region holds observations and rationales of students who do not participate in activity settings; the exterior holds actions of students who did participate.

The following analysis considers each motive separately first before discussing similarities and differences.

Understanding

My teaching experience and my data supports the finding that the number of high school students who authentically want to understand mathematics is tragically small. Considering Figure 3, below, there are a number of immediately obvious features of the behaviour of a student who holds a primary motive of understanding mathematics.

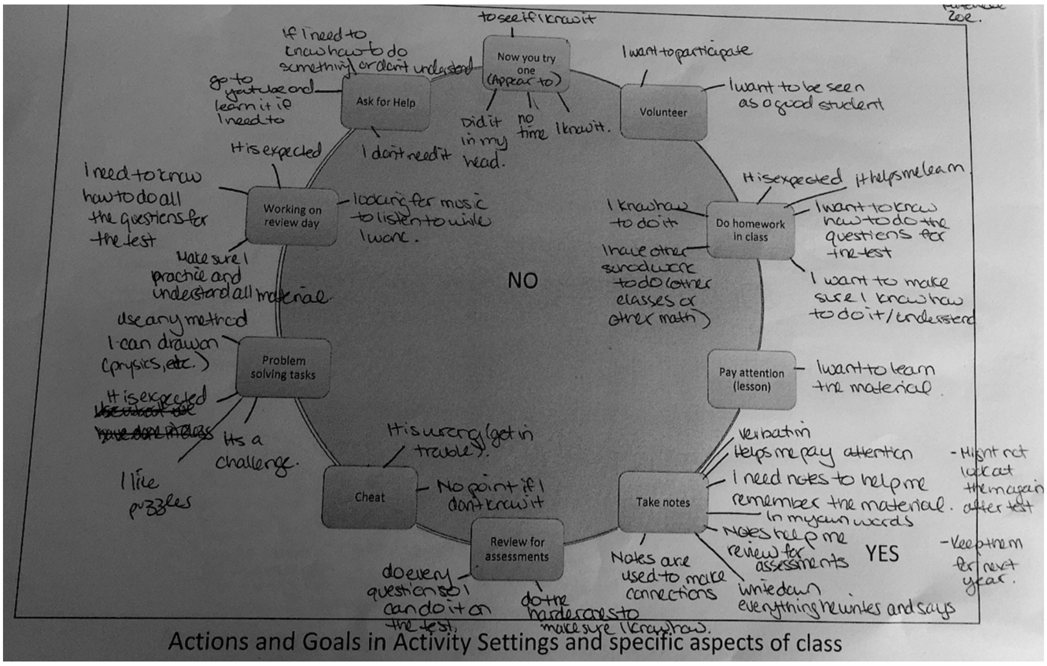

Figure 3: Actions of students with a motive of understanding.

First, there is significant blank space in the inner region; there are very few activity settings, or aspects of classroom life that these students choose not to participate in[2]. Some notable exceptions, for some students, are: *doing homework in class*, *now you try one*, *cheating*, *asking for help,* and *review day*. However, students who chose not to do homework in class communicated that it was because they knew how to do the topic already and didn't need to waste effort practicing more, or because they prioritized other work (from other classes or from mathematics class). Sometimes there was

[2] The inner region, NO, represents actions that align with 'not participating' in an activity setting or activity.

insufficient time to try examples, but other reasons for appearing to not try an example included "*I did it in my head*," and "*I already know how to do it*." Finally, it is not surprising to see that none of the students entertained cheating as a course of action. To a student with a motive of understanding mathematics there is no upside to cheating. When students displayed behaviour that might deviate from expectations, there was usually a rational reason, such as not asking for help because it wasn't needed, and not doing homework or taking notes if the material was well known. In fact, with this motive it would be odd for a student to DO homework if they already understood all of the material and were confident with it, unless he or she was trying to satisfy a secondary motive of getting marks or complying with teacher expectations. Actions that don't comply are usually justified by reasons of efficiency and necessity.

Looking to the outside of the region, some goals or reasons for participating appear quite frequently. One of these is, "*I want to learn*." Notes were taken because they were useful for review, and could be used to build from and make connections with other material at a later date. Students with a motive of understanding engaged with the problem solving tasks. For some it was a challenge and several stated that they enjoyed doing puzzles.

Get a good grade

There is a much larger contingent of students who are motivated primarily to get a good grade in mathematics. At first glance there are two things that are immediately clear in Figure 4. First, there are not only a great number of actions, there is also significant variety among their behaviours. Second, the center region of the diagram is far from empty; there are many actions that don't comply with expectations.

Delving into the content represented in the diagram, although the interior is heavily populated with reasons such as distracted, lazy, and not worth marks, many 'valid' reasons are also present for not exerting effort to pay attention to a lesson, not doing homework in class, or not attempting the *now you try one* examples. Some of these justifications include: "*I know it*," "*I'm doing work for another class,*" "*I'm doing an assignment that needs to be completed for this class,*" "*not enough time is given,*" and "*I'll do the work with my tutor later*," or "*I'll do it at home*."

Another point of interest is the appearance of cheating as an admitted or potential action. Students confessed to cheating as a means to confirm answers, get a hint to help remember, or because the marks were needed. Those who did not cheat explained that it was too difficult to do so and they would get caught; or they would feel guilt; or it wasn't worth the risk of getting caught if the gain in marks was not at a certain threshold; or, it wasn't worth it because they would still have to learn the material later.

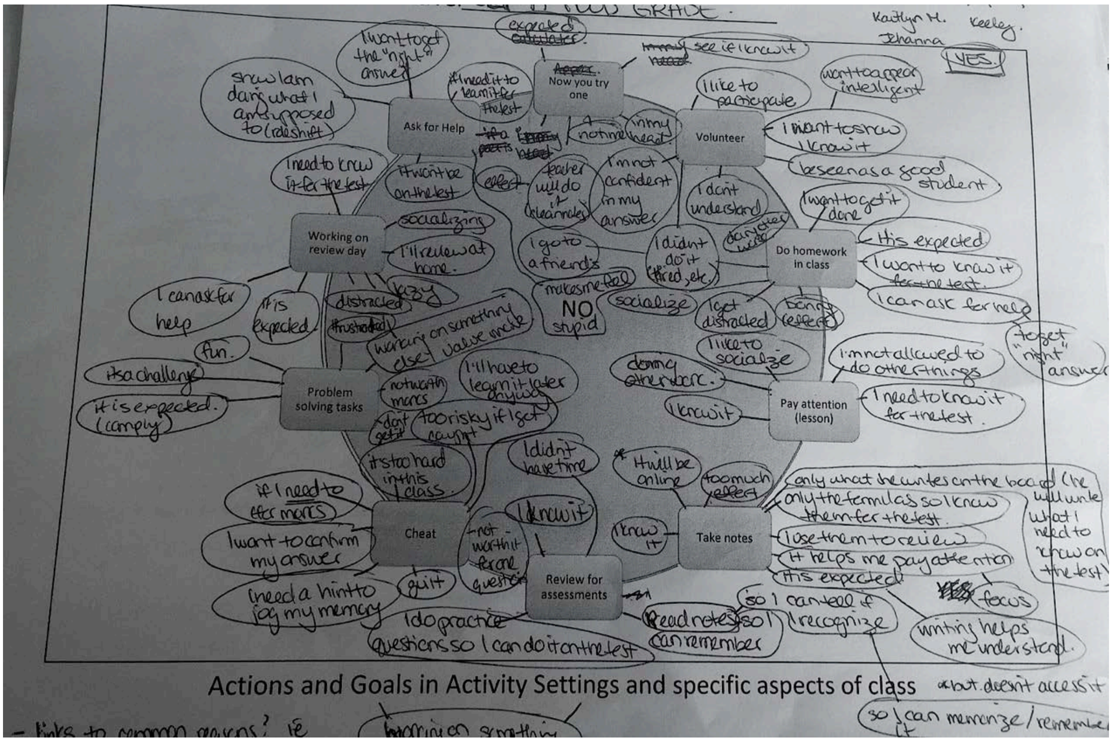

Figure 4: Actions of students with a motive of getting a good grade.

It is notable too that there were many students who would not engage with the problem solving tasks. Reasons for abstaining included frustration from not understanding, "*it's not worth marks*" so there was no purpose in making the effort, and the student was working on something else that was worth marks. Also, at this point there were a greater number of students whose actions complied with expectations *because* it was expected.

Discussion of similarities and differences

Beyond the obvious localized contrasts in cheating behaviour, engagement with problem solving tasks and the similarities in reasons of efficiency or prioritization of other schoolwork for not participating, there are three global findings.

First, students with a motive of understanding tend to put in effort as 'payment' for understanding the topic. Students with a motive of getting a good grade tend to see effort as an exchange for marks; if the payoff is not high enough then it is not worth putting in the work.

Second, what is apparent in the data is that the primary motive *understanding* is more stable than the *getting a good grade*. In line with Kaptelinin's (2005) conclusions regarding conflicting motives, the motive *getting a good grade* is more susceptible to being displaced by environmental conditions. These are often internal, such as being tired or hungry, but are just as frequently external, when other assignments or interests take precedence. But when a student holds a motive of understanding, it is pervasive. It drives all that they do, almost all the time. Referring to Figures 3 and 4 above, we see much more variance in students' behaviour when they have a motive of getting a good grade as compared to the actions of students who primarily want to understand.

The final global finding is tied to the second. Since understanding as a motive is more stable, it can be viewed as continuous, as opposed to discrete. This distinction refers to the actions of students over time. For a student with a motive of understanding, the actions that are consistent with this motive occur continuously, as opposed to only at certain intervals. Discrete, then, describes motives wherein the actions that are consistent with those motives occur sporadically. This is best explained by example.

Consider two fictional students, Jenna and Lisa[3]. Jenna has a primary motive of getting a good grade. That motive holds dominance when assignments (worth marks) are due, and in the lead-up to an assessment such as a test or a quiz. At other times, Jenna may defer her learning (and effort) to another time, or not see participation in certain activities, such as problem solving, as important. Jenna's attention may waver during a lesson and she may not try the examples because she can learn it later, or because she already knows it. In contrast, Lisa holds a primary motive of understanding. She sees opportunities for understanding at all times, not just before an assessment or when an assignment is due. She may not always do her homework in class or try every example, but understanding is most often her primary motive and her actions almost always align with this motive.

CONCLUSION

Although the actions of different students in the mathematics classroom often appear similar, this analysis shows that they are driven by very different motives, with significant consequences for potential learning.

Traditional assessment practices value grades and achievement, which pushes the motive of getting a good grade. What this research suggests is that traditional assessment practices should be reconsidered in light of supporting motives of understanding.

In addition to the distinctions in motives as continuous or discrete, and differentiated by stability, at the very least these results suggest a need for additional research for creating conditions that support student development of a motive of understanding in mathematics.

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USING PATTERNS-OF-PARTICIPATION APPROACH TO UNDERSTAND A HIGH SCHOOL MATHEMATICS TEACHER'S PRACTICE

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In this paper, patterns-of-participation theory serves as a lens to *interpret and understand Saudi high school mathematics teachers' practices. This framework focuses mainly on understanding what practices and figured worlds are significant for the teacher and how the teacher engages in those figured worlds. The data presented is about Noha, a high school mathematics teacher in Saudi Arabia. The data generated suggests that there are five significant practices or figured worlds to Noha's sense of her practice as a mathematics teacher. The paper discusses and explains these figured worlds.*

PURPOSES OF THE STUDY

In Saudi Arabia, the education system has undergone major changes in the past decade. Government agencies involved in education have introduced new policies, standards, programs, and curricula. These changes are accompanied by high expectations that teachers will incorporate the changes seamlessly without consideration of their existing practices. This paper is part of an ongoing study that intends to gain a better understanding of how high school mathematics teachers in Saudi Arabia are coping with recent education reform, including how their practices are evolving in response to the changes that are happening in the education system.

THEORETICAL FRAMEWORK

In this paper, patterns-of-participation (PoP) (Skott, 2010, 2011, & 2013) approach serves as a lens to interpret and understand Saudi high school mathematics teachers' practices during the current reform movement. The PoP framework identifies teachers' practice as being how teachers narrate and position themselves in relation to multiple, and sometimes conflicting, figured worlds (Skott, 2013). Figured worlds are imagined communities that function dialectically and dialogically as if in worlds. They constitute sites of possibility that offer individuals the tools to impact their own behaviour within these worlds (Holland et al., 1998; Skott, 2013).

Traditionally, most research in education that focuses on studying teachers' practices adopt an *acquisitionist* approach, especially those studying teachers' beliefs and knowledge in relation to teachers' practices (Skott, 2013). Recently, more researchers, including Skott (2010, 2013), adopt participationism as a metaphor for human functioning to understand teachers' practices. "The origins of participationism can, indeed, be traced to acquisitionists' unsuccessful attempts to deal with certain long-standing dilemmas about human thinking" (Sfard, 2006, p.153). Skott presents PoP as a coherent, participatory framework that is capable of dealing with matters usually faced in the distinct fields of teachers' knowledge, beliefs, and identity. Therefore, PoP is a theoretical framework that aims to understand the relationships between teachers' practice and social factors. Skott (2010, 2011) initially developed the patterns-of-participation framework in relation to teachers' beliefs. However, in order to develop a more coherent approach to understand teachers' practices, Skott (2013) extended the framework to include knowledge and identity.

The social approach of research in mathematics education has progressively promoted the notion that practice is not only a personal individual matter; it is in fact situated in the sociocultural context. Although the relationships between individual and social factors of human functioning have generated much debate in mathematics education, it is mainly in relation to student learning (Skott, 2013). Therefore, PoP is a theoretical framework that aims to understand the relationships between teachers' practice and social factors. To a considerable degree, PoP adopts participationism as a metaphor for human functioning more than mainstream belief research. Therefore, PoP draws on the work of participationism researchers, specifically Vygotsky, Lave & Wenger, and Sfard.

"The intention of PoP is to take this one step further by limiting the emphasis on acquisition and include a perspective on the dynamics between the current practice and the individual teacher's engagement in other past and present ones" (Skott, 2013, p. 557). This framework focus mainly in understanding what practices and figured worlds are significant for the teacher and how the teacher engages in those figured worlds. A teacher's engagement with these figured worlds inform and adjust the interpretations s/he makes to her/himself and the way s/he engages in on-going interaction in the classroom. These figured worlds work in a very complex system where they could support and sometimes, restrict one another as the teacher contributes to classroom practice.

METHODOLOGY

This paper is part of an ongoing study that intends to develop more coherent understandings of Saudi high school mathematics teachers' practices during the current reform movement. For data analysis, I used a qualitative analysis approach based on grounded theory method as used by Skott (2013). I applied the two fundamental and basic stages of coding identified by Charmaz (2006); the open or initial coding and the focus or selective coding. Through the coding process, I was able to organize, group and reflect on the data. The process includes isolating patterns and categorizing the data to identify practices and figured worlds that are significant for the participant teachers and how they engage with these figured worlds.

The data presented in this paper comes from Noha. Noha is a high school teacher with thirteen years of experience teaching middle and high school. She is currently teaching at a high school. She graduated from university with a Bachelor of Education Degree with a specialization in Mathematics. The education courses Noha had in university focused on general issues related to teaching, such as lesson planning and classroom management. She does not have any experience taking educational courses related to teaching mathematics specifically. After she graduated from university, she started teaching mathematics at a middle school. After four years, she moved to a high school. She has nine years of experience teaching mathematics at the high school.

I conducted two semi-structured interviews with Noha. The first one before I observed her teaching two lessons and the second interview was conducted after the classroom observations. In the first, I asked her to reflect on her experiences with mathematics teaching and learning in school, at university, and during her practicum year. I also asked her questions related to different aspects of the new reform movement in education system in Saudi Arabia. The second interview focused on her experiences with teaching mathematics at her school and on her relationships with the school, her colleagues, and the students. I asked her to reflect on the lesson planning process she had in order to prepare the lessons I observed. During the second interview, I also used a stimulated recall technique by playing audio recordings of parts from the lessons to facilitate her conversation about her own teaching practice in the classroom.

During my visits to Noha's school, I was also able to collect some data from informal observations of staff-room communication between Noha and her colleagues. I also have a copy of Noha's lesson planning notebook and some of her worksheets and tests samples.

DISCUSSION

The aim of this paper is to develop a deeper understanding of the participant teacher's significant practice and figured worlds and how she engages with these figured worlds.

As a teacher positions herself in relation to her profession as a mathematics teacher, she draws on several, often incompatible, figured worlds. Her engagement with these figured worlds does not only appear in her verbal communication, but also by the choices she makes in her all other actions related to her profession, such as her immediate reaction to certain student behaviors or the way she expresses her view when engaging in a conversation with her colleagues.

Teachers' engagement with figured worlds informs and adjusts the perceptions they make and the way they engage in on-going interaction in the classroom. These figured worlds work in a very complex system where they could support, and sometimes restrict, one another as every teacher contributes to classroom practice

Noha's classroom

Noha has 35-37 students in every class. In her classroom, students are usually quiet and calm, sitting in neat rows of two tables that face the front of the classroom. Normally, Noha starts her lesson by checking students' homework, which is assigned daily. She then reviews previous material. The classroom environment is focused on getting work done. Noha plans her lessons very well and she makes sure to follow the plan very carefully. A measure of time on task indicates that the lesson is going very well and that students are doing what they are supposed to do in her class. According to her, the most effective way to teach mathematics is to use the classroom board to introduce a mathematics concept, explain different mathematics procedures in relation to the presented mathematical concept, and then get students to practice theses procedures individually. In her classroom, the official mathematics textbook is never used. Instead of the textbook, Noha designed a notebook that she and her students use during the lessons. This notebook replaces the official textbook in her classroom.

RESULTS

After thirteen years of teaching, the data generated about Noha suggest that there are six significant practices or figured worlds to Noha's sense of her practice as a mathematics teacher. These figured worlds are mathematics, textbooks vs notebook, students' achievement, reform, relationship with students, and voluntary work.

Mathematics

According to Noha, mathematics is a body of knowledge centred on specific concepts, and learning these concepts means knowing how to use them. For Noha, mathematics is all about doing; if you are able to do mathematics, then you know mathematics. During a typical class session, Noha spends 10-15 minutes on whole-class instruction in order to introduce the new concept by using the board. Then she does an exercise that demonstrates how this concept is used and explains very clearly the methods and procedures used to do the exercise. The students' main role during this part of the lesson is to listen carefully to the teacher. Noha makes sure while she is presenting the new material that the students are paying attention to what she is doing by saying phrases like "listen carefully to what I am saying" or "focus your attention on me". After introducing the new material, she gives her students a few minutes to copy into their notebooks what is written on the board. Then she asks the students to do a similar exercise to the one she did.

According to Noha, a basic part of understanding mathematics involves memorization and repetitive practice. She clarified why memorization plays an important role in mathematical understanding by saying,

> some facts in mathematics need merely to be accepted as true and memorized, I can't explain some mathematics to my students in a way that they really understand. Maybe some people would not agree with my view, but I really see that there is a place for memorization of basic facts in mathematics learning

Noha argued that although her teaching style is considered traditional, her approach plays an irreplaceable role in helping all students, regardless of their level of ability and learning style, to gain high level of conceptual understanding of mathematics and acquire strong mathematics problem-solving and reasoning skills.

The textbook vs notebook

One notable practice in Noha’s classroom is the absence of the textbook. Neither Noha nor her students use the textbook during the lesson. Noha explained her history of using the official textbook in her classroom by saying, “during my first year of teaching, I based much of my classroom activities on the textbook. In my second year, I used very little from it. Finally, in my third year, I got rid of it altogether and I haven't used the textbook during my lessons since. I started to rely on the notebook I design”. After using the textbook as a main source for her practice for two years, Noha realized the textbook's deficiencies and substituted with an alternate version of the textbook. “The textbook failed to arouse my students’ interest and keep them on track”.

Noha designs a notebook each year to use with the students in her classroom. This notebook replaces the textbook. During the summer, when schools are closed, she plans her notebook. She organizes the notebook by chapters and lessons based on how they appear in the official textbook. At the begging of the school year, Noha photocopies the notebook and distributes one to each of her students. According to Noha, the notebook provides learning situations that guarantee keeping students engaged in learning activities during the lesson. She added, “Without a textbook, I can create lessons that engage students by relating mathematics to their needs. Lessons become clearer when I present the topic in an organized way, using a language that my students understand”. Noha talked about her notebook very proudly and has no intention of changing this aspect of her teaching practice.

Reform

It goes without saying that though out the years Noha has been recognized by school inspectors as an excellent teacher of mathematics because she represented the culturally accepted values of effective mathematics instruction. However, after the reform movement started, her teaching practice is not appreciated any more. Noha indicated that when the reform movement started, especially with the introduction of the new textbooks, the school inspector told her she needs to reconsider her role as a mathematics teacher with regard to student learning and choosing mathematical activities. Noha was also asked by her school inspector to stop using the notebook in her classroom and to mainly use the new textbook as a part of her classroom activities.

Noha complained that the reform curriculum materials, including the new textbooks, new teacher guide and the circulated notes of recommendations that teachers receive regularly from the Ministry of Education, do not prescribe or describe practice for teachers, but rather offer new visions of mathematics teaching practice. Noha explained that her teaching practices are the result of her own adaptation to existing circumstances; those existing circumstances have not changed enough in a way that allow teachers to make effective changes. She claimed that teachers face so many obstacles if they decide to change their practice. She noted, “In high school, we don’t have the tools and ability to teach mathematics as a subject of figuring things out or making sense of things. The content is getting harder and more abstract. And we don't have the tools and resources to teach this way".

Students' achievement

According to Noha, there is a strong connection between successful and effective teaching and student achievements. Noha indicated that teaching must lead to improvement in students' academic performance; She stated, "Student achievement is always the result of successful mathematics teaching. A teacher will never be considered successful if her students' achievement is low".

Student achievement appears to be the ultimate goal of Noha's job as a mathematics teacher. In her practices, she relies mainly on two sources to evaluate student achievement, written tests and homework. Besides the midterm and final exams, Noha gives her students a quiz at the end of every chapter. The end of the chapter quiz helps her assess the effectiveness of her instruction, as well as students' understanding of the concepts taught. Noha also explained that she does not support weekly testing because it destroys students' interest and motivation to study for tests. Noha also pointed out that she relies on homework as a daily formative assessment tool in class in order to measure the level of student knowledge and understanding of the previous lesson.

Relationship with Students

In Noha's teaching practice, it is crucial to connect with her students in a positive way. She said, "A positive teacher-student relationship can make my classes run easily. Without it, nothing will. Students need to feel that their teacher cares about them". Noha makes sure to demonstrate respect towards her students by using a kind voice and appropriate language when speaking with students. According to her, teachers who treat their students with respect will have active learners in their classroom. It is very important to Noha that her students know she cares about them. She explained some of the strategies she uses, such as stressing the things that she and her students have in common. She noted, "I always explain to my students that I have the same goals as they have and I make it clear to them that my job is to help them achieve their goals". She also communicates positive expectations letting her students know that she is proud of them. Noha likes to show her kind side to her students by using terms of endearment when calling her students in classroom. Terms like sweetie, honey, and my dear are used a lot by Noha.

Noha uses an incentive system using points to motivate her students to participate and engage in class. She uses a notebook to keep track of the points. When she gives her students a task to do, she rewards every student who finishes the task one point. When a student collects five points, the student gets ¼ of a mark. According to Noha, the technique helps keep her students excited and energetic during the lesson.

Voluntary Work

Noha is a very active teacher. She is willing to do any work that could benefit students. She has no problem volunteering to do extra work even if it is not related to mathematics teaching. She explained, ". All I want is to help create a more positive and productive school environment for all students". While I was walking with her to the teachers' room, Noha showed me some posters on the walls that she designed and printed as a part of her volunteer activities. The posters were about topics not related to mathematics, such about the benefits of eating healthy food and the importance of time management skills. Noha is also one of the few teachers who agree to go on field trips with students. Field trips are very rare for girls in high school because of the cultural restriction in Saudi Arabia. Noha feels obligated to support taking her students on field trips because "students need to do something different once in a while".

A major part if Noha's volunteer work is designing and conducting free workshops for students at her school. Noha is one of few teachers in the district who conducts such workshops. The workshops focus on offering students skills and knowledge to help them score better on the General Aptitude Test (GAT) which is a standardized test students at high school take for university admission. The workshops are open to all grade 11 and 12 students attending her school, not only the students in her classes. Noha is not happy that some private institutes are trying to take advantage of the importance of this test for students and offer paid courses to teach students skills that are supposed to be learned at school. Noha indicated that during the workshops, she helps her students understand the nature of the GAT exam and how it is different from tests they usually take in school. She explains the mathematics facts, rules and formulas that students must know.

CONCLUSION

Noha is a very active teacher and has a strong commitment towards her teaching practice. It is very important to Noha to build a strong relationship with her students. She demonstrates interest in extending her relationships beyond the classroom by voluntarily participating in extra-curricular activities with her students. An important part of her commitment towards her teaching practice is her students' achievements. She is experiencing huge stress to help her students achieve well in school tests and raise their scores on standardized tests. Although Noha has a strong sense of duty and obligation toward her students and cares about them, her negative perception of reform and the new mathematics textbook is a result of many factors. These factors include the shortage of resources and professional support she received during the process of implementing new curriculum. Although Noha seems to reject reform ideas about mathematics teaching, some parts of the interview indicate that she seems to admit that there is another way to teach mathematics. This way could work if there were more resources available for teachers.

As Noha positions herself in relation to her profession as a mathematics teacher, she draws on several, often incompatible, figured worlds. Her engagement with these figured worlds does not only appear in her verbal communication, but also by the choices she makes in all other actions related to her profession such as her immediate reaction to certain students' behaviour, or the way she expresses her view when engaging in a conversation with her colleagues. It is also important to clarify that I am not claiming that these are the only figured worlds that contributes to Noha's sense of her practice as mathematics teacher. It is very challenging to get access to all the practices and figured worlds that are possibly significant for Noha's classroom interaction. For instance, challenges could occur if the figured worlds are related to the teacher's experience in schools and university (Skott, 2013).

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MAXIM ADHERENCE IN SECONDARY SCHOOL MATHEMATICS TEXTBOOKS

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This paper extends H.P. Grice's four conversational maxims of Quality, Quantity, Relation, and Manner into the genre of secondary school mathematics textbooks. In order to determine if and how the supervening Cooperative Principle may be applicable to the author-reader relationship, the general notion of division by zero is explored across Grade 9, 10, and 12 mathematics textbooks from a single publishing group. Examining a formal definition of the rational numbers, considerations associated with simplifying rational expressions, and an introductory discussion of the tangent function in these respective textbooks reveals likely authorial assumptions regarding students' prior mathematical knowledge, as well as certain inconsistencies in representing undefined cases connected to potential divisions by zero.

INTRODUCTORY OVERVIEW

School mathematics textbooks represent an interesting space within the larger mathematics discourse, for they are somewhat removed from the textual contributions of the scholarly/academic community and involve more direct interaction with *students of school mathematics* (who are less firmly ensconced within the discourse itself). As with other textbook resources, school mathematics textbooks are deeply embedded within the educational climate, and can be seen as representative of the predominant aims and values that motivate the educational endeavour at the time of their writing (barring mismatches between ideology and practice). Educational policy, curriculum mandates, and greater social factors influence not only the contexts within which school textbooks are produced, but also the content that is ultimately included within (and excluded from) those same textbooks. Such judgments make an important, though perhaps under-acknowledged, contribution to the mathematics discourse.

The impetus for the current paper stems largely from an interest in the scope and applicability of the four conversational maxims of *Quality*, *Quantity*, *Relation*, and *Manner* characterized by H.P. Grice in 1975 and utilized by numerous subsequent researchers to explore particular facets of the discourses manifested within a range of different fields/disciplines (i.e. linguistics, pragmatics, sociology, mathematics, education, et cetera). Considering the potential Grice's Maxims might have for revealing more about core values, ideological stances, and tacit assumptions underlying the broader mathematical discourse, I seek herein to more closely examine some of these factors by exploring the extent to which Grice's Maxims and their supervening *Cooperative Principle* may be relevant to the genre of school mathematics textbooks.

THE GRICEAN FRAMEWORK

As explicated by Grice (1975), "talk exchanges do not normally consist of a succession of disconnected remarks, and would not be rational if they did" (p. 45). To some degree, at least, they are "cooperative efforts; and each participant recognizes in them, to some extent, a common purpose or set of purposes, or at least a mutually accepted direction" (ibid.). These conjoined notions of cooperative effort and recognition of common purpose or direction set the foundation for the *Cooperative Principle* upon which Grice's Maxims of Conversation are constructed. While I recognize that school textbooks do not necessarily evoke "talk exchanges" in the sense intended by Grice, I will suggest that there *is* a sense in which the Maxims of Conversation are still applicable in the analysis of written textbook content.

Textbook resources may not, at first glance, seem an appropriate genre into which to extend the Gricean framework, in large part because textbooks are not exactly a conversational medium. They do not involve speech acts exchanged by two or more participants in real-time, nor do they adapt to an emergent conversational flow between interlocutors. They do, however, entail a form of linguistic interaction between author(s) and reader(s), wherein the intratextual content itself serves as the interface between those parties. This is akin to what John Fauvel (1988) would refer to as the "triangle of writer, text, and reader, whereby the writer is taken to be trying to communicate something to the reader via a text" (p. 25). Similarly, textbooks can also be considered as having voices, or as *giving voice* to their respective contents, if even those voices are to be "heard" only within the minds of readers, or issuing forth from readers' lips when spoken. Thus, I would caution against a view of textbooks merely as vessels for the *transmission* of information, for they do embody and voice aspects of mathematics discourse, and exist within a larger corpus of disciplinary communications.

Indeed, by virtue of the educative intent underlying textbook publications, one could presume that observance of the *Cooperative Principle* would be inherent to the design of *all* textbook materials, for textbooks are developed, produced, and disseminated as informative/didactic tools with specific pedagogical aims at their cores. This latter notion in particular has led me to wonder if Grice's Maxims of Conversation (supervened by the *Cooperative Principle*) might actually underlie the writing of school mathematics textbooks as well or if Grice's Maxims are in fact violated by such publications in particular ways. To that end, this paper focuses upon the question of how closely school mathematics textbooks *adhere* to Grice's *Cooperative Principle*.

Grice's Maxims of Conversation are spread across four primary categories. The maxims of *Quality*, *Quantity*, *Relation*, and *Manner* each embody different facets of the *Cooperative Principle*, and offer guidelines for the interpretation of conversational implicatures (or suggested meanings) that arise through speakers' interactions. Grice (1975) articulates the *Cooperative Principle* itself, as follows: "Make your conversational contribution such as is required, at the stage at which it occurs, by the accepted purpose or direction of the talk exchange in which you are engaged" (p. 45).

From this foundational imperative, the four categorical maxims are then derived. I briefly summarize them below, giving full acknowledgement and credit to Grice (1975, pp. 45-46) for their basic wording and structure.

> The Maxim of Quality: Try to make your contribution one that is true. Do not say what you believe to be false. Do not say that for which you lack adequate evidence.
>
> The Maxim of Quantity: Make your contribution as informative as is required (for the current purposes of the exchange). Do not make your contribution more informative than is required.
>
> The Maxim of Relation: Be relevant.
>
> The Maxim of Manner: Be perspicuous. Avoid obscurity of expression. Avoid ambiguity. Be brief (avoid unnecessary prolixity). Be orderly.

While I mean to extend Grice's conversational maxims toward the author/reader interactions of mathematics textbooks, I also acknowledge that only two of the four will be addressed. As I have already intimated, it seems reasonable to assume that at least some aspects of the *Cooperative Principle* would be innate to the genre of school textbooks. More specifically, I suggest that the maxims of *Quality* (wherein one should strive to make contributions that are truthful) and *Relation* (wherein one should strive to be relevant) may be taken as a priori conditions when discussing textbooks in general. Such resources would be of little use, and essentially antithetical to their intended educational purpose if these guiding principles were not, in fact, heeded. That said, I do not believe that adherence to the maxims of *Quantity* and *Manner* can be as easily assured. It does not seem a trivial matter to determine how much/little information can be deemed *sufficiently informative* in a given context, nor does it seem entirely clear what criteria might be used in deciding whether or not *ambiguity* and *obscurity* have been avoided whilst *brevity* achieved. Thus, the maxims of *Quantity* and *Manner* invite much broader interpretations of their core meanings. It is for these reasons that the ensuing discussions focus on the paired maxims of *Quantity* and *Manner* whilst offering little further commentary on *Quality* and *Relation*.

THE RESEARCH APPROACH

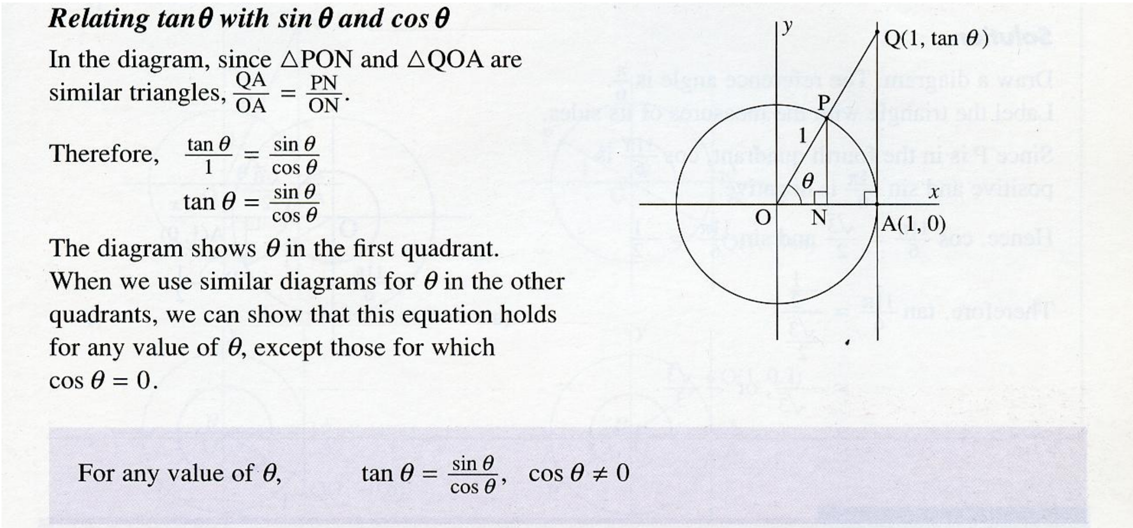
Deliberations concerning the selection of textbook materials to be used as sources of data were influenced by three primary factors: the choice of a mathematical concept/topic upon which to overlay the Gricean framework, the age range and grade level(s) at which to explore representations of the chosen concept/topic, and the option to utilize textbooks produced by multiple publishers or a single publisher. In terms of selecting a mathematical concept/topic that could be fertile ground for the investigation, I gravitated toward mathematical themes that students might encounter and revisit multiple times throughout their junior high and high school experiences with mathematics (potentially in different contexts or extended forms). The intent was to target mathematical material that I anticipated appearing with textual descriptions and explanations, and not simply diagrammatic or pictorial visualizations within the textbook pages. Where possible, I hoped to compare key descriptions and definitions of the chosen concept/topic across the selected texts, so as to determine whether or not the Gricean Maxims of *Quantity* and *Manner* were adhered to or violated in each case. The chosen concept was ultimately the general notion of *division by zero*. Not only is division by zero a concept that recurs in a variety of contexts throughout students' experiences with school mathematics, it also presents a foundational problematic with significance both to associated number theoretical issues and practical action. In addition to its relevance to rational number construction (which students are likely to encounter via simple fraction work in the primary grades and revisit in the Grade 9 curriculum), the notion of division by zero also relates to work with rational expressions (which can appear in the Grade 9 and Grade 10 curricula), and resurfaces with more complex explorations of trigonometric functions/expressions (in Grade 12). Many other instances of its recurrence are likely.

This choice of mathematical topic informed subsequent decisions regarding age range, grade level, and publisher specifics. As opposed to framing my investigation within a single grade level across multiple publishers' textbooks, I instead chose to explore recurrences of the selected topic across multiple grade levels and texts whilst keeping the publisher consistent. The decision to remain within a single publisher's line of textbooks was largely motivated by the wish to preserve some amount of consistency in both the structure of and approach to the language used across grade levels, and to (attempt to) reduce the number of parameters that might influence the overlaying of the Gricean framework. Admittedly, it remains unclear if this choice ultimately helped or hindered the analytical process. For the purposes of this provisional analysis, I acquired a set of three secondary level mathematics textbooks from the Addison-Wesley Publishing Group: *Mathematics 9* (Kelly, Alexander, & Atkinson, 1987a), *Mathematics 10* (Kelly, Alexander, & Atkinson, 1987b), and *Mathematics 12* (Alexander & Kelly, 1999). Interestingly enough, differences in publication years notwithstanding, Kelly and Alexander share authorial credits in all three of the chosen textbooks. This repeated authorial collaboration seemed notable, and offered a particular point of commonality that I had not expected yet now believe might have facilitated a more consistent reading of the texts as a result of the fairly uniform stylistic and technical choices on the part of the authors.

Given that the primary features of the textbooks in which I am interested are already in written textual form, I did not find it necessary to apply transcription protocols to the texts, or otherwise "prepare" the data in any real sense. The core analytical approach mainly involved engaging with selected excerpts from the texts in as open and thoughtful a way as possible, whilst continually "bouncing" those excerpts off of the maxims of *Quantity* and *Manner* and attempting to situate the represented content within the larger curricular treatment of division by zero. To that end, three specific focal points around which to construct my data were identified: a brief introductory discussion of the trigonometric tangent function in *Mathematics 12* from Alexander & Kelly (1999), considerations associated with simplifying rational expressions presented in *Mathematics 10* from Kelly et al. (1987b), and a definition of the rational numbers in *Mathematics 9* from Kelly et al. (1987a). A condensed data analysis and discussion of findings appears in the following pages.

CONDENSED DATA ANALYSIS & DISCUSSION OF FINDINGS

While an introductory discussion of the tangent function appears in the initial pages of Section 3.7 in Kelly & Alexander's *Mathematics 12* textbook, the first text directly linking the topic to potential divisions by zero is found shortly thereafter under a heading entitled *Relating tanθ with sinθ and cosθ* (p. 207). See Figure 1 below.



Relating tan θ with sin θ and cos θ

In the diagram, since △PON and △QOA are similar triangles, $\frac{QA}{OA} = \frac{PN}{ON}$.

Therefore, $\frac{\tan\theta}{1} = \frac{\sin\theta}{\cos\theta}$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

The diagram shows θ in the first quadrant. When we use similar diagrams for θ in the other quadrants, we can show that this equation holds for any value of θ, except those for which $\cos\theta = 0$.

For any value of θ, $\tan\theta = \frac{\sin\theta}{\cos\theta}$, $\cos\theta \neq 0$

Figure 1: A textual and diagrammatic treatment of the connections between the tangent function and the sine and cosine functions (Kelly & Alexander, 1999, p. 207)

In addition to being the first instance in the textbook where the sine, cosine, and tangent ratios of a variable angle (θ) are characterized according to their *interrelationships*, the text contained in Figure 1 is also notable for a number of other reasons. At no point before this, do the authors explicitly frame the tangent function in terms of proportional equality (by reference to similar triangles). Both representations rely upon fraction notation that students will presumably have encountered in their previous school mathematics, yet this may well be the first instance in which students encounter a textbook representation of a trigonometric function involving ratios of *other* trigonometric functions. While students may have already dealt with algebraic manipulations where trigonometric functions appear as denominator terms in more complex expressions, it is unlikely that many will have encountered function *definitions* with that same property. Granted, there would seem to be little need for the authors to address links between the tangent function and the reciprocal functions of secant, cosecant, or cotangent (or the inverse trigonometric functions), for these would likely be deemed irrelevant or extraneous to the current exchange. Inclusion of such information at this point in the textbook could also be viewed as a violation of the *Maxim of Manner* (for it would be adverse to the desired perspicuity), as well as the *Maxim of Quantity* (in that it would be providing more informative than required). One would anticipate, then, that the authors felt that the given description of the tangent function was sufficiently informative for the purposes of this exchange and that no further description was required. On these counts, I suggest that the given text does, in fact, adhere to both the maxims of *Manner* and *Quantity*. Not surprisingly, the authors seem to have adopted a cooperative stance.

Interestingly, while the final text of Figure 1 establishes the constraint that $\cos\theta \neq 0$, the rationale behind that constraint is not spoken to in any way. The written information does not offer any insight into what happens if/when $\cos\theta = 0$. Again, presumably, the authors' judgment might have been that students would recognize (from previous work with rational expressions and fractions) that instances involving variable terms could potentially lead to zero-valued denominators yielding undefined cases. In fairness, graphical representations of the tangent function and its asymptotic behaviour do occur in a later section of the *Mathematics 12* textbook, but even then, nothing is offered that directly links asymptotic behaviour to the notion of the undefined case. At this stage in students' mathematics learning, it could be said that explicit discussion of the issue of "undefinedness" resulting from division by zero might be superfluous, and could constitute a violation of the *Maxim of Quantity* (by providing too much information for the current exchange), but I would suggest otherwise. As I shall make clearer in the remaining discussion, a core piece of reasoning seems to have been omitted throughout the iterative treatment of division by zero in this particular textbook series. What interests me is that the nature of the undefined case is not elaborated upon in any way, and the question of WHY an expression with a zero-valued denominator would be considered to be undefined still remains.

Chapter 4 of *Mathematics 10* (Kelly et al., 1987b) introduces rational expressions, and the premiere section of that chapter (4-1) treats the simplification of rational expressions. It is here that the authors delineate key characteristics of such expressions and draw parallels to students' (presumed) past mathematical experiences with polynomial expressions and rational numbers. See Figure 2 below.

4-1 SIMPLIFYING RATIONAL EXPRESSIONS

In arithmetic,
> we saw that the sum, difference, or product of any pair of integers is an integer, but that the quotient of two integers is not necessarily an integer. In order to include quotients of integers in arithmetic, the number system was extended to include rational numbers.

In algebra,
> we saw that the sum, difference, or product of any two polynomials is a polynomial, but that the quotient of two polynomials is not necessarily a polynomial. In order to include quotients of polynomials in algebra, the set of algebraic expressions is extended to include rational expressions.

Figure 2: An introduction to the notion of rational expressions by way of reference to arithmetic and algebraic parallels (Kelly et al., 1987b, p. 110)

One could imagine the authors' perspective to be one in which students are already largely familiar with the mechanics of rational number arithmetic and polynomial algebra. From that perspective, no further explanation of those background concepts would be necessary, and the maxims of *Quantity* and *Manner* will have been properly adhered to. In a sense, that appears to be the unspoken assumption guiding the introduction to rational expressions; however, the veracity of that assumption is difficult to assess. Nevertheless, I would contend that Kelly et al. (1987b) are still in full observance of the *Cooperative Principle* here, for they deliberately direct readers of the text toward the associated arithmetical and algebraic subject matter. This reminder of relevant background contexts helps to ensure that students can relate the current mathematics to previous mathematical experiences. In the event that students are unable to do so, relevant background matter has been identified.

A direct follow-up to the introductory matter of Figure 2 involves a worked example and the simplification of a rational expression, which leads to a crucial addendum: "A rational expression is not defined when its denominator is equal to 0" (Kelly et al., 1987b, p. 111). This *explicit* statement regarding the case of the zero-valued denominator is especially intriguing, particularly in light of the omission apparent in the tangent function description offered in the *Mathematics 12* textbook. That said, the same question of WHY an expression with a zero-valued denominator is considered to be undefined still goes unanswered in this text. Moving further back to *Mathematics 9* (Kelly et al., 1987a) reveals that this lingering conceptual question also goes unaddressed when dealing with the basic definition of the rational numbers.

While the three selected mathematics textbooks are fairly consistent in terms of stating denominator constraints in their respective cases, none of them directly address the conceptual nature of the undefined case, or what it means for a function/expression to *be* undefined. The three scenarios alluded to within this condensed analysis (an introductory discussion of the tangent function in a Grade 12 textbook, considerations associated with simplifying rational expressions in a Grade 10 mathematics textbook, and a formal definition of the rational numbers in a Grade 9 mathematics textbook) seem to observe Grice's *Cooperative Principle* in a very broad sense, provided one makes allowances for likely authorial assumptions about students' prior knowledge and familiarity with terminology, yet there seem to be certain nuanced ways in which judgments about maxim adherence and violation are more ambiguous.

Insofar as the *Maxim of Manner* concerns the selected examples, it would appear that the textbook authors have achieved the broad goal of being perspicuous. I do not question the orderliness of their presentation, nor their brevity of expression. It is difficult to offer any real commentary on the ambiguity or obscurity of their textual contributions, though, for those factors would seem to be very much contingent upon the prior knowledge of the students concerned (or their familiarity with the background mathematics referred to by the authors). That said, there are some instances (as in Figure 2) in which it is clear that the authors have assumed a very particular stance with regard to students' prior mathematical knowledge. Unfortunately, the small sampling of text that I have explored for this provisional analysis is not substantial enough to allow for more definitive statements here. Ultimately, it is the *Maxim of Quantity* that has proven to be the most concerning as a result of these tentative explorations. Again, with respect to the notion of division by zero, I have attempted to characterize a conceptual omission that I can, at best, only describe as a sort of *retroactive violation* of the *Maxim of Quantity* (in that the nature of the undefined case is never fully explicated, but later entries in the textbook series carry forward as though it had been). Although Alexander, Kelly, and Atkinson's contributions to the given Addison Wesley textbooks do, at times, incorporate terminology that describes *when* certain functions/expressions/numbers are and are not defined, it is never made clear to the reader what the nature of the undefined case actually is, what it might "look like", or what its implications are in a number theoretical sense. This might constitute a violation of the *Maxim of Quantity,* by being insufficiently informative, but one could easily question (both as a researcher/analyst and as a pedagogue) if these textbooks constitute the appropriate space within the mathematics discourse to provide that particular informational contribution. As a result, it is difficult to supply a Gricean interpretation of the overall textbook treatment of the notion of division by zero.

As I have previously acknowledged, textbooks are not exactly conventional examples of a conversational medium. By extension, it is debatable whether or not the assertions and textual contributions found within the pages of mathematics textbooks actually constitute speech acts of a type assessable via the Gricean Maxims of Conversation. I do not doubt my earlier presupposition that textbooks can be seen as innately cooperative resources (by virtue of the educative intent underlying their development, production, and dissemination), however, I now suspect that it may not be appropriate to apply the standard Gricean framework to mathematics textbooks when treating them in isolation from the full conversational discourse of the mathematics classrooms in which they are typically embedded. This is not to suggest that the Gricean framework is flawed; rather, it speaks to the possibility of exploring/developing a *non-standard* formulation of the Gricean framework for use within the genre of mathematics textbooks. Precisely what characteristics such a framework might have is not yet clear; however, analyzing a more substantial corpus of mathematics textbook data (from a wider sample of publishers) could be of value in this regard.

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I WONDER AS I WANDER

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In this paper, I describe an informal self-study experiment during which I use an unfamiliar DGE (Geometer's Sketchpad). The use of previously identified dragging modalities in predictable ways is obvious, but I wanted to explore the possible need for an additional modality and examine how it might fit into this familiar landscape. I use the framework of instrumental genesis to explore the differences between the modality known as wandering dragging and a potential new modality I call "wondering dragging". The results of the experiment are discussed, not in terms of squeezing the new term in to the old lineup, but of creating a new "dimension" in which to use this term and making room for others.

INTRODUCTION

Dr. David Pimm, currently of Simon Fraser University, has been heard many times to say that no matter how many classifications have been determined within a set, it is always well worth asking if there are others. The impetus for this investigation was a happy accident. A chance conversation occurred between this author and Dr. Pimm, who was at the time editing an article in which the authors, Muteb Alqahtani and Arthur Powell (2016), had inadvertently used the spelling "wondering" instead of "wandering" while summarizing the different types of dragging identified by F. Arzarello, F. Olivero, D. Paola, & O. Robutti (2002). I found myself considering whether "wandering" dragging might be split into two types, or whether "wondering" dragging might be a completely different type than any so far delineated.

THEORETICAL FRAMEWORK

During the time that you are exploring how a new tool works, you will likely spend some time failing to accomplish what you set out to do, because of your unfamiliarity with the tool. You may know, for example, that you want to break open a hard nut with a hammer. If you have never used a hammer, you might try first to press the nut between the hammer and a hard surface in an attempt to break the shell under pressure, or you might try to use the hammer's claw to pry open the nut. These attempts will not likely reveal the nut's contents, but they are not in vain. You are learning what the tool does, and as importantly, what it does not do, under certain conditions. Once you try hitting the nut with the hammer once, even if by accident, you are likely ever after to tackle cracking a nut with the same approach.

Once you know how it functions, a tool becomes available to you for future tasks as well, and you will come to each new nut with greater confidence than all the ones before; you are no longer experimenting when you pick up the tool; you are applying a known feature to a known task with a known outcome. After the breakthrough, you can approach the breaking of any nut with confidence.

This is the epitome of instrumental genesis. As you use the tool for more tasks, it shapes you; it gives you power and flexibility. By seeing that it can be used for more than one thing, you more readily look for additional features and uses than those you already know. This shaping of the user by coming to know the tool was labelled *instrumentation* by Trouche (2004). We have “instrumented” (equipped) the user with the artefact.

By using it often, you may discover uses which may not have been part of its original design. Arguably, a hammer was not designed to break nuts, but to drive nails. It is well within the realm of possibility however, that one can be used in this way, and Trouche (2004) called this process *instrumentalization*. The user has “instrumentalized” the hammer artefact; given it a (new) instrumental use.

Instrumental genesis attempts to explain the evolution an individual undergoes as she attempts to use a new artefact, and the new uses the item can be seen to have as the individual becomes more familiar with it and with its properties. The way one uses a hammer evolves over time and with use. The way one uses a computer mouse will likewise evolve, and this is the area in which I am interested; considerable investigation has already taken place around the types of modalities available to mouse users (e.g. F. Arzarello, F. Olivero, D. Paola, & O. Robutti (2002), F. Arzarello, M. Bairral, & C. Danè (2014), M. A. Mariotti (2012)), but none appears to have been done about the different possible reasons a modality is chosen. There is a strong symbiosis between instrumentation and instrumentalization; both the user and the instrument change over time. While it is natural to label the processes involved, it is essential to recognize that neither occurs without the other. Moreover, once the user has evolved, the instrument cannot help but be used differently, and in return, the user is changed again, and so on.

Instrumentalization is something that happens to the hammer, and there is no sentience on its part; it changes neither the hammer’s structure nor its makeup. Instrumentation, on the other hand, causes the user to change. One might argue this change occurs at the cellular level; it certainly occurs at the cognitive level. I will leave it to philosophy to discuss whether these are the same or different.

The question, then, is whether instrumentation can happen through the user’s own motivation or whether it can be imposed through external forces (e.g. task instructions for which the user has no particular interest). Perhaps it does not matter, and the learner’s motivation is irrelevant (or differently relevant) to whether an action is performed due to her curiosity or due only to her having been told to do so.

THE STUDY

I recorded a learning session during which I undertook to complete a task in the dynamic software The Geometer’s Sketchpad (GSP) (Jackiw, 2014). I had not used the software before this with one brief exception. No instructions were given save those that came with the activity. The recording was analysed, and segments selected that contained incidents in which I used wandering dragging as well as other types of dragging. The "other" types were then scrutinized, and held up against the definition of "wondering dragging" I hoped to propose. I was initially looking for evidence that "wondering" dragging might be an addition to the types of dragging already extant in the literature, but discovered instead that if it were to be included, it would more likely be in a different way altogether.

The task was to construct a triangle within the DGE Geometer's Sketchpad, then to construct a segment from one vertex to the midpoint of the opposite side, and use the resulting diagram to show that the two smaller triangles can be made to be separable and to determine under what conditions.

RESEARCH QUESTION

Arzarello et al. (2002) define "wandering dragging" as

> moving the basic points [of a shape in a Dynamic Geometry Environment (DGE)] on the screen randomly, *without a plan*, in order to discover interesting configurations or regularities in the drawing [emphasis added]. (p. 67)

Alqahtani and Powell (2016) nicely summarize the definition given by Arzarello et al. as "dragging that aims to look for regularities".

"Wondering dragging" has no such definition in the literature. It is not a term thus far conceived to describe an action not already described by another term. "Wondering" in the most general sense has a distinctly different meaning from "wandering" however. The conversation that started this investigation quickly gave rise to the desire to review the definitions of this word in both the general sense and in its capacity in the world of instrumental genesis.

Wonder is defined in part in the Oxford English Dictionary (OED) as

> to feel some doubt or curiosity (*how*, *whether*, *why*, etc.); to be desirous to know or learn. (OED)

All of Arzarello et al.'s dragging categories are described as dragging, moving, drawing or linking *points* (2002). It is the purpose for which the points are moved that helps to distinguish them one from another. Wandering dragging is done "to discover"; bound dragging is done just to move a point; guided dragging is done "in order to give [an object] a particular shape"; dummy locus dragging is done "so that the drawing keeps a discovered property"; line dragging is done "in order to keep the regularity of a figure"; linked dragging is done to link a point to an object; dragging test is done "in order to see whether the drawing keeps the initial properties" (p. 67).

I initially conceived that "wondering dragging" might be an action that aided conjecture, but Arzarello et al. seem nicely to have encompassed the possibility that one might be dragging for any reason, intrinsic or extrinsic, and still fit neatly into one of the dragging modalities they provide.

If there were a different dimension however, one for which the definition hinged on one's *desire* to learn, there might be room for wondering, and perhaps for many other types of auxilliary states as well. In such a realm, *wondering dragging* could be described as being done not *only* "to discover" or "to see whether" the shape does something, keeps a property or takes on a shape or feature, but because one is "desirous to know or learn" whether a shape did something or took on a feature.

In this light, it is worth investigating Arzarello et al.'s definitions again. Some of them require a prescribed action, while others a combination of an action and a desire to uncover information, but not necessarily an intrinsic desire to learn. For example, bound dragging is simply "moving a (semi-dragable) point", not moving it (necessarily) for any particular purpose or to a specific end. Guided dragging is performed "in order to give [an object] a particular shape", but again, not (necessarily) because the user is intrinsically desirous to know. Wandering dragging is performed "in order to discover"; discovery implies that the dragger actually wants to learn something, while "moving" and "giving a shape" only require an action, and say nothing explicit about the user's desire. This is not to say that she does not have a desire; only that it is not necessary in order to work in that modality.

We should be looking then, not only at the learner's actions, but at her inspiration while performing that action. Let us formally define *wondering dragging* then as any dragging that is performed by the learner due to, or alongside her intrinsic interest in the result produced by the action. Of Arzarello et al.'s dragging types, then, all might be types of wondering dragging if while the learner is performing them, she is forward or laterally thinking about the result of the action she is undertaking not only because she has been told she must accomplish this task not just for a grade, or so as not to lose favor with the teacher, etc., but because she wants to learn.

If she is performing an action because she has been told to do so, and is not interested in the result, her action might be considered *task dragging*, as it only satisfies an extrinsic requirement to perform a task, and not to fulfil her own desire to learn. Thus one can perform wandering task (or task wandering) dragging or wandering wondering (or wondering wandering) dragging, and so it can be said that one can wonder while one wanders (or, indeed, vice versa).

OBSERVATIONS

In the recording, I use The Geometer's Sketchpad version 5. In order to help distinguish between the software as a tool and the tasks each button performs within the software, I shall refer to GSP as the "instrument" and to the buttons as "tools". For example, GSP has a button that, when selected, will produce a point when a user clicks in the workspace; when she clicks again, another point is produced, and so on. This is the Point tool.

A different button, when selected, will produce a point on the first click, but a second click produces not only another point, but a line segment between the two points. A third and fourth click produce a second line segment, and a pair of clicks is always required to create one shape. This is called the Straightedge tool.

A third button, when selected, will produce a point on the first click, a segment on the second, a closed triangle on the third, and will continue to create polygons with more and more vertices until the first point is clicked again to “finish” the shape. This is called the Polygon tool.

Early in the session, I am attempting to create a triangle. On my first attempt I manage to fix the first vertex, then have to experiment in order to learn how to fix the second endpoint of the first side. This experimentation is “wandering dragging”. After discovering the method for creating polygons and fixing their endpoints, I manage to create the triangle, but do not know how to finish it; without anchoring the last point somehow, the tool wants to continue to produce points with the fixing of each new side. Further experimentation, and thus further wandering dragging, is necessary before I learn, seemingly accidentally, that in order to close a polygon, I must select the first point again. With each new function learned, the next part of the task, or the next attempt at it, can be performed more quickly and with more confidence than the times before.

More importantly to our purpose however, the self-narrative captured with the video is very revealing. From the beginning of the session, I relate the physical setup of the activity, and read aloud the instructions. At 00:40:00, I determine that I need to construct a triangle. I am doing so purely because the task is required in order to satisfy the need to produce a recording and I am performing it by following written instructions. My statement, “…so I need to construct that first” indicates that I know what I am required to do, and that I understand my purpose. The action and narrative that follow show that I do not necessarily have a plan, since I am unfamiliar with the instrument (GSP), and I select various tools from the options available and experiment with them to determine what they do and how they can be used to accomplish my aim. Here is an excerpt of an early piece of the session:

1 I could do it with a polygon, I guess. I think I just want the outline.

2 Click, drag, (release mouse button). I guess I need to click again.

3 Oh, and then I can just click; third point.

4 And then if I wanted to, I could put a second point, or I mean a fourth point, but I don’t want that, so I’m going to see what happens if I do that (clicks on arrow tool, and shape disappears from screen.)

5 Well that didn’t work, so let’s try again: One, two, three points (counting out points as they are fixed to the workspace; this is what might be called “*construction dragging*” as I am not exploring, discovering or keeping regularity, but constructing an object with known procedures.)

I hover the cursor over various tools in an attempt to decide which one to try next. I then drag the cursor back onto the workspace and inadvertently pass over the first fixed point and it “highlights” to indicate a live state; wandering dragging.)

6 Oh, there we go. (click on first point) [You] have to click on the starting point.

None of the above actions indicates any desire to learn. One might argue that I am desirous to learn how to do my task, but even that has no particular motivation other than completing an assigned task, and I am clearly not engaged in any mathematics at a cognitive level. I am simply doing what I have been told to do without thinking about why, without wondering about the features of my creation, without caring if I learn anything about triangles. All of the above is task dragging.

Not much later in the session, the following narrative is produced:

1 I'm supposed to consider the point – the midpoint P, on AB, so, how can I figure out … I can put *a* point here (approximately halfway along AB), but how do I know it's the midpoint?

She selects the Point tool, hovers over AB, slides a new, unfixed, semi-dragable point back and forth (bound dragging) along AB

3 It's just going to be somewhere on there; how do I know it's the midpoint?

After some failed attempts at placing a point on AB, I succeed in this task, but am still trying to determine its correct location to be the midpoint of AB.

4 I can put a point on there (AB) and select (selects arrow tool, clicks on new point and on AB, so that both are highlighted, then unselects both and selects only the segment AB, and selects "Measure" from the pull down menu.)

5 Measure (most menu items are unavailable)

Unselects AB and selects new point instead

7 Measure (most menu items are unavailable)

8 [garbled] do this…

After a brief pause, I discover that the new fixed point will move along AB while staying attached to AB (bound dragging)

10 You could eyeball it, but that's [garbled]

The above monologue differs from the first in that while the I am still attempting to perform the assigned task, I have begun to engage with the shape I have produced in a way that indicates that I want to know something. In this case I want to understand how you can guarantee that a point on a line segment is the midpoint of that segment. I have identified a desire to prove (if only to myself) that a given point has a particular feature. While I express this interest, I move the mouse all about the workspace, hover over and select various tools, try, and fail, and try various tasks again, but all the time I am thinking about how the instrument can be used to answer a question. I apply dummy locus, "*phantom*[1]", wandering and guided dragging, but I do so with a desire to learn. I am wondering dragging.

Instrumental genesis is at work here: by using the instrument, and more importantly by failing at my task, I am learning how the instrument behaves and how to make it do my

[1] A term coined during discussion about observations made of different types of dragging; *phantom dragging* can be defined as dragging an invisible, or yet-to-be-selected point. (Rouleau & Sinclair, personal communication, 2016)

bidding. But the instrument is also shaping how I interact with it, and my response it to change how I approach the problem with each new attempt. I try different tools, and in so doing, even when they fail to do what I need from them, I learn more about the instrument. I am shaped by it, and it can be used by me in ways that I could not use it before.

SYNTHESIS

Wondering is a common misspelling of wandering, and I suspect Drs. Alqahtani and Powell of having made this error in the pre-published version of their article. In their defence, in one particular instance they use the term “wondering dragging” followed by a casual definition. In that instance at least, they genuinely mean it to be used differently than the well-established “wandering dragging”. In a section titled “Technological Tools”, Alqahtani and Powell (2016) say:

> The co-action between the teachers and the environment helped the teachers develop an understanding of the dragging functionality in DGE. This event also evidences how the teachers used reactive or wondering dragging (dragging to conjecture about objects and figures) to explore the figure and used the environment’s reactions to reflect on their dragging.

Wandering dragging as defined by Arzarello et al. is not necessary for conjecture, so Alqahtani and Powell have appropriated a term *similar* to one already known, in order to distinguish it from that term. This is not the same way in which they used “wondering dragging” elsewhere. One can only assume that while the other instances were in fact oversights, this one, at least, was intentional.

Types of dragging are discrete. they may occur simultaneously, but you would not be able to perform 3/5 dummy dragging. On the other hand, you might be a little bit desirous to learn or a lot. Thus during a session a subject might perform wandering dragging at a low level of curiosity, and guided dragging at a high level of curiosity, or vice versa, or a bit of both.

If we assign wondering (or curiosity) a scale, say from 0 to 10, and if we can measure a subject’s level of curiosity, then instead of a timeline such as Figure 1, we might see a timeline such as Figure 2, where in addition to labeling intervals of time with the modality used, we can indicate the level of curiosity the subject experiences. In addition to the way in which a student carries out the physical and even the mental demands of a task, there is a psychological element, a motivation. If we are to understand learning fully, we must explore the reasons students do things as well as how they do them.

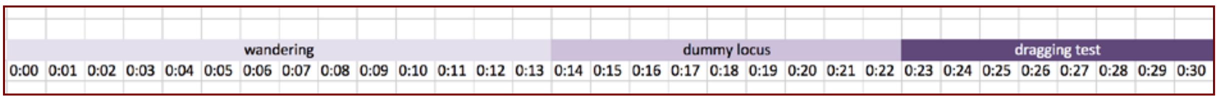

Figure 1: A sample timeline that records intervals of modality use

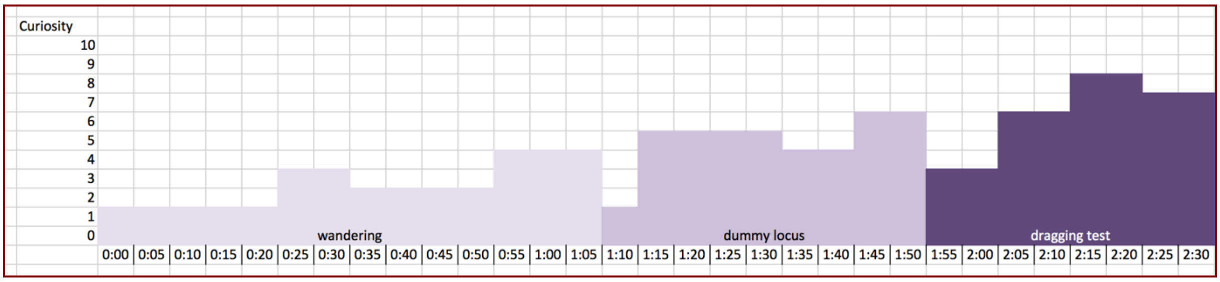

Figure 2: A sample timeline that records intervals of modality use as well as levels of curiosity

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WORDS IN CONTEXTS: 'PROOF' AND 'PROVE' IN A COURSE OF MATHEMATICS LECTURES

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This paper addresses the question: "what do the lexemes PROOF and PROVE mean when they are uttered in a course of undergraduate mathematics lectures". 35 lectures of a third-year abstract algebra course were videotaped and transcribed. The transcript was broken into units called stanzas, and the stanzas into units called lines. A corpus linguistic approach to the transcript is taken, and we use the surrounding stanza as the context for our lexemes. We find that: 1. 'Proof' gets explicitly defined by the professor. 2. Two written proofs are often explicitly compared and contrasted. 3. Whether or not some argument constitutes a proof is contested on a few occasions by the students and the professor. 4. 'Proof' gets contrasted with conceptually close but distinct notions, including illustration, model for a proof, outline, and main idea.

THE MATHEMATICS LECTURE AND PROOF

Speer, Smith, and Horvath (2010) lamented how few research studies there were of the undergraduate mathematics lecture. They highlighted the work of Weber (2004) as a rare exception: he performed a case study of a lecturer in real analysis, with a focus on his proofs. Nardi (2008) conducted interviews with a number of mathematicians, eliciting from them fascinating and thoughtful comments on their pedagogy and strategies in the classroom. The work in this paper is derived from a larger case study of a lecturer in abstract algebra. The topic of proof has been central to mathematics education research at the undergraduate level for many years (Mejia-Ramos & Inglis, 2009).

THEORETICAL FRAMEWORK

The work of John Sinclair, arguably the leading proponent of corpus linguistic theory and methodology when the field was in its nascent state, emerged out of a tradition of empirical linguistic analysis associated with figures such as John Firth and Michael Halliday. Three principles can be said to have guided his work, which we adopt here. Firstly, he preferred as often as possible not to mark up or tag his texts, but to leave them clean. His argument was that it is too easy to find the structures that our linguistic theories predict ought to be there if we tag the text using these theories. This principle he refers to as "trusting the text" (Sinclair, 1991). Secondly, he stressed that patterns that operate on the level of phrases, which with smaller data sets are very hard to find, with larger data sets are possible to spot. Thirdly, he underscored the importance of frequency of co-occurrences of words, which indicate special meanings of words or phrases - many more than were found in the dictionaries at that time (Sinclair, 2004).

Our goal in this paper is to tease out the meanings of *proof* for a member of the community of research mathematicians who teach undergraduates.

Chafe (1994) has emphasized the importance, in spoken discourse, of local contexts where a specific topic is being discussed. He notes that beginnings and endings of these contexts are communicated by speakers in a variety of ways: intonation, volume, pitch, tempo, pauses, body movements, eye gaze, gestures. In the context of mathematics education research, Staats (2008) defended the division of transcripts of classrooms into poetic lines, citing workers in linguistic anthropology such as Dell Hymes and Dennis Tedlock who helped develop the field of study known as ethnography of communication.

DATA COLLECTION AND METHOD

35 50 minute lectures in group theory were videotaped. A 240 000 word corpus was constructed from this data. The transcript was divided into 3004 stanzas, using the tools described by Chafe. Searches for the words *proof*, *proofs*, *prove*, *proving*, and *proved*, yielded 378, 50, 240, 27, and 48 hits respectively. The stanzas that these words appeared in were carefully examined to determine the local meaning of the word, and any frequently co-occurring words.

Quotations from the transcript contain the following special characters: the backslash character, /, in the positions that indicate the end of a line; opening and closing square brackets, [and], surrounding words spoken by a student; a short dash, -, at a moment when the speaker cuts themselves off in order to start their thought again. Following the quotation, in parentheses, will be the lecture number followed by the stanza number within that lecture, separated by a period.

FINDINGS AND DISCUSSION

Definition of *proof* offered by the professor

The professor himself offers, on multiple occasions, a definition of the word *proof*. The first occasion is in the opening lecture: "ok so we're gonna be working on constructing convincing arguments / also known as proofs / for the statements that we make" (01.13). Here the phrase "also known as" is a strong marker for definition. The definition contains two parts: a proof is an argument; it is an argument that convinces.

Surrounding the two key words "convincing arguments" are a cluster of words that reveal more about the nature of proofs beyond their definition. We learn that proofs are constructible and that they as a class will be constructing them. Although it is not quite said outright, he appears confident that proving is more or less a single verb, and not an activity that is wildly different every time you encounter a statement you want to prove. What they will be proving is some statements they will make, which suggests some questions: which statements will need to be proved, and which not; and whether different values are attached to these different sorts of statements. It would be misleading to limit the professor's definition of proof, then, to 'convincing argument'.

Indeed the professor returns to his definition a few minutes later:

> you know imagine having a really belligerent conversation / with the textbook author ok / as you read everything say / are you sure / convince me ok / if you have that kind of attitude to the way that you learn / and the way that you write / you're gonna come out with clear persuasive prose ok / clear persuasive convincing arguments also known as proofs. (01.22)

Two new adjectives make an appearance: "clear" and "persuasive". “Clear” tells us something about the manner in which some argument can be convincing; it can be clear, and therefore not confusing, not ambiguous, not complicated, not hard, not too detailed. These five negatives are obviously not synonymous with each other, and it is notable that the single value clarity can repel multiple dangers. “Persuasive” is interesting in that it not only can connote a persuasion that is full and complete, but also a persuasion that is in progress, and whose full and complete success is not yet clear. A persuasive argument may have fully convinced someone, but to another it might just be starting to push them in that direction. All in all, "convincing" seems like the stronger and more committed word, and the professor switches to it in the next line.

In this passage the professor also reveals how he believes an ability to write proofs can develop: by reading the textbook aggressively. He advises a manner of reading that is combative and sceptical, and not congenial and forgiving. This manner of reading is to be kept up at all times: as the student reads everything they should say "are you sure", they should doubt. The student, as reader, is advised to imagine themselves in conversation with the textbook author, and uttering the key phrase "convince me". The direct implication is that if one playacts the role of someone who sets a high barrier to being convinced, then one will develop whatever the power is that allows one to write arguments that convince others.

Who are these others? On two other occasions the professor returns to his conscious attempt to characterize the word *proof*: "proof is an argument that convinces someone who's never seen it before" (05.70); "whereas I say that proof is an argument that convinces someone who's never seen it before" (14.24). Here the power of the argument is truly made clear: even someone who has never seen the proof before, will, when they read it, become convinced.

Explicitly contrasting two written proofs

In the opening lecture the professor advises the students to carefully compare their written solutions to homework problems to his written solutions:

> compare what you wrote with what I wrote / comp- ok first of all did I get the same answer / secondly which argument is more convincing / mine or his / why is that / ok what's different about what my instructor thinks is a convincing argument / and what I submitted / ok is mine better than his / maybe it is / is his wrong / yeah that happens too. (01.26)

Comparing two written arguments side by side, and deciding which one is to be preferred, is a theme that runs through the course. There are numerous references to rewriting: "now obviously I've cooked this proof up / this is not my first attempt or even my second / I've worked through it" (05.72); "that's like the third rewrite or whatever" (18.72).

Rewritings involve the writer comparing two of their own writings side by side. On three prominent occasions the professor compared his own written proof to the proof offered in the official textbook of the course. On each occasion, the professor highlights a specific contrast:

> Gallian gives a short little argument / which I blithely followed the last time I taught it / and when I came to look at it this time I thought 'says who?' / I think he skipped a big step. (21.16)

The contrast here is that Gallian considered some statement in his proof as not requiring explicit justification, expecting the reader to supply it. Presumably, Gallian expected such a justification to be quickly forthcoming in the reader, either instantly, or relatively instantly by jotting down a couple of lines. To the professor the omitted justifications seemed to him to be significant enough to be written down in full in the proof. The proofs are otherwise of course very similar.

The second occasion features two proofs that are quite dramatically different: "let me show you Gallian's proof of theorem 5.2 / proof starts here and ends here / and look how many symbols he's got / now what is he doing?" (14.23). His own proof is a single sentence that is 2 lines long, contains only words and no equations and only 5 symbols. He goes on to speculate that in his experience students, when they first write proofs, get seduced into thinking that enough symbols thrown around will eventually coalesce into a proof. He stresses on the other hand the "argument that convinces", and repeatedly asks the class if they are convinced by his argument.

The next class, a student returns having carefully thought about his own reactions to both proofs:

> [I agree with your proof / I just think you have to think about yours a lot more] really ok / [than the one in Gallian / Gallian's just like- / grabs your hand- / and then takes you through / the whole thing] right / [whereas yours you have to like / think about it and convince yourself / that's what I found]. (15.03)

The professor's response is to ask for what the student thinks needs to be added to the professor's proof in order to make it a clear proof. After some back and forth, and with the help of another student, the professor proposes a new clause to insert into his one sentence proof, which the student accepts as being a marked improvement. The professor then repeats his contrast with Gallian's proof, by saying that even with the additional clause, his proof emphasizes a verbal argument as opposed to a symbol-heavy approach.

We have seen two written proofs being contrasted in order to indicate that one has a gap, and also two written proofs contrasted in order to show a difference between a proof dominated by words and one dominated by symbols. A third type of contrast occurs much later in the course, and serves as a more sophisticated example. Here the contrast is between a proof that refers to a previously proved lemma, whose statement he can look up quickly and unambiguously, and a proof that appeals to a similar argument that occurs within another proof (which forces the sceptical reader to look through that other proof and figure out what is similar and what is not, and make the relevant adjustments, and decide whether the argument stays sound):

> I remember working very hard on this proof / trying to make it as clean as I could / I found it uh- / and again that went back to how we had- / that's why this lemma- / cause we didn't just use this once / we used it twice / that's why I pulled it out as a lemma / I got fed up with Gallian saying / oh by the innards of some working- / by a previous argument we know or whatever / I was like no / state it explicitly. (33.90)

To extract from a proof a portion of material that can be arranged into a lemma statement and proof of that statement is a standard part of the work that a research mathematician does, and it is noteworthy that some modelling of this practice occurs in this upper year undergraduate course. The students hear their professor valuing explicitness (and therefore checkability), and if they imitatively adopt his value system, will learn to avoid arguments that include justificatory lines of the sort "by the same argument as in the proof of Theorem 2.1, we have…".

Students contest whether a proffered proof actually constitutes a proof

It is not surprising that after so many lectures of observing their professor assess how convincing a particular argument is that episodes developed where it was the students turn to flag a proposed argument (that their professor was fine with) which they believed required more justification. The finest example of this occurred when the professor drew a quick diagram on the board in order to justify a certain set theoretic statement. It was obvious that for the professor this was convincing, as he likely felt very confident in his ability to translate the diagram into the required few lines of proof – so much so that the diagram itself constituted enough justification for him. Not so for one student, who was capable of performing the same translation of diagram into words and symbols, but felt morally required to include these lines in the actual proof:

> [shouldn't we prove that?] yeah I just- / well I just did / [proof by picture?] / are you convinced? / who's convinced? / hands up / see a proof is an argument that convinces- / I don't know / should I / maybe / I don't know / I think it's ok here / I th- I think that's ok / but I grant you that's a little bit borderline. (29.51)

Throughout the course there were ongoing comments about the sorts of lines they used to include in proofs near the beginning of the course, when they were first getting accustomed to properties like associativity, and cancellation, and which gradually and then more rapidly began to be dropped from proofs as too obvious to mention.

Matters of Style in Writing Proofs

Although an absolute distinction can be difficult to draw, it seems clear that while there is a category of comments, considered above, that deal with values, and in which the professor has a strong opinion about which value to cherish the most, there is also a category of comments that deal with matters of style, and in which he allows that there can be legitimate disagreement, even though he has a marked preference for one style over another:

> so I want someone to see where I'm going at the outset / as quickly as possible / I want the reader to try and understand / the structure of my argument / I find it really difficult and annoying to read a proof / where I'm in the nitty gritty detail / and I have no idea why they're doing what they're doing / I find it very very hard to follow / so I try when I'm writing a proof / to give the big picture up front if I can. (05.72)

The stylistic preference is for the main structure of the proof to be presented at the beginning, so that the reader is clear on the overall strategy as soon as possible. This is a theme the professor returns to frequently. Here he defends this practice by noting that he himself can easily get lost trying to follow a proof if the architecture of the argument is unclear, and he doesn't understand on a global level where the argument is headed or why. A little later he elaborates on his reasons:

> but- so one reason I like to write proofs like this / is I find it easier to follow / but the second way- / reason I like to write my proofs like this / is cause I find it easier to do the proof right? (05.83)

So to the ease of reading a proof whose structure is set out at the beginning is added the ease of actually coming up with a correct proof when this writing procedure is followed. The point is later made more forcefully using the contrapositive:

> a different style would be to go all the way through to the end and say / hey look! we did it! / and then you have to say / we did? really? when? how? (11.21)

Moments of needless surprise in reading a proof are to be avoided. This sounds so reasonable that one might wonder why anyone else would develop a different writing style. However, this neglects the important role of compression in writing proofs noted above. All things being equal, the shorter punchier denser proof, to the more experienced reader, will be preferred to a longer one that includes too many lines which that reader finds self-evident. So to those readers to whom the sort of structure of the ensuing proof is tediously familiar, an opening line or paragraph that states this obvious-to-them structure will be unnecessary and perhaps even get in the way.

In addition to setting the structure out up front, the professor frequently discusses the visual layout of the proof on the board, or on the page, with a view to making transparent the structure of the proof: "and there we go / the structure of the proof jumps out / even if we glance at it / because of the two headings / and the fact that I'm- / indent my writing" (14.37). The word "glance" is significant, as it repeats the theme of speed and economy in reading and understanding a proof. A longer proof can be read in a shorter time given the appropriate typography. Comments about headings, titles, bulletpoints, numbered claims, vertical alignment using indentation, and underlining, occurred often in the course: "and as usual / notice that I'm trying to make it easy to read the proof / with my case 1 and my case 2 and my underlining" (09.48).

Whereas the professor confidently asserts his belief in the importance of these stylistic choices, he is less declarative about the stylistic choice between a direct proof and a proof by contradiction:

> so how do we want this / I'm gonna write this as a contradiction / suppose for a moment for- / you don't have to do this as a contradiction / you could go forward way / but I just thought that this was clearer for- for- / for my reading it was clearer to set up a contradiction. (10.55)

He seems to see both approaches as valid options, although he remarks that one is slightly preferable. Later he comments that a former student suggested he proved too many theorems by contradiction, and he acknowledges the truth of this: "I was like yeah you're right / I turned that whole thing back to front for no good reason." (18.07).

Proof/proving distinguished from conceptually nearby noun/verb

Some forms of writing on the board serve a purpose close to the purpose that a proof would serve, but can be conceptually distinguished from proofs. For example, if a result is true for all integers, and if the statement is carefully studied in the case of a single integer, then the professor terms this as "illustrating" rather than "proving":

> if I want to prove that in general / I need to make a statement for every possible case / I'm just gonna illustrate one case / ok so I'm not proving this statement / I'm illustrating this statement. (04.50)

Sometimes the consideration of a single case does not lead to enough insight so as to prove the general statement. In this example it turned out however that the argument justifying the single case could easily be rewritten into a proof of the general statement, so the professor introduces another term: "but the example I give you will give you a model / which you could use to prove the statement in general" (04.50). So sometimes a mere illustration can be promoted to a model for a proof (while still not constituting a proof itself, for this professor).

Another distinction introduced is that between a proof and an outline of a proof:

> and the proof / well it's an outline / I'm not actually gonna- / I'm gonna define two maps / and then I'm going to say they are isomorphisms / and I'll leave you to check that they are isomorphisms / it's just a bit painful to do it all in class. (23.20)

Note that the professor's preference for setting out the structure of a proof up front dovetails nicely with leaving oneself the option of writing only an outline of a proof rather than the entire proof. Details that would back up one or another claim can either be proved later by the author, or left for the reader to fill in if there is no time. In practice, as the professor himself comments, the parenthetical command "(check!)" can substitute for the writing which would have appeared in that location and which would have constituted a justification for the current claim in question. As the course goes on, and the theorems become harder and the time left grows shorter, more of these shorter justifications are left for the student to provide.

The last key distinction the professor draws is between a proof and the main idea or key idea of the proof. Often this main idea can be communicated, perhaps relying more on intuition and less on rigor:

> the proof according to Gallian is long and difficult / and it certainly looks long and difficult the way he writes it out / we're gonna skip that and focus on applying the theorem / but I will note in passing that Nathan Carter has a field day with this / and says you know the proof of the fundamental theorem of finite abelian groups / is pages long and it looks so complicated / but it's such a simple idea / if only you would start thinking about things visually. (31.72)

The professor is referring here to a second book that though not the official textbook is a book he recommended at the start of the course and one that he frequently refers to in the course (and indeed some of the material he presents in class originates in that book, which he tells that class he worked through carefully in preparation for teaching an earlier incarnation of this course). So again, as the course gets further along, another choice the professor can make is to substitute a careful proof of a theorem (in this case the Fundamental Theorem of Finite Abelian Groups) with a discussion (along with accompanying diagrams) of why one might have expected this result to be true. Here the belligerent sceptical reader invoked in the first lecture would of course not be satisfied, but this manner of reading is not appropriate for a discussion of a main idea, nor for a discussion of other situations in which one might expect to see more applications or consequences of this main idea.

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YOUNG CHILDREN'S UNDERSTANDING OF BENCHMARK ANGLES IN A DYNAMIC GEOMETRY ENVIRONMENT

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This paper examines young children's thinking about benchmark angles in a dynamic geometry environment. Using the dynamic sketches in Sketchpad, kindergarten children were able to develop an understanding of angle as "turn", that is, of angle as describing an amount of turn. Children experienced different realizations about the benchmark angles and showed a shift from context specific descriptions to more general descriptions. Children's gestures, motion and environment played an important role in their thinking.

INTRODUCTION

Angle is a basic concept that is used by humans in analysing their spatial environment. Even though children show sensitivity to the concept of angle from very early years (Spelke, Gilmore, & McCarthy, 2011), the multi-faceted nature of angle concept can pose challenges to learners, even into secondary school (Mitchelmore & White, 1995). Angles are normally introduced to children quite late in formal school settings. For example, in British Columbia, they are introduced in grade 6 (11-12 years old). The strong capacity of young children to attend to and identify angles in various physical contexts motivated the present study which aims at exploring the learning of angle concept at the K-1 grade levels. This paper reports on the working of Kindergarten/Grade 1 split class children on the concept of angles using sketches pre-constructed with *The Geometer's Sketchpad* (Jackiw, 1991/2009). The focus of this research is to study how the use of this Dynamic geometry environment (DGE) affects the children's thinking about angles as a "turn". This paper, in particular, discusses the episodes where children attempt to explore the benchmark angles, after the 'angle as a turn' notion was developed in the first few episodes.

CHILDREN'S UNDERSTANDING OF ANGLE

In the research literature, the concept of angle is shown to have different perspectives, namely: angle as a geometric shape, union of two rays with a common end point (static); angle as movement; angle as rotation (dynamic); angle as measure; and, amount of turning (Henderson & Taimina, 2005). An understanding of angle incorporating all three definitions is a complex task that can be slowly developed over a long time (Lehrer, Jenkins, & Osana, 1998). Much research has been conducted on the development of the concept of angles, focusing at the grades 3, 4 and higher levels. Mitchelmore and White (1995) suggest that angles occur in a wide variety of physical situations that are not easily correlated, which makes it difficult to understand the concept of angle. Students also think that the length of the arms is related to the size of the angle (Stavy & Tirosh, 2000). Most studies on benchmark angles are with older students, fourth and higher grades (Browning, Garza-Kling, & Sundling 2007; Millsaps, 2015; and Crompton, 2015). These studies mainly focus on comparing the relative size of angles using the terms acute, obtuse and right with reference to the benchmark angles and they used the standard units of angle measurement (degrees). The present study is different from the above studies in the sense that it does not focus on the standard measurement using dynamic protractor and classification of angles; rather it focuses on promoting and developing the interpretation of benchmark angles as a full turn or part of a full turn using dynamic sketches in Sketchpad.

THEORETICAL PERSPECTIVE

This study draws on the participationist view of learning as proposed by Sfard (2008) that recognizes a close relationship between thinking and communication. Sfard (2008) offers a communicational approach in her discursive framework, which is well suited to this study and has been shown to be effective by other researchers (Sinclair & Moss, 2012; Kaur, 2015) because it enables researchers to make claims about students' thinking in terms of how students communicate. Sfard views thinking as a form of communication and knowing of mathematics as synonymous with the ability to participate in mathematics discourse. Thus, mathematical learning is the development of a mathematical discourse. According to Sfard, the mathematical discourse has four characteristic features: word use (vocabulary), visual mediators (the visual means with which the communication is mediated), routines (the meta-discursive rules that navigate the flow of communication) and narratives (any text that can be accepted as true such as axioms, definitions and theorems in mathematics). Learning geometry can thus be defined as the process through which a learner changes her ways of communicating through these four characteristic features. In this paper, it will be interesting to see whether or not there is a change of discourse about benchmark angles when children, along with their teacher, interact in a DGE. Sfard's four features of mathematical discourse are described mainly in terms of verbal discourse. Since spoken discourse is multimodal, sometimes it fails to account for the full set of resources used by young children to communicate. Given the importance of gestures in communication of abstract ideas (Cook & Goldin-Meadow, 2006), and their potential to communicate temporal conceptions of mathematics (Núñez, 2003), it is necessary to focus not just on the words or the visual mediators that the children use, but also on their gestures.

RESEARCH CONTEXT

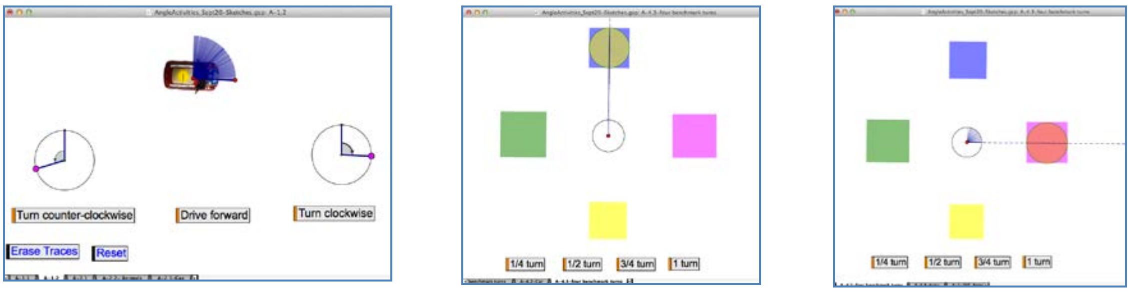
Participants and tasks

We (research team and class teacher) worked with 22 kindergarten/grade1 children (aged 5-6) from a school in a rural low SES town in the northern part of British Columbia. The lessons related to angle were designed along with the classroom teacher, who has been developing her practice of using DGEs for a couple of years.

The teacher and children worked with angles in different ways, using *Sketchpad* for ten sessions (30-40 minutes each) in a whole class setting with children seated on a carpet in front of an IWB (Interactive Whiteboard). The children have worked with the sketchpad prior to the instruction of angles, where they explored the concept of symmetry using the sketchpad. All the sessions were videotaped and transcribed. This paper will focus only on the two (seventh and eighth) sessions, where students explored the benchmark angles.

Dynamic angle sketches

Using *The Geometer's Sketchpad*, different sketches were designed to explore the concept of angle with the children during ten sessions. The concept of angle was introduced using the 'driving angle sketch'. The 'driving angle' sketch (Figure 1a) shows both a static as well as dynamic sense of angle. It includes a car that can move forward as well as turn around a point. The turning is controlled by two small dials (each of which has two arms and a centre) - one dial allows clockwise turns and other counter-clockwise turns. No numbers are used. Five action buttons (*Turn counter-clockwise*, *Drive Forward*, *Turn clockwise*, *Erase Traces* and *Reset*) control the movement of the car. In this sketch, the turn of the car is associated with the amount of angle adjusted in the small dials. The traces of a turn offer a visible, geometric record of the amount of turn. So, in the first few sessions the children explored and understood the turning of the car and its association with the dial.



1(a) Turning trace of car after pressing Turn clockwise button

1(b) Benchmark angles sketch

1(c) Traces after pressing ¼ turn button

Figure 1(a, b, c): Dynamic sketches created in Sketchpad

After the children developed some sense of angle as a turn, they were presented with benchmark angles sketch. For this purpose, a sketch shown in Figure 1(b & c) was designed using four different coloured squares positioned in a way so that there is a 90° angle between any two consecutive squared boxes. There is a circle in the centre, which is attached to another circular ball using a ray. There are four action buttons (¼ Turn, ½ Turn, ¾ Turn and 1 Turn) that control the movement of the ball. When you press any action button, the ball moves clockwise from one square to another, tracing its associated turn in the central circle. For example, pressing ¼ Turn button moves the ball by 90° (to next square) and generates the traces in the central circle (see Figure 1c).

The purpose of this sketch was to help children understand the commonly used benchmark angles without referring to any standard measurement units of angles such as degree, radian etc.

EXPLORING THE BENCHMARK ANGLES

To begin the seventh session, the teacher hid the ¾ turn button and presented the benchmark angles sketch with three buttons (1 turn, ½ turn and ¼ turn buttons). The following excerpt outlines the initial interaction with the benchmark sketch.

No.	*Who said/did*	*What was said/what was done*
887	Teacher (T)	How many… I see a blue box, a purple box, a yellow box, a green box (*pointing one by one at each box in Figure 2a*)? How many boxes does a full turn going to touch?
888	Larry	Four
889	Sss	Four
890	T	Four. Okay let's see if you are right. (*Neva presses the 1 turn button*)
891	Larry, Kia	Five
892	T	Why five?
893	Pat	Because it touch the same one
894	T	Because it touch the same one…two times.
Then the teacher asks children to guess how many boxes will a half turn touch.		
897	Pat	Three
Teacher asks Larry to press ½ turn button.		
899	Larry	(Larry presses ½ turn button) <Figure 2b>
900	T	How many boxes did it touch?
901	Sss	Two, two, three
902	T	It was touching that one too (pointing at the top blue box and gesturing half turn clockwise with right hand, Figure 2c), so we will go three

To begin with, the teacher drew children's attention to all the four coloured boxes and asked, "how many boxes a full turn is going to touch"? Larry and other children's responded with utterance "four" as in [888], [889]. The utterance "four" might be due to the fact that there were only four boxes on the screen. It is worth noting that the children showed interest in turning the car by full turn during the first session. So, they were familiar with the notion of full turn to some extent. So, maybe Larry and other children visualized the one turn and concluded that the ball would touch the four boxes. When Neva pressed 1 turn button, Larry and Kia uttered the word "five" [891], hence changing the initial utterance of "four" [888]. It seems like the rotational movement of the ball from blue box to purple, then yellow, then green and finally blue again acted as a visual mediator and provoked Larry and Kia to change their response to five. This is confirmed by Pat's utterance, "Because it touch the same one" [893]. Thus, the traces of turning ball provided a visual mediator for the children to count the boxes touched by the ball to five, although there were only four boxes present on the screen. This description of full turn in terms of number of boxes is very situated and is the result of the design of the sketch.

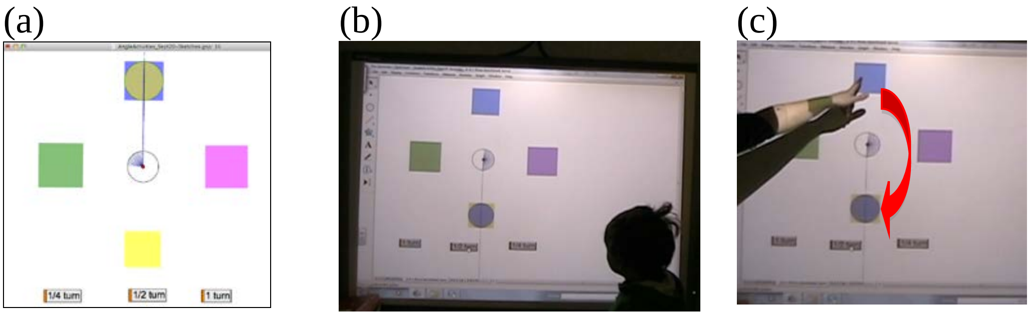


Figure 2 (a-c): (a) The benchmark angles sketch with three buttons (¼ turn, ½ turn and 1 turn); (b) Traces after Larry presses ½ turn button; (c) The teacher's gesture of half turn

Later, when teacher asked the children to guess the number of boxes touched by a half turn, Pat responded with "three" [897]. This shows that Pat had developed a routine of counting the starting box. The children's mixed responses "two, two, three" [901] on seeing the half turn taken by the ball suggests that some children counted the starting box for the ball, while others did not. Both the responses are correct depending upon the inclusion or exclusion of the starting box. The teacher's utterance "It was touching that one too, so we will go three" [902] with her gesture (Figure 3c) suggests that she endorsed the routine of inclusion of the starting box while counting the number of boxes for a particular turn. Her half turn gesture from blue box, then purple and finally yellow box provided another visual mediator for the children to count three boxes for a half turn. Later, for quarter turn, the children gave mixed responses of "one" or "two" depending on their inclusion or exclusion of starting box. Thus, in the above episode, the children developed a routine of counting the boxes for describing the different benchmark turns. This routine is first initiated by teacher's specific questions about how many boxes would be touched by a particular turn.

DEVELOPING CONCEPTIONS OF ONE FULL TURN

After the initial discussion of the benchmark turns in terms of number of boxes, the teacher asked the children to predict the colour of the box on which the ball would land after a particular turn. To start with the ball was at blue and the teacher pressed the 1 turn button and the ball went back to blue again. The following excerpt describes the episode, when the ball was at the green box (Figure 3a) and the teacher asked the children to predict its position after one turn. The children gave mixed responses of yellow, purple and green.

No.	*Who said/did*	*What was said/what was done*
965	T	Why purple, why green, why yellow, if I move it by one full turn?

Maria's response for green is as follows:

967	Maria	Because last time when it went around, it went to the blue again (*making a full turn gesture with her right hand, Figure 3b*), it might go back to the green again.

After a little class discussion, teacher asks Peter to press the 1 turn button to verify the thoughts of everybody. The ball lands on green.

978	T	It landed on the green. So what does one full turn mean?
979	Maria	It means it goes back to the same colour.
980	T	It means it goes back to the same colour. What else can one full turn mean? Maya
981	Maya	Full (*Gesturing a full turn with her right hand 3 times repeatedly, Figure 3c*)

After little more discussion, the teacher moves the ball to the purple box (Figure 3d) and asks what colour will it land on?

990	Sss	Purple
991	T	Why?
992	Maria	Because it goes back to the same colour (*making a full circle gesture with right arm, same as Figure 3b*)
993	Larry	It goes back to the start

Maria predicted that the ball might land on the green box. Her utterance, "Last time when it went around, it went to the blue again, so it might go to the green again" [967] along with the full turn gesture with her right arm (Figure 3b) suggests that she projected the ball on different starting position (colour) and she was using her embodied gesture as a visual mediator which helped her to predict the position of the ball after one full turn. She associated her description of the full turn with the words like "around" and "again". The word "around" suggests the movement and the word "again" suggests the recurrence of something. Thus, Maria's description of one full turn is dynamic in nature, which might be partly due to the dynamic nature of the learning environment. Maria's later statement about full turn, "It means it goes back to the same colour" [979] is more general and is independent of any particular colour.

Upon asking about other ways to describe a full turn, Maya did a repeated circular gesture (Figure 3c) with her right arm. Maya's this description of full turn can be considered as a general gesture for the full turn, as she didn't start her circular gesture from the side (pointing towards the green colour) rather she started her circle from the top and finished towards the top direction after going around the full circle. Maya's circular gesture is associated with the motion and hence suggests a dynamic sense of angle.

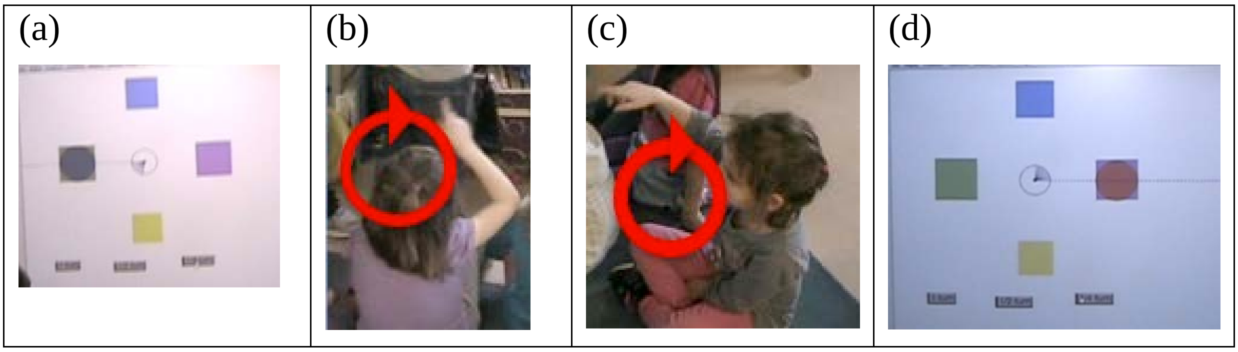


Figure 3(a-d): (a) Ball on green box when teacher asked the question; (b) Maria's gesture for one full turn; (c) Maya's description of a full turn; (d) Ball at purple

If one compares Maria's utterance [992] and Larry's utterance [993], it can be noticed that Maria's description is context specific as she uses the words "back to the same colour", whereas Larry's utterance, "It goes back to the start" is independent of any context.

After the initial review of the realizations of the full turn, quarter turn and half turn in terms of number of squares passed by them, the teacher proceeded to the discussion of three quarter turns.

1213	T	One quarter touches how many squares?
1214	Sss	One
1215	T	One more. So it touches two of the squares. This one (pointing at blue square) and this one (pointing at purple square)
1216	T	Okay, so a half-turn goes past how many… touches how many?
1217	Sss	Two
1218	T	It goes past two. Okay. And then three quarters is going to go past how many?
1219	Sss	Three, three

The teacher's question, "then three quarters is going to go past how many?" (1218) invited the children to predict the number of boxes covered by the three quarters turn, to which the children responded "three" (1219). The response "three" might be initiated due to different factors for different children. First, it might be due to the sequence in which the teacher asked the questions. The teacher asked about the number of boxes covered by a quarter turn (1213), half turn (1216) and three quarters turn (1218) respectively. The increasing number of boxes for each subsequent response (1214, 1217, 1219) might have triggered the use of word "three". Secondly, the presentation of the four turn buttons in the sketch follows the increasing sequence (¼ turn, ½ turn, ¾ turn and 1 turn), which might have initiated the use of word "three" for three quarters turn. Thirdly, the utterance "three" might be occurred due to the number "3" on the ¾ turn button in the sketch. Fourthly, because the children had enough practice with the quarter turn, half turn and full turn in the previous session where they saw one, two and four boxes covered by these turns respectively, so the only choice left was "three". Lastly, the children might be visualising a quarter turn three times to see how many boxes would be covered by a three quarters turn.

After some practice with the use of the ¾ turn button, the teacher asked the children if they can reach at a particular colour with the use of two turn buttons.

No.	*Who said/did*	*What was said/what was done*
1237	T	Okay come on up and do it Maria. I want it to land on the purple, but I want you to take two turns to do it. (*The ball was on the yellow box, Figure 4a)*
1238	Maria	(Maria presses the ½ turn button first (*Figure 4b*) and the she presses the ¼ turn button (*Figure 4c*)
Later the teacher asks the children to articulate the combination turns for three quarters turn.		
1242	T	So the three quarters is the same as what? It is same as what other ones?
1243	Maria	Half and quarter

To make the ball turn from yellow box to green box, Maria used the combination of ½ turn and ¼ turn buttons (1238) respectively. After Maria's physical manipulation on the sketch, the teacher invited the children to verbalise her actions. The teacher asked about the other turns that are equivalent to three quarters turn. Maria responded "half and quarter" (1243) were the same as the three quarters turn. Thus, for the realization of a three quarters turn as a combination of a half and a quarter turn, Maria's physical manipulation was followed by her verbal actions.

(a) (b) (c)

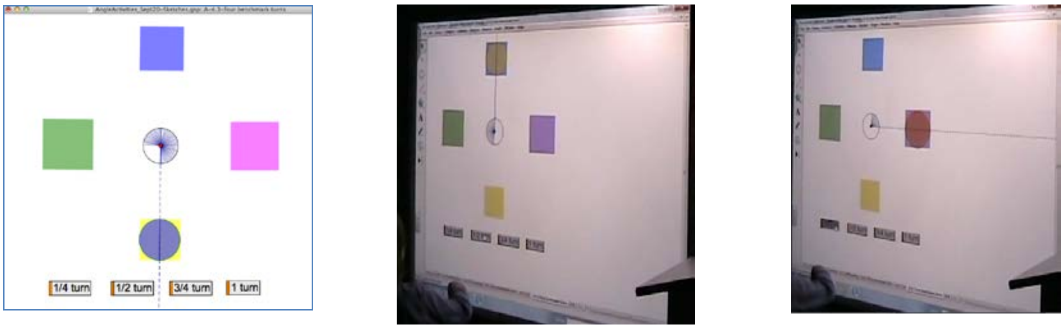


Figure 4 (a-c): (a) the position of the ball when Maria started the task; (b) the position of the ball after Maria pressed the ½ turn button; (c) the position of the ball after Maria pressed the ¼ turn button.

DISCUSSION AND CONCLUSION

Thus, in the above episodes, initially the children developed a routine of counting the boxes for describing the different benchmark turns. They formed a mixed routine of inclusion or exclusion of the starting box for their counting. This description of benchmark angles was very context specific. The realization of signifier one full turn unfolded in a series of steps. Maria first described one full turn as starting at a particular colour and then ending at that particular colour again. Later, her description of full turn was replaced with more general description as going "back to the same colour". The use of words "same colour" in [979, 992] encapsulates all the four colours (blue, purple, yellow and green), thus turning the different coloured boxes in a single entity. And finally Larry's context free description of full turn as going "back to start" covers all instances of a full turn, thus reifying the discourse about one full turn to some extent. Maya and Maria also described one full turn through their embodied actions. It seems like their gestures (Figure 3b, c) are perfect descriptions for one full turn and these can be treated as "embodied narratives" for the signifier full turn.

The teacher's move of not presenting the ¾ turn button in the sketch initially was helpful in arousing the need to use the combination of two turns together and in setting the stage for introducing the concept of three quarters turn. The children demonstrated that a three quarters turn was same as a half turn and a quarter turn together.

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DISCURSIVE PATTERNS IN THE MATHEMATICS TEACHER BLOGOSPHERE

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Teacher collaboration is essential for improving teaching, but is often difficult to establish and sustain in a productive manner. Despite this, an unprompted, unfunded, unmandated, and largely unstudied mathematics teacher community has emerged where mathematics teachers use social media to communicate about the teaching and learning of mathematics. This paper presents an analysis of one episode where teachers engage in a prolonged exchange about responding to a common mathematical error. Analytical tools drawn from variation theory are used to explain generative moments of interaction. Results indicate that discursive patterns signal taken-as-shared pedagogical approaches, which can extend the space of possible variation while establishing a range of permissible change.

INTRODUCTION

Teacher collaboration, community building, and networking are indispensible components of effective teacher professional development (Lerman & Zehetmeier, 2008). However, due to time, funding, and facilitation constraints, teacher professional development initiatives are commonly limited to sparse one-time workshops held in face-to-face synchronous settings. Such workshops, due to their temporal nature, are generally unconducive to building sustainable communities in which teachers collaborate daily.

In contrast to centrally organized and synchronous professional development initiatives, teachers from across North America are participating in decentralized, virtual, and autonomous professional communities. One such community involves hundreds of geographically separated mathematics teachers who regularly use Twitter in conjunction with blog pages to publicly communicate their musings and practices, and have come to be identified as the Math Twitter Blogosphere (MTBoS) (Larsen, 2016). These teachers participate frequently, and hold prolonged conversations about mathematics teaching that are longer than typical interactions on Twitter. Most Twitter reply threads do not extend past two replies deep, meaning that most Twitter posts are not likely to get a reply followed by a reply to that reply (Sysmos, 2010). However, conversations among mathematics teachers on Twitter can at times extend more than 20 replies deep. As such, Twitter is providing space for asynchronous mathematics teacher collaboration in an unmandated virtual environment.

This unprompted, unfunded, and unevaluated teacher community is a rich phenomenon of interest that is largely unstudied. Only one empirical investigation into the nature of this community has been undertaken by Parrish (2016), who identifies the

MTBoS as "a promising avenue for providing support to teachers in selecting and implementing cognitively demanding tasks" (p. ii). Parrish (2016) indicates that further study is needed to explore the nature of interactions in the MTBoS community, and to identify what receives attention in this community. As such, this paper aims to investigate the nature of prolonged interactions between mathematics teachers on Twitter that extend past three replies deep. This investigation is part of a larger study on the affordances of the mathematics teacher blogosphere for teachers who engage in it regularly. This paper is not concerned with teacher knowledge (Ball, Thames, & Phelps, 2008; Shulman, 1986), but rather with the discursive elements that promote prolonged social interaction around issues of mathematics teaching in the Twitter environment and the issues that these interactions illuminate.

THEORETICAL FRAMEWORK

The underlying theoretical framework in this study is that of *communities of practice* (Wenger, 1998) because it is a mid-level theory that accounts for situated participation. *Communities of practice* is a social theory of learning where learning is considered as increasing *participation* in the pursuit of valued enterprises that are meaningful in a social context. *Practice* is at the heart of Wenger's (1998) *communities of practice*, and a key aspect of *practice* is the ability to motivate the social production of *meaning*. The continuous production of *meaning* is termed as the *negotiation of meaning*, which in the blogosphere happens through continued interactions (Larsen, 2016). In the search for *meaning* within mathematics teacher interactions on Twitter, a more specific lens is necessary to examine how and what teachers co-constitute when engaging in prolonged interaction on Twitter. To this end, a theory that reveals the space of possible learning within a set of interactions, such as *variation theory* (Marton & Booth, 1997; Runesson, 2005; Watson & Mason, 2006), is desirable.

Variation theory is fundamentally interested in identifying the *space of possible variation* and the *range of permissible change* (Watson & Mason, 2006). Watson and Mason (2006) explain that by identifying aspects that are kept constant and aspects that are varied (*space of possible variation*), as well as how they are varied (*range of permissible change*), the *object of learning* may be revealed. According to Runesson (2005), "the enacted object of learning [is seen] as a space of variation . . . [and] as a potential for learning" (p. 83). That is, by looking for *variance* and *invariance* within exchanges, it is possible to identify what is available for participants to notice.

Further, Watson and Mason (2006) note that sensing the possible variation in a relationship is considered an act of *generalization*, which requires a heightened level of awareness. Mason and Pimm (1984) note that such awareness develops through experience, and may be brought forth through *stressing* and *ignoring* various features through *variance* and *invariance*. Although these notions are theorized within mathematical contexts, I suggest they may be extended to examples of mathematics teaching. As such, constructs from *variation theory* are used to analyse moments in which teachers engage in prolonged moments of *negotiation* in the MTBoS.

METHOD

Given that the MTBoS began developing as early as 2006 when mathematics teacher bloggers began to incorporate the use of Twitter into their blogging practice, and that there are over 500 self-identified MTBoS members, many of whom post multiple times a day, the sheer mass of data that has accumulated over the past few years makes the phenomenon too large to study within the confines of this paper. As such, a very specific subset is chosen as the data set for this paper. This subset contains all responses to a given Twitter post made by one particularly well-followed member. This conversation reflects the breadth and depth of MTBoS because it includes both very brief responses that do not continue conversation, and responses that initiate further conversation, both of which are generally encountered within the MTBoS.

Since Twitter is an ultra-personalized environment where users only see posts made by members they subscribe to as ‘followers’, I have taken an ethnographic approach as participant observer by immersing myself in the MTBoS community and subscribing to over 400 mathematics teachers who engage in the MTBoS. Without such an immersion, noticing and identifying the data set would be near to impossible. In addition, Twitter offers a feature which gives updates on the most relevant and most replied-to tweets one has missed. This feature enabled me to identify a particular post that generated a significant number of replies from mathematics teachers around the world. This post was made by Michael Fenton, who has over 4000 followers, and asked users about how they would respond to a mathematical error (see fig. 1).

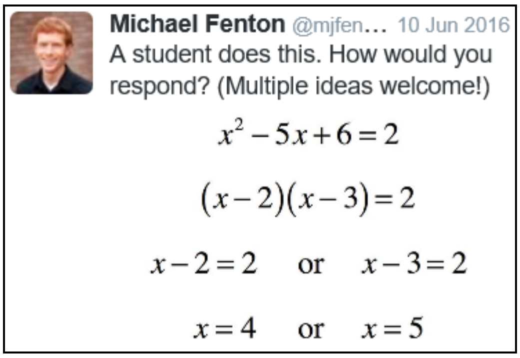

Figure 1: Fenton’s initial math mistake query

Fenton’s post elicited 254 replies from a total of 87 users, 52 of whom identify themselves as mathematics teachers. Replies included explaining the error, explaining why the error could have been made, describing a teaching approach to help the student come to a deeper understanding about the nature of the error, and generating activities to use with students to help mitigate this error. With an effort to maintain the reply structure and the chronological order of posts, the data was organized into threads. Some of these threads were considered as *non-continuing* replies because they were made by one user and spawned little to no discussion. Other threads were considered as *continuing* because they included conversation between at least two users and elicited more than four subsequent replies. Out of the total 254 tweets, 84 were identified as non-continuing, 155 were identified as continuing, and 15 were irrelevant. Since prolonged negotiation is of interest in this study, the 155 continuing tweets were reconstructed based on both chronology and logical conversation order into ten threads ranging from 5 to 45 tweets in depth. A thread of depth 29 was chosen for analysis because it consisted of discussion pertaining directly to Fenton's original inquiry and was adequately deep. These 29 tweets form the data set for this paper.

With an aim of understanding the *meaning* that is made within prolonged interactions among mathematics teachers on Twitter, this data was analysed for *variance* and *invariance* (Marton & Booth, 1997; Runesson, 2005; Watson & Mason, 2006), which in turn helped identify moments of *stressing* and *ignoring* (Mason & Pimm, 1984), and the enacted *object of learning* (Runesson, 2005). In what follows, a reduced version of the selected conversation thread is presented, and is then reviewed in terms of the analytical framework. Conclusions are then drawn to illuminate the nature of prolonged interactions between mathematics teachers on Twitter.

RESULTS

On June 10th, 2016, Michael Fenton (@mjfenton) asks his followers to think about how they would respond to a mathematical error made by a student as shown in Figure 1. One of the reply threads is initiated by Max Ray-Riek (@maxmathforum). Ray-Riek begins exploring the ideas around the prompt, and doesn't end at his first response, but rather, continues his thinking in a journal-like fashion.

June 10	07:50	@maxmathforum	@mjfenton (x-3)(x-2) = 2 still only has 2 answers ... there is only one set of factors of 2 that make this true. Why those? Hmm ...
June 10	07:56	@maxmathforum	@mjfenton I think the direction I'd go is to look at solving a bunch of quadratics that = 2. They all have different factors. Compare to = 0
June 10	08:22	@maxmathforum	@mjfenton I think I'd look at (x+8)(x+4) = 12, (x-1)(x-2) = 12, and (x-6)(x-10) = 12. Analytically we could come up w/ different sol'ns …

Five days later, Michael Pershan (@mpershan) replies to one of Ray-Riek's tweets with an example of an instructional routine activity referred to as an 'equation string'.

June 15	17:19	@mpershan	@maxmathforum @mjfenton How does the approach this equation string aims at compare to what you'd be aiming for? $(x-2)(x-4)=15$ $(A-3)(A-5)=15$ $(A-3)(A-5)=35$ $(Y-3)(Y-10)=0$
June 15	19:19	@maxmathforum	@mpershan @mjfenton not thinking of it as eqn string. Idea is that each has solns at different factors of 2.

June 15	19:30	@maxmathforum	@mpershan @mjfenton oh now I see the string you are talking about. Is the idea here that 1) is easy and 2) is not b/c hard to get 7*5?
June 15	19:33	@maxmathforum	@mpershan @mjfenton oh! Now I see the whole string. X=7, A=0, A=10,Y=3 or 10 ... No, I don't think your string gets at the same idea I had.
June 15	19:56	@mpershan	@maxmathforum @mjfenton Interesting. I thought it might be the same idea because I'm urging you to go back to multiplication here?
June 16	03:35	@maxmathforum	@mpershan @mjfenton definitely related, but it matters to me that I'm focused on three problems that all = 12 but in different ways.
June 16	03:48	@mpershan	@maxmathforum @mjfenton Do you think we're aiming to support learning the same sort of thinking? Different paths to same goal?

After another set of interactions between Ray-Riek and Pershan where Pershan proposes another equation string example, they discuss their intentions more broadly.

June 16	05:06	@maxmathforum	@mpershan @mjfenton I see value in exploring multiple examples that = x and also multiple examples that factor easily in only one way
June 16	05:08	@mpershan	@maxmathforum @mjfenton I don't quite yet understand why we want multiple examples that = x, but I think I'm getting there.
June 16	05:55	@maxmathforum	@mpershan @mjfenton mainly bc kids ignore the negative/non-obvious solutions, hence different problems that obviously factor differently

Ray-Riek then suggests an instructional routine called ‘which one doesn’t belong’.

June 16	05:55	@maxmathforum	@mpershan @mjfenton I wonder about a #wodb with A: (x-2)(x-1)=12, B: (x-2)(x-1)=0, C: (x-5)(x+2)=0 What might kids notice?

This sparks a discussion that draws Kate Fisher (@K8Fisher) into the exploration.

June 16	08:15	@K8Fisher	@maxmathforum @mpershan @mjfenton B/C are equal to 0, A/B have same factors
June 16	10:21	@maxmathforum	@K8Fisher @mpershan @mjfenton Is there a reason why B doesn't belong?
June 16	10:29	@K8Fisher	@maxmathforum @mpershan @mjfenton B's solutions are both positive.
June 16	13:33	@maxmathforum	@K8Fisher @mpershan @mjfenton How do you know that A's solutions aren't both positive?

June 16	13:45	@K8Fisher	@maxmathforum @mpershan @mjfenton Factors have a diff of 1, so pairs must be 3,4 or -3 ,-4. x can either be 5 or -2.
June 16	14:42	@maxmathforum	@K8Fisher @mpershan @mjfenton Oh, that's neat -- that means A and C have the same solutions, too.
June 16	13:46	@K8Fisher	@maxmathforum @mpershan @mjfenton I hope my Ss would see the difference in the factors to simplify solving.

ANALYSIS AND DISCUSSION

In the initial journal-like thread where Ray-Riek tweets out his thoughts in response to the original prompt, Ray-Riek *stresses* factors that have a product of 2 by keeping it *invariant*. He introduces *variance* in the quadratics, and *generalizes* that he would solve "a bunch of quadratics that = 2". Next, he produces an example according to this approach of generating a "bunch of quadratics" equal to a constant value: "I think I'd look at (x+8)(x+4) = 12, (x-1)(x-2) = 12, and (x-6)(x-10) = 12". Here, Ray-Riek is *stressing* the factored quadratic equation structure as well as the notion of product. Ray-Riek introduces a *space of possible variation* by holding the equation format and product *invariant*, and *varying* the factors. He also exemplifies an approach for responding to the original query with a sequence of examples, contributing to a *range of permissible change* and setting the *object of learning* as that of producing examples.

In the next passage, Pershan works within the confines of this *range of permissible change* by holding the format of the equations *invariant* and maintaining an 'equation string' approach. Within this structure, he varies the values of the factors and the products in a way that *stresses* the importance of 0 as a product. Pershan not only *varies* the values, he also *varies* the pedagogical approach and associates it specifically with an *instructional routine* referred to as a 'problem string'. A 'problem string' is known as a practice where "students answer related questions, the teacher models student thinking, [and] students construct relationships and connections" (Harris, n.d., para 3). It is a structure that exists elsewhere in the data and is used by members who are relatively active in the MTBoS. Using the term 'equation string' signals an associated implementation and structure. It also continues to play a role in the subsequent tweets, forming a discursive pattern of *invariance* around a notion that is *taken-as-shared* between Ray-Riek and Pershan.

Although Ray-Riek entertains the idea of Pershan's suggested 'equation string', he concludes that there is *variance* not just in the specific values, but also in the intention of what the string of equations is designed to elicit for a hypothetical student. Ray-Riek draws attention to this *variance* around intention by stating, "No, I don't think your string gets at the same idea I had," which prompts further conversation about approaches towards guiding student thinking. Pershan then responds by asking, "Do you think we're aiming to support learning the same sort of thinking? Different paths to same goal?" At this moment, it seems that they are *negotiating* what is *variant* and what is *invariant* in terms of their pedagogical approaches. This prompts Ray-Riek to restate his intentions of keeping the product the same so that students can potentially have their attention drawn to the notion of how to get products to be the same in different ways. They establish that their *object of learning* is to guide student thinking through a sequence of related mathematical examples, but that there is *variance* in what they are working to illuminate within the examples.

Eventually, Ray-Riek proposes an alternate example, introducing another *instructional routine* that is commonly referred to as 'which one doesn't belong', a discursive pattern used by many members of the MTBoS signalled by the hashtag #wodb. A 'which one doesn't belong' problem typically has three or four different problems or images from which students are asked to identify similarities and differences with an aim to prove how each option could 'not belong'. In this case, Ray-Riek holds the format of the equations and the general structure of a sequence of equations *invariant* as compared to previous examples, but strategically *varies* the product and the factors in such a way that every pair of equations has some *variance* and some *invariance*. He also chooses to *vary* the *instructional routine* in a way that remains within the *range of permissible change* that has been *negotiated* through these interactions. In doing so, Ray-Riek has taken the ideas he began expressing initially around holding the product *invariant*, while also incorporating Pershan's ideas around drawing attention to the product of 0. Through their continued *negotiation* in which various features and intentions are *stressed* and *ignored* using *variance* and *invariance*, Ray-Riek and Pershan have extended the *space of possible variation* and developed novel examples for helping guide student noticing in relation to solving quadratic equations. Interestingly, the use of the #wodb discursive pattern attracts a new member to participate in further *negotiation*. In subsequent tweets, Fisher explores the *variance* and *invariance* between equations provided by Ray-Riek in his #wodb example. In doing so, she *stresses* what students may see and reinforces the *taken-as-shared* understanding of what it means to work through a #wodb example. The *object of learning* is therefore not only to produce examples, but also to interpret them.

CONCLUSION

The above transcript and analysis illustrates a prolonged *negotiation* in response to a specific query. By observing the sources of *variance* and *invariance* between interactions, it is evident that these members are extending the *space of possible variation* while inadvertently establishing and working within a *range of permissible change*. Sources of *invariance,* such as holding the structure of the equations constant, allow for attention to be drawn to sources of *variance*, such as the *taken-as-shared* pedagogical approaches used to enhance the tasks. These *taken-as-shared* pedagogical approaches, including 'equation strings' and 'which one doesn't belong', form discursive patterns that signal a pattern of interaction that shapes the conversation and allows for novel ideas to develop and continue within the parameters of the familiar. In this way, the *space of possible variation* is extended while maintaining a sense of a *range of permissible change* and establishing the *object of learning* as that of innovating and interpreting. The key implications of this are that interactions between mathematics teachers in this virtual social media space can foster novel ideas for teaching mathematics in classrooms. An important aspect found in the case presented in this paper is the use of *taken-as-shared* pedagogical practices signalled by discursive patterns that are recognized by MTBoS members. Further investigation into the roles of such discursive patterns as they occur within prolonged exchanges between mathematics teachers of the MTBoS is necessary because they seem to be a contributing factor in the continued collaboration among teachers and the development of novel approaches to teaching mathematics.

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REPRESENTING MATHEMATICAL LEARNING DISABILITIES: AN ANALYSIS OF A CBC RADIO INTERVIEW ON DEVELOPMENTAL DYSCALCULIA

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On October 23, 2015, CBC radio host Rick Cluff conducted an interview with cognitive neuroscientist Daniel Ansari on developmental dyscalculia, discussing what it is, its effects and its treatments. The purpose of this paper is to apply the methods of critical discourse analysis to examine the interaction between host and interviewee to see what lines of inquiry emerge. The intent is to demonstrate how the nature of the medium positions the host and interviewee in relation to dyscalculia, and how the medium represents developmental dyscalculia, those who have it, and its treatments to the CBC audience. Analysis suggests that the radio interview enables certain traditional storylines regarding developmental dyscalculia to be told while also allowing some alternative ones to emerge.

INTRODUCTION

This study is part of a larger research project aimed at analyzing the various representations of mathematical learning disabilities (MLD) in various media such as the Internet, textbooks, policy documents, and academic articles. This paper is a brief analysis of a CBC radio interview conducted the morning of October 23, 2015. Rick Cluff is the host and his interviewee is cognitive neuroscientist Daniel Ansari. Daniel would be speaking that day in Vancouver at the Eaton Arrowsmith Neuroplasticity and Education Conference. The topic of the interview is developmental dyscalculia (DD) and the interview can be listened to in its entirety at http://www.cbc.ca/news/canada/british-columbia/programs/theearlyedition/dyscalculia-needs-more-attention-says-neuroscientists-1.3286224 (5:13 in length).

The purpose of this study is to examine how various discourses on MLD used in this interview help shape our "common sense" understandings of what MLD, particularly DD, are and if there are also any competing meanings. While much has been written about reading disabilities such as dyslexia, much less has been written about dyscalculia (Mazzocco, 2007). By applying the methods of critical discourse analysis (CDA) to this interview, I aim to explore how language in use creates certain identities, relations, and realities around the concept of MLD. My concern is less about finding the "truth" about DD than about investigating how competing "truths" about the nature of DD struggle for hegemony within the same social domain.

THEORETICAL AND METHODOLOGICAL FRAMEWORK

Norman Fairclough's (1992) CDA is a useful framework for analyzing discourse as social practice and, in particular, the discursive practices used in the CBC interview. The main objective of Fairclough's approach to language analysis is to study social and cultural change. Shifts in language use play a central role in the understanding of changes in social phenomena. Fairclough's CDA synthesizes two different senses of discourse—the social-theoretical sense (such as Foucault's) and the "text-and-interaction" sense—and forms a three-dimensional model in the following way:

> Any discursive "event" (i.e. any instance of discourse) is seen as being simultaneously a piece of text, an instance of discursive practice, and an instance of social practice. The "text" dimension attends to language analysis of texts. The "discursive practice" dimension, like "interaction" in the "text-and-interaction" view of discourse, specifies the nature of the processes of text production and interpretation, for example which types of discourse (including "discourses" in the more social-theoretical sense) are drawn upon and how they are combined. The "social practice" dimension attends to issues of concern in social analysis such as the institutional and organizational circumstances of the discursive event and how that shapes the nature of the discursive practice, and the constitutive/constructive effects of discourse referred to above. (Fairclough, 1992, p. 4)

This synthesis of the socially and linguistically oriented views of discourse is what Fairclough calls a "social theory of discourse." His multi-dimensional approach emphasizes the importance of text and language analysis such as systemic functional linguistics (see Halliday, 1994) in discourse analysis and has developed an explicit and operational approach for researchers to analyze discourses. There are two key focal points of any analysis in Fairclough's CDA: the order of discourse (the totality of discursive practices of an institution, and the relationships between them) and the communicative event (an instance of language use). Due to the small scope of this paper, the focus will be on the communicative event (the radio interview), and Fairclough's three-dimensional model of discourse (as text, discursive practice, and social practice) will be applied to it.

My analysis of these three dimensions for the radio interview will be done separately. Firstly, the analysis of discourse as text involves a focus on the linguistic features of a text (linguistic analysis) and looks at eight analytical properties: interactional control, modality, politeness, ethos, connectives and argumentation, transitivity and theme, word meaning and wording, and metaphor. I will examine each of these eight properties in turn. The analysis of discourse as discursive practice concerns how authors draw on existing discourses and genres to create a new text. A key question to ask is: Are discourse types (genres and discourses) used conventionally or creatively? Conventional discourse practice involves a normative use of discourse types and helps to reproduce the relationships in the order of discourse. On the other hand, creative discourse practice often mixes together a number of genres and discourses and helps to restructure the boundaries of the order of discourse. The last dimension to be considered when analyzing a discourse sample is discourse as social practice: What is the relation between this discourse practice and the larger social structure to which it belongs (is it conventional and normative or creative and innovative)? Does it transform or reproduce existing social practices?

TEXT ANALYSIS

Note that prior to analysis, the radio interview was transcribed in detail noting any pauses, interruptions, changes in intonation and other things of possible significance.

Interactional Control

This interview is a cordial exchange between Rick Cluff and Daniel Ansari and follows a formulaic interaction. Both participants play their assigned roles: Rick as the interested host who asks the types of questions an audience member might ask, and Daniel as the expert who must describe this disability to the lay public in a way they may understand. The line of questioning follows a similar pattern to that of other interviews with medical experts that often begin with a definition of the condition, followed by its characteristics and treatments. Rick has full control of the line of questioning and goes through his list of questions and Daniel his responses with little interruption.

There is no veering off topic and little follow-up or evaluation of responses. The only follow-up that Rick makes to one of Daniel's responses is when he points out that "'Awareness' is the key word there." Similarly, Daniel's only evaluation of one of Rick's questions is when he says, "That's absolutely right Rick," early on in the interview when he agrees that there is a reason for many people not getting math. Rick and Daniel must "stick to script" as both are aware there is limited time in the radio interview and that the purpose of the interview is to raise some awareness of the disability and to promote the upcoming conference. While Rick introduces all topics, the interview feels so formulaic from both parties that questions of control and policing seem irrelevant and there seems to be no motivation for controversy or to delve too deeply into the disability.

Modality

Modality refers to the affinity one has with a particular proposition. Daniel stands strongly behind his statements and says them clearly and confidently. However, there are two notable instances of modality that indicate some degree of uncertainty. The first occurs when he talks about the causes of dyscalculia: "We and others think that developmental dyscalculia is caused by deficit in the ability to deal with very simple magnitudes." The cause of dyscalculia is not one hundred percent certain but his research suggests that it may be due to this particular deficit. It is not clear however what causes this deficit in the ability to deal with very simple magnitudes in the first place. Is it due to a particular disorder in the brain? This is unclear and Rick does not pursue the matter any further.

The second notable instance of modality occurs when Daniel talks about the interventions for dyscalculia. In his response to Rick's question, "Is this something that can be treated?", Daniel hedges in almost every sentence of his response: "I believe so"; "You know, we are still working on the best interventions"; "We need more randomized control trials"; "I do think you can remediate developmental dyscalculia to some extent"; "There's no reason to expect that that might not be the case for math." Daniel does not suggest any specific treatments for dyscalculia either, although he seems hopeful there will be interventions developed in the future. His response is reminiscent of a medical specialist couching his response with an element of cautious optimism.

Politeness

Politeness strategies used in discourse implicitly suggest particular social and power relations. As mentioned, this is a very cordial exchange. Right from the beginning of the interview, Rick defers to the expert by asking, "Am I saying that correctly?" with regards to the pronunciation of "dyscalculia." Throughout the entire interview, Rick never interrupts the professor and demonstrates his respect towards the professor through the interested tone of his questioning. This respect seems mutual however. The addressing of each other by first names "Rick" and "Daniel" suggests a friendliness and equal power relation between host and interviewee. Rick only addresses Daniel as "professor" in the introduction and at the end of the interview. This has the affect of making the neuroscientist seem more accessible and likable to the layperson compared to the stereotypical stodgy and impenetrable professor. This friendly tone of the interview is carried right to end in the closing salutations. Rick: "Daniel, fascinating topic. Thank you for joining us this morning. Enjoy your stay here in Vancouver." And Daniel's response: "Thanks very much for having me."

Ethos

Ethos considers not just discourse but the whole body in constructing "selves" or social identities. While this is a radio interview and audiences only hear the audio it is worth noting Rick's comment that Daniel "joins us here in studio ten this morning." Rick's use of the word "us" is ambiguous and can imply the workers in the studio or the entire listening audience. The latter would suggest that it isn't just Rick who's interviewing Daniel but Rick as a representative of the wider CBC audience hence metaphorically shortening the distance between listener and interviewed. Moreover, while this is a formal interview setting in the sense that the interviewer is on one side of the table and the interviewed is on the other side, the interview takes place "in studio" as opposed to over the phone which makes the interview seem more intimate and lowers the barrier posed by such a technical topic such as dyscalculia.

Connectives and Argumentation

This analytical property is related to cohesion. Fairclough (1992) notes that "text types differ in the sorts of relation that are set up between their clauses, and in the sorts of cohesion they favour, and such differences may be of cultural or ideological significance" (p. 174). Fairclough distinguishes three main types of relation between clauses: elaboration, extension, and enhancement. He also distinguishes four main types of explicit cohesive markings: reference, substitution and ellipsis, conjunction and lexical cohesion (e.g. word repetition or the use of synonyms). (See Halliday, 1994 for a more thorough discussion.) Analyzing all such relations and markings in this interview would be beyond this paper, although I will note a couple of them. Firstly, this interview is a very cohesive interview overall in the sense that Rick asks a question and Daniel proceeds to elaborate on it (e.g. "Tell us about dyscalculia," "What are some of the tell-tale signs?" or "Is this something that can be treated?"). There is little in the way of probing or critiquing on the part of Rick and neither does Daniel do any qualifying or enhancing of his questions. One exception is when Rick repeats Daniel's use of the word "awareness." This affirmative speech act suggests that Rick is keen to listen and that Daniel is worthwhile to listen to. Indeed, the very tight and logical structure of this interview is consistent with interviews where the interviewer (as lay person) is asking an expert about a foreign subject with little interruption or veering off topic.

Transitivity and Theme

Fairclough (1992) notes that "the ideational dimension of the grammar of the clause is usually referred to in systemic linguistics as 'transitivity,' and deals with the types of process which are coded in clauses, and the types of participants involved in them" (p. 178). He distinguishes four types of clauses: relational, action, event and mental. In the interview, the individuals with dyscalculia are often talked about, but are rarely described as agents or knowing and feeling individuals. When dyscalculics are the subject of a clause, they are often described as deficient (e.g. "individuals with dyscalculia will struggle" or "individuals with DD will calculate, they will use their fingers, they will use strategies other than retrieving it from their memories.") The "thinkers" and "knowers" are the researchers or educators often denoted with the ambiguous use of the pronoun "we" (e.g. "we and others think that DD is caused by," "we've done a lot of research into the reasons behind," or "we need more randomized control trials"). The voices of individuals with dyscalculia remain unheard throughout the interview, but the voices of the psychologists and scientists dominate the interview. Even the feelings of parents are hinted at ("I get emails all the time from parents who are concerned.") This imbalance is not unexpected as this is an interview with a researcher of dyscalculia. A panel interview including parents and students with dyscalculia may contain different varieties of clauses.

In his discussion of transitivity, Fairclough discusses two other aspects of grammar: theme (referring to the initial part of a clause) and nominalization (the conversion of processes into nominals often leaving who is doing what to whom implicit). However, I will omit a discussion of these two aspects due to space limitations.

Word Meaning and Wording

Fairclough (1992) points out that "the meanings of words and the wording of meanings are matters which are socially variable and socially contested, and facets of wider social and cultural processes" (p. 185). The choices that text producers and interpreters make with words and word meanings may hint at some of these "wider social and cultural processes." Consider first the various ways in which the concept of a MLD is labelled throughout the interview. The medical/technical terms "dyscalculia" and "developmental dyscalculia" are both used throughout the interview although there is no distinction made between them. It is also interesting to note at the beginning of the interview how Rick asks if he is saying "dyscalculia" correctly suggesting the opacity or unawareness of the concept perhaps to the general public.

Later in the interview, dyscalculia is described as a "learning disability" and then a "specific learning difficulty." It is worth noting here the use of the term "difficulty" rather than "disability" perhaps suggesting a shift in the way researchers think about or categorize the concept. The medical connotation of dyscalculia is furthered when Daniel begins talking about the treatments for the condition. Terms and phrases such as "formally diagnosed," "treated," "remediated," "interventions," and "randomized control trials" all suggest that dyscalculia is not unlike other medical conditions that have specific definitions, causes and treatments. Despite the use of this pseudo-scientific terminology throughout the interview, dyscalculia is sometimes described generally and less formally using everyday language as merely "being bad at math" or having "trouble with math."

Metaphor

Fairclough (1992) notes that, "When we signify things through one metaphor rather than another, we are constructing our reality in one way rather than another. Metaphors structure the way we think and the way we act, and our systems of knowledge and belief" (p. 194). There are some notable uses of metaphor throughout the interview. According to Daniel, individuals with dyscalculia "*struggle* with the most basic aspects of math." They have a "*problem* with simple arithmetic." Rick relates dycalculia with people who have "*trouble* with math" or "*being bad* at math." (Italics are my addition.) This common metaphor associates dyscalculia with a deficit model of learning. Alternative ways of thinking about dyscalculia as a learning disability are overshadowed by this metaphor. Can dyscalculics sometimes exhibit strengths that may overcome their deficiencies? Is there something in the way that school mathematics is taught or assessed which exacerbates the mathematics difficulties for dyscalculics? Alternative realities can sometimes be masked by the metaphors that we use in our everyday language without even thinking.

The metaphor of mathematics as something that resides within the individual versus something more social is evident in Daniel's response to Rick's question: "What are some of the tell-tale signs?" Daniel responds that the signs are "an inability to *retrieve* the solutions to very simple problems from *memory*." For individuals who don't have dyscalculia, solutions to simple problems "will *pop* into your head. It's *stored* in your long-term memory." (Italics added.) The terms "retrieve" and "store" suggest much of basic arithmetic is a matter of memorization and solutions are retrieved and stored like objects in a box. It is interesting to note that Daniel suggests people with dyscalculia "will use their fingers, they will use strategies other than retrieving it from their memories." These alternative "strategies" used by dyscalculics are made to seem inferior to that of memory retrieval.

ANALYSIS OF DISCOURSE AND SOCIOCULTURAL PRACTICE

Discourse Practice

Analysis of discourse practice focuses on how a text is produced and consumed. I will focus only on text production in this section as text consumption requires research into reader response which is beyond the scope of this paper. One approach to text production is to identify the types of discourses that the text draws upon. There is a strong interdiscursive mix within the interview. For example, a traditional academic discourse runs throughout the interview when Daniel talks about the causes of DD, diagnostic criteria, how DD is in the DSM, its relation to dyslexia and treatments. But there is also the discourse of "everyday life." For example, Daniel explains how individuals with DD will struggle with "how much change they might get back when they buy their coffee." He describes how "all hands go up" when he asks a group of educators "how many of you know about developmental dyslexia?" Moreover, he notes how poor math skills may lead to such things as "unemployment and imprisonment and mortgage default." This use of the language of everyday life mixed with technical/academic language in radio interviews is not uncommon as it is difficult for a purely academic discourse to be taken up in the media. According to Fairclough a high level of interdiscursivity is associated with change, while a low level of interdiscursivity signals the reproduction of the established order. While this brief interview in itself cannot indicate wider societal change, there is the suggestion perhaps that the "traditional closed off university" is making better attempts to communicate what goes on there to the general public.

Sociocultural Practice

According to Fairclough (1992), "the general objective here is to specify: the nature of the social practice of which the discourse practice is a part, which is the basis for explaining why the discourse practice is as it is; and the effects of the discourse practice upon the social practice" (p. 237). A key question is what kinds of institutional and economic conditions are the discursive practice subject? This CBC interview was conducted the morning of the 3rd annual Neuroplasticity and Education: Strengthening the Connection Conference hosted by the Eaton Arrowsmith School. So the interview serves as a bit of a promotion for the conference and ultimately the school itself. According to the eatonarrowsmith.com website, the Eaton Arrowsmith School follows a particular teaching method:

> We teach our students a series of exercises to help them strengthen their brains and address the cognitive weaknesses that cause their specific learning difficulties. All of our schools operate on the principle of neuroplasticity – the brain's ability to be strengthened over time with targeted training. This sets us apart from other learning intervention programs. Traditionally, a student who struggled with handwriting, for example, would learn to use a keyboard or be given more time to write tests. We don't teach students to work around their difficulties; we help them address them.

The philosophy of the Eaton Arrowsmith school helps to explain some of the discursive practices used in the interview and is consistent with the academic discourse that Daniel uses throughout (e.g. that dyscalculia is a disability located within the individual brain and that such a disability can possibly be remediated or treated much like dyslexia can). The fact that the interview serves as a bit of a promotion for the school is also consistent with Daniel's use of everyday language as such a discourse will be accessible to a wider audience beyond just educational professionals but also parents whose children the school would like to attract. Thus, the interview reproduces the traditional medical interview that defines and characterizes certain disorders or illnesses to the lay public. Yet it is also transformative in the sense that it implicitly touches on the relative new field of neuroplasticity and its application to education and dyscalculia in particular.

SUMMARY AND CONCLUSION

This preliminary analysis of a CBC radio interview applying Fairclough's three-dimensional social theory of discourse combines the analysis of text, discourse practice and sociocultural practice. The analysis of text and discourse practice suggest a high level of interdiscursive mix of academic/medical discourse with the discourse of everyday life. Moreover, at the level of sociocultural practice there is the suggestion that there may be a promotional/consumerist element at play in the interview. Finally, this study shows that this interview contains a mix of traditional storylines (dyscalculia as a medical condition located in the brain), new storylines (neuroplasticity), and also suggests omitted storylines (voices and lived stories of individuals with dyscalculia). Only by analysis of further texts, however, will it become clearer how these types of discursive practices take part in constituting and changing (if indeed they do) particular aspects of the social world.

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STUDENTS' MODELLING PROCESS – A CASE STUDY

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In this paper I discuss a case study where two grade 8 students worked collaboratively to solve a modelling problem. Their modelling process shows that rather than closely following the modelling cycle suggested by modelling literature, where they develop a real model, a mathematical model, a mathematical solution, and a real solution for the entire situation and repeat the modelling cycle to improve their solution, these students broke down the modelling problem into smaller pieces and went through the modelling cycle multiple times in order to generate a realistic solution to the modelling problem.

INTRODUCTION

Modelling tasks are problems situated in the real world. They require students to approach the problem from a real-world perspective and to use mathematics as a tool to produce a mathematical solution. The process of which students solve modelling problems can be described by modelling cycles (Figure 1). Students begin with understanding the situation. They then simplify the situation and create a real model to represent the situation, mathematize the real model in reality into a mathematical model in the world of mathematics, determine a mathematical solution using the mathematical model, and verify the mathematical solution by comparing it to the original situation (for example, see Borromeo Ferri, 2006; Blum and Borromeo Ferri, 2009). Research has shown that students may repeat the modelling cycle multiple times to improve their solution (Stender and Kaiser, 2015).

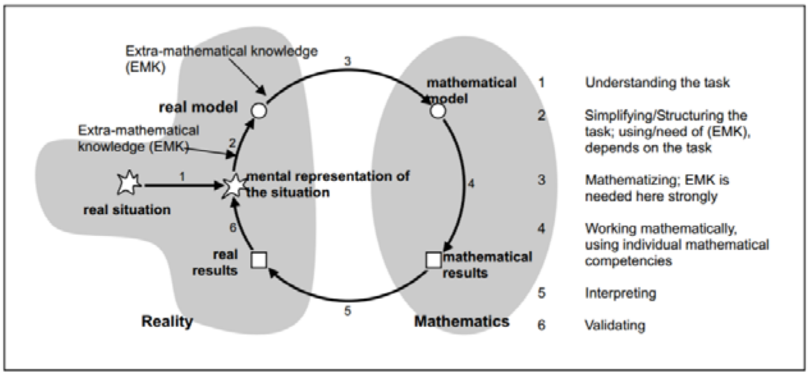

Figure 1: Modelling Cycle proposed by Borromeo Ferri (2006)

This study investigates the process of which students generate a realistic mathematical solution to a modelling task and their ability to draw on their extra-mathematical knowledge (EMK) to evaluate their solution from a real-world perspective. More specifically, I intend to take a closer look at students' modelling process when the modelling problem involves a number of aspects. Do students work on the problem situation holistically, consider all aspects of the problem at once to develop a solution, and repeat the modelling cycle to improve their solution? Conversely, do they break up the problem into something more manageable and put together a complete solution afterwards?

PARTICIPANTS AND METHODS

This study is part of a larger project which looks at students' developments in mathematical literacy skills through a modelling practice. In this article I present a case study of two grade 8 (age 13-14) mathematics students' modelling process. The two students are Amy and Anna[1]. At the time of this study, these students have little experiences with such tasks.

This case study took place in a grade 8 mathematics class, where students were given a modelling task and were asked to work as a group to solve the task, "Designing a new school"[2]. Data include in-class observations, field notes, impromptu interviews, post task interviews, and audio recordings of students' work in their group. The following is a summary of "Designing a new school":

> Your city is building a new 11000m^2 school building, an all-weather soccer field (100m×75m), 2 tennis courts (15m×27.5m each), and a 30 car parking lot on a 200m × 130m lot. Create a design and layout of the school grounds using the grid provided.

The mathematics skills required for students to complete the task is fairly minimal: basic computations, conversions between actual measurements and measurements on the grid, etc. However, to be successful, students also need to draw from their extra-mathematical knowledge (EMK) and make various assumptions about the situation. These EMK include an understanding that buildings could be taller than 1 floor, that it is possible to incorporate some of the facilities mentioned in the question in other buildings, etc. In what follows I present these 2 students' modelling process, and analyze their modelling process using Borromeo Ferri's (2006) modelling cycle.

STUDENTS' MODELLING PROCESS

Amy and Anna began their modelling process by carefully reading the instructions, paid specific attention to the area on the gird which they could use and the length each square on the grid represents: All buildings should be at least 12.5m away from the property lines; each square on the grid is 10m×10m. They divided 12.5 by 10 and got "one and one-fourth", and outlined a rectangle 1.25 squares away from the edge of the grid to represent this space.

Afterwards, they used the measurements given in the instruction to draw the soccer field and the tennis courts on the grid. They divided the length and width of the soccer field and those of the tennis courts by 10, and drew a rectangle that is 10 by 7.5 squares to represent the soccer field.

[1] Pseudonyms are used to protect students' identities.
[2] The task is taken from: http://www.peterliljedahl.com/teachers/numeracy-tasks

After pencilling in the soccer field, Amy and Anna realized they could not fit the tennis courts beside the short edge of the soccer field. In order to accommodate for the tennis courts, they rotated the soccer field 90° and put the tennis courts (two rectangles that are 2.75 by 1.5 squares long) beside the soccer field towards the bottom of the grid.

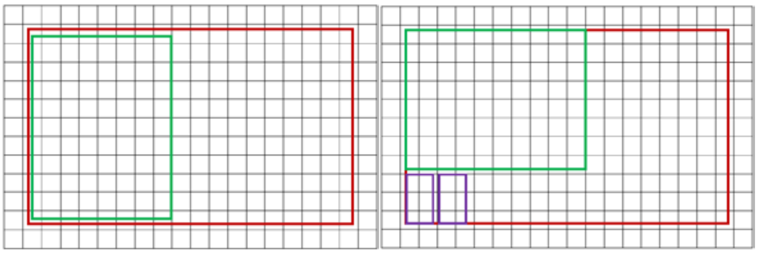

Figure 2: Students' design of the soccer field and the tennis courts. Left: Amy and Anna put the soccer field on the left side of the grid (green) inside the "usable space" (red). Right: They later on rotated the soccer field (green) to accommodate for the tennis courts (purple).

After drawing 3 rectangles to represent the soccer field and 2 tennis courts, Amy and Anna read the instructions again and focused on the parking lot. They drew a quick sketch of a few parking spots and tried to visualize what the parking lot might look like (Figure 2). However, they had difficulties visualizing the parking lot and were not certain what else they needed to consider other than the areas taken up by parked vehicles.

Figure 2: Amy and Anna drew 5 parking spots based on the assumption that each parking spot is 4m long and 2m wide.

A brief discussion with the researcher led Amy and Anna to realize that there are more to consider other than the area each parked vehicle takes up.

Amy	The car space… 4 by 2. Because, there are bigger cars.
R	Okay. So that's the size of a car. But once I parked the car…
Amy	You can't get out.
R	Ahuh. I need to get out. So… would I need… what does that mean?
Anna	You need some extra space!
R	You need some extra space! So, how much is that extra space?
Amy	Like, 0.5 metres between each car.
Anna	Oh okay.
Amy	You [also] need to be able to go behind a car
R	You need to go behind the cars and to…

Amy/Anna	Drive

In this conversation, Amy pointed out that cars in parking lots do not park right next to each other. Rather, there is a gap between each car to allow for drivers and passengers to enter and to exit their vehicles. Amy and Anna also pointed out they needed to include a driveway for vehicles to drive into and out of the parking spots. After discussing their work with each other, Amy and Anna created an outline of their parking lot. It is 60m long and 5m wide. All 30 parking spaces are lined up along the long edge of the parking lot, and each parking space is 2m wide and 4m long. The driveway, which runs along the long edge of the parking lot, is 1m wide and 60m long. They divided these measurements by 10, and drew a rectangle that is 6 by 0.5 squares on the grid. Afterwards, they began to work on the school building. Very soon, they were stuck. Anna complained that she couldn't fit the school building on the grid because there was not enough space. Amy described Anna's frustration as a "mental breakdown", and called the researcher over for help. During their discussion with the researcher, Amy had an "AHA" moment (Liljedahl, 2005) and realized that she could "stack" the school building because in reality, it is possible to have buildings taller than one floor.

Anna	I have no more space left!
R	You have no more space left to fit what?
Anna	Um… the rest of the school.
R	The school is not big enough and you don't have any more space. Oh my… Oh no… Oh no… so we need more space.
Anna	But there is no space!
R	Oh there is always space.
Amy	Stack them!
R	What do you mean stack them?
Amy	Two floors!

While Amy's "AHA" led them to realize that they could design a school building taller than 1 floor, Amy and Anna have not quite grasped what the building might look like and how much space they wanted for each floor. They joked about creating a "110 floors" building, to which the researcher took the opportunity and discussed with them building shapes and floor area. During the discussion, Amy and Anna explored the idea of having a two floors tennis court building, and the idea of incorporating the tennis courts into the school. They also discussed with each other the possible designs of the school building. They toyed with the idea of a three-floor school building where each floor takes up 36.7 squares. They interpreted their solution, 36.7 squares, as a rectangle with an area of 36.7 squares, but quickly dismissed their solution as they couldn't see an immediate solution, a length and a width that would result in this area. As they further chatted with each other, they drew from their experiences and concluded that the areas on each floor of their school building do not need to be the same. They eventually settled on a two floor building. The main floor is 100m by 70m, and the second floor is 80m by 50m, a total of 11000m^2. They then drew 2 rectangles to represent the school building, and assigned the remaining space on the grid as green space. Looking at their work, Amy and Anna wanted to make modifications to improve students' school life, and added a garden next to the parking lot. The garden is 20m wide and 60m long, and has a gate (~3m wide) on one side of the garden. As a final touch, they added two 20m wide doors to the school building, and a path that leads to the school's front entrance (Figure 3). The 2 doors are located on the second floor of the school building, and the path connects the edge of the school grounds to the second floor of the school building rather than the ground floor. This could simply be an oversight as they mistook the second floor of the school building as the first. After installing these additional features, Amy and Anna submitted their solution.

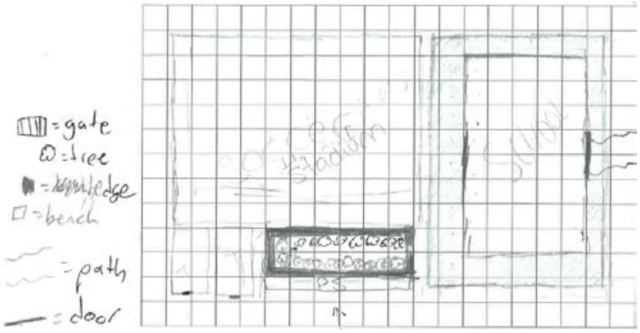

Figure 3: Amy and Anna's final solution

A BRIEF DISCUSSION OF AMY AND ANNA'S MODELLING PROCESS

Amy and Anna began by reading the instructions carefully and focused on one of the rules of the situation: All fields, courts, buildings, and parking lots must be no closer than 12.5m to any of the property lines. They interpreted this rule as a restriction to the "usable space" on the grid. This interpretation represents their mental representation of the situation (MRS) of the usable space aspect of the problem. To solve the problem of usable space, they drew from the instructions the information they needed (every square on the grid is 10m × 10m), and recognized the need to convert 12.5m in into number of squares in order to outline the usable space on the grid. Together, these recognitions and decisions of what needed to be done formed their real model. Amy and Anna converted the distance away from the property line into number of squares on the grid by dividing 12.5 by 10 (mathematical model), carried out the calculations to determine the number of squares that they needed to stay away from the property line (mathematical solution), interpreted the solution as an outline on the grid (interpretation), and then drew the outline on the grid to represent the space they could use (real solution). This represents Amy and Anna's first modelling cycle (MC1), in which they focused on the usable space of the grid. The outline helped them organized and built real models for the rest of the problem situation. Since Amy and Anna focused only on the usable area here, they were forced to repeat the modelling cycle a few times.

After outlining the usable space, Amy and Anna focused on the soccer field (MC2) and the 2 tennis courts (MC3). They understood that they needed to create outlines of these structures on the grid (MRS) and took a similar approach as the usable space: they took the measurements of the soccer field given, combined this with the understanding that the length of each square on the grid represents 10m, and decided to draw a rectangle on the grid to represent the soccer field (real model). They then divided these measurements by 10 (mathematical model) to get a mathematical solution, interpreted this as the length and width of the rectangle they needed to draw (interpretation), and drew a rectangle on the grid to represent the soccer field (real solution). Afterwards, they worked on the tennis courts (MC3) and repeated the process, and made changes to their real solution of the soccer field to accommodate for the tennis courts.

Afterwards, Amy and Anna worked on the parking lot (MC4), but experienced difficulties in visualizing the relationships between the vehicles, the parking spots, and the parking lot. It seemed that they were not able to visualize what the parking lot might look like. It is possible that both Amy and Anna had troubles accessing their EMK in this area since they were under the legal driving age at the time of the study. They never experienced parking lots from a driver's perspective and therefore never paid much attention to the parking spots and the driveways in parking lots although they understand the existence of these features. A brief discussion with the researcher helped them realize that they needed to extend their measurements of their parking spots to allow for drivers and passengers to enter and to exit their vehicles. This discussion also helped them see the need of a driveway in the parking lot to allow for vehicles to reach these parking spots. All these understanding and realizations formed a new MRS. They then created a real model to represent the designs of their parking lot based on this understanding, a design that includes 30 parking spots and a driveway; and a mathematical model, in which the length of the parking lot (in terms of number of square on the grid) is the width of a parking spot × 30 ÷ 10, and the width of the parking lot is the length of a parking spot plus 1 and then divided by 10. They then generated a mathematical solution, 6 and 0.5, based on their mathematical model. They interpreted 6 and 0.5 as the length and width of the rectangle to represent the parking lot (interpretation). Afterwards, they drew the rectangle on the grid to represent the parking lot (real solution). This concludes their fourth modelling cycle.

Unfortunately, despite their hard work, Amy and Anna did not create a realistic parking lot design. They put the parking lot on the school grounds close to the property line, but did not include any driveways that connect the edge of the property line to the parking lot. Also, the driveway is only 1m wide. Based on Amy's measurements, a vehicle is at least 1.5m wide. Therefore, the driveway they created is not wide enough to let vehicles through. Finally, Amy and Anna recognized that they needed additional space between vehicles, but did not consider the extra space when they designed the parking spots.

After they finished their parking lot design, Amy and Anna proceeded to work on the school building (MC5), and determined the area of the school building using the information given in the question. This understanding served as their MRS. As they converted 11000m^2 into the number of squares on the grid, they quickly got stuck, as Anna believed they had “no more space left” for the school (MC5 incomplete). During their discussion with the researcher, Amy had an “AHA” moment (Liljedahl, 2005) and realized that they could extend the vertical height of the school building to satisfy the floor area and to decrease the construction area required (MC6). This realization allowed Amy and Anna to draw from their past experiences to think about the various shapes and possible height of the school building. Their conversation also led them to examine the possibility to design a two floor tennis courts building and to in-cooperate the tennis courts into the school building. These discussions formed Amy and Anna’s MRS and these understanding helped them to re-organize their MRS of the entire problem situation and eventually allowed them to build a real model of the school building.

As they moved forward, they settled on the idea of a three floor school building where each floor has an equal floor area. This idea represents their first real model of the school building. They then built a mathematical model to determine the number of squares they required on the grid by dividing 11000 by 100, and then by 3. They carried out the calculations and generated a mathematical solution (36.7). Amy and Anna interpreted this solution as a rectangle with an area of 36.7 squares. Upon discussing with each other the possible length and width of this rectangle, they realized that their school building did not need to have an equal floor area on each floor, and they went back and modified their real model. They eventually settled on a two floor building where the main floor (100m by 70m) is larger than the second floor (80m by 40m). They did not explain how they arrived at these measurements. The researcher speculates that they first reduced 11000m^2 to 110 squares, picked 10 and 7 squares as the length and width of the main floor to generate an area of 70 squares, and chose 8 and 5 squares as the length and width of the second floor to make up for the 40 squares they needed to make up for 110 squares and to satisfy the 11000m^2 area requirement. Finally, they outlined two rectangles on the grid, one within the other to represent the two floors of the school building. These two rectangles represent their real solution to the school building aspect of the problem situation.

Finally, Amy and Anna verified that they satisfied all the requirements and added additional features to their solution: green space, a garden, a path that leads to the school building, and 2 doors to the school building. These doors are approximately 20m wide. The width of these doors was out of proportion, as a 20m door is about the height of 10 interior doors. It seems that they lack experiences and knowledge in spatial orientation and did not use their EMK to help them design these doors. After making all these adjustments and verifying that they have satisfied all requirements, Amy and Anna submitted their work and this concluded their modelling process.

CONCLUSION

Modelling literature suggest that students would spend time understanding the problem situation (MRS), create a real model, a mathematical model, a mathematical solution, and a real solution, and repeat the modelling cycle to improve their solution if necessary. If students were to follow the modelling cycle closely, they would have read and focus on all the instructions given, recognized that they need to draw outlines of all building structures, created a design for both the school building and the parking lot, converted all measurements provided into number of squares on the grid, and finally drew the outlines of these building structures on the grid. However, this was not the case.

In this case study, Amy and Anna broke down the problem into its individual aspect, and worked on one aspect at a time. This resulted in them repeating the modelling cycle 6 times, and they pieced together these real solutions from each modelling cycle to create an overall solution for the problem. These results are different from what modelling literature suggest. This could be the results of the problem involving multiple aspects. While the mathematics skills required to solve the problem is simple, piecing together all the information or real solutions all at once is rather difficult. Therefore, it becomes logical for students to break the problem down into smaller pieces and worked on one aspect of the problem at a time, generate a solution for one aspect at a time, and piece together everything at the end to generate an overall solution for the problem.

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UNDERGRADUATE STUDENTS' PERCEPTION OF TRANSFORMATION OF SINUSOIDAL FUNCTIONS

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Trigonometry is one of the fundamental topics taught in high school and university curricula, but it is considered as one of the most challenging subjects for teaching and learning. In the current study Mason's theory of attention has been used to examine undergraduate students' perception of the transformation of sinusoidal functions. Two types of tasks – (A) Recognizing sinusoidal functions and (B) Assigning coordinates – were used in this study. The results show that undergraduate students participating in this study experienced difficulties in identifying a period of a sinusoid, especially when it was a fraction of π radians.

BACKGROUND

Trigonometry has a long history. Ancient people used trigonometry for different purposes. For example, Egyptians applied trigonometry to determine the correlation between the lengths of the shadow of a vertical stick with the time of day. Astronomers also used trigonometry to find the longitude and latitude of stars, as well as the size and distance of the moon and sun. However, trigonometry was not an essential part of mathematics textbooks until a Persian mathematician named Khwarizmi introduced trigonometric functions to the world. Since then, trigonometry has become one of the main topics in high school and university mathematics books and students are required to assign time for learning trigonometry, especially trigonometric functions. This is the case since a strong foundation in trigonometric functions will likely strengthen their learning of various mathematical topics, such as Fourier series and integration techniques (Moore, 2010). It is also shown that understanding calculus and analysis is dependent on the learning of trigonometric functions (Hirsch, Weinhold & Nicolas, 1991; Demir, 2011). However, learning and understanding trigonometric functions is a difficult and challenging task for students, compared to other mathematics functions, such as polynomial functions, and exponential and logarithmic functions. While other functions (e.g., logarithmic functions) can be computed by performing certain arithmetic calculations expressed by an algebraic formula, trigonometric functions involve geometric, algebraic and graphical concepts and procedures, simultaneously (Weber, 2005; Demir, 2011).

Despite its importance and its complexity, research on trigonometry is sparse and quite limited. In the literature, only a small number of studies concentrate on students' learning of trigonometric concepts, and in particular trigonometric functions (e.g. Brown, 2005; Weber, 2005; Moore, 2010). Moore (2010) and Weber (2005) indicated that students often have difficulty using sine and cosine functions defined over the domain of real numbers. Thompson (2008) also noted that students are unable to construct understanding of the trigonometry of right angles and the trigonometry of periodic functions. In a study of undergraduate students, Weber (2005) agreed that students could not rationalize various properties of trigonometric functions or reasonably estimate the output values of trigonometric functions for various input values. Kendal and Stacey (1997) concluded that students had difficulty interpreting trigonometric functions in the unit circle, recognizing that x and y coordinates of a point on the unit circle are cosine and sine values of corresponding angles compared with other determined trigonometric functions in terms of a right triangle.

In spite of all the research efforts in the area of teaching and learning trigonometry, especially trigonometric functions, there are still gaps in the literature. There is no research study that focuses on the concept of the transformation of sinusoidal functions; the current research attempts to fill this gap.

In order to deal with the transformation of sinusoidal functions, students need to understand the notion of the 'period of a function.' The period of a function is the distance (x value) in which function values repeat themselves. In the case of the canonical sine function $f(x) = \sin x$, the period is 2π, the circumference of the unit circle. Considering the standard format for the sinusoidal function $f(x) = A\sin(Bx \pm C) \pm D$, students are required to identify the relationship between the coefficient of x (B in the function) and the period when dealing with the transformation of sinusoidal functions. As such, the research questions are: How do undergraduate students identify period? How do they recognize the period on the graph of the sinusoidal functions?

DATA COLLECTION AND ANALYSIS

This study is part of a bigger project which examines undergraduate students' perception of the transformation of sinusoidal functions. In the larger study, seven undergraduate students from a large North American university participated They were selected from among students who had either completed a Calculus I course and were enrolled in Calculus II (3 students) or they were in a Calculus I course (4 students) in the Mathematics Department. Participants volunteered their time to contribute in the study right after I made a general request from all the classes (Calculus I and II). For the purpose of this research report, I focus only on the performance of one of the participants, Emma. She was studying Applied Science and was enrolled in a Calculus II course at the time of her interview.

A 60-minutes task-based interview was conducted and Emma was required to complete two types of interview tasks: **A)** Recognizing sinusoidal functions and **B)** Assigning coordinates. Both types of tasks were presented with the help of the Dynamic Geometry software, Sketchpad. For the 'Recognizing sinusoidal function' tasks, the sketches indicating the sinusoidal graphs were given and the student was asked to identify the sinusoidal functions represented in the given graphs (see Figure 1). For the 'Assigning coordinate' tasks, a wavy displace (see Figure 2) along with the sinusoidal functions were given and Emma was required to assign coordinates on the wavy curve such that it described the given functions. Type A tasks comprised of Task 1: $f(x) = sin(2x)$, Task 2: $f(x) = sin(\frac{2}{3}x)$ and Task 3: $f(x) = cos(\frac{2}{5}x - \frac{\pi}{5})$. Type B tasks included Task 4: $f(x) = sin(4x)$ and Task 5: $f(x) = cos(3x - \frac{\pi}{4})$.

THEORETICAL FRAMEWORK

The collected data in this study were analyzed and interpreted using the *theory of shifts of attention* (Mason, 2008). Mason's theory provides opportunity to study the critical role of attention and awareness in learning and understanding mathematics and in particular the concept of the transformation of sinusoidal functions. Mason (2008) distinguishes five different structures of attention: 1) Holding wholes; 2) Discerning details, 3) Recognizing relationships; 4) Perceiving properties; and 5) Reasoning on the basis of agreed properties. Mason's framework of shifts of attention is appropriate for analyzing the collected data in my research. Applying this framework supports me in gaining insights not only into '*what*' Emma attended to when completing mathematics tasks related to the transformation of sinusoidal functions, but also '*how*' she shifted her attention in identifying the period of sinusoids. Mason's terms for different structures of attention also provide a language for analyzing students' work. For example, when a student considers a particular graph and recognizes its shape as representing a sinusoidal function, s/he is *holding holes*. A student who looks for particular details from the given sinusoidal functions or the given sinusoidal curve (e.g., she is seeking for the point the graph intersects the y-axis), she is, in fact, *discerning details*. The student is *recognizing relationship* when she able to find a connection between the graphical representation of the sinusoidal functions and their symbolic representations. When a student determines the particular parameters that determine the given sinusoidal curve by considering its periods, she is *reasoning based on perceived properties*. To investigate how the participant realized the transformation of sinusoidal functions, and in particular, how she identified period from the given graphs/functions, I reviewed the student's answers and the transcripts several times.

Please note that in all the five interview tasks the participant was required to connect the period of the given sinusoidal function or the given sine curve to a coefficient of x in the standard formula for sinusoidal functions (considering the sinusoidal function in the standard form : $f(x) = Asin(Bx \pm C) \pm D$). For brevity, we refer to this connection as 'recognizing the period' (see Figure 3).

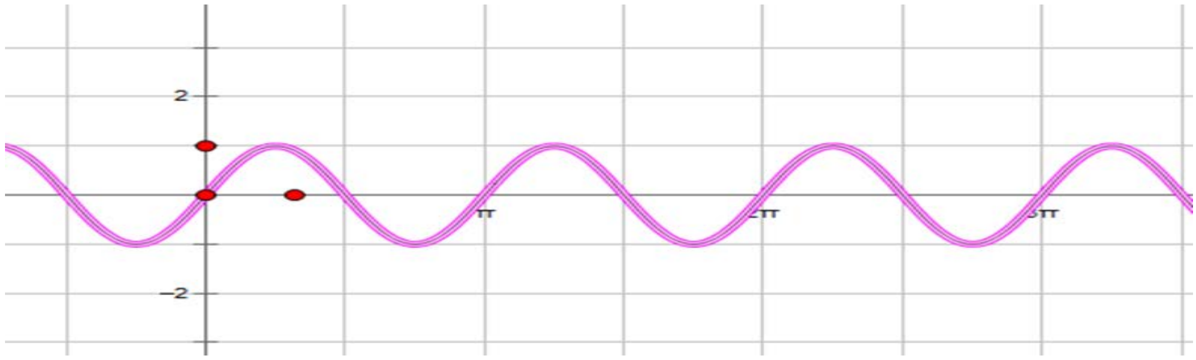

Figure 1: Graph presented in Task 1

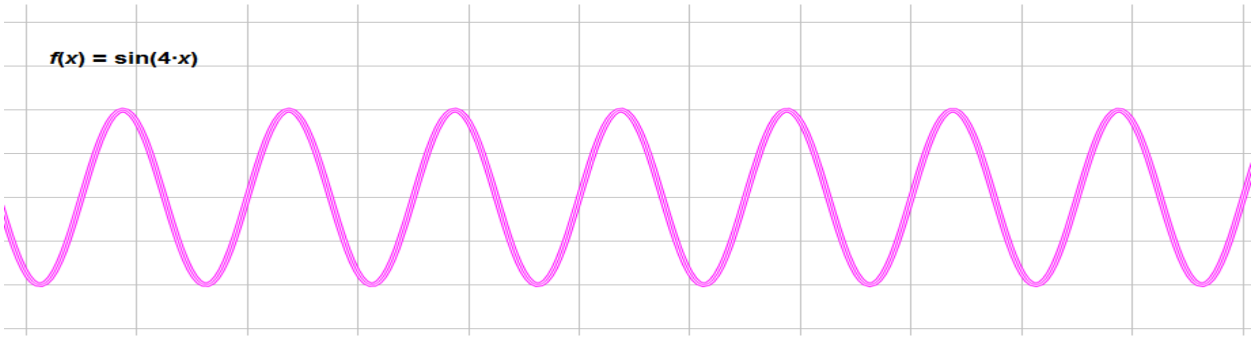

Figure 2: Graph presented in Task 4

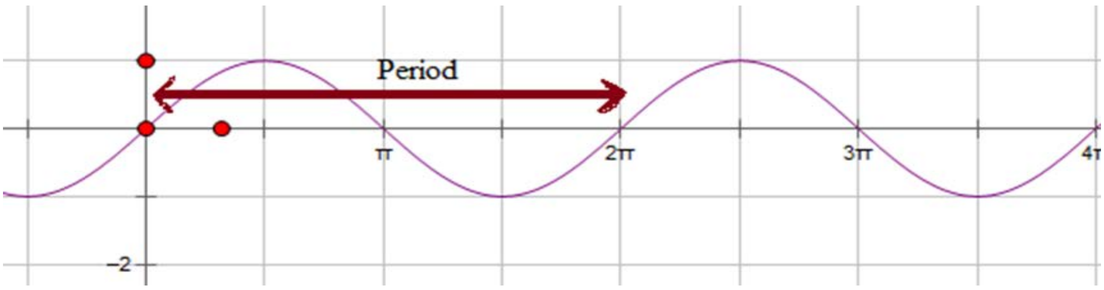

Figure 3: Recognizing the period

RECOGNIZING PERIOD (COEFFICIENT B OF X)

At the beginning of the interview, I showed Emma Task 1 in which the graph of the function $f(x) = sin(2x)$ was given (see Figure 4) and she was asked to identify the sinusoidal function represented by the graph. In order to complete Task 1, Emma first focused her attention on the given graph and waited for visual feedback from the graph (her attention was on *holding wholes* according to Mason's classification). Emma stated:

> "It is $f(x) = sin(\frac{1}{2}x)$. It is a sine graph because it starts at 0 and it should be $sin(\frac{1}{2}x)$. The sine graph start at 0 and then π and 2π, but in this one is 0, π, $\frac{2\pi}{3}$. This is half of sine graph, because the period here is π, while it is 2π in the original sine curve."

The above statement indicates that the participant recognized incorrectly the function for the given graph, determining it to be $f(x) = sin(\frac{1}{2}x)$. Analyzing the situation using Mason's (2008) framework it can be concluded that Emma *reasoned on the perceived properties* of the sinusoidal functions and from there she *determined (*incorrectly*) relationships* between the visual representation and the symbolic representation. Emma recalled the fact that the period of a canonical sine function is 2π, whereas the period of the curve given in Task 1 was π . She thus concluded that the given curve represented the function $f(x) = sin\left(\frac{1}{2}x\right)$. Emma then connected the period of the sine curve, which was π radians, with the coefficient of x in her suggested sinusoidal function. Her statement illustrates that Emma, in fact, divided the argument x by 2 because the period of the canonical function (2π) was divided by 2 in the given graph (the period was π).

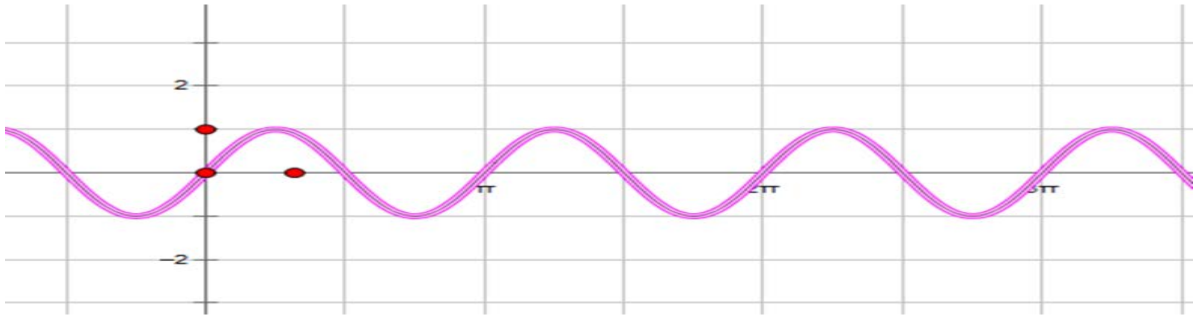

Figure 4: Graph of function $f(x) = sin(2x)$

Detecting Emma's mistake in recognizing the proper function for the given graph in Task 1, I showed her the graph of $f(x) = sin(\frac{1}{2}x)$. Observing the graph of the function $f(x) = sin(\frac{1}{2}x)$ made the participant realize that the graph of the suggested function did not correspond to the given curve. At this time Emma stared at both graphs #1 and #2 (see Figure 5) for a while and she *held the graphs (#1 and #2) as wholes*. She then began to *describe in detail* the given graph (#2 in Figure 5) in respect to the graph of $f(x) = sin(\frac{1}{2}x)$. Emma stated:

> "….so, if $f(x) = sin(\frac{1}{2}x)$ is like this, so it is going to finish at 4π. So this is going to be the whole graph. So it should not be $\frac{1}{2}x$, it should be 2x. Because when we have $\frac{1}{2}x$ we can see that it ends at 4π. But if I put here 2x, I compressed it and I can…have this curve finishes at π π...The period of sine graph is 2π but this one is compressed, so it is $f(x) = sin(2x)$, but $\frac{1}{2}x$ is expansion in fact."

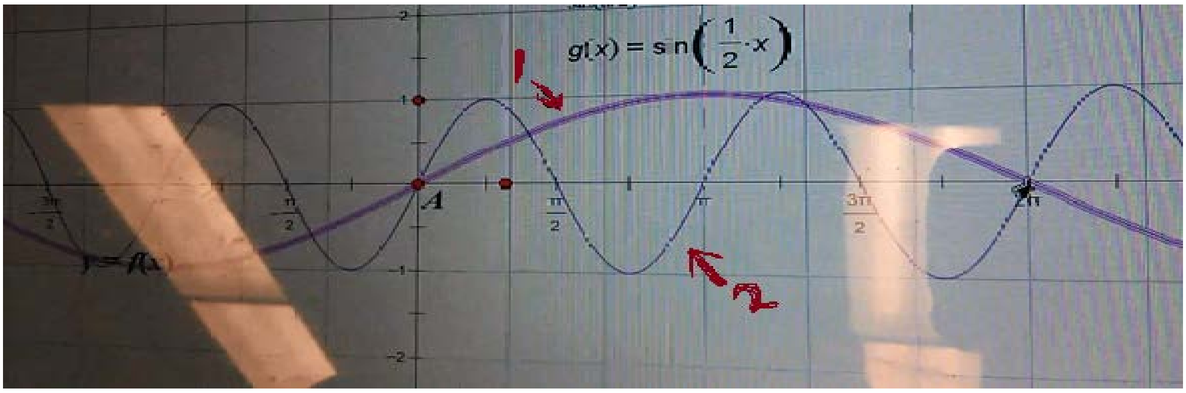

Figure 5: Graphs of $f(x) = sin(\frac{1}{2}x)$ and $f(x) = sin(2x)$

As it is indicated from the above statement, Emma compared the end point (or the length of a full cycle) of the curve #2 with that of curve #1, considering the origin (0, 0) as a beginning of a cycle ("…. $\frac{1}{2}x$ we can see that it ends at 4π. But…I can…have this curve finishes at π...”). In other words, by linking the end points of the full cycles (in both curves and comparing them with the graph of the canonical function), Emma was able to find relationship between the visual representation and the symbolic representations. She chose the number 2 (which was the reciprocal of the coefficient of $\frac{1}{2}\pi$) as a coefficient for x in the sinusoidal function. As such, she eventually recognized the correct period and thus the proper function for the given curve.

Emma's proper realization in Task 1 directed her to complete successfully similar tasks having a whole number for the coefficient of x. As an example, when approaching Task 4 in which the function was $f(x) = sin(4x)$ and a wavy curve was given, Emma was able to assign correctly the coordinates in the given wavy displace such that it represents the graph of $f(x) = sin(4x)$. After gazing at the given function in Task 4, she expressed:

> "…I know that 2π is here [see Figure 6] because 1, 2, 3, and 4 periods is between 0 and 2π and here are 1 and -1…"

The above excerpt shows that Emma perceived *properties of sinusoidal functions* ("*...period is between 0, 2π and here are 1 and -1*"). The feedback she received from Task 1 (the fact that there is a direct relationship between the coefficient of x in the sinusoidal functions and the number of repeated full sine cycles between 0 and 2π) allowed Emma to assign coordinates properly in Task 4. In other words, Emma was able to realize period from the given function and therefore assign axes successfully on the sinusoidal curve. Considering Emma's success in Task 1 and Task 4, one might conclude that she was able to recognize period and also sinusoidal functions, from their graphs, and vice-versa, successfully. However, Emma performed differently on the other interview tasks.

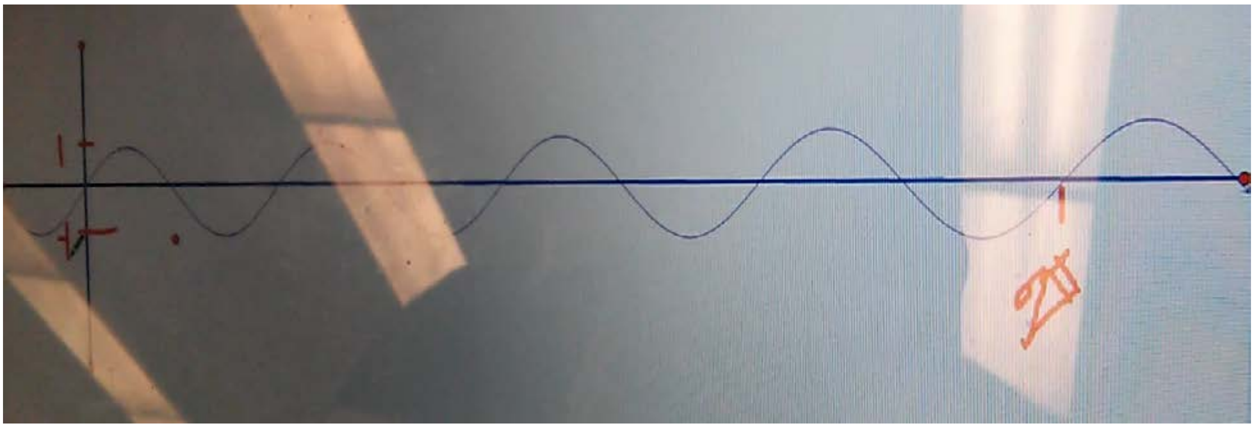

Figure 6: Emma adjusts coordinates for Task 4

As an example, when completing Task 2, in which the graph of the function $f(x) = sin(\frac{2}{3}x)$ was given, after *holding the graph as whole* for a long pause, Emma did *discern some details* from the x-axis. She then stated:

> "…It is sine of x over something because if it is sine of x it would end here [at 2π]…ok, it is $f(x) = sin(\frac{x}{3})$ because there are one, two and three spaces here between 0 and this point and again one, two, three here…" (see Figure 7).

As it appears from the above statement, Emma counted the number of 'blocks' between 0 (the point A in Figure 7) and the point in which the curve intersected the x-axis (point B) and again from point B to another point in which the graph intersected the x-axis (point C). Since the distance between the points A and B, and B and C was 3 blocks, Emma put the fraction $\frac{1}{3}$ for the coefficient of x in the suggested sinusoidal function. It appears that she was eager to find an opposite relationship between 3π and the coefficient of x which was $\frac{1}{3}$. This evidence illustrates that Emma was unable to *recognize appropriately the relationship* between graphical representations of the sinusoidal function and its symbolic representations.

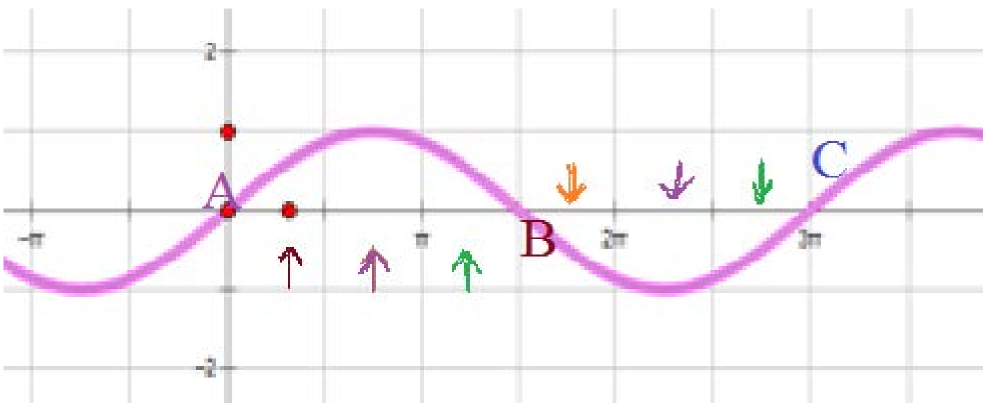

Figure 7: Emma counting the blocks between the points

Emma's unsuccessful attempt in recognizing period and its relation with the coefficient of x in the sinusoidal function in Task 3 was typical of further errors in the other tasks having fractions for the coefficient of x. In other words, in Tasks 3 and 5, as in Task 2, Emma was unable to recognize period successfully. As it was mentioned previously, it seems that the fractional coefficient was problematic, because Emma often attempted to reverse the point in which a full curve was finished (which was 3π in the Task 3) in order to find a coefficient for x in the sinusoidal function. Although applying this method directed Emma to determine proper functions when the coefficient of x was a whole number, it did not work for the other tasks. Emma, in fact, should find the relation between the period of a canonical function which is 2π and the point 3π in the given graph ($3\pi=\frac{2\pi}{B}$, so $B=\frac{2}{3}$) in order to identify the coefficient of x in the sinusoidal function.

DISCUSSION

The findings of this study show that the student recognizes period and transformations in different manners when the coefficients of x in the sinusoidal function are whole numbers and when they are fractions. The data from this research demonstrates that Emma is capable in matching the algebraic representations with the graphical representations when the coefficient of x was a whole number. These results are in contrast with the findings of Leinhardt, Zaslavsky, and Stein (1990), Yerushalmy and Schwartz (1993), and Knuth's (2000) studies in which a group of undergraduate students were unable to use graphical representations to complete mathematics problems in the symbolic form. The contribution of this research is in connecting together the participant's understanding of transformations, graphs and periodicity, whereas the previous research studies focused distinctly on the concepts of transformations (e.g., Yerushalmy & Schwartz, 1993), graphs (Brown, 2005) and periodicity (van Dormolen & Zaslavsky, 2003).

The findings, however, illustrate that Emma was unable to recognize period correctly, when the factor of x was not a whole number in the sinusoidal functions. That is, she was unable to connect the graphs with the sinusoidal functions when the factor of x was a fraction. As such, further research studies are required to investigate how undergraduate students interconnect the three concepts of transformations, graphs and periodicity when the coefficient of x is a fraction.

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IDENTIFICATION OF SURFACE MARKERS FOR POSITIONING OF MATHEMATICS IN STUDENT WRITTEN DISCOURSE

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This study explores the actualization of mathematics learned within the secondary school classroom into extracurricular experiences of its students. The Positioning theory developed by Wagner & Herbel-Eisenmann (2014) is used to explore what student mathematical discourse reveals about the students' relationship with mathematics. British Columbian students enrolled in a Workplace Mathematics 10 class were given the task to pose a math question of personal interest and relevance yet the student responses mimicked the discourse of the dominant curriculum resources. Understanding student positioning within the discipline of mathematics is critical for guiding the application of mathematics beyond the classroom.

INTRODUCTION

Mathematical reform curriculum recognizes mathematical reasoning and mathematical skills as fundamental to an educated and engaged citizen (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010; British Columbia Ministry of Education, 2015). Skovsmose (2011) emphasizes "the meaning of a classroom activity is constructed by the students, and this construction depends on what the students may see as their possibilities; it depends on their foregrounds and intentions". Mathematical processes are interwoven in the extracurricular experiences of all students yet students express a profound incapability with subject. This phenomenon illuminates the underlying motive for this study and the importance of recognizing students positioning of mathematics within the context of their daily activity.

The locus of this study is the belief that student positioning with respect to classroom mathematics is fundamental to the employment of mathematics beyond the secondary school classroom. In this context extracurricular experience is the students' daily activity outside academic studies; home life, social situations, decision-making. Ways mathematical knowledge move out of the classroom into the students' lived experiences is central to the effective application of mathematics in active citizenship.

THEORETICAL FRAMEWORK

This study takes the perspective that student positioning of mathematics is directly influenced by the intersection between the discourse inherent in curricular resources, the discourse of the teacher, and the discourse of the students' extracurricular experience. This perspective is grounded in prior research attending to the positioning of mathematics with respect to the teacher, with respect to the textbook, as well as with respect to the spoken mathematical discourse of the student (Kaur, Anthony, Ohtani, & Clarke, 2013). This study places the students' written mathematical discourse at the centre of the exploratory lens. Attending to student written discourse enhances awareness of the role the student plays in the construction of their individual mathematical knowledge within the traditional assessment practices of the secondary school mathematics classroom. The students' written discourse magnifies the intersection between the mathematics classroom and the lived experiences of the students revealing inconsistencies between the two contexts.

Student authority, specifically *personal latitude* (Wagner & Herbel-Eisenmann, 2014) in relation to mathematics is the overt phenomenon of interest in this piece. The "characterisation of the practices of mathematics classrooms must attend to the learners' practice with at least the same priority as that accorded to the teacher's practice" (Kaur, et al., 2013, p. 7). This research offers insight into the movement of mathematics into the larger social context of its students recognizing that *personal latitude* is required of students to see possibilities for the application of mathematics beyond the classroom.

The nature of the mathematical research in this study was an activity based in written language. The use of stance-bundles/lexical bundles offered by Wagner & Herbel-Eisenmann (2014) offered opportunity to access authority though student written discourse. Within the mathematics classroom their study identified four ways authority was marked: 1) *personal authority*, 2) *discourse as authority*, 3) *discursive authority*, and 4) *personal latitude*. The first three categories: personal authority, discourse as authority, and discursive authority exist externally with respect to the student. The fourth way authority exists, *personal latitude*, recognizes that classroom participants make decisions thus demonstrate authority (Wagner & Herbel-Eisenmann, 2014, p. 873). This study emphasizes *personal latitude* in an attempt to explore different ways students of mathematics merge their classroom experiences into their community and social interactions.

The ideas in this research are explored through an analysis of students' mathematical discourse in a classroom environment. This study focuses on the identification of superficial markers in student written discourse. With the markers identified the positioning of the students in relation to mathematics is hypothesized. This research attempts to instantiate authority and positioning within the limitations of students' written work during a summative assessment. The research question central to this study is: *How might a student position mathematics in relation to their extracurricular experiences?*

METHOD

This paper draws on empirical data collected during a secondary school workplace mathematics course. The students were enrolled in their tenth and eleventh years of public school education. The course minimally met the requirement for high school graduation and is not applicable towards university entrance requirements. Students were assigned the task to offer a question they were interested in solving. Mathematical questions authored by the students were analyses for insight into how mathematics learned in the classroom transferred to extracurricular contexts.

In advance of data collection the student's were offered a *typical* mathematics word problem printed on the classroom whiteboard. The problem read, "Judy drives 43 km to a campground and Robert drives 32 mi to the campground to meet Judy, who drove further?" The teacher allowed 10 minutes to pass after which "Who cares?" was offered as the solution. The resulting disruption of the classroom norms opened space for discussion of the disconnect students experience with many textbook questions. A discussion parsing the mathematics in the question (unit conversion) apart from the assumptions inherent in the question (students drive, have access to a car, etc) emerged from the discordant nature of the comment "who cares". After this group discussion students were informed of the task to offer a problem of personal interest on the written assessment scheduled for two days following.

The data used in this research project was student generated during an individual pencil-and-paper summative unit test for the topic of Unit Conversions. The activity, a unit test, is a familiar mathematical experience in the high school mathematics classroom. The test was developed from an electronic test bank of questions purchased with the curriculum resources. The presence of English as-a-second-language-learners in the classroom motivated a repetition of the most common terms so as not to create confusion during the administration of the test. The dominant grammatical marker on the test was "*convert"* appearing in seven of fifteen questions. The superficial markers best distinguishing question type were: express, convert, how many, what is the, and which […] has higher. The final page of the test contained one question from which the data was drawn, "Since many word problems in math are not interesting to students I would like you to please share a question you are interested in answering."

Gathering data for this research directly from a written test contains inherent bias. Throughout this research there was underlying recognition that written summative assessments are a means of reinforcing student enculturation to the mathematics community dominant discourse. In turn, any attempt to have students author unique written mathematical application problems in the context of a unit test could be viewed as problematic. Therefore, these questions must be considered as an initial glimpse into the influences of mathematics both inside and outside the classroom.

The initial data was expected to reveal students' authority relations with respect to mathematics. After the initial analysis as it seemed as though students had no individualized relationship with mathematics and thus the research was a failure. Stance bundles as identified by Wagner & Herbel-Eisenmann (2014) seemed to have more commonality with the mathematical discourse of the textbook and the teacher. Out of this frustration, new elements of language reminiscent of stance bundles entered into the foreground as a better unit to characterize the positioning of mathematics through the student-generated written data. Once these superficial grammatical markers of the student mathematical discourse were identified as anchors different ways of student positioning emerged. The markers included the pronouns and adverbs: what, why, how, how many, which one, and convert. The grammatical markers in combination with contextual clues suggested five categories of student positioning.

DATA

Mathematics as facts

Six out of forty-one students responded with mathematics questions reflecting math facts. A close study of the questions revealed the positioning of mathematics as only a fact-based effort. This was evoked from the lack of any grammatical markers. Student authored questions such as “3 680 214 x 65% = 2, 392, 139.1” were indicative of what appeared to represent mathematics as limited to facts. For these students mathematics did not have any socio-cultural connection to their experience.

Mathematics as a tool

Mathematics as a tool was identified through its utilization as a solution to a task. Two types of tasks were identifiable; 1) tasks without context reflected in straight forward conversions, and 2) tasks with a context in which there is a comparison between two different quantities or units.

Seventeen out of forty-one students responded with mathematics questions that use mathematics as a tool for comparison. The concept of mathematics as a tool was identified by the use of mathematics in both contextual and non-contextual situations.

Fourteen out of the seventeen students created questions marked by the verb *convert.* It was also notable that these questions had no context. “Convert 25000 ft to yards, feet, and inches” was indicative of these question types. These question types were also a reflection of the questions within the body of the test. This supported the hypothesis that students socio-culturally distanced themselves from the mathematics.

Three out of the seventeen students offered questions prompting a calculation for best buy. These problem types were marked with a combination of adverb and adjective, *How much*, and *How many*, were the most common grammatical markers. When there was a clear comparison of two quantities this was viewed during data analysis to reflect a closer positioning of mathematics socio-culturally. This was especially since mathematics was a social connector between two distinct individuals.

Mathematics as a problem

Six out of the 41 students represented mathematics open problem to solve. These questions were marked grammatically by the adverb and adjective grouping of *How many.* The openness of the question was identified from the nature of having greater than one possible answer. These criteria set the questions apart from the questions which represented *Mathematics as a tool* that otherwise might have been marked by the same grouping *How many*.

Mathematics as a tool to simplify complex problems, single Truth

Six out of forty-one students represented mathematics at a distance. These questions were grammatically marked by isolated nouns and pronouns, *Why* and *What.* "Why is the world so complicated?" and "What is the meaning of life?" both contain the markers of this category.

Category	Example	Superficial Grammatical Markers
Disengaged (distant) from mathematics	Blank entry; "I don't know"	Absence of grammatical markers
Math as facts	"$72 + (8 \times 3) \div 4 \times 4^2$" "$3\ 680\ 214 \times 65\%$" "$1 + 1 = 2$"	Absence of grammatical markers; mathematical symbols only markers
Math as a tool (calculation; no social context)	"Convert 1000 miles into yards" "Convert 500 km into millimetres and metres"	convert
Math as a tool (calculation; comparison in context)	"Laurel lives 18 miles away from her school. Ron lives 3520 yards away from his school. Who lives the closest to their school?" "Both pizzas are $1.75. Which one is a better deal? (2 slices: diagram one 2" crust 5" long; diagram two 3" crust 4" long."	How much How many Who […] closest Which one is a better deal
Math as a problem	"How many pieces of bacon wold we eat if we eat 4 pieces for every mean for 20 months?" "How many of our schools would you have to stack on each other to reach the moon?"	How many
Math a simplifier; Truth	"Why is the world so complicated?" "What is the meaning of life?"	Why What

DISCUSSION

To be thoughtful citizens students are expected to fluidly apply the mathematics learned within the four walls of a classroom into the contexts of their extracurricular experiences. Inherent to the success of this transfer of knowledge is an expectation that students see mathematics as relevant to their lives. The adaptation of Herbel-Eisenmann & Wagner's (2007) positioning theory framework offered an effective instrument to initiate play with the complex layers of the data.

This research suggests students position themselves with respect to mathematics in five ways; these ways seem to parallel the precision of calculation. Mathematics is closest when mathematics is represented as "best buy" and "calculate" problems found within the discourse of the classroom resources. The mathematics classroom tradition offered "all necessary information […] and the students […] solve the exercise while remaining seated at their desks. An exercise establishes a micro-world, where all measures are exact, and where the information given is both necessary and sufficient in order to calculate the one and only correct answer"(Skovsmose, 2011, p. 8). As the mathematical context shifts between the classroom-based applications to extracurricular experiences, the students' relationships with mathematics lost precision. The questions outside the walls of the classroom seemed incompatible with a single solution, yet students' posing of those questions suggested that they felt mathematics still held the single solution.

CONCLUSION

These findings make a direct statement about the effects of the simplified goals set within the mathematics classroom. A student's "working knowledge" of mathematics is the point where school mathematics merges with the student's daily life. Any lack of continuity in the space where the two contexts approach one another contributes to the barrier students face regarding the thoughtful application of mathematics necessary for engaged-citizenship. It seems that students distanced from mathematics might not take any responsibility for utilizing mathematical work to solve problems. Students are not only disconnected from mathematics but completely disengage from efforts to problem solve. It might be suggested that this disengagement is the direct result of the simplification of mathematics application problems in the classroom to achieve one single correct answer. Outside the classroom application problems are never so simple but perhaps students have been trained to seek the single response.

This research will contribute to studies of how students move the mathematics outside the classroom. Surface markers of the mathematical discourse were useful in the discussion of student-formulated questions. It would be helpful to seek a broader context for building a broader catalogue of questions. This would enable consideration of both similarities and differences between student-authored questions and questions drawn from the mathematics test bank and those question students ask in their daily experience. Furthermore, it would be of interest to create complex mathematics problems within the classroom and reflect contexts students might use mathematics in their extracurricular experiences.

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STARTING FROM SCRATCH: AN INVESTIGATION OF POLYGONS

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In this paper, I report on the work of one student in a computerized environment - scratch programming - and consider ways in which children learn to assign and internalize meanings to geometrical ideas, specifically polygons. I use the Vygotsky inspired theory of semiotic mediation as an analytical lens, to show how technology tools (as mediators) enable the invention and use of signs as auxiliary means of constructing mathematical meanings. I argue that scratch programs and the potentialities provide a deeper understanding and educe creative innovations which may not be possible in paper-and-pen environments.

INTRODUCTION

Despite the importance of Geometry to our physical surroundings and emphasis in mathematics curriculum, students face challenges in understanding the topic; polygons. They can find it difficult to express understanding about shapes, their angles and the properties that bind them together. In an effort to curtail this tension, many educators turn to technology as a redeemer for infusing deeper understandings about polygons in the mathematics classroom. Seymour Papert (1980) seems to have steered the ship of technology into the right harbour and decades later there is an influx of technological tools being used to enhance student's learning of the subject. Scratch programming is one of many such tools which provide learners with the potential to be both innovative and interactive. Recently there is a call for computational thinking (CT) to be seen as important to a child's development. "To reading, writing and arithmetic, we should add computational thinking to every child's analytical ability" (Wing, 2006, p. 33). According to Wing (2006) "computational thinking involves solving problems, designing systems, and understanding human behaviour, by drawing on the concepts fundamental to computer science" (p. 33). Devlin (2012) proposed a perspective on mathematical thinking (MT) as a way of thinking about the world logically and analytically. He claimed that MT is not about "doing math" but rather involves quantitative reasoning as well. The aim of this paper therefore, is to investigate how children assign and internalize meanings to polygonal shapes. In carrying out this investigation I provide a brief description of scratch programming and report on the practical experience in which the student became engaged with the technology. I also use a scratch programming environment as a tool of semiotic mediation to better understand how children learn about polygonal shapes and their properties. In doing so, I show that the semiotic process promotes the emergence of mathematical and computational thinking (MT and CT) skills in ways that new mathematical meanings become possible.

BACKGROUND

This mini-research was motivated by an assignment for a course and my participation at the CMESG 2016 conference where I became intrigued by the functionalities of the scratch programming. I was a member of the study group "Computational Thinking and Mathematics Curriculum". In this forum, I gained experience with an innumerable amount of computer-based applications and was particularly drawn to Scratch because it was new to me. I explored its basic affordances on my own and found that while it had similar features to LOGO, it also offers multi-level dimensions of creating stories, games and animations which are different from LOGO. My first instinct was to become familiar with scratch for conversational purposes, but then, I realized the unlimited possibilities that it offers through my readings. In addition, I am a private tutor for a student and my partial role is to revisit the topics she pursues at school on a weekly basis. This important role, her parents believe, will assist her in maintaining a good pace in her regular class lessons. There was one predicament however; I had little knowledge about using scratch programming and Danielle had no experience with the computer as a mathematical learning tool, she has never used scratch and was hearing about it for the first time.

SCRATCH

Scratch is an innovative, visual programming environment which allows users to create interactive, media-rich projects. It has possibilities of allowing users to develop computer programming and thinking skills. Scratch was built on the ideas of LOGO where specific instructions are required to create actions. Unlike LOGO, where the instructions are "typed codes", scratch uses "drag-and-drop codes" with features of building blocks to produce an object. This drag-and-drop approach was inspired by Lego blocks and was specifically chosen because of the "intrigued and inspired way children play and build with Lego bricks" (Resnick et al., 2009, p. 61).

The scratch interface is a self-directed, constructive learning environment which strives to provide visual and instantaneous feedback and requires indirect interaction with the users, where input of data is dependent on the click of a mouse. In an effort to make interaction easy and aesthetically appealing, scratch uses colour-coded icons and a single-pane window with a multi-pane design to ensure that key components are always visible. (Maloney et al., 2010). In the single-pane there are four rectangular shaped blocks called "panes" as shown in Figure 1 below, each of which carries out different functions.

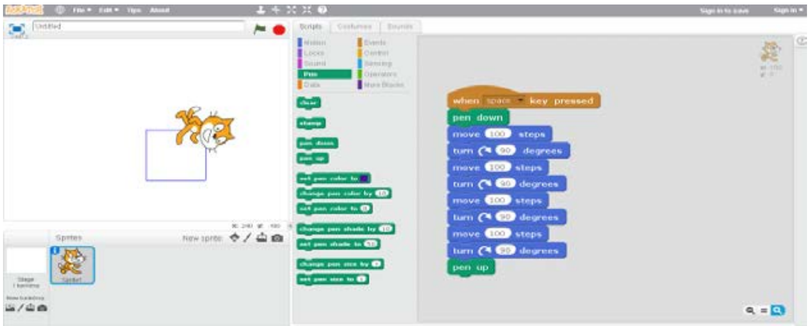


Figure 1: Single pane window with four operating panes along with script and action displayed.

The middle pane (blocks palette) is used to select commands, the right pane (scripts area) displays the codes, the left upper pane (stage) where actions are carried out while the bottom-left pane (sprite list) displays all the "sprites" or "characters" within a project. In this example, the sprite is the tiger-shaped character and it functions similarly to the turtle in LOGO. Though scratch shares similar features to LOGO, one of the main differences is the colourful appearance of the scripts that can usually run in parts. That is, the script is able to display motion even when it is incomplete. Another built-in feature of Scratch which was adapted from LEGO blocks is that it has no error messages.

Similar to parts fitting together in certain ways when building with LEGO, the code blocks fit together only in ways that are logical as shown in Figure 1. In this sense, if a script does not perform the correct action, it does something. Usually, what it does allows the user to think and make adjustments to the code. This process is referred to as 'debugging' (Papert, 1980) and involves situating and fixing errors within a code which is useful in developing computational thinking skills.

THEORETICAL FRAMEWORK

Semiotic mediation, developed by Bartolini-Bussi and Mariotti (2008) is a term which emerged from Vygotsky's (1978) idea that the use of tools (technical [1] and psychological [2]) is essential in the learning process which contrasts with the behaviourist perspectives, that the direct link between subject (the learner) and object (knowledge) is a cognitive one. Vygotsky, instead presents the tool, with a dualistic role of artifact and sign, as a mediating (a middle man) force to enrich the cognitive activity between the subject and the object. According to Vygotsky (1978) "like words, tools and nonverbal signs provide learners with ways to become more efficient in their adaptive and problem-solving efforts" (p. 127). In other words, Vygotsky is adamant that it is not sufficient for genetic and social or cultural development by themselves to account for individual's development-*tools* and *signs*-are also contributing factors. In

[1] Technical tools (artifacts) are usually the physical man-made object that is used in human activities. These are externally oriented

[2] Psychological tools (signs) are the internal representation of technical tools.

this framework, Bartolini-Bussi and Mariotti explained that, enfolded in an intertwining relationship between two dimensions (natural and socio-cultural) is the development of human cognition and within the socio-cultural dimension Vygotsky positions two main constructs; *Zone of proximal development* (ZPD) and *internalization*. ZPD refers to “the distance between the actual developmental level as determined by independent problem solving and the level of potential development through guidance from an adult” (1978, p. 86). Rooted in the process of ZPD is *internalization*; where the transformation from one level to another occurs. It is at this point that new mathematical ideas which are internally oriented are formed. The process of semiotic mediation then, involves a part of knowledge to be mediated. The teacher usually sets the task and chooses the mediating artifact which will aid in the emergence of this knowledge. The learner then uses the artifact along with the task to produce a solution that is usually within his or her experience to produce artifact signs which are later transformed to mathematical signs by the teacher.

METHOD

Participant and Setting

Danielle[*3] is a fourteen year old female student who is currently enrolled in a high school in the Coquitlam school district. I interact with her once per week - as a private math tutor. Each session was two hours long and is usually done without interruptions. Our sessions were done in her home, in a spacious, ‘classroom’ like room with various educational resources. Our interaction was done in two distinct ways, firstly, we met twice in our regular face-to-face meeting and secondly, online through Google Drive and Hangouts to share possibilities of shapes that were constructed outside of our meetings. I took on the role of participant observer due to the interactive nature of the scratch program.

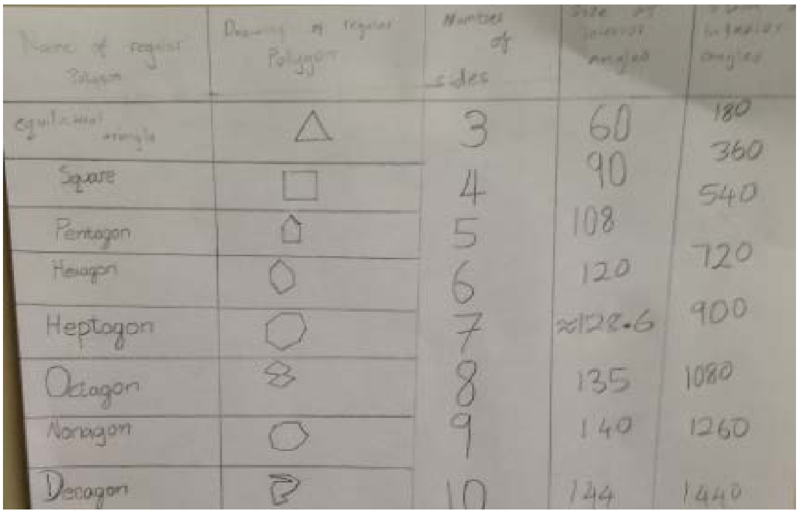
Data source

This task was chosen because Danielle was doing her school assignment which involves using patterns to investigate the properties of regular polygons. I recognized that she was efficient with identifying the pattern but was uncertain about what it means to be a regular polygon (Figure 2 below). I used scratch program as a mediating tool to bridge the gap in her knowledge. The corpus includes field notes from data collected through observations and screenshots of her interactions with scratch. Work that was done outside of our face-to-face meetings was stored in Google Drive. In addition, Google Hangout was used four times to discuss new observations and share insights. The corpus for this mini project comprises field notes, the students work and a record of conversations in the social media forum (Hangout).

[*3] Danielle is a fictitious name given to conceal the student’s identity.

Task

I found that Danielle demonstrated challenges with understanding concepts such as, "What it means to be a regular polygon?" She was unable to differentiate between the relationship of *the sides* of a regular polygon, its *interior angles* and the *sum of the interior angles*. She was also not capable of producing appropriate visual representation of some regular polygons. I saw these tensions as teachable moments where scratch could be introduced. She was particularly good at identifying patterns and was able to complete the table shown in Figure 1, except for drawing the pictorial representations of some shapes. I decided to have her examine what would happen to a regular polygon if the sides should increase to 100. This I believe would help her better understand the idea and properties associated with regular polygons and ultimately lead her to later mathematics experience of Archimedes contribution to mathematics, with the polygonal approximation of pi.



Name of regular polygon	Drawing of regular polygon	Number of sides	Size of interior angles	Sum of interior angles
equilateral triangle		3	60	180
Square		4	90	360
Pentagon		5	108	540
Hexagon		6	120	720
Heptagon		7	≈128.6	900
Octagon		8	135	1080
Nonagon		9	140	1260
Decagon		10	144	1440

Figure 2: Danielle working with polygons in paper-pen environment

Data analysis

Danielle was completing her homework (Figure 2) as is customary before I commence teaching. I watched her as she repeated line after line, making the necessary changes, when I noticed that she was merely following the pattern and did not comprehend what she was doing. I asked:

S: What would happen to the polygon if you continue to increase the sides?

D: I am not sure! What do you mean? Like eleven, twelve?

S: Yes! What if you add another side here? (Pointing to the dodecagon)

D: I am not sure!

S: Let us explore it!

Scratch is an online programming environment and at this point I provided her access using https://scratch.mit.edu. We went through the basic functionalities (all I knew) and asked her to find a strategy that would move the sprite. After many attempts she was able to do so and demonstrated accomplishment in her ability. With the barrier of possible rejection of the technology out of the way, I proceeded.

S: Let's make a code to draw a triangle. What do you know about a triangle?

D: It has three sides, three angles and three vertices?

This represents the starting point of the dialectic cycle of the semiotic process where Danielle engages with the digital tool to produce her own signs. That is, mapping a particular code to the construction of a triangle. After several failed attempts, she scribbles the following algorithm on a piece of paper and expresses that "turn 360/3" will produce the three interior angles of the triangle. This is evidence of the role previous experience plays in the construction of personal signs. She did not realise that this angle represents the opposite exterior reflex angles and not what she expects. Her code is the emergent of a possible sign representing the geometric shape-triangle.

Move 100 steps turn 360/3

Move 100 steps turn 360/3

Move 100 steps turn 360/3

She knew it must be done three times but was uncertain of why the sprite was spinning around on the stage and was not drawing the triangle. I introduced her to the "wait one second" and "pen down" blocks (figures 3 and 4 below). She was able to insert them appropriately and the triangle was produced visibly showing the drawing of the lines.

The next phase was utilizing the "repeat" command to ensure that the codes were shorter and more efficient as shown in Figure 4. This brought to the fore one of the possibilities of scratch to show that mathematics task can be achieved from varying strategies.

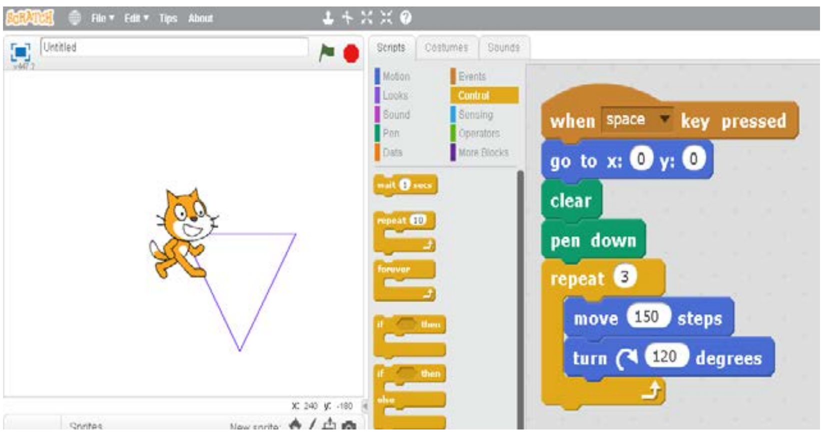


Figure 3: Completed triangle in scratch

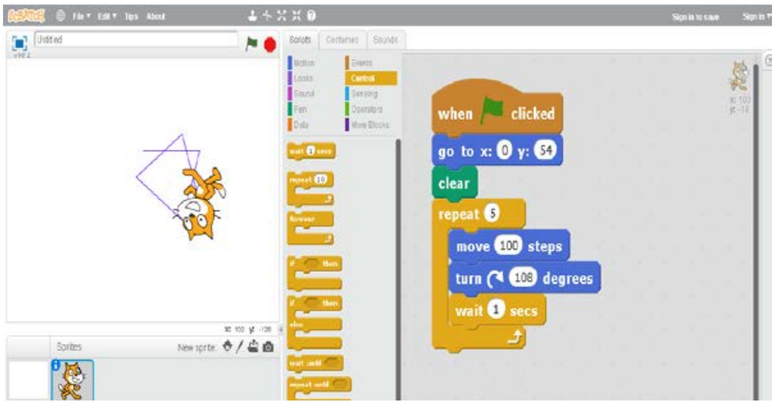


Figure 4: Failed attempt of the regular pentagon

Danielle suggested that we utilize the repeat command and do random numbers to increase the sides each time. This resulted in a failed pentagon as shown above in Figure 4. Neither of us was pleased with this outcome. As a result, we revisited the code to see what was missing. I was perplexed about the simple detail of the sprite blocking my visibility to examine the shape in details, while Danielle's focus was mainly on the code. She was convinced that it was what we did to the code that produced the object on the screen, demonstrating Bartolini-Bussi and Mariotti (2008) idea of the tool "embodying meanings". This was the first indication of the existing possibilities of Scratch and the development of computational thinking skills, as Danielle and I tried to comprehend where in the code the error was located and how it should be corrected. This is the process of debugging which is involved in developing computational thinking skills. There was a great deal of mathematical thinking at play as well. In order to amend the error Danielle revisited the code for the triangle and square, which revealed the relationship(s) she saw between polygons. She recognized that there is a relationship with a full turn (360 degrees), the number of repeats and the angle through which the sprite turns. In completing the task Danielle transitioned from basic codes into more complex ones where she personalized variables and her coding algorithm changes. That is, sliders were introduced for side lengths and number of sides. At our second interaction Danielle knew that increasing the number of sides of a regular polygon is "getting closer" to a circle but not exactly a circle as she could still see impressions of straight lines. When asked if the polygon had 1000 sides if the polygon would eventually become a circle she said "I don't know, perhaps not because you are just increasing the sides of the polygon with straight lines and not replacing curved ones."

DISCUSSION AND CONCLUSION

In keeping with the semiotic process in order to investigate the properties of regular polygons, Danielle was given the opportunity to use the scratch program in two ways. Firstly, as an artifact to produce the regular polygons and secondly, as a sign to internalize the properties (equiangular and equilateral). In addition, the computer, Scratch codes and the student written work all function as mediating tools throughout the semiotic process. The data revealed that her initial drawings of regular polygons were in fact representations of the psychological tools from her previous experience. By interacting with the codes she produced solutions that were either correct, partially correct, or wrong. This process is the construction of personal meanings which are situated in a category of signs called the artifact-signs. When I asked her what would happen if the sides of the regular polygons were to increase, I utilized the process of ZPD and Danielle was able to internalize that the polygons tend to a circle as the number of sides increase, demonstrating the emergence of a new mathematical meaning (mathematical signs) - that of limits.

Today, technology-rich classrooms mean different things to different people. If we examine the past connections of mathematics education and technology we will find that over the decades they both share dependency on each other. In this paper, I used Scratch programming environment as a tool of semiotic mediation to determine how new mathematical meanings were constructed. I argued that through the dialectic cycle of the semiotic process the student was able to visualize and internalize the physical appearance of regular polygons. Most importantly, through the affordances of the digital tool, scratch provides opportunities for deeper MT and CT skills that would not be possible in static environments.

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CREATING TENSION BETWEEN ACTION AND INTENT

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Pre-service teachers come to mathematics methods courses with well-established conceptions of what it means to teach and learn mathematics. Images of teaching reinforced by their own lived experiences shape their pedagogy. This can be problematic for a teacher educator for whom it may be necessary to offer a way of reframing traditional notions of teaching and learning. The research presented here examines that process of reframing. In this study we deliberately introduce a tension in pre-service teachers' conception of timed drills and examine the resulting process of transition they undergo. Using a tension pairing from Berry's (2007) framework, our findings suggest that the introduced tension provided the means for reflection on intent and resulted in a subsequent change in action.

MOTIVATION FOR THE STUDY

> "Math facts are a very small part of mathematics but unfortunately students who don't memorize math facts well often come to believe that they can never be successful with math and turn away from the subject… For about one third of students the onset of timed testing is the beginning of math anxiety." (Boaler, 2015)

First author narrative:

It was my first foray into teaching an elementary mathematics methods course for pre-service teachers. Wanting to gauge their thoughts around the teaching and learning of basic facts, I broached the subject of timed drills. With timed drills being a common practice among elementary teachers, it was not surprising that a survey of my pre-service teachers revealed that they had all experienced timed drills as students and that the majority intended to utilize them in practice. While not unexpected, this was problematic for me. With their emphasis on memorization and speed, timed drills are at best ineffective, and at their worst, potentially harmful to students.

Mindful that 'telling' my pre-service teachers about the detrimental effects of timed drills would have little impact on their practice, I decided to replicate a learning experience designed to reframe conceptions of timed drills (see Liljedahl, 2014). Gathering my pre-service teachers around me, I told them we were going to have a multiplication drill — they would be required to respond verbally to a random multiplication question within an allotted time frame. Correctly answering the question would allow them to 'sit out', essentially completing their role in the intervention; incorrect or slow answers meant the pre-service teacher had to continue playing. By default, the last person standing would be the 'loser'. Although a handful appeared excited by the prospect of competing, the majority of the pre-service teachers were visibly anxious. The overwhelming relief in the room when I announced that this was a ploy and they would not actually have to answer was palpable. And the ensuing discussion was rich and reflective — many underwent a transformative experience that they felt compelled to share.

The comments during the debrief reflected a newfound awareness that the fear and anxiety experienced by the group would likely be the same emotions that the majority of children in their classroom would feel in a similar situation. This was expressed by both those who feared the intervention and by those who were excited by it. Their later journal entries reaffirmed that theme and also revealed that they would no longer consider using timed drills in their classrooms. This was despite the majority previously indicating that they had 'no issue' with timed drills.

This shift from planning to use timed drills to an avowal never to use them was intriguing. It seemed clear that the intervention had achieved its intended goal of raising an awareness in the pre-service teachers that caused them to reflect and reconsider implementing timed drills in their future classrooms. What was less clear, however, was the mechanism by which this transition occurred. The aim of this paper is to explicate that transition process. Thus, our research question is: What is the process through which pre-service teachers shift from acceptance of an established teaching practice to a determination never to use it?

THEORETICAL BACKGROUND

Mathematics for many people is commonly associated with being able to get the correct answer quickly without the need for conceptual understanding (Boaler, 2015). It is not surprising then, that timed drills, in which students are required to answer basic fact questions in timed tests, are an accepted practice among elementary teachers (Kling & Bay-Williams, 2014). Drills are completed individually or, more commonly, as a group activity where the students are required to answer in front of their peers. With their emphasis on memorization and speed, these drills are unnecessary and damaging (Boaler, 2015; Harper & Daane, 1998). Instead, research suggests that effective teaching practices are those that promote conceptual understanding (NCTM, 2014; Tirosh & Graeber, 2003).

Yet the use of timed drills persists and a visitor to any Canadian elementary classroom is likely to encounter timed tests. Timed drills are iconic of established teaching practices, which have come to be commonly accepted, and little conscious thought is put into their continued use and implementation (Buchmann, 1987). Referred to as 'folkways' of teaching, they cut across the experiences of students and are built up through collective participation. Accepted as the way mathematics is taught and learned, folkways are "capable of being practiced without understanding their point or efficacy, the folkways are widespread and emblematic, expressing in symbol and action what teaching is about" (Buchmann, p. 7). In order to disrupt the universal acceptance of folkways such as timed drills, it is necessary to provide a means of raising awareness and reflection.

One way to achieve this is to introduce a tension. Typically a byproduct of teaching, tensions are described as the inner turmoil experienced by teachers. They are the unintended yet inevitable consequence for teachers who find themselves pulled in differing directions by competing pedagogical demands; however tensions can be useful for those who accept the conflicts and use them to shape identity and practice (Lampert, 1985). It is tension that often propels teachers towards professional development and provide the impetus to improve their practice (Rouleau & Liljedahl, 2015).

However, teachers are not always attuned to these tensions and subsequently there is no reflection to provide that impetus (Berlak & Berlak, 1981). As with the pre-service teachers, whose previous experience with the established folkway of timed drills had deflected any awareness of a tension, a tension may need to be deliberately introduced for a change in practice to occur (Liljedahl, 2014). Berlak and Berlak (1981) suggest that because a person is capable of being made aware of tensions, they are capable of altering their practice. However, a caution regarding change is necessary here. As Mason (2002) suggests, "Effective change is something that people do to themselves; more radically, but more aptly when investigated closely, change is something that happens to people who adopt an enquiring stance towards their experience" (p. 143). Essentially, a teacher educator can provide the opportunity for change, but the agency of change must lie with the pre-service teacher.

A framework for both identifying and understanding tensions emerged from the work of Berry (2007). Isolating six pairs of interconnected tensions, Berry used these as a lens to examine her practice. These pairs of tensions are: (1) telling and growth (2) confidence and uncertainty (3) safety and challenge (4) valuing and reconstructing experience (5) planning and being responsive (6) intent and action. The last of these — the tension "between working towards a particular ideal and jeopardising that ideal by the approach chosen to attain it" (p. 32) — is the tension that is most relevant to the phenomenon introduced at the beginning of this article. Until participating in the timed drill intervention, the pre-service teachers had expressed no conflict between their *intent* of having students learn basic facts through their chosen *action* of timed drills. It was difficult for them to recognize the pitfalls inherent in habitual ways of practice even though these ways were actually working against their intended goals for their students' learning. The implementation of the intervention introduced the tension, which resulted in them making the transition from wanting to utilize timed drills to never wanting to use them.

In what follows we use Berry's (2007) tension pairing of intent and action to frame the transition process experienced by the pre-service teachers. In doing so, we examine the process of moving from their acceptance of an established practice to a determination that they will never use it in their practice.

METHOD

The participants for this study were sixty-nine pre-service teachers enrolled in two sections of a fourth year elementary mathematics methods course taught by the first author. As the lead up to a lesson on the teaching and learning of basic facts, the pre-service teachers experienced an intervention designed to cause tension between their professed intent of having their students learn basic facts through the action of incorporating timed drills into their teaching practice. A whole group debrief of the experience followed immediately. The pre-service teachers were then required to complete a reflective journal entry composed of prompts modelled on Gibbs Reflective Cycle (1988). First, they were asked to describe the experience, then describe what they were thinking and feeling, then provide an evaluation and analysis of the experience. The prompts were assigned at the end of class with the expectation that they would be submitted, along with all their other journal entries, at the end of the semester.

The core of the data comprises entries from sixty of the pre-service teachers' written journals, which were submitted electronically. The remaining nine pre-service teachers' journals were submitted in paper format and returned prior to data collection. Other data sources were a simple three question pre-intervention survey asking whether the pre-service teachers had participated in timed drills as a student, had used them (or observed them being used) during their practicums, and finally, whether or not they expected to use timed drills in their future classroom. Notes were also taken of the in-class discussion that occurred prior to the intervention and of the activity debrief.

The data were coded and analysed using the methodology of modified analytic induction, which requires a phenomenon of interest and a working theory that can illuminate other similar situations (Bogdan & Biklen, 1998). It requires that data are coded and analysed for themes in order to develop or disconfirm the working theory (Gilgun, 1992). In this study, the phenomenon of interest was the pre-service teachers' process of transition and the theory began with the assumption that introducing a tension creates a consequence that can alter one's actions.

The themes were generated using NVivo analysis and coding software. For example, for indicators of tension, we initially looked for utterances with emotional components such as mentions of anxiety, panic, or anger. Noticing that these utterances frequently contained phrases like "I never will" or "I remember", we further divided the theme into tensions around future intentions and past memories. The latter underwent a subsequent iteration which resulted in subcategories of positive and negative memories.

RESULTS AND ANALYSIS

In what follows, results from the survey as well as excerpts from the pre-service teachers' journals are used to exemplify the tensions that they are experiencing (or not) and how these tensions evolve as a result of the intervention. These results, and the accompanying discussions, are broken into the three salient stages of the pre-service teachers' evolution.

No tension between intent and action

Prior to participating in the intervention, fifty-seven of the pre-service teachers ($n = 69$) indicated that they would likely be using timed drills in their future classrooms and thirty-six ($n = 69$) had used them during their practicums. When questioned, the majority felt timed drills were an effective way of learning basic facts. They expressed no tension between their intent and their action; this was an accepted practice that they fully anticipated utilizing in their classrooms. This is exemplified in the following two excerpts:

> Julianne: I have grown up doing them [multiplication drills] and I don't see them as an issue.

> Cate: I have memories of having to spew out math facts as fast as possible. I hated it but I think it's a good way to learn math facts.

In the journal entries, seventeen of the pre-service teachers ($n = 60$) mentioned the enjoyment they experienced as young students participating in timed drills. They excelled at it and expressed positive emotions regarding the activity. It is not unexpected then, that their initial surveys indicated that they would be using timed drills in their future classrooms. What was interesting was the twenty-six pre-service teachers who wrote about their negative experiences with timed drills as young students. Despite this, in their initial surveys, they too, indicated their intention of using timed drills in their future classrooms. Their personal experiences were not enough to overcome their ingrained acceptance of this common yet pedagogically unsound teaching practice.

Creating a tension between intent and action

The intervention used to create a tension was a mock timed multiplication drill. Reliving the familiar experience of timed drills as an adult brought to bear not only intense anxiety but also all the negative feelings this type of intervention had caused them as a child. The journal entries contained vivid descriptions of the experience as seen in the following excerpts:

> Cate: The minute you told us to stand up and that we will be doing multiplication questions; I went into a panic. My heart was racing, my stomach was clenching and I felt as if my brain was freezing.

> Jennifer: It's definitely eye opening, having that memory from almost 20 years ago, and then the feeling of panic that I had when I thought that it was going to happen all over again in a university class.

> Meryl: I came home thinking about all of those students who were in my own grade three class years ago that must have just been riddled with anxiety. There is something incredibly disturbing about that. Moreover, there is

something even more disturbing that this is still a very, very commonly used practice. My own SA (mentor teacher) did it throughout my practicum.

Natalie: After you revealed that we actually weren't going to do this activity, and we debriefed it, I realized just how unhealthy it was for me to think that this was a normal way of teaching.

What emerged from the journals was that introducing the timed drill in an authentic manner was vital to the success of the intervention. Experiencing the activity as an adult learner highlighted the disconnect between their intent and their action. The excerpts reflect the recognition that the action of a timed drill interfered with their intent to have students master basic facts. In revealing the folkway of timed drills, we made room for doubt and uncertainty to creep into the pre-service teachers' mental image of timed drills. This emerging awareness can ultimately create what we call a useful tension in that it can lead to reflection and change in practice.

Consequence of the Tension on Action and Intent

The anxiety experienced by the pre-service teachers during the intervention was intense and this was reflected in the journal entries where forty-seven of the pre-service teachers (n = 60) wrote about the negative effect they felt when asked to participate in a timed drill. Consequently, they were able to redirect this self-awareness to an understanding that children in their future classrooms would likely experience the same feelings — as we see in the following two excerpts:

Sandra: After we debriefed this activity I realized how many people in our adult class felt uncomfortable with timed drills and being out on the spot in front of the rest of the class. This definitely was comparable, over even more, to the type of feelings and nerves we may see in our own classroom while teaching children. Only a small amount of students would love this activity while the rest of the class would face nervousness, anxiety and worry.

Marion: When debriefing, I found it relieving and surprising to know how many other people felt the same way I did. Standing in a room full of adults who are becoming teachers, looking around at how much anxiety was caused by this one activity, I can only imagine in a room full of young students how they would feel.

The journal entries revealed that the intervention served as a means of reflection on the pre-service teachers' own future practice. The newfound pedagogical tension resulted in fifty-one of the pre-service teachers (n = 60) stating that they no longer felt that timed drills had a place in their classroom. As one pre-service teacher wrote:

Reese: As a teacher of mathematics, I will never force my students to do timed drills. After experiencing anxiety when you suggested we do this and seeing the anxiety it provoked in my peers, I was able to understand the anxiety that this causes in our students when we do the same to them.

DISCUSSION AND CONCLUSION

In answer to our research question, the shift from acceptance of an established teaching practice to a determination never to use it began with the introduction of what Berry (2007) describes as a tension between intent and action. This resulted in a consequence that disrupted the balance between the pairing, and we suggest that this will cause the pre-service teachers to seek out a new action.

It was readily apparent that, prior to the intervention, there was an absence of tension between the pre-service teachers intent to help students learn basic facts through the action of timed drills. They believed it to be an effective technique that would help them reach their goal. Considering that tension can be the impetus for change in practice (Lampert, 1985); a lack of tension is a strong indicator that the pre-service teachers would utilize timed drills in their future classrooms.

Participating in the intervention provided a new lived experience and created a tension that unseated the folkway of timed drills. In reflecting on that tension, the pre-service teachers realized that the action they were considering using to help students learn basic facts interfered with that very intention. The end result is that their initial action is no longer satisfactory for reaching their goals. The intent to have students learn their basic facts remains, but they will be searching for a new action to implement that will help them achieve that aim. While we cannot conclude that the pre-service teachers will never use timed drills, the presence of a tension, they have experienced first hand, may be enough to impede or curtail the activity.

Berry's (2007) tension between intent and action offers a way of reframing traditional folkways of teaching and learning through reflection on experience. As pre-service teachers are unlikely to reflect on practices, which they view as common and accepted; it is the teacher educator who must devise a way to make these 'folkways' self-evident. Analysis of the data in this study revealed that purposefully creating a tension was useful in altering pre-service teachers' conceptions of timed drills. The intervention resulted in a useful tension which became a source of reflection and praxis.

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ARE THEY GETTING ANY BETTER AT MATH? REFLECTIONS ON STUDENT EVALUATION AND *MATHING*

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Conversations with stakeholders about students' improvements in mathematics invariably focus on student grades and work habits. Further, research into improvements in mathematical performance focus almost exclusively on the acquisition of mathematical content and improvement in test scores. This narrow focus makes assumptions about what it means to know and do mathematics. By analyzing traditional evaluation data gathered from a year-long grade 10 mathematics class, I evaluate the usefulness of this data in determining student improvement and, by exploring the micro and macroculture of mathematics classrooms, reflect on the role traditional evaluation and pedagogy have in shaping how students "math".

ARE THEY GETTING ANY BETTER?

I was having a conversation recently with a colleague of mine about her assessment practice. At the time, she was teaching English Language Arts, and we were discussing the practice of including marks from student writing from early in the year in the students' overall grades. It was the practice in her department to average student writing grades from September to June. We both had questions about this practice. Shouldn't the student's grade be determined exclusively on how the student was writing at the end of the course? After all, if the student's writing had improved significantly over the year, what was the point of including early writing marks? We both agreed that one of the main purposes of the course was certainly to help students improve their writing, among other competencies, and that, while the student should receive feedback throughout the course, evaluation seemed to make the most sense at the end.

"Well what do you do in Math?", she asked me. "Do you average over the year, or just give them a mark at the end?"

"We almost always average," I said. "It wouldn't make sense to just give them a mark at the end, because our grading is based on discrete units."

"Well, how do you know if they're getting any better at math then?" she asked.

I can't remember exactly how I answered at the time. It's likely I said something about Math being different that English, and that we measure achievement by content acquisition, or something like that, but the question had been asked, and I knew as I answered it, I wasn't satisfied. "How do you know if they're getting better at math?" I had no idea. How could this be? How was it possible that I wasn't able to tell if students were getting better in the way she and I were talking about with respect to her students? Certainly, my students were getting better at math throughout the year, weren't they? Ah, I thought, I've got tests! Yes, that's it. Look at what they can do that they couldn't do before: factoring polynomials, reducing radicals, graphing linear equations. They couldn't do that stuff in grade 9, and now they could (mostly) after grade 10. There, they'd gotten better at math. But somehow, the satisfaction of my answer rang hollow. Was that it? Was math just about learning content? Certainly, they must be getting better at something else? But what? And how did I know?

I decided to examine the evaluation data I had gathered from three students in a year-long Foundations and Pre-Calculus Mathematics 10 class in the hopes that this would help me better understand if and how these students had improved over the year.

MY STUDENTS' EVALUATION DATA

	Volume & SA		Trigonometry		Radicals		Relations	
Student	Quiz	Test	Quiz	Test	Quiz	Test	Quiz	Test
Jesica	88	91	85	89	90	90	83	89
Robbie	65	75	80	87	71	69	57	55
Spencer	45	40	50	47	61	52	35	50

	Graphing		Systems		Final Exam
Student	Quiz	Test	Quiz	Test	
Jesica	87	86	91	98	95
Robbie	68	78	77	81	73
Spencer	47	53	65	52	48

Table 1: Student evaluation data

The year-long Foundations and Pre-Calculus Mathematics 10 course is organized into three major organizers, which are further arranged into topics or chapters. The Measurement organizer includes chapters on surface area and volume, and trigonometry; the Algebra and Number organizer includes operations on rational and irrational numbers, and factoring polynomials; the Relations and Functions organizer includes sections on graphing linear relations, including solving systems of linear equations. The organization of the course into discrete chapters is a familiar model to anyone who has ever taken a high school math course, as is my evaluation of student learning.

In the particular class in question, quizzes were administered after shorter periods of study, and tests given at the end of some units. The tests and quizzes were graded for correctness, with partial marks given for partial solutions. Students were expected to justify their answers by “showing their work”. Table 1 shows the grades of three students. These grades, recorded as percent of total marks available, are representative of almost all the students in the class. Jesica, for example, scored consistently well on all quizzes and tests. Robbie’s scores varied throughout, while Spencer struggled to maintain a passing grade. There is little, if anything exceptional or novel about my evaluation data. In fact, I contend that if we examined most North American high school math teachers’ grade books, with some slight variance around number of quizzes and tests, we could extract virtually identical data.

It is clear from the data, and the method by which it was collected (i.e. quiz and test scores), that no reasonable inferences can be made as to whether or not my students had gotten any better at math. None of the three data sets I’ve chosen to examine show improvement in scores over time; in fact, none of the students from this class showed a consistent improvement over time. Even if the data had shown trends towards improved scores over time, it is impossible to infer whether a student was getting better at math, or whether that student was simply getting better at writing the tests.

The method of data collection, however, has broader implications. This data was used to measure, in large part, the success of students in my class. Parents, students, their next years’ teachers, and other stakeholders, look to this data as, in many cases, the only indication of mathematical competence and proficiency. But what does this data actually suggest about what it means to learn and to do mathematics in my classroom?

MATHING AND MATHEMATICAL CULTURE

The data gathering method for this discussion, or more importantly, the way in which my students were evaluated in my class, was meant, almost exclusively, to measure content acquisition. The “almost” in the last statement acknowledges that paper and pencil tests and quizzes, while intending to measure their acquisition of mathematical content, also act as an evaluative proxy for many other things such as work habits, student motivation, reading skills, etc. While these are important considerations, well worth exploring, they are beyond the scope of this paper. What I am interested in exploring here are the implications of using an evaluation scheme that is based on content acquisition, in terms of understanding how I can come to understand student improvement.

As mentioned above, the evaluation data for students is viewed as a measure of both success and proficiency. What we measure has an implicit and explicit value attached to it, since, in my classroom and many like mine, the extent to which students can demonstrate their content acquisition is the most significant measure of their success. This is not to say that the instructional focus in my class was procedural; in fact, my belief was quite the opposite. I spent a great deal time guiding students towards conceptual understanding by engaging them in discussions, activities, and explorations designed to deepen their appreciation and conception of the content. At some point, however, this focus narrowed, and the measure of their success became what they could do – what questions they could answer, how they could demonstrate their ability to perform skills, and how they could apply those skills to familiar questions on a test. I contend that this narrowing of focus defines for students what it means to know and to do mathematics. For my students, then, in spite of my efforts to focus on conceptual understanding and inquiry, the measure of their success was the degree to which they learned how to do the kind of mathematics that they could reproduce on a paper and pencil test, meaning that to improve in mathematics, one must learn to get better at performing on tests and quizzes.

The difference between engaging in the process of doing math and engaging in the process of learning math content is significant. The question of what Mathematics is, much like the validity and efficacy of paper and pencil testing, is beyond the scope here; however, it seems that if I am interested in trying to determine whether my students are getting any better at math, the question necessarily comes down to distinguishing between the doing of math and the learning of math as content. Both are a kind of "doing" of math or *mathing*, i.e. considering math as a verb, or the act of engaging mathematically. I recognize that these are not mutually exclusive, but it is in the mathing that the processes involved in doing mathematics lie. I contend that what constitutes mathing in a classroom defines for most students what it means to know and do mathematics, and, perhaps, even what Mathematics is.

According to Bauersfeld (1993), mathematical activity depends on social and cultural processes. The classroom itself is certainly a dynamic system, in which social and cultural norms are introduced and reinforced by the teacher:

> [T]he understanding of learning and teaching mathematics ... support[s] a model of participating in a culture rather than a model of transmitting knowledge. Participating in the processes of a mathematics classroom is participating in a culture of using math or better: a culture of mathematizing. (p. 4)

Mathematizing, in this sense, defines, for each classroom, what it means to know and do mathematics. In their work on sociomathematical norms, Yackel and Cobb (1996) identify intrinsic aspects of a classroom's microculture, defined by teachers' and students' activity. They argue that these classroom normative understandings are modified by the ongoing interactions of students and teachers, and are unique to specific classrooms. While Yackel and Cobb (1996) look specifically at those interactions that sustain a culture of inquiry and problem solving, I contend that sociomathematical norms are present in classrooms with a focus on content acquisition also. Further, it is the activities of students and teachers that define both what *mathing* is for a particular classroom, and what mathematics is for students in that classroom culture.

Following the work of Richards (1991), Cobb et al. (1992) identify two classroom traditions: the *school mathematics* tradition, and the *inquiry mathematics* tradition.

While it is tempting to view each tradition as theoretically dualistic, with the *school tradition* exemplified by direct teaching to passive receivers of knowledge, and the *inquiry tradition* as a dynamic environment where students make and test conjectures, solve problems, and actively construct meaning, Cobb et al. (1992) argue that this is an oversimplification. In both traditions, the meaning of mathematical activity is based on mutually agreed upon classroom interactions that constitute acceptable discourse in each tradition. The fundamental difference, however, lies in what is accepted as meaningful *mathing* in the respective traditions. The interpretation of what is meaningful can be connected to Skemp's (1976) discussion of instrumental and relational understanding, with respect to what constitutes normative activity in each classroom. In the school mathematics tradition, students view meaningful activities as those in which they can demonstrate correct procedures, while in an inquiry mathematics tradition, students see activities as meaningful when they facilitate the construction of personal meaning (Cobb et al., 1992).

Both the microculture of sociomathematical norms and the macroculture of classroom traditions are negotiated by teachers and students (Yackel and Cobb, 1996; Cobb et al., 1992) through an interactionist framework. This aligns strongly with my own experience. Classroom culture is a blend of my own philosophical, ideological, and epistemological perspectives, and those of my students. A profound tension exists, then, between an instructional approach that favours a tradition of inquiry and an evaluation plan that is grounded in a school mathematics tradition.

If success is defined by paper and pencil testing of procedural competence, as is the testing culture of content acquisition, this goal dominates and overwhelms efforts towards a discourse that aligns with an inquiry tradition. How, then, does this relate to finding out if my students are getting any better at math? It is clear from the data I gathered that I cannot comment on students' improvement by examining test data if I consider "doing" mathematics, or mathing, to be any more than acquiring content. Students, and teachers in many respects, are pragmatists, and will strive to improve in the areas in which they are measured. To know if students are getting better at mathing, I need to measure the processes students use to engage in mathematical inquiry, and not just measure their content acquisition.

This presents an alternative way of examining the mathematics classroom – by separating the way in which students engage with the content, and the content itself – describing mathing separately from the content that is the context for that mathing. If, as the research suggests, the mathing is a function of the micro and macrocultures negotiated in the classroom, then it follows that this process is relatively stable throughout the course, and defines what it means to math in that classroom. The content, on the other hand, is ever changing. As almost all math teachers know, the content is the clock; teachers and students both need to keep pace to be in certain chapters at particular times in the year. For example, if I don't finish trigonometry by Christmas, I'm behind. Further, content is almost always unidirectional. Apart from review for mid-terms and finals, once we've finished a chapter, we almost never go back to it. So, given that we have an ever changing part of the course, i.e. the content, and a consistent part, that is, the way students math, if I want to know if they're getting any better, why do I measure that which is always changing? If I want to know if students are getting better at doing mathematics, and not just getting better at acquiring mathematical content, I need to find a way to measure and evaluate the mathematical processes that students use to math. Essentially, the normative culture of the classroom, the stable and consistent environment of doing mathematics, is what I need to measure, not the variable content topics.

CONCLUSION

Ultimately, in reflecting on my colleague's question, how do I know if my students are getting any better at math, I can only conclude that, as long as I am only evaluating the dynamic part of the class, the content, my students' evaluations are necessarily focused on narrow, procedural competencies, and not on the kind of *mathing* which is representative of the inquiry tradition I believe to be important. By re-visioning my student's evaluation towards the processes they engage in inside an inquiry tradition, perhaps I can guide students towards getting better at actually doing math and not simply getting better at demonstrating their acquisition of content on a paper and pencil test.

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ALGEBRAIC AND GEOMETRIC REASONING PREFERENCES IN ADULTS ON THE AUTISM SPECTRUM

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This study examines the mathematical learning of an undergraduate student on the autism spectrum. I aim to expand on previous research, which often focuses on younger students in the K-12 school system. I have conducted a series of interviews with one student, recording hour-long sessions each week. The interviews involved a combination of asking for the interviewee's views on learning mathematics, self-reports of experiences (both directly related to courses and not), and some particular mathematical tasks. In particular, I examine observations related to geometric reasoning that I have encountered.

BACKGROUND

The Autistic Self Advocacy Network (2014) states that autism is a neurological difference with certain characteristics (which are not necessarily present in any given individual on the autism spectrum). These include differences in sensory sensitivity and experience, atypical movement, a need for particular routines, and difficulties in typical language use and social interaction. They also list “different ways of learning” and particular focused interests (often referred to as 'special interests'), which are especially relevant to this investigation. Over the past few decades, there have been many research studies about learning in students on the autism spectrum, such as those reviewed by Chiang and Lin (2007). A large portion of these studies focus on K-12 students, and particularly elementary students, pointing to an important gap in research which I am focusing on.

THEORETICAL FRAMEWORK

In Vygotsky's writing, there is some work that directly addresses the study of “defectology”. At the time, this was used to refer to studies involving children with certain disabilities (of a narrower scope than we might consider today) (Gindis, 1995). One of the main ideas Vygotsky used for this was overcompensation (sometimes referred to as just 'compensation', possibly due to shifting connotations of the original over time). Vygotsky's idea of overcompensation was explained initially in a framework of physical overcompensation, such as a kidney or lung necessarily strengthening when the other one is missing or by analogy to vaccination. He argued that overcompensation also occurred in psychological development, both in its general course and in particular in the presence of various disabilities. However, he could not have made any assertions about it in terms of autism specifically, since the diagnosis did not exist when he wrote.

While my views are informed by the Vygotskian framework, there are some issues with using it directly. Some parts that are particularly relevant in autistic people, such as the ideas about atypical development and concept formation, particularly concern things that have already must have occurred far before starting university coursework, and thus cannot be observed in my interview subjects. Thus, I have introduced additional tools for the analysis of these interviews.

One of these tools is to compare the observed performance of the students with traits of people on the autism spectrum that have been noted in the literature. For instance, there is often an association between the spectrum and visualization or spatial reasoning (Grandin, Peterson, and Shaw, 1998). In the Vygotskian framework, such an association may be related to overcompensation, using a stronger ability in spatial reasoning in place of the typical modes of reasoning that would be expected based on typical performance.

PARTICIPANTS

The first participant in this case study, Joshua (a pseudonym) received an Autism Spectrum Disorder diagnosis at age 18 (changed from a previous diagnosis of Obsessive Compulsive Disorder). He reported a strong interest in chemistry as well as a particularly low level of interest in subjects unrelated to his major and a strong inclination to work alone. He also reported a preference for visual interpretation and explanation that was reflected throughout the interviews. He was taking integral calculus and linear algebra courses, and I conducted interviews every week for the term of those courses.

The second participant, Cyrus, received an ASD diagnosis at the age of 13. His mathematical background included a bachelor's degree in mathematics. In contrast to Joshua, he showed a fairly strong preference toward algebraic solutions rather than geometric ones, both in his stated preferences and in his given solutions. Since he was not taking any courses at the time, the focus of my interviews was on particular tasks (including ones used in the first set), which included many mathematical paradox-related items.

INTERVIEW PROCEDURES AND TASKS

All of the interviews were in person and audio recorded. In the transcripts, I use 'I' for the interviewer and 'S' for the interview subject. I have tried to capture how the dialogue went. '[?]' indicates something that was not clear, but by context is most likely not significant, and '(…)' indicates a significant pause. A '…' on a line in brackets indicates that some lines have been omitted.

In the part of my case study I am presenting here, I used the first of the Magic Carpet problems used by Wawro, Rasmussen, Zandieh, Sweeney, & Larson (2012). The Magic Carpet tasks were designed for and used on students in a linear algebra course who had completed at least two semesters of calculus. Since they were used at the beginning of their course, the students had not been previously exposed to linear algebra instruction, although all had been introduced to the idea of a vector in some capacity. The first Magic Carpet task presents a situation where 'you' have two tools (a hoverboard and the magic carpet of the problem's name), each with movement restricted to a particular vector ((3,1) and (1,2), respectively). The question presented is whether and how one can get to a cabin at (107,64) using these tools, and to explain either how it can be done or why it cannot be done. Instructionally, the intent of the problem in the context of a linear algebra context seems primarily to introduce the idea of linear combinations, and to lead into other problems in the setting which introduce span and linear independence of vectors.

My intent is to ask how students on the autism spectrum will respond to these tasks. The guiding idea for this research is to find particular areas of similarity in difference in their responses, and find possible explanations through my theoretical framework.

INTERVIEW AND ANALYSIS 1: MAGIC CARPET TASKS WITH JOSHUA

Despite the intent of the first Magic Carpet task, Joshua's initial idea to solve it is entirely geometric:

J.1.1	S: 107- okay, so 107 x, 64 y, okay. (produces a drawing, using a ruler) So... could- isn't the 107 and 64- couldn't we just, uh, that would be equal to the determinant, wait, you can't really use- no, because what I'm thinking, Jeffrey, we could do is we could literally draw a parallelogram, so are we allowed to do that?
J.1.2	I: Uh, okay.
J.1.3	S: 'cause here's how I would do it. ...draw a sketch, Jeffrey, a sketch. (drawing) So, I mean, I don't know if this has to be explicitly done mathematically or whether it can be done, you know, by drawing a sketch but in physics I know that we always drew sketches. That way, whoever- the person knows how to, uh, knows what is going on in our heads. …
J.1.4	S: So now we've got the vector that we wanted to, which... look like so... 1, 2 we could scale up to 10, 20, so that's what I did here, and we have the vector 3, 1, which we could scale up to 30, 10, ... shit. Oh, that's right, the- and so now, how we could approach this is we could follow this vector here, ...we could follow the vector, this is the vector 30, 10, and this is the vector 10, 20, I just scaled it by 10,
J.1.5	I: Okay.
J.1.6	S: Each vector scaled by 10, then from this point here, what we could do, is we could slide this vector up, slide it up all the way up to here and then, so what we could do, is from this point here, we could draw, so if we have 64, 107, we could- go down 20, 20 units, and 10 units to the left, like so, and then, we have the same vector, we just literally transformed

	the vector, [?] I guess not trans- we moved the vector, the point is that it's the same thing, 10, 20, and then, this point of intersection, is where we would shift our- we would change what instrument we were using, so I don't know- you want to look at that.
J.1.7	I: Okay. Interesting.
J.1.8	S: So that's how I would do it. I- I'd approach it literally geometrically. Yeah, there you go. (handing over paper)
J.1.9	I: Hm.
J.1.10	S: And so we've got two lines there, and that little point of intersection is where- and you can find that quite easily, on the x-axis, uh, and that point of intersection is where you would switch over, switch your instruments, ...and again, we have the vector 1, 2, um, we can find the slope of that vector, uh, and then we can move it over to the point 64, 107 or what- [?] 107, 64, and then, you know, it- it wouldn't really be that hard to find that point of intersection but that's how I'd do it, literally just play around with those vectors.

Here, Joshua provides an entirely geometric solution by sketching the vector corresponding to each tool (one from the starting point and one from the end point) and finding their intersection. This solution is unlike those of any of the students observed by Wawro et al. in their use of this task. The drawings used by Joshua were produced using a ruler and were very precise, enough to give a correct solution. I measured the drawings after they were produced, and the point of intersection found by the drawings was the correct point. The interpretation as the location where the person in the problem changes from one instrument to the other is also correct. Thus, this solution accomplishes the stated goal of the problem perfectly well (although avoiding the intent to push the student toward a standard linear algebra solution, and providing nothing related to linear combinations of vectors).

Later in the interview, I ask Joshua to find an algebraic way to solve the problem:

J.2.1	I: Can you think of the algebraic way to solve the problem?
J.2.2	S: Well, like I said, I'd probably find the slope of the vector 1, 2, uh, make that, once I have the slope, make a line with slope 1, 2, so literally it would be y is equal to m x plus b, um, so, our y and our x values would have been 64 and 107, um, six [?] we have 64, 107 with some slope, you find the b, you find- the b, the y-intercept, you'd have an equation y equals m x plus some y-intercept, you'd know the slope, you'd find the other slope, ah, you'd find the other equation, and that point of intersection is where the x and the y values are the same. Do you know what I mean?
J.2.3	I: Mm... okay. Wait... hm.

J.2.4	S: You find two equations for both lines, so I found two lines there, find equations for both lines and then find a common- find the common solution for both lines. (…)
J.2.5	I: So you're finding an equation that's sloped on one of the vectors and hits this point,
J.2.6	S: Yes.
J.2.7	I: and the other equation that's sloped on the other vector hits the origin.
J.2.8	S: Yeah. Exactly. And then find a common solution to those. And that's where you would switch your implements. (writing) …
J.2.9	S: Is there an easier way? […]
J.2.10	I: The... both, the linear algebra algebraic way, is essentially to set up, ah, (writing)
J.2.11	I: a system like this.
J.2.12	S: Hm.
J.2.13	I: Like, "what linear combination of vectors gives this?"
J.2.14	S: Hmhm, okay, yeah.
J.2.15	I: And then you have your two equations, two unknowns, system.
J.2.16	S: Oh, that's a lot easier. See, I didn't think of that application. I literally, it's easier for me to just literally draw it out, Jeffrey. Yeah no, that didn't even come to mind. Goes to show you what I'm getting out of this class, [laugh]. Which I'm not saying it's his fault, it's- it's just the way it is.

At J.2.2 in this part of the interview, we see Joshua's own algebraic solution to the problem, which is essentially his geometric solution put into algebraic form. As he acknowledges later, he had been presented with the linear algebra material that one could use to solve it in the intended linear algebra way, but it appears that he adapted his geometric solution to its algebraic counterpart rather than trying to find an algebraic solution. For comparison, Wawro et al. (2012) state in their paper that the student solution attempts they observed fell into three categories of “guess and check”, “system of equations first”, and “vector equation first”. The third category fits most closely with the standard linear algebra solution, and its presence for the original study's students highlights the differences in Joshua's approach (which fits into none of the three): some students with no prior linear algebra instruction presented a vector-based solution, while Joshua did not despite linear algebra instruction.

Another possible effect of the unusual tendency seen here is that it may pose a difficulty for an instructor's plan to confront students with a problem that would ordinarily necessitate another approach, as happened with this problem. The intent seen in the problem design by Wawro et al. was to push students into the use of a vector-based approach, which notably did not occur here (note particularly J.2.16).

One possible line of explanation is that this tendency results from an instance of overcompensation, as defined by Vygotsky (1993). Joshua may have particular strengths in areas related to the geometric reasoning he uses here that he is using to compensate for weaknesses in areas related to the algebraic reasoning that would be involved in the 'standard' solution to the problem. However, since this is an idea relating to individuals' development, the observations in interviews with students of this age will most likely be of the end result of the compensation process that Vygotsky describes (and not the process itself). It may also be possible to craft future interviews in order to investigate this possibility more closely.

The use of a geometric/visual approach notably fits with what we see in other sources, such as a description of the thought process in Temple Grandin's work. Grandin (1995) describes her own memory as being based on remembering static or moving images, and being able to both understand others' information and express her own better in writing than verbally (which may suggest an issue with the interview process). She also describes thinking of abstract ideas in terms of images or sequences of images. However, the range of variation in autism as well as other interview experiences lead me to believe that the underlying mechanism is more complex and does not always push toward a geometric approach (although it may be common). I plan to conduct interviews with several other participants to more fully investigate this.

Also, the concluding remarks by the student (J.2.16) not only point to a need for instructional intervention in the linear algebra class, but also suggest that there could be a more general pattern across multiple courses of using unexpected approaches that may avoid the general intent of the lesson. I would suggest that while this can certainly be a problem if it goes unnoticed, with a well-tuned approach it could be turned to an advantage, much like with Vygotsky's conception of compensation (although strictly speaking Vygotsky's original conception of compensation was for development of more general reasoning abilities, rather than something like specific linear algebra skills).

INTERVIEW AND ANALYSIS 2: MAGIC CARPET TASKS WITH CYRUS

While Cyrus had taken courses in linear algebra, for him it had been several years. However, he immediately went to trying an algebraic method of solution. In fact, at no point did he produce a sketch of any kind, even though I presented the problem initially with a drawing of the destination point; all written work was algebraic.

C.1.1	S: No, you c- I think you only can with the second one, because a hundred and seven is not divisible by- that's a hundred and seven, right?
C.1.2	I: Yeah.
C.1.3	S: A hundred and seven is not divisible by three.

C.1.4	I: We can use these in combination.
C.1.5	S: Oh, we can use,
C.1.6	I: Like we can use part of this one and part of that one. …
C.1.7	S: Okay, um, so then that means... hmm... I'm not sure it can be done, hold on, think, ...okay. [?] am I allowed to write down on this, do some calcul-

With this first attempt at thinking about the problem, Cyrus erroneously believes that the tools cannot be used in combination. However, for someone believing this, it is somewhat surprising to see the concept of divisibility invoked more readily than thinking about an extended form of the vectors.

Once Cyrus has done some of the calculations, he explains his thought process:

C.2.1	S: I'm treating it as, a, matrix algebra problem. Solving systems of linear equations. I'm going through the- uh, what do you call it, doing row operations to find out if this system actually has a solution or not.
C.2.2	I: Okay.
C.2.3	S: And for this problem I think both a and b have to be- they both have to be, what is it, they- they both have to be whole numbers, and definitely nothing negative, for it to work.
C.2.4	I: Oh, ah, I should mention that, ah, the context of our problem actually doesn't require that. Uh, we could ride the magic carpet for, like, half a time unit, […] or we could ride it backwards.
C.2.5	S: Okay, um, [writing] […] that's a mistake, alright, so...
C.2.6	I: This is- is this a solution method that you, recall from, a course, or something like that?
C.2.7	S: Yeah, this is a solution method I recall when I took a second year linear algebra course.

Cyrus ultimately gave a solution which was not correct, and noted that the solution was not reasonable. The discussion of what it implied, and what a reasonable solution might imply, marked the only instance of any use of geometric methods of reasoning, although Cyrus did not initiate it on his own:

C.3.1	S: Okay, I got one unique solution, but I'm really not sure it makes sense. Either I did something wrong somewhere along the way, or- so, I don't know, just based on the- I'm just gonna guess it's not possible to do this.
C.3.2	I: Hmm... ah, if you got a solution that did seem reasonable, then would

	your conclusion be different?
C.3.3	S: Yes. If these two numbers were, um, if these two numbers were different, I would say it would make sense.
C.3.4	I: Okay, but the- these numbers that you have, they don't make sense, ah, why in particular?
C.3.5	S: Well they're both- they're both negative, that's why.
C.3.6	I: Okay. So that would take you off, over there.
C.3.7	S: Yeah.
C.3.8	I: Quadrant three. And tha- that's right, that doesn't make sense, the-
C.3.9	S: Mm-hm. I suppose that's like a little bit- I suppose that's a little bit of geometric thinking on my part there, 'cause you'd be all the way in quadrant, I forget it's one, two, three, four I think isn't it?
C.3.10	I: Yeah.
C.3.11	S: You'd be all the way in quadrant three.

The concluding discussion of what sort of solution would indicate that reaching the point was possible returns to purely algebraic terms:

C.4.1	S: Okay, that's fine. Okay, so, yeah, tha- that's pretty much what I make of this problem. If we were able to find- if we were- I th- there would be two cases. If- if these had both been positive numbers in there was a unique solution, then definitely there is- this is- the problem was asking if you can- you can reach this point. Using either the carpet or the hover-thingy.
C.4.2	I: Not either-or. We can use part of one then part of the other. …
C.4.3	S: But can we reach this as our destination and, if there was either one unique solution and it must be positive, or if there was infinitely many solutions, in either of those cases it would be possible to do this.

Here, we see that even approaching the problem from a hypothetical standpoint, the interpretations provided are still entirely algebraic. Since Cyrus stated (C.2.7) that this was a method recalled from a linear algebra course, we can see evidence for both an inclination toward algebraic methods and an inclination for the use of recollection of previous solution techniques, which both stand in contrast to Joshua’s earlier solution.

CONCLUSIONS

From these interviews, there appears to be a strong tendency toward particular modes of thinking and problem-solving that differ from the typical student population, though the particular modes themselves differ. Some of these may initially appear to be a problem for the typical methods of instruction used. However, I believe that if they are taken into account, they can be turned to the advantage of students on the autism spectrum, particularly in light of some cases of unexpected success using unorthodox methods of solution. This fits with the general Vygotskian idea of overcompensation, as well as Vygotsky's original proposals regarding instruction based on that idea. While these interviews were with only two students, the results suggest that interviewing more students could produce useful and interesting results in further research studies.

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ORAL VS. WRITTEN EXAMS: WHAT ARE WE ASSESING IN MATHEMATICS?

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One of the most striking differences between the Canadian educational system and most European educational systems is the importance given to oral examinations, particularly in mathematics courses. In this paper, seven mathematics professors share their views on mathematics assessment, and their thoughts about the types of knowledge and understanding in mathematics that can be assessed on written and oral exams. With the increased emphasis on closed book written examinations, the results in this study show that written exams alone are not sufficient to assess students' conceptual knowledge and relational understanding, and therefore, there is a critical need for implementing the oral assessments in mathematics courses.

INTRODUCTION

For many years the primary method of assessment in mathematics classroom has seemed to be strictly based on closed book written examinations. The USA in particular appears to be dominated by closed book examinations (Gold, 1999; Nelson, 2011). Iannone and Simpson (2011) note that the majority of mathematics students in the UK seem to be assessed predominately using high stakes, closed book examinations at the end of almost every module. Joughin (1998) argues that the structure of the assessments today are either closed and formal, with little interaction between student and assessor(s), or open, with less structure and the opportunity for dialogue between student and assessor(s). Ernest (2016) believes that conversation lies at the heart of mathematics and that mathematical knowledge representations are conversational, consisting of symbolically mediated exchanges between persons as well as claiming that, "the ancient origins and various modern systems of proof are conversational, through dialectic or dialogical reasoning, involving the persuasion of others" (p. 205). When it comes to different types of oral assessment, according to Joughin (2010), they can be categorized into three forms: presentation on a prepared topic (individual or in groups); interrogation (covering everything from short-form question-and-answer to the doctoral viva); and application (where candidates apply their knowledge live in a simulated situation, e.g., having trainee doctors undertaking live diagnoses with an actor-patient). Although oral assessment is used in many areas, there is very little literature examining the use of oral assessments. Hounsell, Falchikov, Hounsell, Klampfleitner, Huxham, Thompson, and Blair (2007) note in their comprehensive review of the literature on innovative assessment that less than 2% of the papers address the oral assessments. They reviewed the recent UK literature on 'innovative assessment' and of 317 papers considered, only 31 dealt with 'non-written assessments.' Within this category, only 13% addressed the use of oral examinations. Today, there are many countries that still maintain an oral assessment as an important part of their assessment diet, such as Hungary, Italy and the Czech Republic (Stray, 2001). Germany is also one of them.

Oral Examination in Mathematics

In most of the cases, students would have to take written exam first, and then after passing the written exam, they would go to the next stage, which would be taking an oral exam. During the oral exam, students would have access to a blackboard, paper, and pen. The exam would be conducted by the course instructor, and each oral exam session could last anywhere from 30 minutes to 1 hour. Occasionally during the oral exam three or four students would be invited at the same time. The instructor would have prepared in advance a set of cards with questions of approximately equal difficulty, so a student would step in, randomly draw a card from the set of cards, and then, he/she would take a scrap paper and go back to his/her desk and start working on the chosen question. After some time working on the question, each student, one by one, would go up to the board and present his/her answer to the instructor. In addition, the teaching assistant would be in the same room, monitoring students and taking the protocol. During the oral exams, usually students would be able to receive some help if needed and would receive a grade immediately following the exam. A typical card would have one theoretical question (for example 'prove the fundamental theorem of calculus') and one exercise (for example 'show that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$').

THEORETICAL FRAMEWORK

Skemp (1976) introduced two perspectives of mathematics, relational understanding as knowing both what to do and why, and instrumental understanding as the ability to execute mathematical rules and procedures. On the other hand, Hiebert and Lefevre (1986) contrasted two perspectives of mathematics, conceptual and procedural knowledge, defining both of them as:

Conceptual knowledge:

> Knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network (p. 3-4).

Procedural knowledge:

> One kind of procedural knowledge is a familiarity with the individual symbols of the system and with the syntactic conventions for acceptable configurations of symbols. The second kind of procedural knowledge consists of rules or procedures for solving mathematical problems. Many of the procedures that students possess probably are chains of prescriptions for manipulating symbols (p. 7-8).

After having numerous discussions with some mathematics professors in Canada as well as in United States, I realized that oral examination in mathematics courses at university level is not present at all even though there is a number of research that indicate that oral assessments have a positive impact on students' learning of mathematics (Boedigheimer, Ghrist, Peterson, & Kallemyn, 2015; Fan & Yeo, 2007; Iannone & Simpson, 2012; Iannone & Simpson, 2015; Nelson, 2011; Nor & Shahrill, 2014; Odafe, 2006). Teachers' views "can provide significant insight into what teachers value and the relative importance they assign to different aspects of mathematics or the teaching of mathematics" (Wilson & Cooney, 2002, p. 131). In this paper, the following research question was investigated: *What are the mathematics professors' views on the nature of mathematics assessment?*

METHODOLOGY

The research design for this study is descriptive/qualitative. Seven participants were interviewed using open-ended questions to gather information about their personal experiences and perspectives on using written and oral assessments in mathematics classroom. These participants were selected based on the following criteria: each participant has been exposed to oral assessment either as a student, teacher, and/or professor. In terms of recruitment, I used a methodology of snowballing, wherein I started with mathematicians whom I knew professionally, and then asked them to recommend others in the mathematics department or elsewhere, for whom they suspected that they may have a history of experiencing or using oral assessment. Seven mathematics professors were selected for interviews: Melissa, Elisabeth, Van, Nora, Dave, James, and Jane. Melissa, Elisabeth, Van, and Nora are currently teaching at a Canadian university while Dave, James, and Jane are currently teaching at a university in Germany. With respect to familiarity with oral assessment, Van, Melissa, Nora, and Elisabeth had been previously exposed to oral examination in mathematics prior moving to Canada while Dave and Jane, who were educated in Canada and the US, had never been exposed to oral examination in mathematics prior moving to Germany. James was born and educated in Germany, and thus, he has had a lot of exposure to oral assessment in mathematics. The audio recordings of interviews were transcribed and transcriptions were used for data analysis.

RESULTS

There are three aspects of the results that will be discussed in this section:

- What do participants value about oral assessment over written assessment?
- Where do participants' views on oral assessment come from?
- What types of knowledge and understanding can be measured using oral assessment as compared to written assessment?

What do participants value about oral assessment over written assessment?

When it came to the nature of mathematics assessment, it seemed that most of the participants valued students' ability to explain their reasoning and understanding of mathematical concepts, in relation to the oral examination. The following comments exemplify this:

> "I would still say that oral examination was better in assessing understanding not just the knowledge… oral examinations were to a deeper extent probing understanding of the concept" (*Melissa).*

> … "when there is an oral exam, there is an ability to show your logical thinking" *(Nora).*

> … "I often doubt if the written exam gives the complete picture… the oral exam can give an opportunity to students to show their knowledge better than the written exam…." *(Van).*

> … "the questions where I need to see if they understand the chain rule, the person has to explain to me in two words. They don't need to solve the problem on twenty lines" *(Nora).*

> "I would say that during oral examination, it is easier to discover the level of your understanding" *(Van).*

Where do participants' views on oral assessment come from?

It seemed that one of the main sources of participants' views of mathematics assessment came from their own prior schooling experience. Oral examinations in mathematics were part of the educational system in some of the participants' prior schooling and teaching experience, therefore, oral exams were considered to be an essential and natural part of examination process, from primary to higher education. The following comments support this:

> … "so, we were used to, it was natural, it was not something that different in high school, it was a continuation of high school" *(Melissa).*

> "Mathematics I think very much lives from discussions. So, for me the oral examination is much more natural and the written examination is just out of necessity" *(James).*

> "I have reasons that I feel are good reasons that I prefer written exams, but, you know, maybe I wouldn't think those things if I had gone through a system with oral exams" *(Jane).*

Another reason for believing that oral exams play an important part in assessment process in mathematics was related to the culture and study program of the university where they are teaching. The following comment exemplifies this:

> … "this is natural because it had this effect of getting to know those students who will continue into the higher level diploma courses, so much like you would

> get to know those master students so to speak that come after…" *(James).*

On the other hand, the oral exams could cause discomfort to those who had never been exposed to it, as being something that is not completely natural or familiar. Dave commented:

> "…[It] is primarily I guess if you like a cultural issue… I think there's going to be a difference between me doing an oral exam and somebody who has grown up with oral exams doing an oral exam… I'm doing something that is not part of my cultural background that I don't have any intuitions about it even if I have knowledge about it."

What types of knowledge and understanding can be measured using oral assessment as compared to written assessment?

Based on the participants' responses on what could be assessed in oral and written exams in mathematics, it seemed that there was a clear division between the views of participants who had previously been exposed to oral assessments in mathematics and the one who had not. Their views were presented in Table 1.

Examination Type	What Are We Assessing?	Participants
Written	Procedural knowledge/Instrumental understanding	Van; James; Melissa; Nora; Elisabeth
	Procedural knowledge/Instrumental understanding Conceptual knowledge/Relational understanding	Dave; Jane
Oral	Conceptual knowledge/Relational understanding	Van; James; Melissa; Nora; Elisabeth

Table 1. Written and oral exams: What are we assessing?

All five participants, Van, James, Melissa, Nora and Elisabeth, who had been previously exposed to oral assessments in mathematics, agreed that written exams could mostly assess procedural knowledge and instrumental understanding while oral exams could better assess conceptual knowledge and relational understanding. On the contrary, the other two participants, Dave and Jane, who had never been previously exposed to oral assessments in mathematics prior coming to Germany, believed that the written exam alone could efficiently assess both procedural knowledge and instrumental understanding and conceptual knowledge and relational understanding.

The following two subsections contain comments supporting each of these views.

Oral Exams: Conceptual knowledge/Relational understanding

Written Exams: Procedural knowledge/Instrumental understanding

> "In the homework written assignments I would say more procedural…. procedural in the sense of computational. So, conceptual in the sense of abstract arguments… more oriented towards prove this and that statement" (*James).*

> "I guess in written maybe you can assess procedural. You can see if they could follow a strategy for solving an equation. But I guess relation, yeah it's more-- You can do that I guess better with oral" (*Elisabeth).*

> "If I have oral assessment even in tutorial, I can very quickly get the picture across the class, how is the class doing… The drill part, the technical part, they can always pick up if they understood the concept" *(Nora).*

> "The oral exam was more of about theoretical questions…. to prove or disprove something or give me example or counter example or justify this or justify that or make a difference between this subject and this subject…. more in-depth. And, the written exam was with the type of question, you know, if this is given and this is given, then find this or find that" *(Van).*

> "In most cases those questions were sort of follow up of the written exam questions both to check understanding or maybe give students opportunity to correct but also to look deeper into student's thinking." (*Melissa).*

> "If I ask you to do a proof of a theorem, and if I ask you to write that on a paper and read it later, I'm not going to get the same idea of your understanding of the stuff if I ask you to do this in the front of the board and if I can ask you the questions why and how at every step…proving and disproving, examples and counter examples, the meaning of the definitions, the oral exam is I would say much precise, a better tool than written one" (*Van).*

Nora felt that conceptual questions could only be assessed orally and when asked for an example, she responded:

> **"**Explain to me what is the derivative…. Can you put this question on the written exam? No…. because nobody has the resources to mark it. It takes forever to read students' poor handwriting and to see exactly what they discussed, from which position, is it a geometrical side… The understanding can be assessed only in oral exam."

Written Exams: Procedural knowledge/Instrumental understanding and Conceptual knowledge/Relational understanding

> … "for mathematics, the questions that can be answered quickly for me are mostly the sort of procedural questions… you need to think for a while to answer those questions and so I'm not sure in a context of an oral exam where you don't have very long whether there's such good questions" *(Dave).*

Jane also felt that conceptual questions, theory, and proofs could be better assessed in writing than orally and when asked for an example of a question that could not be assessed orally, she responded:

> "The ones that take more time to think about. Yeah… the time is a pretty big issue because you're doing advanced Mathematics. You tended to need just more time to think about things."

DISCUSSION AND CONCLUSION

Mathematical problems that could better assess procedural knowledge and instrumental understanding, participants considered types of problems that would require some sort of computational skills. On the other hand, when it came to mathematical problems that would better assess conceptual knowledge and relational understanding, participants considered theoretical type of questions in the sense of abstract arguments that would involve proving, justifying and defining given statements. Another interesting finding is that for Jane and Dave time played an important role in terms of choosing the most appropriate mathematical questions for the exam. Moreover, it was interesting to see that both Jane and Dave were relating "conceptual" types of questions in mathematics to the questions that would take more time to think about, and so, they could be only answered through written exam. On the other hand, they considered the questions that could be answered quickly to be sort of "procedural" questions, and only these types of questions could be assessed orally.

Overall, if we acknowledge that each student learns differently, then having a common approach to assessment would be inadequate. Educators accept the need for differentiated instruction in order to deal with the individuality and variability of students, and thus, they also need to accept the need for differentiated assessment to represent the learning of the fractured student collective (Liljedahl, 2010). Also, it is very important for me to mention that in this paper I am not trying to depreciate written assessment, but merely to argue for a balanced diet of the most appropriate assessment methods for the students. I hope that the ideas and examples that I was able to present in this paper will encourage many mathematics educators to continue or to begin using oral assessment in their mathematics courses as well as to help promote discussion with their colleagues and students on this matter.

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