MEDS-C 2017

PROCEEDINGS OF THE 12th ANNUAL MATHEMATICS EDUCATION DOCTORAL STUDENTS CONFERENCE

November 4, 2017

SIMON FRASER UNIVERSITY | FACULTY OF EDUCATION

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CONTRIBUTIONS

MEDS-C 2017 was organized by members of the Mathematics Education Doctoral Program. The conference would not have been possible without the following contributions:

Conference Coordinators: Judy Larsen and Tanya Noble

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Review Coordinators: Jason Forde and Peter Lee

Lunch Coordinator: Lyla Alsalim

Snack Coordinators: Robert Sidley and Milica Videnovic

Technology Support: Leslie Glen

Timers: Arezou Valadkhani

PLENARY SPEAKER

Ofer Marmur

UNDERGRADUATE STUDENT LEARNING DURING LARGE-GROUP CALCULUS TUTORIALS: KEY MEMORABLE EVENTS

In recent years there has been an increase of interest in research on undergraduate mathematics education. However, relatively little is known on how students learn during frontal tutorials, and how the learning relates to and is shaped by student affect and the teaching that took place. This presentation will examine these issues in a setting of large-group undergraduate-calculus tutorials via a conceptualized theoretical construct termed Key Memorable Events (KMEs). These are classroom events that are perceived by many students as memorable and meaningful in support of their learning, and are typically accompanied by strong emotions, either positive or negative. Theoretical and pedagogical implications will be discussed in terms of lesson design, data analysis, and conceptualization of learning in the undergraduate mathematics classroom.

ABSTRACTS

Lyla Alsalim

UNDERSTANDING HIGH SCHOOL MATHEMATICS TEACHERS' PRACTICE

In this paper, patterns-of-participation theory serves as a lens to interpret and understand Saudi high school mathematics teachers' practices. This framework focuses mainly on understanding what practices and figured worlds are significant for the teacher and how the teacher engages in those figured worlds. The data presented is about Maram, a high school mathematics teacher in Saudi Arabia. The data generated suggests that there are five significant practices or figured worlds to Maram's sense of her practice as a mathematics teacher. The paper discusses and explains these figured worlds.

Sandy Bakos

INTERACTIONS AND TENSIONS BETWEEN COMPUTATIONAL THINKING AND MATHEMATICAL CONCEPTS

This study explores the interactions and the tensions between mathematical activity and Computational Thinking (CT) within the context of a primary classroom. Using a highly visual, geometric programming language, grades 2/3 students were introduced to the concept of looping through a combination of visual media and dynamic sketches. Analysis of interview results suggest that young learners are capable of recognising and identifying looping, however application of this concept can be challenging for them. An examination of the ways in which CT concepts appear to conflict with or facilitate the learning of mathematics for young learners will be explored through the use of a dynamic geometry environment (DGE).

Jason Forde

A CONJOINED ENACTIVIST/QUANTUM MECHANICAL INTERPRETATION OF MATHEMATICAL MODELLING

This paper provides a conceptual overview of a larger program of research that explores the implications of reframing mathematical modelling practices according to a conjoined enactivist/quantum mechanical framework. The research is motivated by the view that reinterpreting the nature of mathematics through a more inclusive materialism has the potential to deepen our understanding of the role that mathematics plays in shaping perceptions of the natural world and the ways in which mathematical models ultimately take on the meanings that they do. Incorporating literature from the disciplines of physics, mathematics education, educational psychology, and philosophy more generally, four core themes are identified, and groundwork laid for a broader perspective regarding the mathematical structure of reality.

Leslie Glen

THE PRACTICAL AND THE THEORETICAL: POLAR OPPOSITES OR COMPANION SKILLS?

It is the ability to make connections between concepts that signals competence in algebra. This study investigates the divide between students who are able to make those connections and those who are not. A task designed to examine one of these connections is described, and a preliminary analysis performed using Sierpinska's framework of "Practical" versus "Theoretic" thinking. The results are evidence that students who have made connections are able to interpolate from a given situation, and those who have not often cannot even use their own answers to draw conclusions.

Victoria Guyevskey

DYNAMIC GEOMETRY AS A SPATIAL PROGRAMMING LANGUAGE IN ELEMENTARY SCHOOL

The purpose of this study was to develop and research the effectiveness of geometric coding environments that can enable students to model the world in which they live, as well as develop spatial reasoning skills. We conducted a two-month classroom intervention with grade 2/3 students, experimenting with computational thinking (CT) tasks, centrally involving geometry and spatial reasoning within a dynamic geometry environment (DGE). In our experiments, we were interested in how the DGE/CT approach gives rise to concepts relevant to both mathematics and coding. We wanted to see (1) what the students would learn, and (2) what the effect of combining CT and geometry would be. Our finding is that DGE-based tasks effectively support teaching of many CT concepts, making Math/CT interdisciplinary integration a realistic goal.

Andrew Hare

LINES, STANZAS, EPISODES: UNITS IN THE UNDERGRADUATE MATHEMATICS LECTURE

Undergraduate mathematics courses very commonly include a few dozen hours of lecturing. This study investigates the structure of a course of 35 50-minute lectures delivered in an undergraduate abstract algebra course at a midsize Canadian university. Influenced by the approach of Halliday in his work on determining useful categories and units in spoken and written texts, this paper describes and characterizes 2 structural building blocks of 'chalk and talk' mathematics lectures of increasing size: lines and stanzas. Beginnings and endings of such units are typically marked in multiple ways: intonation, body orientation, direction of gaze, volume, pitch, position of body relative to the board, and commonly used starting or ending words.

Harpreet Kaur

COMPARING ANGLES IN A DYNAMIC GEOMETRY ENVIRONMENT

This paper examines young children's thinking while comparing geometric representations of angles of different sizes in a dynamic geometry environment (DGE). This study is based on a classroom experimentation, during which the kindergarten/grade1 children worked in a whole classroom setting over nine sessions, in which they could interact directly with Sketchpad on an interactive whiteboard. Using the dynamic sketches in Sketchpad, children were able to develop an understanding of angle as a turn. DGE approach is found to be helpful in focusing children's attention on the quantity of turn rather than on the length of the line segments during angle comparison tasks. This study provides one way of establishing a relationship between angle-as-a-turn and angle-as-a-shape conceptions.

Judy Larsen

WHAT MATHEMATICS TEACHERS SEEK WHEN APPROACHING PROFESSIONAL LEARNING THROUGH SOCIAL MEDIA

Mathematics teachers are using social media to approach professional learning. This sort of activity is often carried out by calling on the digital community to respond to various questions related to the practice of teaching mathematics. This paper presents an analysis of a collection of such queries using a taxonomy of teacher wants. A revised taxonomy is emerged from the data that provides insight into the possible desires mathematics teachers have when using social media for professional learning. Results indicate that the social media space is used most prominently to seek pragmatic solutions for teaching, but that willingness to rethink more significant aspects of practice is not common. These results reveal the limitations of social media as a place for professional learning and inform professional development initiatives.

Peter Lee

MATHEMATICAL LEARNING DISABILITIES AND INTERTEXTUALITY

The study of mathematical learning disabilities (MLD), while relatively new, has received contributions from diverse fields such as psychology and education. The purpose of this paper is to analyze an early 20th century text on special abilities and disabilities to show how the discourses of psychology and education merge to constitute the burgeoning study of MLD and its participants. Norman Fairclough's version of critical discourse analysis will be applied, particularly his notions of intertextuality and discursive change. Findings suggest the sample text reflects the medico-scientific voice of psychological discourse, yet the book as a whole contains elements characteristic of educational discourse.

Minnie Liu

STUDENTS' USE OF EXTRA-MATHEMATICAL KNOWLEDGE DURING THEIR PROBLEM-SOLVING PROCESS

When students work on mathematical tasks situated in reality, they need to connect their intra-mathematical skills with their extra-mathematical knowledge (EMK) to produce a realistic and reasonable solution. As such, EMK plays a crucial role in the quality of the generated solutions of such tasks. This study finds that the quality of students' generated solutions depends not only on students' existing EMK, but also on their abilities to identify, acquire, and apply these EMK.

Tanya Noble

JUSTIFICATIONS SIGNAL SYSTEMIC CONTRADICTIONS AS CONDITIONS CHANGE WITHIN THE SECONDARY SCHOOL MATHEMATICS SYSTEM

Engestrom's Activity Theory is a powerful structure through which to describe the activity of teaching mathematics. A view into the diverse actions and goals among the co-creators of this shared systemic object(ive) are outlined through Leont'ev's Activity Theory as actions are influenced directly by the conditions in which the broader system is operating. Each part of the broader system both influences and reacts to changes in conditions. As used in this paper, the theory allows insight into consistencies and contradictions as the co-creators of the object(-ive) adjust to changing conditions imposed by changes in the British Columba education system.

Annette Rouleau

TENSIONS IN IMPLEMENTING MATHEMATICS JOURNALING: AN ACTIVITY THEORY APPROACH

Research suggests there is a strong connection between mathematical writing and mathematical learning. As a result, many educators are implementing journaling in their mathematics classroom, which can be a challenging process. This paper identifies the tensions faced by an individual teacher implementing journal writing for the first time and interprets those tensions through the lens of activity theory. The results suggest that pinpointing the areas of tension within an activity system may provide a means of mitigating the challenges.

Robert Sidley

ARE THEY GETTING ANY BETTER AT MATH?

Conversations with stakeholders about students' improvements in mathematics invariably focus on student grades and work habits. Further, research into improvements in mathematical performance focus almost exclusively on the *acquisition of mathematical content and improvement in test scores. This narrow focus makes assumptions about what it means to know, do, and improve mathematically. By exploring the micro and macroculture of mathematics classrooms and analyzing interview data with two secondary school mathematics teachers, I explore what it means to get better at math inside an inquiry tradition.*

Arezou Valadkhani

DOES THE GEOMETRIC SOFTWARE WE USE INFLUENCE GEOMETRIC THINKING?

The study reported here is dedicated to research about our own way of thinking mathematically. It investigates the possibility of the existence of different such ways through using different tools and tries to explain how thinking mathematically could be realized by physical reactions. In doing so, the exposition will centre on a specific geometrical problem in two different environments, paper/pencil and the Geometer's Sketchpad. We identify dragging trajectories and other physical reactions in GSP as well as a solution based on theories in paper/pencil environment, consequently resulting in different visions to the problem (ways for attacking the problem). It is also suggested that a geometric software can prompt users to pay attention to certain features of a geometric problem that are ignored in a paper/pencil environment.

Milica Videnovic

EVIDENTIAL VS NON-EVIDENTIAL BELIEFS IN THE CASE OF ORAL ASSESSMENTS

One of the most striking differences between the Canadian educational system and most European educational systems is the importance given to oral examinations, particularly in mathematics courses. In this paper, seven mathematics professors share their views on positive and negative aspects of oral and written assessments in mathematics. Four of the seven professors were born and educated in Bosnia, Poland, Romania, and Ukraine, and they are currently teaching in Canada. The other three professors were born and educated in Canada, the United States, and Germany, and they are all currently teaching in Germany. The results in this study show that non-evidential beliefs can affect views on oral assessments in mathematics.

UNDERSTANDING HIGH SCHOOL MATHEMATICS TEACHERS' PRACTICE

Lyla Alsalim

Simon Fraser University

Patterns-of-participation theory is used in this study as a lens to *interpret and understand high school mathematics teachers' practices in Saudi Arabia. This framework focuses mainly on understanding what practices and figured worlds are significant for the teacher and how the teacher engages in those figured worlds. The data presented is about Maram, a high school mathematics teacher in Saudi Arabia. The data generated suggests that there are five significant practices or figured worlds to Maram's sense of her practice as a mathematics teacher. This paper discusses and explains these figured worlds.*

PURPOSE OF THE STUDY

In Saudi Arabia, the *education system has undergone* major changes in the past decade. Government agencies involved in education have introduced new policies, standards, programs, and curricula. These changes are accompanied by high expectations that teachers will incorporate the changes seamlessly without consideration of their existing practices. This paper is part of an ongoing study that intends to gain a better understanding of how high school mathematics teachers in Saudi Arabia are coping with recent education reform, including how their practices are evolving in response to the changes that are happening in the education system.

THEORETICAL FRAMEWORK

In this paper, a patterns-of-participation (PoP) (Skott, 2010, 2011, & 2013) approach *serves as a lens to* interpret and understand Saudi high school mathematics teachers' practices during the current reform movement. The PoP framework identifies teachers' practice as being how teachers narrate and position themselves in relation to multiple, and sometimes conflicting, figured worlds (Skott, 2013). Figured worlds are imagined communities that function dialectically and dialogically as if in worlds. They constitute sites of possibility that offer individuals the tools to impact their own behaviour within these worlds (Holland, Skinner, Lachicotte, & Cain, 1998; Skott, 2013).

Traditionally, most research in education that focuses on studying teachers' practices adopts an *acquisitionist* approach, especially those studying teachers' beliefs and knowledge in relation to teachers' practices (Skott, 2013). Recently, more researchers, including Skott (2010, 2013), adopt participationism as a metaphor for human functioning to understand teachers' practices. "The origins of participationism can, indeed, be traced to acquisitionists' unsuccessful attempts to deal with certain long-standing dilemmas about human thinking" (Sfard, 2006, p. 153). Skott presents PoP as a coherent, participatory framework that is capable of dealing with matters usually faced in the distinct fields of teachers' knowledge, beliefs, and identity. Therefore, PoP is a theoretical framework that aims to understand the relationships between teachers' practice and social factors. Skott (2010, 2011) initially developed the PoP framework in relation to teachers' beliefs. However, in order to develop a more coherent approach to understand teachers' practices, Skott (2013) extended the framework to include knowledge and identity.

The social approach of research in mathematics education has progressively promoted the notion that practice is not only a personal individual matter; it is in fact situated in the sociocultural context. Although the relationships between individual and social factors of human functioning have generated much debate in mathematics education, it is mainly in relation to student learning (Skott, 2013). Therefore, PoP is a theoretical framework that aims to understand the relationships between teachers' practice and social factors. To a considerable degree, PoP adopts participationism as a metaphor for human functioning more than mainstream belief research. Therefore, PoP draws on the work of participationism researchers, specifically Vygotsky, Lave and Wenger, and Sfard.

"The intention of PoP is to take this one step further by limiting the emphasis on acquisition and include a perspective on the dynamics between the current practice and the individual teacher's engagement in other past and present ones" (Skott, 2013, p. 557). This framework focuses mainly in understanding what practices and figured worlds are significant for the teacher and how the teacher engages in those figured worlds. A teacher's engagement with these figured worlds inform and adjust the interpretations s/he makes to her/himself and the way s/he engages in on-going interaction in the classroom. These figured worlds work in a very complex system where they could support and sometimes, restrict one another as the teacher contributes to classroom practice.

METHODOLOGY

This paper is part of an ongoing study that intends to develop more coherent understandings of Saudi high school mathematics teachers' practices during the current reform movement. For data analysis, I used a qualitative analysis approach based on grounded theory method as used by Skott (2013). I applied the two fundamental and basic stages of coding identified by Charmaz (2006); the open or initial coding and the focus or selective coding. Through the coding process, I was able to organize, group, and reflect on the data. The process includes isolating patterns and categorizing the data to identify practices and figured worlds that are significant for the participant teachers and how they engage with these figured worlds.

The data presented in this paper comes from Maram, a teacher with eleven years of experience teaching high school. After she graduated from university, she started teaching in a private school. She worked there for three years and taught grades 10 to

12. After that, she received an offer to teach in a public high school. She has been teaching in the public high school for eight years. When I met Maram, she was teaching 18 lessons per week to students in grade 11.

I conducted two semi-structured interviews with Maram. The first one before I observed her teaching two lessons and the second interview was conducted after the classroom observations. In the first, I asked her to reflect on her experiences with mathematics teaching and learning in school, at university, and during her practicum year. I also asked her questions related to different aspects of the new reform movement in education system in Saudi Arabia. The second interview focused on her experiences with teaching mathematics at her school and on her relationships with the school, her colleagues, and the students. I asked her to reflect on the lesson planning process she had in order to prepare the lessons I observed. During the second interview, I also used a stimulated recall technique by playing audio recordings of parts from the lessons to facilitate her conversation about her own teaching practice in the classroom.

During my visits to Maram's school, I was also able to collect some data from informal observations of staff-room communication between Maram and her colleagues. I also have a copy of Maram's lesson planning notebook and some of her worksheets and tests samples.

DISCUSSION

The aim of this paper is to develop a deeper understanding of the participant teacher's significant practice and figured worlds and how she engages with these figured worlds. As a teacher positions herself in relation to her profession as a mathematics teacher, she draws on several, often incompatible, figured worlds. Her engagement with these figured worlds does not only appear in her verbal communication, but also by the choices she makes in her all other actions related to her profession, such as her immediate reaction to certain student behaviours or the way she expresses her view when engaging in a conversation with her colleagues.

Teachers' engagement with figured worlds inform and adjust the perceptions they make and the way they engage in on-going interaction in the classroom. These figured worlds work in a very complex system where they could support, and sometimes restrict, one another as every teacher contributes to classroom practice.

Maram's classroom

Maram's classrooms are relatively crowded for two reasons. First, there are around 42-48 students in her classrooms, which is more than the average number of students in a regular high school classroom in the Al Khobar district. Second, the classrooms in Maram's school are small. Students' desks are in rows of three facing the front of the classroom. She occasionally assigns her students homework. Once every week or two, she gives her students a quiz, or as she calls it, a "timed test". She mostly relies on lecturing to introduce the new concepts but tries to use additional teaching techniques to encourage deeper and meaningful discussions in her classroom. She follows the textbook sometimes but not always. For example, she might ask students to read a lesson introduction from the textbook for a particular chapter, but then ignores the textbook introduction other times.

RESULTS

After eleven years of teaching, the data generated about Maram suggests that there are five significant practices or figured worlds to Maram's sense of her practices as a mathematics teacher. These figured worlds are mathematics, the textbook, the reform, students' achievement, and social network engagement.

Mathematics

Maram is very passionate about mathematics. She expressed her love for mathematics as a subject by saying, "As far back as I can remember, I have always loved mathematics. It is just something I enjoy. When I was a student at school, mathematics classes made me happy and I was relaxed in mathematics classes because I was doing something that makes sense to me". Although Maram loves mathematics as a subject, she does not consider that main objective of her job as making her students love mathematics; she sees her job as making her students do mathematics. For Maram, teaching mathematics is about helping students build a strong understanding of mathematical concepts and practice mathematical procedures. According to her, the best way to learn mathematics is to encourage students to talk about mathematical concepts, explain their thinking process and do as many exercises as they can. She stated, "Practicing mathematics problems is a major part of learning mathematics. You can't learn to do mathematics without actually doing it a lot".

Maram noted that it is very common in high school to see that most students do not like mathematics. "I think students mostly don't like mathematics because they don't like that they have to struggle to be able to do mathematics. I always tell my students not to expect mathematics to be easy. It is actually hard". Maram explained that struggling is normal while doing mathematics. Students usually expect to know how to solve the problem as soon as they read it. She encourages her students to not get frustrated when they do not know how to solve the problem and look for ways to help themselves by drawing a picture, looking for the important words in the problem, and trying to remember a similar problem. According to Maram, most of her students do not expect mathematics to be meaningful and make sense. Students are mostly content working with mathematical symbols and doing routine problems without ever grasping a real understanding of the problem. According to Maram, students' view of mathematics learning is a direct result of traditional teaching of mathematics. Although Maram was comfortable learning mathematics in a traditional environment, as a teacher she tries to do something different. She does not consider her teaching style far from the traditional style but tries to find ways to move away from the traditional approach.

Reform

Current reform ideas in education remind Maram of the teacher she wanted to be when she started her teaching carrier. When she began teaching, she was determined to be a "nontraditional mathematics teacher", meaning a teacher who does not rely on the traditional lecture format of teaching. However, Maram considers her approach to mathematics teaching to be more on the traditional side. She implied that she feels she is under pressure to be a good teacher. When I asked her to explain to me her view of a good teacher, she replied that a good teacher is the one who has a positive impact on student understanding, quality of learning, and student achievement.

Generally, Maram supports reform and articulated clearly that changing how teachers teach mathematics in schools is a necessary step. She supports reform recommendations regarding concrete exploration and meaningful representation of mathematical concepts. However, she finds some reform ideas too challenging and hard to translate into practices. According to Maram, in high school, mathematics is presented in a very abstract and formal way. It is very difficult, and sometimes impossible, for teachers to create a learning environment for students where they can experience mathematics in a meaningful way that is related to real world. Maram also finds reform recommendations to be misleading, sometimes providing teachers with mixed messages about best and effective practices. She explained that reform focuses on ensuring that high school students are equipped for university. According to Maram, most parents, high school students, and teachers interrupt "preparing students for university" to help students to achieve higher grades. She stated, "In recent years, there is too much emphasis on preparing students for university. This emphasis forces teachers to adopt what I call teaching for entering university practice or teaching for the tests; …it is the practice that focus on doing routine problems and never having a proper understanding of the principles behind it". Maram's opinion is that teachers should not be concerned with students getting high grades as much as helping students to achieve deep and real understanding

The textbook

Maram finds that the new textbooks are generally better than the old ones at providing more opportunities for student engagement and participation. She likes that the new textbooks offer different levels of exercises that aim to develop mathematical problem solving and communication skills. While she understands the vision of the new textbooks about what school mathematics should be, she still admits that mathematics in her classroom and in all her colleagues' classrooms "Is still generally taught using lecturing, whole class teaching, and regular testing". Maram explained the reason why the new textbooks did not have the expected effect in her teaching practices could be due to the lack of support and preparation she received during their implantations.

Using open-ended problems in her teaching is a new practice that Maram started to use after the implementation of the new textbooks. However, Maram usually makes changes to the challenging open–ended problems presented in the textbook when she presents them to her students. Most of the time, she re-writes the problems or activities, so they contain more structure, direction and clues to help students engage more with the problem. One of the techniques that Maram is trying to effectively implement in her classroom is group work. Using group work is one of the new textbooks recommendations. She usually asks students to do group work with their neighbours. Therefore, groups usually contain three students.

From her experience teaching with the old and new textbooks, Maram explained that she has learned that any textbook is merely as good as the teacher who uses it. While during her first years of teaching, she would follow the textbook entirely without even thinking about making any changes, now she understands that "The textbook is just a tool". She elaborated "[the textbook is] possibly a very important tool, but still a tool that I can use the way I find appropriate".

Students' achievement

Maram explained that the main goal of high school mathematics teachers is to create a supportive environment, so students can learn the necessary concepts for academic achievement. However, Maram finds that for most mathematics teachers this view of academic achievement does not go beyond the classroom and the written tests given to students. She understands students' achievement in a way that is different from how other teachers, and students, understand. She indicated that most teachers and students limit student achievement to their grades. "Teachers and students are not concerned with learning as much as they are concerned with achieving higher scores". In her opinion, the grading system used in schools, which is mainly based on written tests, does not reflect or communicate the level of actual academic progress or achievement that a student has developed during her time in school.

Maram also stated that teachers usually evaluate their teaching practices based on their students' grades. This practice leads teachers to focus on increasing their students' grades by teaching to the test. Maram blamed reform recommendations for not trying to adjust teachers' view about students' grades and academic achievement. She also claimed that some reform changes, such as introducing standardized tests, have emphasized the culture of testing in schools. Maram wishes to be able to eliminate some of the mandatory written tests students take in her classroom, such as the midterm and final.

Social network engagement

Maram is very active on social media, especially on Twitter. She is a social media enthusiast with a passion for sharing ideas about mathematics and mathematics teaching. She uses Twitter to interact with others who share the same interests. She described her social media interaction, especially on Twitter, as her best hobby. She likes to post mathematics problems and get people's responses. She discusses their ideas about the problems she posts, corrects their answers and finally provides the right answer. According to Maram, this interaction helps her to learn more about the thinking process when engaging with mathematics. She enjoys responding to the questions people ask and the comments they make.

In her Twitter bio, she identifies herself as a person who loves mathematics and does not mention that she is a mathematics teacher. She stated, “I didn’t want to describe myself in my Twitter bio as a mathematics teacher because my Twitter reflects me as a person not merely as a teacher. Of course, being a teacher is part of who I am, but it is not all of who I am”. She indicated that she mainly uses Twitter to “Share the love of mathematics with others” and do “Fun mathematics stuff without being restricted to an official school curriculum”. While Maram stated that she mainly uses social media to share her love of mathematics, she also uses social media to connect with other mathematics teachers. Maram follows several mathematics teachers; and teachers, both teachers of mathematics and other subjects, from different schools follow her. Twitter allows her to keep in touch with some mathematics teachers she met outside of her school.

CONCLUSION

PoP suggests that Maram regenerates meanings of her participation in classroom practices by drawing from different figured worlds. Through her engagement in the figured worlds of mathematics, the textbook, the reform, students’ achievement, and social network engagement, she constructs meaningful practices for the classroom context.

Maram is concerned about her practices and is trying to be a good teacher. She is trying to find ways to improve her students’ learning experience. She supports reform ideas in education, but at the same time, is struggling to adopt reform-oriented teaching. Although she is trying to adopt some changes in her practices, such as incorporating group work and unfamiliar activities such as writing, she still relies on traditional methods of teaching mathematics. Her engagement in social media suggests that she is willing to learn new things to improve her practices. Maram is approaching the way she is using the new textbook cautiously, looking for ways it fits with her students’ needs and abilities. She understands that the purpose of introducing the new textbooks was to encourage teachers to develop learning environments where their students have more time and room to reflect as well as discuss and investigate on their own. She admitted that the new textbooks did not have the expected effect on her teaching practices

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INTERACTIONS AND TENSIONS BETWEEN COMPUTATIONAL THINKING AND MATHEMATICAL CONCEPTS

Sandy Bakos

Simon Fraser University

This study explores the interactions and the tensions between mathematical activity and Computational Thinking (CT) within the context of a primary classroom. Using a highly visual, geometric programming language, grades 2/3 students were introduced to the concept of looping through a combination of visual media and dynamic sketches. Analysis of interview results suggest that young learners are capable of recognising and identifying looping, however application of this concept can be challenging for them. An examination of the ways in which CT concepts appear to conflict with or facilitate the learning of mathematics for young learners will be explored through the use of a dynamic geometry environment (DGE).

INTRODUCTION

With the increasing availability and use of digital technologies in mainstream society, there is an increasing recognition that students require exposure to and instruction related to computational thinking (CT), in order "to function, problem-solve, engage in digital innovation and advance a society already heavily technology-based" (Kotsopoulos et al., 2017, p. 155). First defined, then further refined by Jeannette Wing (2008), CT is "an approach to solving problems, designing systems and understanding human behaviour that draws on concepts fundamental to computing" (p. 3717). Concepts such as variables and loops, and practices such as abstraction and decomposition are shared across many other disciplines (Kafai & Burke, 2013; Lye & Koh, 2014). It is claimed that CT is "a powerful cognitive skill that can have a positive impact on other areas of children's intellectual growth" (Horn, Crouser, & Bers, 2012, p. 380) and Wing (2008) argues that this learning is best done with children, to ensure a solid foundation for understanding and applying CT.

There are many potential connections between CT and mathematics "in terms of logical structure and in the ability to model and investigate mathematical relationships" (p. 77). As mathematics educators, we are interested in examining the ways in which CT concepts appear to conflict with or facilitate the learning of mathematics. Although various misconceptions and difficulties related to looping commonly encountered by middle school students have been reported (Grover & Basu, 2017), research about the teaching and learning of looping with primary school children has not figured prominently. With the emergence of research suggesting that geometric computer programming language is an appropriate language for use in school mathematics at the high school level (Sinclair & Patterson, in press), we were curious to see if geometric computer programming language is an appropriate medium of learning for primary school children as well. Using a DGE, we wanted to investigate concepts of looping with young learners, using visual rather than alpha-numeric methods, to better understand the tensions and interactions between mathematical activity and CT in this context.

THEORETICAL FRAMEWORK

The main theoretical perspective utilised to frame this exploration of mathematical activity and CT is the computational thinking pedagogical framework proposed by Kotsopoulos et al. (2017). The four pedagogical phases of learning that comprise this frame include: unplugged, tinkering, making and remixing, all of which may relate to physical, digital, computer-based or conceptual objects. Each phase is distinct, yet overlapping, and although not necessarily sequential, it is proposed that exposure to each learning phase is necessary for learners to fully experience CT. However, it is suggested that a sequential approach to CT experiences may be beneficial when exploring certain concepts or when working with novice learners.

Activities that are approached and completed without the use of computers are considered unplugged. Foundational in developing CT, unplugged experiences provide a means of introducing "preliminary and overlapping concepts related to CT which can then be explored…either conceptually or technologically" (p. 159). Tinkering experiences involve taking existing objects apart and making changes and/or modifications for the purposes of exploration and consideration of the implications of these changes. Through the use of existing objects, the cognitive demands of object building are relieved during tinkering, allowing learners to focus their attention on "[a]pplication, simulation and problem solving" (p. 161). When tinkering with computer programs, students can easily "see the connection between changes in the program and the outcome and… know immediately that an error has occurred" (p. 161). Tinkering experiences, with their focus on 'what if', provide a rich "context for conjecturing, problem-solving, generalising and predicting – all which can lead to deeper mathematical understanding" (p. 161-162). Making experiences involve constructing new objects and require learners to make use of their foundational skills of knowledge and understanding to "problem-solve, make plans, select tools, reflect, communicate, and make connections across concepts" (p. 162). Remixing activities "involve the appropriation of objects or components of objects" (p. 154) which are then modified, adapted and/or embedded within another object, and used for substantially different purposes.

PARTICIPANTS AND CONTEXT OF THE RESEARCH

The children were introduced to the concept of looping over the course of two 90-minute lessons, each held one week apart. The eight- and nine-year-old children were in a composite grade 2/3 classroom, located in an affluent urban neighbourhood in British Columbia, Canada.

In the initial activity, pairs of students shared an iPad displaying a sketch of four moveable cards and were asked to sequence the cards to tell a story. Each card depicted a swimmer engaged in different activities, which included diving into a pool, swimming towards the right, swimming towards the left and stopped at the pool's edge. Once engaged in tinkering to create and explain a logical sequence of swimming activity, we began questioning students about how far the swimmer travelled and how the sequence might look if the swimmer travelled twice as far. At this point, an online sketch (see Figure 1) was projected onto the SmartBoard and was used to explore these longer sequences with the whole class. Our goal was to utilise this dynamic sketch to visually represent the concept of looping in a way that would allow students to easily identify the repeating parts of the sequence, and to recognise that the repeating sequence of steps are often buttressed by initial and terminal events. The four pictures were displayed on the screen, along with a large coloured "bracket" or C block, which surrounded the repeat block of the pattern and the number of desired repeats could be typed into a small box. The swim generator sketch would then arrange the pattern created in a comic strip format that enabled students to view the repeating pattern.

Conscious of the mathematics available in the lesson as well as the CT, questions were included in an attempt to keep the mathematics an important part of the children's learning experience. For example, students were asked what it would look like for the swimmer to travel five times as far. If each lap was 25 m, how many repeats would be needed for 200 m, 1 km, 10 km? Using the same format used by the swim generator, the children were to draw what it would look like to have the swimmer swim 350 m, using a C bracket to indicate the repeated actions.

The lesson was concluded with two unplugged activities. First the children were asked to identify other situations in their everyday lives where they participated in activities that repeat like this. One student shared that in badminton you pick up the racquet, bat the birdie back and forth, and at the end put the racquet down. Another stated that when you play basketball at recess, you get a ball, then continue to bounce it and throw it in the hoop, until you put the ball back. A third student discussed interacting with her pet hamster, and how she opens the cage, and keeps patting, patting, and then closes the cage. This response provided an opportunity to reinforce the difference between

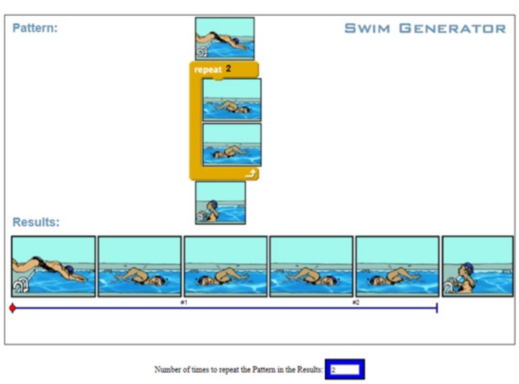


Figure 1: Swim Generator Sketch

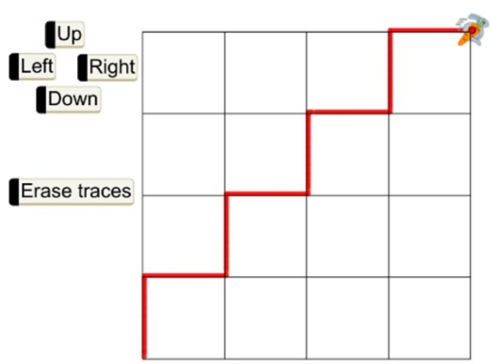


Figure 2: Rabbit and Carrot Sketch

one loop and a loop that repeats, as well as to reinforce the vocabulary for initial action, loop and final action. The children were then asked to individually think of a story of something they could do in their life that repeats. The story needed to introduce a loop that could be drawn in their notebooks with a beginning, a repeat and an ending, in a similar fashion to the swim generator sketch.

In the second lesson, we decided to take the context of looping and place it on the coordinate grid, while exposing the children to a making experience. We were very intentional in our choice of doing looping activities in the context of a grid, since the grid is such an important part of both CT and mathematics. A dynamic sketch (see Figure 2) was designed involving a rabbit and a carrot that were placed on a grid, which was projected onto the SmartBoard. Students were given the challenge of figuring out how to move the rabbit to the carrot, using the up, down, right and left buttons. Individual students would give verbal directions, then press the corresponding buttons on the SmartBoard to move the rabbit. Each time the rabbit moved, it left behind a trace that could be seen on the grid. Another student was responsible for keeping a record of student suggestions on chart paper for future reference.

There were also opportunities for tinkering, if the making experience was not to the students' satisfaction. After successfully moving the rabbit to the carrot and reviewing the sequence of instructions written on the chart paper, students were challenged to make or tinker with finding other paths that the rabbit could take to the carrot. After experiencing multiple path options, we transitioned back to an unplugged activity, where students were asked to identify a faster way of saying one of the patterns recorded earlier on the chart paper. Students initially identified the pattern as *right, up, right, up*, and when prompted to say it using a faster way, they suggested: *r, u, r, u*. With further teacher questioning, one student suggested using repeat, but the other students immediately objected. However, the student persisted with the idea of repeat, stating that "we could say right up and repeat it four…three." After questioning her if it would be a repeat three or four, the child responded, "Four, I counted."

At this point, we explicitly demonstrated how this pattern could be a repeat four and then modelled that this would be written as *right, up (repeat 4)*. Changing the orientation of the rabbit and the carrot on the coordinate grid, the class continued to experiment with these ideas through both making and tinkering experiences. Reinforcement was required regarding the looping aspect and use of the repeat terminology after a student stated the sequence and then "do it again" rather than repeat. Transitioning back to an unplugged experience, students were given various rabbit-carrot grid scenarios in a booklet and asked to either draw the path on the grid based on the given procedure or to provide the procedure for a given drawing.

A group of three students were observed spontaneously playing their own unplugged game of rabbit and carrot on an imaginary grid. Taking a carrot from her lunch, one child placed it on the carpet. The trio then decided to use a wet water spot on the carpet for their starting point and took turns telling the "rabbit" which direction and how many steps to take in order to get to the carrot. Although challenged to use repeat in their directions, the girls preferred not to do so, perhaps because there was not a visible grid on the carpet.

DATA COLLECTION

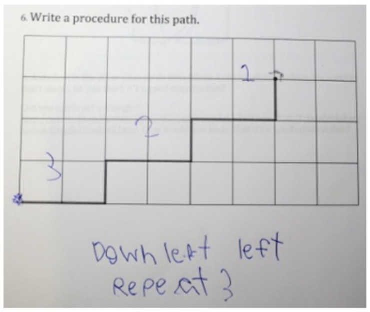
Throughout the teaching lessons related to this project, extensive written notes and photos were taken, and records of student work were retained. To ascertain student understanding of the looping concepts introduced and explored, four interview questions were devised by the research team. Student interviews were conducted on two separate occasions, the first session was a week after, and the second session occurred two weeks after the final looping lesson. A total of sixteen children were interviewed by one of three interviewers. Data consists of student written responses and related teacher notes, the use of screen capture software to record student actions on the computer screen, and audio recordings of the interviews, which were transcribed for data analysis.

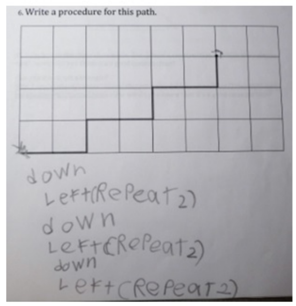
DATA ANALYSIS

The first question was an unplugged activity that asked the children to physically demonstrate a given procedure, which included a repeat component [stand up, look right, look left (repeat 3), sit down]. The procedure was read aloud for students, but each child could refer to a written copy in front of them. The majority of students were able to successfully demonstrate the procedure, although two of the successful students indicated that they were unsure about left/right directionality. Given the age of the children, and the fact that we were looking for understanding of the looping concept versus knowledge of left/right directionality, the interviewers decided to indicate which direction was right or left, if asked by a child. Of the three children who were unsuccessful with the activity, one of them appeared overwhelmed by the question and it was deemed best to move on, and the remaining two had difficulty with the looping component. One child completed the entire sequence once, without any acknowledgement of the repeat component, and the second child demonstrated the entire procedure three times, clearly demonstrating a misconception regarding the repeat function.

The second question was also an unplugged activity, in which students were shown a grid with a rabbit and the red trace of a path and were asked to write a procedure for the given path (see Figure 3). There appeared to be some confusion as to whether the rabbit indicated the starting or the ending point. Therefore, when analysing the data, it was decided that if the written procedure reflected the course of the red trace, either starting or ending with the rabbit, that it would be considered correct. There were four students who had errors related to right/left directionality in their written procedures. The interviewers noted these errors when they occurred and asked the children to verbalise and trace their written path on the grid, in an attempt to confirm the intended direction. In each case, it was determined that the child erred in naming the direction, and therefore the response was analysed in relation to the use of looping and not penalized for the incorrect directional indicator.

There were three children who wrote out the entire procedure without any use of looping. One of the children explained verbally that, "Instead of doing repeat, I just like writing it." It is unfortunate that the interviewer did not question the child further to determine how the procedure would change if repeat was used. The child clearly indicated an awareness of looping, but the opportunity to determine if she could accurately apply this knowledge was lost without further probing. There was one student who indicated 'x 2', and this response was classified as an indication of repeat two. When initially discussing looping, a student suggested using 'x 2' as a more efficient way of writing a repeating sequence. It was explained that repeat two is a more formal way to indicate 'x 2', and that this is the type of language that computer programmers use. Therefore, with this background in mind, the 'x 2' response was classified as an indication of repeat two for the purposes of these results.





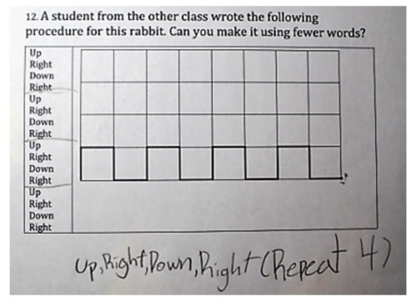


Figure 3: Sequence looping *Figure 4: Unit looping* *Figure 5: Question 4*

The majority of students correctly utilised looping in their written responses. However, there were two very distinct categories of responses, one of which we will term sequence looping and the other unit looping (see Table 1). Sequence looping is considered the ideal response and involves identifying the entire loop and indicating how many times this sequence would repeat (see Figure 3). Unit looping involves identification of a smaller unit that repeats within the sequence (see Figure 4). There were two students who demonstrated misconceptions relating to the number of repeats that would be included in the loop. One student verbalised the initial sequence of the path as down, left, left and then pointed to the remaining two sections of the sequence and wrote repeat two, rather than including the down, left, left as part of a repeat three sequence. A second child appeared to use a similar logic with unit looping, when recording down left (repeat once). Rather than including left as a repeat two, the initial left indicator is step one and the repeat once is the second left movement.

Use of sequence looping	Use of unit looping	No use of looping
43.8%	37.5%	18.7%

Table 1: Question 2 results

For the third question the students were given a written procedure [up (repeat 4), right, down (repeat 4), up (repeat 4)] and asked if this procedure will allow the rabbit to successfully reach the carrot. The children could choose to use the dynamic sketch of the rabbit and carrot on the grid to confirm their response. All children, except for one, confirmed that the rabbit would reach the carrot. Most children approached this as a making activity (see Table 2), by creating the path on the iPad screen to confirm that the path was, indeed correct. Many children transitioned back and forth between making and tinkering, often due to self-correction of an error. There were two children who did not correctly complete the procedure. When the rabbit did not reach the carrot, one child stated that "No, I don't think so. It's not going at the ending" and moved on to the next question. The second child, after making an error, simply "adjusted" the procedure to ensure that the rabbit would reach the carrot. It is unknown whether this adjustment was intentionally made or simply subconscious on the part of the child.

Unplugged	Making	Making & Tinkering
6.7%	40%	53.3%

Table 2: Question 3 results

Also, an unplugged activity, the final question provided students with a written procedure and the accompanying visual trace of the path (see Figure 5) and asked them to rewrite the procedure using fewer words. The majority of children used sequence looping to write the procedure using fewer words (see Figure 5). There were two children that demonstrated misconceptions related to the number of repeats that would be included in the loop. Perhaps this was due to recording the original sequence, and then counting each sequence following it to get repeat three, rather than correctly including the original sequence as part of the count, to end up with repeat four. There was a single student who used 'x4' instead of repeat four. There were only two students who attempted to use unit looping when rewriting the procedure, and both did so incorrectly. One appeared to count and 'collect' how many ups (4), rights (8) and downs (4), then wrote his procedure to reflect this. The other student seemed to notice some patterns and attempted to identify units and the looping that would go with them. Even though the unit looping recorded could be correct for parts of the procedure, what was written does not accurately address the entire procedure. There was one child who rewrote the entire sequence without the use of looping and another who indicated that she was unsure what to do and therefore did not attempt the question.

Use of sequence looping	Use of unit looping	No use of looping
78.6%	14.3%	7.1%

Table 3: Question 4 results

DISCUSSION AND CONCLUSION

In the above analyses, we have shown that the visual aspects of DGEs and the use of a geometric computer programming language are appropriate for and can facilitate the learning and understanding of CT concepts with young learners. Although the application of these concepts may be challenging for them, primary students are clearly capable of learning looping concepts. Throughout this project, the children have demonstrated an ability to independently choose and move between various pedagogical experiences in order to develop their understanding. Choosing unplugged or making experiences and shifting between making and tinkering were all strategies effectively utilised by students to determine answers to CT questions. Another interesting strategy that emerged when the children were still grappling with how to apply looping ideas to procedures that contain multiple repeating steps, was the use of unit looping almost as a precursor phase to sequence looping. There were also tensions related to spatial awareness and directionality that emerged. Although the students were engaging in CT activities, the geometric context provided by the DGE privileged the spatial aspects of the activities, providing a more prominent place for the mathematics. There is a resonance between patterning and looping which would allow CT and mathematics to potentially complement each other. Although this was not pursued or elaborated in our activities, it is potentially an area for further exploration. In future work, it would be interesting to determine how mathematical activity can be further enriched through CT experiences in primary classrooms.

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A CONJOINED ENACTIVIST/QUANTUM MECHANICAL INTERPRETATION OF MATHEMATICAL MODELLING

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This paper provides a conceptual overview of a larger program of research that explores the implications of reframing mathematical modelling practices according to a conjoined enactivist/quantum mechanical framework. The research is motivated by the view that reinterpreting the nature of mathematics through a more inclusive materialism has the potential to deepen our understanding of the role that mathematics plays in shaping perceptions of the natural world and the ways in which mathematical models ultimately take on the meanings that they do. Incorporating literature from the disciplines of physics, mathematics education, educational psychology, and philosophy more generally, four core themes are identified, and groundwork laid for a broader perspective regarding the mathematical structure of reality.

INTRODUCTION & THEORETICAL BACKDROP

In an effort to inquire further into how mathematical modelling practices allow human beings to characterize aspects of the natural world, and to perturb present notions of what mathematics is, this paper reports on a current literary exploration that draws together an enactivist philosophical view with elements of quantum mechanical theory. It is through a combination of these respective orientations that the larger program of research aims to shed light upon certain ontological and epistemological themes relevant to research in mathematics education. Alongside other works, mathematician and theoretical physicist Eugene Wigner's seminal 1960 publication *The Unreasonable Effectiveness of Mathematics in the Natural Sciences* acts as something of an impetus for this exploration and offers up specific insights into the mathematical structure of reality, whilst presenting a less conventional view of what mathematics is, of what physics is, and of the mutually informative roles they play in shaping one another. Widely known for his contributions to atomic theory, solid-state physics, and mathematical formulations of the wave theory that would become integral to later developments in quantum mechanics, Wigner (1960) noted that:

> … it is important to point out that the mathematical formulation of the physicist's often crude experience leads in an uncanny number of cases to an amazingly accurate description of a large class of phenomena. This shows that the mathematical language has more to commend it than being the only language which we can speak; it shows that it is, in a very real sense, the correct language.
>
> (Wigner, 1960, p. 8)

Not only do Wigner's assertions embed the language of mathematics squarely within an ontological and epistemological discourse, but they also draw out deep connections that exist between the mathematical and scientific pursuits enacted by human beings. Indeed, how is it that mathematics is so "amazingly accurate" at characterizing (or modelling) the various phenomena of the natural world? What epistemological significance might be surmised from such consideration, and what ontological implications might accompany it in turn? Though subtly worded, this characterization of mathematical language as "the correct language" is quite significant. The core idea put forth by Wigner suggests that the robustness of mathematics as a means of describing physical phenomena and the great depth to which mathematical language is applicable to such description are not simply coincidental. Rather they are the case entirely *because* mathematics itself is an ontological *fundament* and not simply a tool for mediating the human experience of reality (although it may well serve that purpose also). Through Wigner's perspective, there is a sense in which mathematics may actually be seen as *underlying* the very structure of reality, as opposed to merely *representing* it.

Décio Krause (2000), professor of logic and the philosophy of science at the Federal University of Santa Catarina, Brazil, addresses related themes by attending to connections between logic, mathematics, and reality at a quantum mechanical level:

> It is generally agreed that we cannot describe the world in the absence of any theory or of any prior way of understanding it. Even in the particular case of the microscopic world, despite the well known discrepancies between the kind of 'objects' that inhabit that world and the standard objects described by classical physics, we still keep our prejudices based on classical logic and mathematics when considering the stuff on which to base the theory we are considering.
>
> (Krause, 2000, p. 155)

Together, Wigner and Krause's remarks seem to advocate a re-evaluation of the classical views and biases that continue to shape modern mindsets in mathematics and science research (physics specifically). They also underscore the inherent connectivity between these fields, which can be seen as sharing common philosophical roots/lineage (Campbell, 1999). Interestingly enough, though, Krause levies the criticism that the persistent "prejudices" of our *classical* worldviews are very often at odds with our more sophisticated theoretical understandings. This raises the question of how our mathematical descriptions of the world might differ should we endeavour to escape from such classical biases. Considering the sense in which Wigner sees mathematics as being the *correct language* with which to describe the natural world, and Krause's recognition of long-standing hang-ups that might nevertheless hinder our theoretical perspectives, I am thus led to wonder how appealing to a more inclusive materialism might allow us to deepen our understanding of the role that mathematics plays in shaping our perceptions of the natural world and the ways in which mathematical models ultimately take on the meanings that they do?

Modern quantum mechanical theory comes as, perhaps, the most successful and revealing theory in the history of physics (Barad, 2007), and presents an ontology that extends well beyond the classical principles that preceded it. It has ushered in a new era of conceptual understanding and revolutionized human comprehension of the natural world in many respects. Subatomic theory bolstered by quantum perspectives has prompted advances in microprocessing and computing science, materials design and engineering, and fostered developments in microscopy, thermometry, and precision time measurement (Barad, 2007; Faye, 2008). Reconceptualized cryptographic procedures and even new insights into certain biological processes have also accompanied the evolution of quantum theory (Palmer & Mansfield, 2013).

In even more profound ways, though, quantum theory has troubled longstanding classical views concerning the determinacy of matter and the nature of spacetime. In much the same way that Gödel's incompleteness theorem raises issues about the nature of knowledge and the logical systems from which we derive meaning, so too do quantum mechanical principles like Heisenberg uncertainty and quantum entanglement raise issues about the nature of reality, our perceptions of it, and the manners in which we characterize it. Both mathematicians and physicists are confronted with analogous (and equally substantial) ontological and epistemological problematics as a result. While the various implications of quantum mechanical theory are undeniably counterintuitive, perhaps even disquieting at a certain level, they are nevertheless intrinsic to the human experience of both space and time, whose phenomenological and, indeed, *mathematical* relevance was espoused by Kant centuries ago (Sutherland, 2004). Considering the successes of modern quantum theory in the realm of the physical sciences, and the philosophical/historical parallels between mathematical and scientific thought, I propose that a quantum mechanical view of matter, coupled with a particular philosophical perspective (namely an extended enactivist monism), might serve to foster rather different, potentially revealing, insights into the nature of the human experience with (and embodiment of), mathematical processes and structures. To return to Wigner (1960):

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse…, to wide branches of learning.
>
> (Wigner, 1960, p. 14)

METHODOLOGICAL APPROACH & CORE THEMES

Although the research discussed here is driven by a qualitative exploration of literature at a thematic level, and not a quantitative analysis or empirical investigation, certain statements about methodology can still be expressed. Most notably, because the program of research engages with both an extended enactivist framework and a quantum mechanical framework (i.e. the *standard* quantum mechanical model devised primarily by Niels Bohr, Werner Heisenberg, and Max Born), it is important to acknowledge which of these is prioritized in the overall interpretation of the literature, and which is to be read *through* the other. As with the composition of functions, which can depend greatly upon the ordering of the composition, this particular thematic exploration is also seen as depending in large part upon the ordering and prioritization of the given frameworks. That said, it is a monist enactivist interpretation (to be introduced shortly) around which the discussions are formed, with a quantum mechanical perspective providing additional insight. Thus, while a number of quantum mechanical formalisms are presented, reference to them is ultimately intended to support the more primary enactivist interpretation. This is to say that the quantum mechanical shall be in service to the enactivist, and not the reverse.

The enactivist stance proposed within the larger program of research initially draws from the works of Merleau-Ponty (1962, 1968), Varela, Thompson, and Rosch (1991), Campbell and Dawson (1995), Campbell (2001, 2003), Barad (2007), and Malafouris (2008), all of which speak in some way to notions of embodiment and aspects of the classical Cartesian dualism. However, the proffered view also extends beyond these by offering a slightly different approach to navigating past the mind-body problem, ultimately establishing the groundwork for a monist enactivist view *atop which* a quantum mechanical monism (based on the Bohr model of the atom) is overlaid. These two monisms are drawn together by appealing to intersections that arise from: a) comparing the Cartesian dualism with wave-particle duality in quantum mechanics, b) aligning Merleau-Ponty's notion of double-embodiment with the notion of quantum entanglement underlying Karen Barad's agential realist theory, and c) offering a synthesizing perspective through which mathematics is itself seen not only as "the science of organization" or "the science of processes and structures" (Campbell, 2001, p. 4), but as the *science of material assemblage* (where assemblage refers to a *dynamic bringing together of elements*, and not a static configuration as might be assumed).

Thus, it is the exploration and articulation of perceived connections between an extended enactivist monism and specific aspects of the standard quantum mechanical model around which the core discussions evolve. Based upon the literary investigation that informs the larger program of research, four core themes would seem to underlie the connective discourse being sought out. It is entirely possible that a continued survey of the scholarly landscape might reveal additional salient themes; however, the following have been frequently and strongly represented in the literature, and are perceived as being foundational to the matters at hand:

1) New materialisms and issues of dualism (by way of material indeterminacy, wave-particle duality, and Cartesian dualism)

2) Matters of agency (by way of Merleau-Ponty's ontological monism and Karen Barad's agential realism)

3) Emergent systems (by way of probabilistic interpretations and complexity theory)

4) Epistemological uncertainty (by way of Heisenberg uncertainty and Gödelian incompleteness)

When approached from the standpoint of a monist enactivist view, each of these themes carries implications that can also be re-framed according to quantum mechanical perspectives derived from the Bohr model of the atom, and all have some bearing on the ontological and epistemological assertions that can be levied with respect to mathematical modelling practices. In the interest of paying proper respect to the relevant subject matter, it is anticipated that only the first two of the four themes outlined above will be addressed in the larger program of research. While implications associated with the remaining two themes are likely to be incorporated into the discussions where appropriate, any comprehensive treatment of them will be deferred to future research efforts. This is not in any way meant to diminish the importance of these particular themes to the broader exploration. Rather, it is an acknowledgment, or admission, that these themes are markedly complex, and likely too expansive to be fully addressed within the scope of a study that also includes the preceding pair.

OVERVIEW OF DISCUSSION

The renegotiations of enactivism with respect to the Cartesian dualism are evocative in much the same manner as the renegotiations entailed by quantum mechanical theory with respect to the nature of wave-particle duality. Just as Merleau-Ponty (1968) makes reference to a monist metaphysical notion of "flesh" as an ontological primitive, or some "general thing, midway between the spatio-temporal individual and the idea" (p. 139), the "objects" referred to by the standard quantum model might also be seen (through an analogous monism) to manifest as some uncertain, yet general things midway between the material form and the theoretical construct. At least in classical terms, it is not entirely obvious how to go about characterizing the referents of these new materialisms.

New Materialisms and Issues of Dualism

In the quantum realm, ontological and epistemological assertions very similar to those of enactivism arise from the unusual duality exhibited by elementary particles. Most notably, photons and electrons demonstrate seemingly indeterminate characters. Depending upon the mode of interaction or the circumstances to which they are subjected, these "objects" may reveal either particulate or wave-like characteristics, which is to say that they simultaneously contain within their being the capacity for both *localized* corpuscular tendencies as well as *distributed* wave-like tendencies. This leads to a classically perplexing oddity, for the traditional (and still quite pervasive) view is that such particulate and wave-like qualities should manifest exclusively. The common sensibility is that matter and energy should be somehow more "decisive" in their representations of themselves at any given time. This is one way in which *classical modes* of thought perpetuate issues similar to those of the Cartesian dualism. Under the quantum mechanical lens, though, the so-called indeterminacy of matter is a direct (and entirely acceptable) consequence of a unifying principle that circumvents wave-particle duality in much the same way as enactivism does Cartesian dualism. In so doing, it fits nicely with the remarks from Krause expressed earlier in this paper, and gives further justification for invoking a more *inclusive* materialism when contemplating how it is that mathematical modelling practices allow human beings to characterize aspects of the natural world. Indeed, the indeterminacy of matter deftly illustrates how the classical materialisms onto which we routinely hold can be fairly myopic in their scope and restrictive in the ideations they permit.

In reference to the inherently mathematical language of quantum mechanics, Wigner (1967) claims that "… quantum mechanics teaches us to store and communicate information, to describe the regularities found in nature" (p. 170). With a very similar sentiment driving his own perspectives on the nature of mathematical modelling, Campbell (2001) puts forward a view of mathematics as "the science of organization", or "the science of processes and structures" (p. 4). Together, these perspectives entail a representation of mathematics as a means of managing or systematizing information in order to describe or access the structures that constitute the natural world. Nevertheless, while Wigner's perspective proves to uphold the Cartesian dualism, the enactivist standpoint put forth by Campbell (2001) rejects it outright, re-characterizing both cognition and sensory experience as complementary aspects of an equal-parts mutual and reciprocal relationship between the thinking, knowing consciousness and its environment. Through that interpretation, we (as thinking, knowing, sensing beings) both shape and are shaped by our experiences in/of the world. By extension, it might also be said that we (as thinking, knowing, sensing, *mathematical* beings) organize and are simultaneously *organized by* the processes and structures of the natural world. While deeper implications of this perspective are still being explored, other implications regarding notions of agency can now be foreshadowed.

Matters of Agency

As with the monism put forth by Merleau-Ponty and extended by Campbell (2001, 2003), Malafouris (2008) stresses the importance of material engagement in the understanding of the self as an active agent that both contributes to and is influenced by the structures of the world (I extend these discussions to *mathematical structures* more specifically). In so doing, Malafouris (2008) also presents "a view of selfhood as an extended and distributed phenomenon that is enacted across the skin barrier" (p. 1993). He suggests that the self is not situated inside or outside any *single* material object (be it brain or body). Rather, it is "constantly enacted in-between brains, bodies and things and thus irreducible to any of these three elements taken in isolation" (Malafouris, 2008, p. 1997). Carrying forward from this idea, Malafouris chooses to characterize the self as a "self enacted through the act of embodying" (ibid.), which emphasizes the nature of embodiment as a process of *becoming* or *emergence* (as opposed to a more traditional static state of being). Malafouris's comments are of particular interest, for they seem to advocate revisions to notions of agency and self-extension in ways that place greater priority on the relationship between the embodied self and the world in which it comes into being. By implicating other material objects in the enactments that lead to an understanding of selfhood, Malafouris suggests that the ways in which we come to know the self-emerge *simultaneously* with the ways in which we come to know the world as well.

This discussion of the extended self and principles of embodiment draws upon similar philosophical bases as the notion of *intra-action* that underpins Karen Barad's agential realism (a relational ontology rooted in Bohr's atomic theory). Notably, Barad's agential realism also hinges upon processes of mutual emergence within larger material systems, and describes such emergence by appealing to the concept of quantum mechanical entanglement. Through this view, we are agentially inseparable from the world in which we enact (and experience) that agency, and for Barad, matter and meaning are necessarily entangled with one another. It is this entanglement of matter and meaning that ultimately leads me to an interpretation in which mathematics is itself seen as a *science of material assemblage* (alluded to earlier in this document). As offered previously, there is a sense in which Wigner's characterization of mathematical language as "the correct language" suggests that mathematics may actually *underlie* the very structure of reality, as opposed to merely *representing* it. Considered alongside Barad's agential realist perspective and the extended enactivism that I have attempted to outline here, seeking out a means of characterizing the nature of that entanglement may be extremely valuable in terms of reconceptualizing mathematical modelling practices via a conjoined enactivist/quantum mechanical interpretation.

CLOSING REMARKS

Considered together, Wigner (1960) and Krause (2000) present an interesting impetus for revising certain core assumptions that underpin both our mathematical and scientific worldviews. Their respective remarks concerning the *correctness of mathematical language* and the *pervasiveness of classical prejudices* have, here, been taken as an invitation to rethink the role that mathematics plays in shaping our perceptions of the natural world and the ways in which mathematical models ultimately take on the meanings that they do. Re-framing perspectives according to a more inclusive materialism is certainly not a trivial matter; however the four core themes identified in the early pages of this paper would seem to offer valuable starting points for that endeavour. Although these discussions of the first two themes have been *highly* compressed, it is hoped that they have offered some sense of the broader explorations being conducted in the larger program of research. Significantly more nuance lies behind the perspectives presented within the space of this paper, and I am aware that this nuance is not necessarily evident. As offered earlier, new materialisms and issues of dualism, as well as matters of agency are two of four themes that have been strongly represented in the literature surrounding the topic at hand. This brief paper offers only an introductory account of a much more involved discussion.

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THE PRACTICAL AND THE THEORETICAL: POLAR OPPOSITES OR COMPANION SKILLS?

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It is the ability to make connections between concepts that signals competence in algebra. This study investigates the divide between students who are able to make those connections and those who are not. A task designed to examine one of these connections is described, and a preliminary analysis performed using Sierpinska's framework of "Practical" versus "Theoretic" thinking. The results are evidence that students who have made connections are able to interpolate from a given situation, and those who have not often cannot even use their own answers to draw conclusions.

INTRODUCTION

It is critical that students understand connections between representations of functions; certainly, if they do not understand the component parts, there is nothing for them to connect, but it is the connecting of concepts that signals competence in the discipline of algebra (Moschkovich, 1996). Students often struggle with making those connections and a great body of work exists that indicates students have trouble learning the various components of the algebra of linear functions. In this paper, I will describe a task designed to examine one of these connections and analyze the results.

THEORETICAL PERSPECTIVE

In an influential chapter on how students think during tasks in linear algebra, Sierpinska (2000) makes the distinction between what she calls "modes of thinking … synthetic-geometric, analytic-arithmetic and analytic-structural" (p. 233). While Sierpinska's chapter was written from a curriculum in linear algebra, the text could be about any algebra-based course, so familiar are the patterns of student reasoning. For courses at college level (e.g. linear algebra, but also college algebra, trigonometry and calculus), Sierpinska's analysis of students' "reluctance to enter into the structural mode of thinking" (p. 209) is a valid and necessary deliberation. "In linear algebra courses," she explains, "even at the most elementary level, the number of new concepts grows very fast and the links between them become more and more intricate" (p. 218).

Sierpinska attributes her distinction between "practical" and "theoretic" thinking to the work of Vygotsky; specifically, his "distinction between spontaneous and scientific concepts" (p. 211). She acknowledges that her adaptation of Vygotsky's partition "is very general and does not capture the specificity of the thinking in linear algebra" (p. 211), which is why her own work includes the finer-grained filter of synthetic-geometric, analytic-arithmetic and analytic-structural. The third "mode" in Sierpinska's triad, "analytic-structural", may be too fine a filter for use in an exploration with students of elementary (pre-college) algebra, but the balance of this insightful chapter has a great deal to offer in that exploration. A large part of Sierpinska's discourse details the difference between "practical thinking" and "theoretical thinking" (p. 211).

While a coarser filter than the one she ultimately uses to analyze linear algebra responses, Sierpinska's division into three modes of thinking is actually a subdivision of "theoretic" thinking. "[T]the synthetic mode belongs to the practical way of thinking, and the analytic – to the theoretic way of thinking" (p. 233). Thus "practical" versus "theoretic" serves as a strong introduction to the discussion about the three modes of thinking; more valuably for us, "practical" versus "theoretic" may adequately serve as the only necessary filter at the pre-college level.

Sierpinska has taken Vygotsky's theory and adapted it to mathematics education, so while her particular application of it may be too exacting for our needs, she has been kind enough to outline thoroughly the transition of the model from Vygotsky's general application to learning and development, to its use in mathematics education, to its application in a particular college-level course. If we do not need the entirety of Sierpinska, we can still be quite content to apply the intermediate level of analysis that is her extension of Vygotsky.

Students can "[fail] to think theoretically" in several ways: "transparency of language", in which the student does not translate written language from the way in which she would express herself verbally, including failure to express what abbreviations or references might mean; "lack of sensitivity to the systematic character of scientific knowledge", whereby students produce work that contradicts earlier work (often including their own); "thinking of mathematical concepts in terms of their prototypical examples rather than definitions" and "reasoning based on the logic of action, and generalization from visual perception" (Sierpinska, 2000, p. 229). It is this last form of "practical" thinking, "generalization from visual perception" that I will investigate in this paper.

The idea of "links" or connections between concepts has been a recurring theme for this author, the absence of which is a death knoll for success in algebra. Moschkovich (1996) explains that:

> Linear functions is a complex domain where the development of connected pieces of conceptual knowledge is essential for competence…. Conceptual understanding in this domain involves more than using procedures to manipulate equations or graph lines; it involves understanding the *connections* between the two representations. (p. 242; emphasis added)

The current investigation attempts to blend Moschkovich's observation about the necessity of making connections in mathematics to competence in the discipline with Sierpinska's partitioning of responses into "theoretic" or "practical".

PHENOMENON OF INTEREST

A line is the graphical representation of the relationship between two variables given by the equation of the line (a relation).

> [I]n the synthetic [practical] mode, a straight line is seen as a pre-given object of a certain shape. One can speak of the properties of the straight line, but these properties will only describe the line, they will not define it. In the analytic [theoretic] mode, the straight line is defined as acertain [*sic*] specific relationship between the coordinates of points. (Sierpinska, 2000, p. 233)

Many students, when asked to identify a point on a line, resort to observation and select a point that appears to be on the line, even if it is not.

In the experiment described here, the goal was to see whether students would be able to connect the pieces of information they did have (the two given points and the illustration of the graph) with the idea that another point could also be on that line. In other words, I wanted to determine whether:

> …the visualizations provided by the teacher are taken holistically by [“practical”] students without their trying to discriminate between the mathematically relevant features and those contingent on the technical support used to create them (Sierpinska, 2000, p. 229).

The observation that led to the investigation outlined in this paper was that students appeared to be seeing a graph as the only necessary source of information about points on the line. I had noticed that even if a line does not pass precisely through the “crosshairs” of a grid, students often assumed that their observed values were “close enough”, either in form or in value, and did not consider algebraic means by which they might confirm their choices, or else they approximated the non-integer values, but still did not try to verify them.

The Instrument

A task was designed to diagnose students’ understanding of what it means for a point to be on a line. Two points were given, and the introductory statement informed students that these two points were included in a line graphed for them, and an image of the graph was presented alongside the introductory text. The graph provided included the two points given in the introductory text (represented as dots) and the line through them, with a sufficient interval of the domain to indicate that the line continued beyond the two points.

The Task

The students’ task was, given a fixed value for the independent variable, to find a corresponding value of the dependent variable for a point on the same line. The task appears on the surface to be quite straightforward, as the value given for x was an integer, and the corresponding integer value for y appeared also to be an integer. In fact, the task was written to be somewhat misleading in that, without the use of algebra, students who made assumptions from observations could still arrive at a conclusion that seemed reasonable, although that conclusion would, in most cases, be incorrect.

Correct resolution of this task required considerable analysis and synthesis for students at this level. Since the image, but not the equation of the line was given, students who would resort to observation could do so easily, but would likely fall into the trap, if they did so, of assuming that the graph passed through integer coordinates where it did not, necessarily.

A slope of $\frac{5}{13}$ for the line was selected because of a peculiar phenomenon that occurs with ratios of Fibonacci numbers[1]. The numbers 5 and 13 are still small enough to be relatively easy to work with, but their ratio produces a graph that appears to be what it is not, in several places.

It should be noted that the task did not include explicit instructions to establish the equation of the line. The point of the experiment was to see whether students would do so voluntarily, either in part or in whole, without prompting.

METHOD AND RESULTS

The expectation was that students would either answer with the use of algebra (theoretic; group 1) or by observation (practical; group 2). Initial observations returned exactly this result, but simply partitioning responses into these two groups said little about the differences between or within the groups. In an effort to examine the correlation more deeply between the theoretic approach and the way in which students used the information they accumulated, responses were scrutinized more closely and the steps taken in responses by "theoretic" students' were noted and catalogued. A category of "inadequate analysis or no response" was added to capture data from students who were unable to approach the task (group 3).

This secondary analysis revealed a clear distinction between those who could write and use an equation from the given points and those who could not. A few within the latter group wrote an equation but did not know how to use it, and so fell back on observation; several students did not write an equation at all, and so had only observation available as a means to approach the task.

The Practical Approach

The graph appears to include the point $(8,1)$, and 16 students (47%) gave this point as their response with no additional explanation, or gave one like it; for example, Mason's response was "$(8, 1.25)$"; Miranda's was "$(8,1.5)$ not exactly at 1 a little over one," and Charles explained "y would have to be around 1.2 because the line doesn't go through the exact point." This handful of students recognized that $(8,1)$ was not accurate, but they did not employ an algebraic approach to close the gap, and their responses were therefore assigned to the "practical" category.

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[1] Ratios of nearby Fibonacci numbers appear to be close to more familiar fractional values. For example, $\frac{5}{13}$ can look like $\frac{2}{5}$, $\frac{5}{8}$ or $\frac{1}{3}$.

[2] All student names in this report are pseudonyms that preserve gender.

The Theoretic Approach

17 students used an algebraic approach overall to answer this task, and the category was subdivided in order to capture a snapshot of those students who found the slope of the line and by what means, those who could identify the y-intercept and those who were able to assimilate this information into a useful summative equation.

The slope

Fully 26 students correctly found the slope of the line, 20 by the conventional algebraic formula and six by "counting" up and over from one of the given points to the other. Two additional students used algebra, but found the reciprocal of the slope, and two others began using the slope formula but did not complete the work. These last four responses were assigned to the theoretic approach category for the determination of slope, since although the equations these students ultimately produced were incorrect, nevertheless they had used a theoretic approach in finding the slope component.

Not present in this result are the responses of some students who found the slope incorrectly by observation. Although they, too, "counted", they did so based on assumptions of points that are not on the line, and concluded that the slope was $\frac{2}{5}$.

Thus, the result of finding the slope was assigned to either the "practical" or the "theoretic" category based not on whether students used the conventional formula or a "counting" technique, but whether they used algebra or counted using the given points (assigned to theoretic) or whether they counted using incorrect, observed points (practical).

It is interesting to note that of the six students who found the slope by counting, only two of them (33%) used the slope they found to answer the task, while 15 out of the 20 who used algebra (75%) went on to use their computed slope in completing the task.

The y-intercept

All of the students who found the slope algebraically were able to recognize the point $(0, -2)$ as the y-intercept, and all of these went on to utilize these two components to write an equation for the line.

Of the six students who correctly found the slope by counting, only two correctly found the correct y-intercept and correctly constructed the equation of the line, although neither of these ultimately used their equation to answer the task but answered instead by observation (practical). Another two found a y-intercept, but not the correct one; one of these used some valid reasoning to find her value, but incorrect computation, and the other seems to have pulled a value out of thin air.

The group of four students who found the slope algebraically, but did so incorrectly, contains only one individual who identified the y-intercept. He did not use the equation of the line to answer the task; like the "slope by counting" group, he was able to construct an equation for the line but did not use it.

ANALYSIS

The task had the effect of separating students into the three expected groups: the theoretic thinkers, the practical thinkers, and the non- (or insufficient) responders. Since the creation of an equation seemed to be what separated the theoretic from the practical, the data were sorted with the priorities, in order, of students who

1. created an equation and used it to answer the task,
2. wrote an equation and tried to use it, but resorted to observation
3. wrote an equation, but made no obvious attempt to use it

Figure 1 shows the responses to the task sorted in this way. It should be noted that entries do not reflect the correctness of a student's answer, only their approach.

wrote and used eqn	e	e	e	e	e	e	e	e	e	e	e	e	e	e																				
wrote/tried to use eqn; fell back on obs.															e	e																		
wrote eqn but didn't try to use																	e	e	e	e	e													
never wrote eqn																						0	0	0	0	0	0	0	0	0	0	0	0	0
found the slope	a	a	a	a	a	a	a	a	a	a	a	a	c	c	a	a	a	a	a	c	c	a	a	a	c	c								
recognized the y-int	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y	y										
used obs to answer task	t	t	t	t	t	t	t	t	t	t	t	t	t	t	p	p		p	p	p	p	p	p	p	p		p	p	p	p	p	p	p	p

Figure 1: Data on student methods; each column represents an individual.

There is no surprise in the top three rows of this table, as these are simply ranked by method. What is instructional is that the last row, which is not sorted in any way but whose data has "followed" the other data for each student, shows a clear correlation between students who wrote *and used* an equation and those who approached the task theoretically. The students who wrote an equation but did not use it did not make the connection between the equation of a line and what that equation tells you about the graph of the function; they were able to perform mechanical algebra-like computations but were unable to assimilate how what they had done might be useful.

There is a remarkably clear delineation of two distinct groups: Those who were able accurately to write the equation of a line from two points *and use it* to find the accurate location of a third point approached each part of the task theoretically; those who were unable to use the equation they wrote, unable to write an equation from its component parts, or unable even to identify those parts, ultimately approached the task practically. Some students in the practical approach group used appropriate techniques to find the equation of a line (if not the correct line), but still resorted to observation to answer the task; others found the correct slope by "counting" but used a different slope elsewhere in the assessment, which may be an indication of the reason for the assumption that the desired y-value is 1. In Eliza's case, she appears to have "counted" to find the slope as evidenced by the presence of marks on her graph; she even wrote down her slope in the space for the answer, but she did not use it; if she had, she would not have arrived at $(8,1)$. It is possible that she tried to use her slope, starting, for example, from $(0,-2)$, but bypassing in the process the point whose x-coordinate was 8, was unable to reconcile this issue and resorted to observation.

CONCLUDING REMARKS

The distinction is not between students who think theoretically and those who think practically, but between those who have made connections between various representations and those who have not. It is certainly not true that there is no divide between the theoretic and practical approaches; rather, this divide is the *result* of having made the connections critical to "conceptual understanding", not its *cause*. In short, we can partition students into two groups: those who have made algebraic connections between the slope and the intercept of a line, their places in an equation and the use of that equation in a larger context; and those who, although they may be able to perform some algebra-like manipulation, have not made the connection to how they may be used to construct an equation, let alone how to use such an equation.

Having made a connection, not one of the "theoretic" students relied on observation at any point (which is not to say that they never *used* it), trusting instead the more rigorous methods they had at their disposal. "Practical" students, some of whom even wrote down linear equations, and some of whom did so correctly, could not use the equation they had written. These students did as much work as the "theoretic" students, but they were unable to use the work they had done.

Students either made connections between the component parts of a linear function, between the function and between an equation and its use to interpolate and/or extrapolate about the function and its graph, or they have not. This is not to say that the latter group could not go on to do so. On the contrary, this observation is a clear indication that alongside teaching the mechanics of algebra we should be teaching students about those connections. Those who have managed to make the connections, readily made use of the information afforded by the different representations in an integrated way.

While the ability to make a theoretical assessment is valuable, the practical perspective is also important:

> The distinction between theoretical and practical thinking is a methodological distinction, not a distinction between actual processes of thinking, which seem to always have both a practical and a theoretical component. Scientific knowledge is theoretical, but scientists do not always think in the theoretical mode. Most of the time scientists think in 'practical' ways, where the practice has to be understood, not as everyday life practice, but their practice as scientists working in a familiar paradigm. They switch to theoretical thinking in situations which challenge their common mathematical sense. (Sierpinska, 2000, p. 232)

While we are all able to approach a problem practically (by Sierpinska's definition), there are times that the practical approach is insufficient to the level of analysis necessary. Being able to think theoretically as well as practically means being able to switch *between* practical and theoretic, and not being stuck in a world where solutions are often out of reach. The difference may be compared to being bilingual, or not.

Those who are, regularly "code switch" (Heredia & Altarriba, 2016) when speaking to others who are fluent in the same languages. Scientists "code switch" when they communicate with other scientists, using elements of both practical and theoretical ways of thinking and speaking. Not having a second language means only being able to communicate in one; where one's only language is informal, there is no way to communicate formal ideas.

Mathematics education has long been searching for better ways to convey mathematics, to understand what happens when learning takes place and why it sometimes does not, and to help students make connections. Making connections is, as Moschkovich has told us, vital to competence. Without connections, there is no mathematics. In addition to teaching mathematical concepts, perhaps we should be teaching mathematical connections.

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DYNAMIC GEOMETRY AS A SPATIAL PROGRAMMING LANGUAGE IN ELEMENTARY SCHOOL

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The purpose of this study was to develop and research the effectiveness of geometric coding environments that can enable students to model the world in which they live, as well as develop spatial reasoning skills. We conducted a two-month classroom intervention with grade 2/3 students, experimenting with computational thinking (CT) tasks, centrally involving geometry and spatial reasoning within a dynamic geometry environment (DGE). In our experiments, we were interested in how the DGE/CT approach gives rise to concepts relevant to both mathematics and coding. We wanted to see (1) what the students would learn, and (2) what the effect of combining CT and geometry would be. Our finding is that DGE-based tasks effectively support teaching of many CT concepts, making Math/CT interdisciplinary integration a realistic goal.

INTRODUCTION

While geometry has been an academic subject ever since Plato introduced his Quadrivium of Mathematical Sciences at the Athenian Academy in 387 B.C.E. (Campbell, 2004), the idea of computational thinking (CT) as a curriculum focus did not emerge until very recently, even though it is not brand new: Dr. Seymour Papert (1980) used this term when he invented Logo programming language to help improve the way children think and solve problems. CT is a set of cognitive skills and problem-solving processes that Dr. Jeannette Wing (2006), a computer science professor at Columbia University, envisioned as part of every child's education.

In British Columbia's (BC) new curriculum, which was officially introduced in September 2016 in all K-9 classrooms, a CT module has been added under the subject of Applied Design, Skills, and Technologies for all middle grades. Similarly, as of 2016, Nova Scotia has started Information and Communication Technology and Coding curriculum integration in both primary and intermediate grades. Kotsopoulos et al. (2017) emphasize that England, Finland, Estonia, and the United States have all mandated CT curriculum. Gadanidis (2017) adds that "the trend of adding some form of computer coding to curriculum is an international phenomenon", and also point out the benefits of integrating CT and mathematics in school:

> At the heart of computational thinking – and mathematics – is abstraction. When children write code, they come to (1) understand in a tangible way the abstractions that lie at the heart of mathematics, (2) dynamically model mathematics concepts and relationships, and (3) gain confidence in their own ability and agency as mathematics learners. (p. 1)

Our research project was aimed at studying the use of dynamic geometry as a spatial programming language at the primary school level. We were experimenting with CT activities centrally involving geometry, spatial reasoning and the use of dynamic geometry environment (DGE). In our experiments, we were interested in how the DGE/CT approach gives rise to concepts in both mathematics and programming.

THEORETICAL FRAMEWORK

In order to first design quality tasks – tasks that would effectively support teaching of both geometry and CT skills – and to better understand the outcome of our experiments, we relied on three theoretical frameworks: Duval's Semiotic Approach (in Sinclair et al., 2016), Bartolini Bussi and Mariotti's (2010) Theory of Semiotic Mediation, and Hoyles and Noss' (2015) Computational Thinking Taxonomy. Each of these frameworks has a distinct role of illuminating a different facet of the dataset. While Duval's approach encouraged emphasis on the process of dimensional decomposition in task design – in order to foster non-iconic visualization, which is a characteristic of geometric thinking – Bartolini Bussi and Mariotti's work provided insight into the use of tools as geometric concept mediators, and Hoyles and Noss, by assigning categories to CT attributes, helped inform both task design and data analysis. When braided together rather than melted into one another, each framework retains a distinct role in helping make sense of the findings, whilst offering a greater potential for a more accurate and comprehensive interpretation of the interplay between teachers, students, tools, and content, than if used in isolation.

Duval's Semiotic Approach

Duval's semiotic approach and his theory of dimensional decomposition emphasize "ways of seeing" and consider the visualization component to be fundamental in geometrical thinking. Duval argues that the cognitive power of visualization lies in its tendency to fuse units of inferior order into one unit of superior order. Duval distinguishes four apprehensions of a geometrical figure: (1) perceptual (recognition of shapes), (2) sequential (construction of shapes), (3) discursive (property-based recognition of shapes), and (4) operative (processing of shapes, as in transforming and reconfiguring).

Duval views geometry as having two representation registers: the visualization of shapes, and the language for describing its properties. He argues that it is important to achieve synergy between the visual and discursive registers of geometry. This can be done through tasks that engage a "constructor" way of seeing, which is supported by the process of dimensional decomposition. This process involves the flexibility of shifting perception between the two-dimensional view of the surface of a polygon and the one-dimensional view of its sides. Thus, dimensional decomposition involves two aspects: seeing the basic shapes as constructed from lines and points, and seeing that many two-dimensional shapes could emerge from networks of lines.

Duval proposes that the learning of geometry could begin with an exploration of the different configurations that could be formed with lines. To support dimensional decomposition, he recommends construction as the point of entry – when a shape is being constructed, the dimensional decomposition is done with the geometric tools, which puts an emphasis on one-dimensional elements of the shape:

> The shape is no longer a stable object but one that evolves over time, capable of being decomposed and reconfigured. This non-iconic visualization is characteristic of geometric thinking… It is a way of seeing that eventually enables all the discursive procedures in geometry (pp. 459-460).

Theory of Semiotic Mediation

While Duval is helpful in getting insight into effective task design, we think it is important to also account for the use of virtual artifacts in the development of geometric knowledge. Theory of semiotic mediation (TSM) not only emphasizes the use of tools when designing tasks, but also interprets how DGE in general, and virtual circle tool in particular can mediate geometric concepts.

Taking its origin from the work of Vygotsky, TSM describes how cultural artifacts can be used by teachers to support teaching and learning of mathematics. Bartolini Bussi and Mariotti (2010), the Italian scholars who formulated the theory, propose that cultural artifacts have semiotic potential: both personal and mathematical meanings may be related to the artifact and its use. Because of this double relationship, the artifact may function as a semiotic mediator, but this function is not automatically activated.

Any artifact can become a tool of semiotic mediation, if it is intentionally used by the teacher to mediate mathematical content through a designed didactical intervention. Bartolini Bussi and Boni (2003) describe what the process of semiotic mediation looks like in real life. They report on a mathematics lesson taking place in a grade 5 classroom, where students were asked to draw a circle with a particular radius, that was tangent to two other circles. Students were only familiar with the use of compass to draw precise circles. However, through hands-on activity, group discussion, and careful guidance by the teacher, who was aware of the semiotic potential of this artifact and chose to use it as a tool to mediate mathematical content, students began seeing a compass as a tool to find a point at a given distance from another point. Bartolini Bussi and Boni explain that integration of two ways of thinking of circles took place: mechanical/dynamic/procedural of Hero, and geometrical/static/relational of Euclid; the geometric compass is no longer only a material object – it has become a mental object as well. The two ways of using the compass are gesturally the same whether a student wishes to produce a round shape or find a point at a given distance, but the senses given by the students to the processes and products are very different (Bartolini Bussi & Boni, 2003).

Computational Thinking

Hoyles and Noss (2015) identified three groups that all CT attributes fall into: processes, concepts, and practices. CT processes include abstraction, algorithmic thinking, decomposition, and pattern recognition. CT concepts involve loops, conditions, subroutines, and variables, while CT practices include debugging and decomposition. Based on this classification, Kotsopoulos et al. (2017) proposed the Computational Thinking Pedagogical Framework, which includes four pedagogical experiences: (1) unplugged, (2) tinkering, (3) making, and (4) remixing. These experiences are not hierarchical, although such progression may prove beneficial when working at the beginner level. Unplugged experiences focus on activities implemented without the use of computers. Tinkering experiences involve activities that take things apart and engage in modifications to existing objects. Making experiences involve activities where constructing new objects is the primary focus. Remixing refers to those experiences that involve the appropriation of objects or components of objects for use in other objects or for other purposes. Mathematics education researchers have found the definition proposed by Hoyles and Noss (2015) to be the best fit for mathematics.

METHODOLOGY AND RESEARCH QUESTIONS

This research project was carried out in an urban elementary school in North America. We aimed to design tasks that would combine CT and geometry, and we wanted to see (1) what the students would learn, and (2) what the effect of combining domains would be on how the concepts emerge.

We conducted a two-month classroom intervention with grade 2/3 students. In our lessons, we opted for a combination of whole-class Smartboard discussions, followed by students working in pairs on iPads. Each lesson has been documented through note-taking, photographs, and examples of students' work, both in static and dynamic format.

In our tasks, we examined properties of dynamic circle as a semiotic tool that enables students to determine whether two segments are of the same length, and to construct two segments of equal length.

We conducted a series of seven lessons around this topic. Each session was about an hour-and-a-half in length. We used The Geometer's Sketchpad (Jackiw, 2012) and more specifically, a custom-built "Point, Segment, Circle" online web sketch (Sinclair, 2017). Research has demonstrated that the use of Dynamic Geometry Software (DGS) can significantly enhance the learning process, being a powerful tool in dissolving prototypical thinking in students, which is consistent with Duval's dimensional decomposition process. It also helps promote spatial reasoning skills (Sinclair & Bruce, 2014).

Sixteen students were interviewed afterwards, in order to enable researchers to gain understanding into the coding process that students undertook, and into the evolution of geometric concepts. Interviews were comprised three components: static (paper questionnaire), dynamic (construction tasks using Sketchpad), and auditory (dialogue between a student and a researcher). The dynamic component of the interview was recorded using a screen-recording software. Interviews were audiotaped, transcribed, and analysed for instances of CT in students' cognitive processes.

DATA AND ANALYSIS

All data has been collected via lessons and interviews, with lessons being documented in the form of notes, photographs, and students' work, and with interviews – in the form of audio recordings, screen captures, and paper-and-pencil questionnaires.

Lessons

Participants had never worked with Sketchpad before, so the first couple of sessions were introductory. It was observed that students had more difficulty drawing the designs than actually making them. During the third session, a special web sketch that contained only three tools – segment, circle, point – was introduced, and students were encouraged to build triangles from scratch and then assemble them into a house. This task was the first instance of developing a procedure, and carrying it out afterwards.

In the fourth session, a circle with two radii was constructed on the Smartboard, and two questions were raised: Are these two radii of the same length? and How many more radii can we fit into one circle? First, students thought the right radius was longer than the left one, but after a quick demonstration that it can move and approach the other radius, the class came to consensus that they were of the same length.

The question about the potential number of radii that could fit into the circle caused hesitation, so students were invited to participate in a role-play demonstration, where everyone had to be a point on a circumference, and at the same distance from the centre. As the circle became tight and another person tried to join in, one of the students protested: "There is no more space left". The researchers suggested to make the circle bigger by stepping back. This seemed to have caused a shift in understanding, and once students returned to the Smartboard, the answers to how many radii could fit into a circle were multiple: "ten", "twenty", "a bunch", with someone adding that "it will take us three hundred years to find them all".

Finally, we asked the class to construct a stickman with two equal arms and two equal legs. This task was very rich in its potential to build fluency in such CT processes as abstraction, algorithmic thinking, decomposition, and debugging – before students could build a stickman, they had to create an algorithm to build two segments of equal length, iterate it to build arms, legs, and fingers, combine parts into a whole step by step, perform a dragging test, and debug, if the design fell apart.

Interviews

The interviews portion related to the circle tool, featured the following questions:

Q1: Which of these line segments are the same length? How do you know? (Fig. 1)

Q2: Does this triangle have any equal sides? How do you know? (Fig. 2)

Q3: What does the circle tool let you do? Can you use it to make this? (Fig. 3)

Q4: Can you make two segments of exactly the same size, without measuring them?

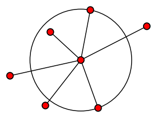

Figure 1

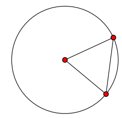

Figure 2

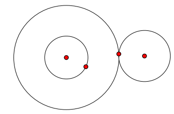

Figure 3

Students' answers to the first two questions yielded 32 responses, which fell into four categories: (1) reference to both the centre and the circumference (7 out of 32 responses), (2) reference to the circumference only (14 out of 32 responses), (3) physical measurement (7 out of 32 responses), and (4) visual estimation (4 out of 32 responses). Based on these results, approximately one-third of all responses did not demonstrate understanding of the concepts taught, while the remaining two-thirds made a reference to either the centre, or the circumference, or both, thus demonstrating good grasp of the concepts taught, i.e. they could use the properties of a circle to compare two lengths, when provided with a visual.

The remaining two questions involved construction using Sketchpad. While all sixteen students were able to use the circle tool to produce the required design (Fig. 3), more than half required multiple attempts, which could either indicate lack of confidence with the concept, or fine-motor difficulty with the digital tool itself. In either case, all students without exception needed to make modifications to their models when creating concentric and tangent circles. This task required students to use CT practices of algorithmic thinking and decomposition, and also apply the process of debugging.

In the last question, when requested to construct two segments that were of the same length, one-third of all interviewees made use of the circle tool, while the remaining two-thirds were either using the segment default function (which also generated correct results), or visually estimating lengths when dragging end points. This could mean that only one-third of all interviewees mastered the concepts taught and demonstrated pattern recognition; this could also mean that the wording of the question did not insist on using the circle tool.

Overall, when comparing results from two sets of questions, it was concluded that while two-thirds of all interviewees could identify equal segments when provided with a visual, only one-third was able to transfer their understanding to hands-on tasks.

DISCUSSION

Duval (in Sinclair et al., 2016) sees visualization as fundamental in geometrical thinking, and proposes to start with construction tasks that require exploration of different configurations that could be formed with lines. When students, for example, were constructing triangles in the warm-up phase of the project, using virtual segments in Sketchpad, they were focusing on how the 2D shapes that they were constructing are in fact comprised of one-dimensional objects, thus practicing "a constructor way of seeing", which is very important in order to achieve "synergy between visual and discursive registers of geometry", according to Duval.

When students were contemplating how to construct a triangle to build a house, they first had to solve a smaller problem. In order to construct a triangle first, students needed to use segments, thus practicing dimensional decomposition. When moving between dimensions, on one hand, and moving between smaller and larger problems on the other, students blended together geometrical and CT thinking practices within one task. Ability to see one-dimensional elements ultimately helped to solve the larger problem of constructing a house: students were able to correct crooked houses, because they were able to focus on an isolated segment and see whether it was perpendicular to the ground or not. As was evident in previous experiments with prefabricated triangles, it was often challenging to achieve the desired orientation. To add to the complexity of the tasks, we introduced a circle. However, it was not used as an object, but as a mental tool that students used to build segments of equal lengths, serving as semiotic mediator.

Each geometric task that we presented to students was both geometrically and CT-rich. In addition to exploring geometric shapes, it required abstraction, automation, and analysis, using Papert's (1980) terminology. For example, in the series of warm-up tasks, students were offered a particular design for examination (a square filled with triangles), and they had to extrapolate from that design an abstract procedure of how to fill any shape with any number of triangles. Then, in the automation phase, students were creating an algorithm that would help solve any problem of the same type (e.g. drag triangles into the confinement of a shape, then adjust the size of the triangles by dragging vertices until the shape gets filled). Finally, in the analysis phase, students were testing out their solution and debugging, if necessary (e.g. realizing that there is no space left to fit the last triangle, and either re-doing the design, or shrinking it almost into a line to create an illusion of fitness).

CONCLUSION

Our finding is that, in addition to supporting the teaching of geometry, the DGE-based tasks effectively support teaching of many CT concepts, which seems to provide sufficient ground for Math/CT interdisciplinary integration. CT tasks feature low-floor-high-ceiling-wide-walls design, abstraction and automation, dynamic modelling, tangible feel, student agency, and aesthetic experience, in addition to many other affordances (Kotsopoulos et al., 2017).

Many of these affordances are also present in DGE, which provides good reasons to use it in teaching school geometry. Since nearly every geometric construction task in DGE involves creating a procedure and testing if it works, with likely adjustments afterwards, there is a natural connection to CT. While DGE tasks can benefit from CT structure in that they have to be problem-based and procedural to help develop abstraction, automation, and analysis skills, CT tasks in their turn become enriched by

DGE affordances to promote spatial reasoning skills and non-verbal intelligence through the use of geometric (as oppose to alpha-numeric) programming language.

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LINES, STANZAS, EPISODES: UNITS IN THE UNDERGRADUATE MATHEMATICS LECTURE

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Undergraduate mathematics courses very commonly include a few dozen hours of lecturing. This study investigates the structure of a course of 35 50-minute lectures delivered in an undergraduate abstract algebra course at a midsize Canadian university. Influenced by the approach of Halliday in his work on determining useful categories and units in spoken and written texts, this paper describes and characterizes 2 structural building blocks of 'chalk and talk' mathematics lectures of increasing size: lines and stanzas. Beginnings and endings of such units are typically marked in multiple ways: intonation, body orientation, direction of gaze, volume, pitch, position of body relative to the board, and commonly used starting or ending words.

INTRODUCTION

Mathematics educators have called for a closer and deeper examination of teaching practice at the university level, noting a relative dearth of empirical work in the area (Speer, Smith & Horvath, 2010). For example, a bibliographic study of research conducted on proving activities in mathematics classrooms found that of 96 articles that discussed at least one proving task, 82 were concerned with students constructing arguments, 24 were concerned with students reading arguments, and none were centered on presentation of arguments by the instructor (Mejia-Ramos & Inglis, 2009).

Lecturing in mathematics has received a slowly growing share of interest from researchers in mathematics education in recent years. The lectures of a real analysis instructor were analysed and classified into styles: logico-structural, procedural, semantic (Weber, 2004). Also performing a case study, Bergsten (2007) searched for factors he could correlate with a notion of 'quality' in lectures. The commognitive approach to mathematical communication has been used as a framework to discuss 'routines' in lecturers' pedagogical discourse, such as explanation routines, motivation routines, and question-posing routines (Viirman, 2015). Hannah, Stewart & Thomas (2011) adapted Schoenfeld's framework for teacher decisions in the school classroom to the university level, examining the resources, orientations and goals of a mathematics lecturer and how they interact in the complex decision-making process. This study joins these and others in addressing the practice of mathematics lecturers, focusing on the structure of the spoken component of the lectures.

THEORETICAL FRAMEWORK

Mathematical writing

The first observation made in this study that is regarded as fundamental is that mathematics lectures contain a critically important written component. This importance is revealed in various mutually reinforcing ways. For example, classrooms in which students do not have a line of sight to the board would clearly not be tolerated. Mathematics instructors who only talk and who refer to no written material that is visible to the students are profoundly rare. The typical webpage for an undergraduate mathematics course names a textbook that is the classroom text; the short summaries of each day's lecture often correlate with section headings from this text. and the lectures to some degree contain within them written components that bear some relationship to this text. Anecdotally, mathematics professors often develop personal lecture notes that they carry to class to lecture from.

The first feature I wish to emphasize is that mathematical writing is highly segmented. It is a commonplace that undergraduate and graduate mathematics textbooks are explicitly segmented, with headings largely coming from the following list: Definition, Theorem, Proof, Exercise, Example, Fact, Remark, Historical Comment, Corollary, Lemma, Proposition (Bagchi & Wells, 1998). Beginnings and endings of each of these textual environments are typically unambiguous and clear. For example, the "Halmos symbol" ends a proof, and a proof begins with the heading: Proof. Konior (1993) discusses the role of what he calls "delimitators" in mathematical text: phrases of the form (these are his examples) "It remains to show that g is one-to-one" and "We have thus proved formula (32)".

Even within environments such as those listed above, there are sub-environments that are clearly demarcated. An if and only if statement is established in one direction, then there is a paragraph break, then the other direction. Or the result falls into two cases, when $n = 0$, and when n is a positive integer. Or a conclusion is heralded to follow from three claims: then each claim is proved in turn, separated typographically.

One would expect that whatever software mathematicians use to typeset their papers would efficiently reflect the structures of the mathematical texts they write. Almost to a person the mathematical community uses LaTeX (a language built on the typesetting language TeX, developed by Donald Knuth) to typeset their mathematics, for everything from lecture notes to journal articles to conference presentations to books. LaTeX indeed has a large number of standard environments that can be called upon by the author: for example, "\begin{theorem}" and "\end{theorem}" are the commands to initiate and end a theorem environment.

It is plausible to expect that someone lecturing on mathematics, who will be writing such segmented text on the board, will exhibit segmentation in their speech as well.

Halliday on constituency

The work of Michael Halliday on linguistics that he terms systemic functional grammar has found a large number of applications in education, including mathematics education (O'Halloran, 2008; Herbel-Eisenman, Wagner & Cortes, 2010; Herbel-Eisenman & Wagner 2010). Halliday begins his textbook introduction to the subject with a chapter on constituency: "If we look at a passage of writing in English, we can see clearly that it consists of larger units made up out of smaller units. These smaller units, in their turn, are made up of units that are smaller still...This kind of layered part-whole relationship which occurs among the units of a written text is referred to as *constituency*" (Halliday, 1994, 3). Halliday calls the periods at the ends of sentences and the spaces at the ends of words *structure signals*. He observes that constituency can be visually rendered as a tree diagram, because "sentences follow sentences, words follow words, and letters follow letters in a simple sequence; they do not overlap, nor does anything else occur in between". Finally, he argues that "if discourse can be adequately represented by sequences of written symbols arranged like this in constituent hierarchies, it is reasonable to assume that language is inherently organized along something like these lines" (6). He cautions us not to naively expect that constituency is the only structure, or the most basic structure, of language. Nevertheless, he considers it probable that constituency will play some role in the organization of language. I am adapting this claim to the context at hand. The constituency of mathematical writing leads us to expect that constituency in mathematical speech during lecturing will play an important role in understanding lecturing.

Linguistic anthropology - lines and stanzas

Staats (2008), beginning from the standpoint that "at times, the form of a student's statement can convey meaning as much as the isolated definitions of the words themselves" (26), turned to work done in linguistic anthropology in order to analyze student discourse in terms of structures she called 'lines' and 'stanzas'. She drew on the theoretical framework developed by Tedlock and Hymes in their attempts to transcribe and analyze Native American oral poetics and folktales.

Tedlock and Hymes come paired in a great deal of literature that refers to at least one of them because of the perception of later researchers that they each stand for separate and distinguishable paradigmatic methodological choices they made in their transcriptions of the narratives that they heard and taped. Tedlock (1983) was mostly guided by *pauses* in his transcription of speech and divided his written lines at these pauses. Hymes (2003) was mostly guided by certain repeated words, or *particles*, which to him announced the ending of a line or the beginning of a line. An alternative scheme of contrasting these researchers is that Tedlock comes down on the side of "how do you perform this text" and Hymes comes down on the side of "what is the structure of meaning of this text". This dichotomy both captures something true and is also an oversimplification, as Hymes himself notes: "It might be a fair summary to say that Dennis is concerned most of all with the moment of performance, and I am much concerned with the competence that infoms it...What is particularly not so is the equation Tedlock: Hymes = pause: particle. Dennis has sometimes attended to particles as relevant, and I have never attended to particles alone." (37) What I adopt from both authors is the conviction that an accurate rendering of the constituents of speech such as the word, the line, and the next unit size up (which they call verses and which I call stanzas) is necessary to a full understanding of the meaning and function of the lecture itself, and that this accurate rendering can only be gained by a close attention to the rhythms, beats, tones and repetitions of this speech.

The lecturer, the board, and the students

There is one more factor that is necessary to consider before the mathematical lecture can be properly segmented, and that is the positioning and orientation of the lecturer. Barany and Mackenzie (2014) detail numerous interactions which frequently occur between the lecturer and the board: the lecturer can touch previously written statements to refer to them; can block terms with their hands to indicate that their present exposition doesn't require them; can annotate what has been written with further marginal comments, and so on. Rodd (2003) emphasizes the role of the students as audience participants and witnesses to a piece of theatre. Following these researchers, I will include the position and orientation of the lecturer with respect to the board and the students as potential structure signals in segmenting mathematical speech by the lecturer.

METHOD

A course of 35 lectures was videotaped. A transcript was made of these lectures. It is in the process of making this transcript that segmentation first occurred naturally. I aimed for a naturalistic approach and wished to impose as few external structures to the words being spoken as possible. It was very hard to type what was being said into sentences, even though so much of the lectures, especially near the beginning of the course, were monologues. The lecturer would self-correct himself, and I would face the decision: is this the end of sentence and the beginning of a new one? Was this pause worth a comma? Was that one a semi-colon? The pressures of keeping a good pace of transcription led to natural improvements. I noticed that I was typing to a regular rhythm: a spurt of a few words, perhaps 8 to 10, then another such burst, and then another. I took to putting line breaks at these moments when my lecturer took a bit of a breath.

Later, when the transcript was finished, but I was doing a second pass through watching the videos, I also adopted the method of playing the video at a faster speed, 1.5 - 2 times the original speed. At this pace the pauses between bursts of words come out into sharper relief, and it becomes perceptually easier to distinguish between the continuous flow and the little silences in between.

It also became quickly clear that there was another sort of segmentation occurring at a larger time scale. The lecturer would complete a step in an argument, and then he would begin the next bit of business. Or he would finish the writing of a sentence and then step back from the board and look at these notes to remind himself of what to do next. As I continued on with the transcription I made a gradually increasing list of the sorts of sounds, moves, behaviours that were correlating with such segmentations. On the second pass through the videos, I now had a list of such potential structure signals to check against the videos (whereas on my first pass I was still developing this list): this allowed me to more uniformly apply these signals to the transcript.

For this study I will present the robust structure signals that were determined at the stanza level.

RESULTS AND DISCUSSION

I have chosen 4 stanza transitions from lecture 4 to indicate the use the lecturer makes of these structure signals. (04.17-04.18 means the transition between the 17th and 18th stanza of lecture 4).

04.17-04.18

1. Sharp intake of breath.
2. Opens mouth that had been closed.
3. Very quick rotation of the head and
4. Very quick switch of eye gaze from his notes to the board.
5. Initiates large pointing gesture to what he has already written on the board.
6. His volume had been low. It jumps sharply to a higher volume.
7. He says the word 'ok'.
8. He says the word 'other' - indicating a change in subject.

04.62-04.63

1. Waves his hands at a result as if to say he wants nothing more to do with that.
2. Walks to desk and flips over to latest page of notes.
3. Says 'good' 'thank you' (closing particles).
4. Picks up latest page of notes.
5. Turns to face the board.
6. Picks up eraser to erase board.
7. Says 'ok' (opening particle).
8. Erases left half of board 2.
9. Says 'ok' again.
10. Tone elevated.

11. Volume elevated.
12. Says word 'theorem'.
13. Begins writing on the board. He writes a Heading ('Theorem 2.2').

04.67-04.68

1. Breaks away from sustained eye contact with students.
2. Turns to face the board.
3. Erases right half of board 2.
4. Ends on lowering tone and volume.
5. Extended silence.
6. Takes page of notes back to desk. Picks up new page of notes.
7. Says the word 'corollary'.
8. Tone elevated.
9. Volume elevated.
10. Begins writing on the board. He writes a Heading ('Corollary').

04.89-04.90

1. Writes Halmos box symbol on the board.
2. Walks away from board.
3. Returns to board and draws vertical dividing line.
4. Begins writing.
5. Changes his mind and walks away from the board. False start.
6. Stands in front of class and begins a monologue addressed directly at them.

The lecturer uses a variety of structure signals to indicate the ending of one stanza and the beginning of a new one. First of all, there are signals that have to do with tone: a lowering tone at the conclusion of a stanza and an appreciably higher tone to begin a new one. Second, there are signals to do with the volume of his voice which mirror tone: lower at endings of stanzas, markedly higher (with renewed gusto) at the beginnings. Third, there are particles: 'ok' as both opening and closing particle; 'and now', 'and then' 'so' as opening particles; 'good' as closing particle. Fourth, there are sudden changes in body orientation: from facing the students to facing the board or vice versa. Fifth, there are sudden and rapid changes in body position: from near the board to a step or two back; from a step or two back to three or four steps away; from the board to the desk. Sixth, there are sudden changes in eye gaze: from the board to his notes, from his notes to the students. Seventh, beginning new environments in the mathematical writing signal new stanzas; similarly, with closing environments.

It is also clear that such stanza segments are often over-determined. Two or three or more of these signals combine together.

CONCLUSION

One of the observations that has served as an inspiration for this research is due to Thurston (1990, 847): "Mathematics is amazingly compressible: you may struggle a long time, step by step, to work through some process or idea from several approaches. But once you really understand it and have the mental perspective to see it as a whole, there is often a tremendous mental compression. You can file it away, recall it quickly and completely when you need it, and use as just one step in some other mental process. The insight that goes with this compression is one of the real joys of mathematics." If this is true, then it must follow that what gets lectured on in the form of 15 stanzas as the proof of some result about cyclic groups might be compressed into the line "intelligently apply the Euclidean algorithm". It is only by focusing attention on the lines and stanzas of lecturers that we can begin to accurately and systematically trace this important process of compression, and its counterpart process of unpacking - the ability to at a moment's notice spool out a 15-stanza version if needed.

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COMPARING ANGLES IN A DYNAMIC GEOMETRY ENVIRONMENT

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This paper examines young children's thinking while comparing geometric representations of angles of different sizes in a dynamic geometry environment (DGE). This study is based on a classroom experimentation, during which the kindergarten/grade1 children worked in a whole classroom setting over nine sessions, in which they could interact directly with Sketchpad on an interactive whiteboard. Using the dynamic sketches in Sketchpad, children were able to develop an understanding of angle as a turn. DGE approach is found to be helpful in focusing children's attention on the quantity of turn rather than on the length of the line segments during angle comparison tasks. This study provides one way of establishing a relationship between angle-as-a-turn and angle-as-a-shape conceptions.

INTRODUCTION

Angle is an important topic in geometry. It is a concept that children find challenging to learn, in part because of its multifaceted nature (Mitchelmore & White, 2000). The purpose of my main study is to understand how children's thinking about angles evolves as they participate in a classroom environment featuring the use of a dynamic geometry environment (DGE) in which the concept of angle as turn was privileged, a concept that does not require a quantitative dimension. I report on the working of Kindergarten/Grade 1 split class children on the concept of angles using sketches pre-constructed with *The Geometer's Sketchpad* (Jackiw, 1991, 2011). This paper, in particular, focuses on children's thinking while comparing geometric representations of angles of different sizes in a dynamic geometry environment.

NOTIONS OF ANGLES AND CHILDREN'S UNDERSTANDING OF ANGLES

Research literature shows classification of angle definitions in different categories those points to the duality of dynamic and static aspects (Mitchelmore & White, 2000; Henderson & Taimina, 2005). Henderson and Taimina (2005) defined angles from three perspectives (a) angle as movement (a dynamic notion of angle) (b) angle as a measure and (c) angle as a geometric shape. Freudenthal (1973) makes it clear that there is more than one concept of angle and argues that the appropriate meaning of angle is directly related to the mathematical context for which it is used. More recently, Zazkis and Kontorovich (2016) note that some perspectives of angles are incompatible and reported that these incompatibilities were problematic even for secondary school teachers. It seems like no formal definition of angle can capture all aspects of our experiences of what an angle is. Also, children have difficulty seeing a turn as an angle and a static angle as a turn (Mitchelmore, 1998; Clements, Battista, Sarama & Swaminathan, 1996). Students also think that the length of the arms is related to the size of the angle (Stavy & Tirosh 2000; Clausen-May, 2005; Munier, Devichi & Merle, 2008). The goal of this paper is to see what attributes children focus on while comparing angles in a DGE.

THEORETICAL PERSPECTIVE

This study draws on the participationist view of learning as proposed by Sfard (2008) that recognizes a close relationship between thinking and communication. Sfard (2008) offers a communicational approach in her discursive framework, which is well suited to this study and has been shown to be effective by other researchers (Sinclair & Moss, 2012; Kaur, 2015) because it enables researchers to make claims about students' thinking in terms of how students communicate. Sfard views thinking as a form of communication and knowing of mathematics as synonymous with the ability to participate in mathematics discourse. Thus, mathematical learning is the development of a mathematical discourse. According to Sfard, the mathematical discourse has four characteristic features: word use (vocabulary), visual mediators (the visual means with which the communication is mediated), routines (the meta-discursive rules that navigate the flow of communication), and narratives (any text that can be accepted as true such as axioms, definitions and theorems in mathematics). Learning geometry can thus be defined as the process through which a learner changes her ways of communicating through these four characteristic features. Sfard's four features of mathematical discourse are described mainly in terms of verbal discourse. Since spoken discourse is multimodal, sometimes it fails to account for the full set of resources used by young children to communicate. Given the importance of gestures in communication of abstract ideas (Cook & Goldin-Meadow 2006), and their potential to communicate temporal conceptions of mathematics (Núñez, 2003), it is necessary to focus not just on the words or the visual mediators that the children use, but also on their gestures.

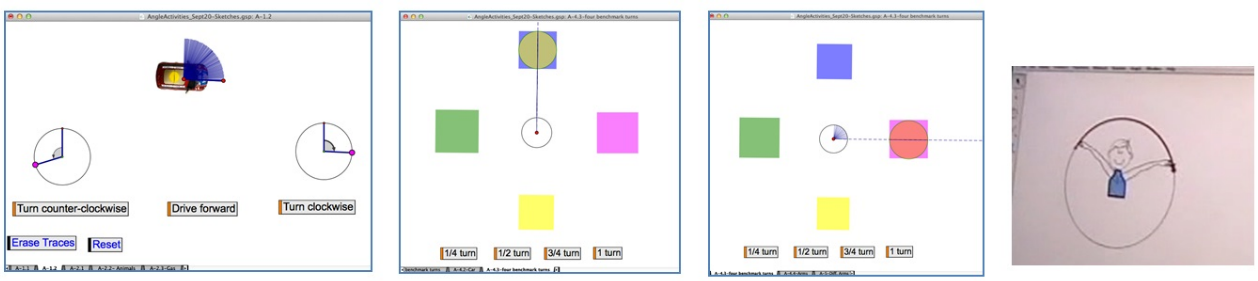
RESEARCH CONTEXT

Participants and tasks

We (research team and class teacher) worked with 22 kindergarten/grade1 children (aged 5-6) from a school in a rural low socio-economic status town in the northern part of British Columbia. The lessons related to angle were designed along with the classroom teacher, who has been developing her practice of using DGEs for a couple of years. The teacher and children worked with angles in different ways, using Sketchpad for ten sessions (30-40 minutes each) in a whole class setting with children seated on a carpet in front of an IWB (Interactive Whiteboard). The children have worked with the Sketchpad prior to the instruction of angles, where they explored the concept of symmetry using the sketchpad. All the sessions were videotaped and transcribed. This paper will focus only on the ninth session, where children explored and compared geometric angle representations of different angle sizes.

Background

Prior to working on angle comparison tasks, the children had worked with different angle sketches made using *The Geometer's Sketchpad* to explore the concept of angle. The concept of angle was introduced using the 'driving angle sketch'. The 'driving angle' sketch (figure 1a) shows both a static as well as dynamic sense of angle. It includes a car that can move forward as well as turn around a point. The turning is controlled by two small dials (each of which has two arms and a centre) - one dial allows clockwise turns and other counter-clockwise turns. No numbers are used. Five action buttons (*Turn counter-clockwise*, *Drive Forward*, *Turn clockwise*, *Erase Traces* and *Reset*) control the movement of the car. In this sketch, the turn of the car is associated with the amount of angle adjusted in the small dials. The traces of a turn offer a visible, geometric record of the amount of turn. So, in the first few sessions the children explored and understood the turning of the car and its association with the dial.



1(a) Turning trace of car after pressing Turn clockwise button

1(b) Benchmark angles sketch

1(c) Traces after pressing ¼ turn button

1(d) Arms sketch

Figure 1(a, b, c): Dynamic sketches created in Sketchpad

After the children developed some sense of angle as a turn, they were presented with a benchmark angles sketch. For this purpose, a sketch shown in figure 1(b & c) was designed using four different coloured squares positioned in a way so that there is a 90^0 angle between any two consecutive squared boxes. There is a circle in the centre, which is attached to another circular ball using a ray. There are four action buttons (¼ Turn, ½ Turn, ¾ Turn and 1 Turn) that control the movement of the ball. When you press any action button, the ball moves clockwise from one square to another, tracing its associated turn in the central circle. For example, pressing ¼ Turn button moves the ball by 90^0 (to next square) and generates the traces in the central circle (see figure 1c). This sketch helped the children understand the commonly used benchmark angles without referring to any standard measurement units of angles such as degree, radian etc. In *Arms sketch* (figure 1d), the arms of a boy could move to form different angles in a circle. The teacher also invited the children to make different benchmark angles with their arms.

Previously all the sketches (figure 1) had used circles, so the arm lengths were equal for the angles. The next sketch comprised of angles with different arm lengths so that the children could compare the size of the turns. The goal was to see if children considered the length of the arms as a factor while comparing the angle sizes.

COMPARING THE GEOMETRIC ANGLE REPRESENTATIONS OF DIFFERENT ANGLES SIZES

The teacher presented the sketch with angles shown in figure 2a on the screen. Below is what follows:

No.	Who said/did	What was said/what was done
1341	T	Okay if we look up here (*figure 2a*), which is the bigger angle? Maya?
1342	Maya	(Pointing towards the screen) That one.
1343	T	Can you go up and touch it for me?
1344	Maya	(*Maya comes to IWB and points at the second angle, figure 2b*)
1345	T	Why is that the bigger angle?
1346	Maya	Because that one is smaller (*touching the first angle*) and this one is bigger (*touching the second angle*).

The teacher changed the angles by dragging, as shown in figure 2c.

1355	T	Now which one has the bigger angle? (*Figure 2c*)
1356	Pat	Still that one.
1357	T	Still that one. Why?
1358	Larry	Because it is wider.
1359	T	Because it is wider. It is much wider. Isn't it?

Upon being asked which of the two angles in figure 2a is bigger, Maya at once responded that second angle is bigger. Her response, "that one is smaller" [1346] suggests that she is not attending to the length of the arms; otherwise she might have thought the other one was bigger.

(a) (b) (c)

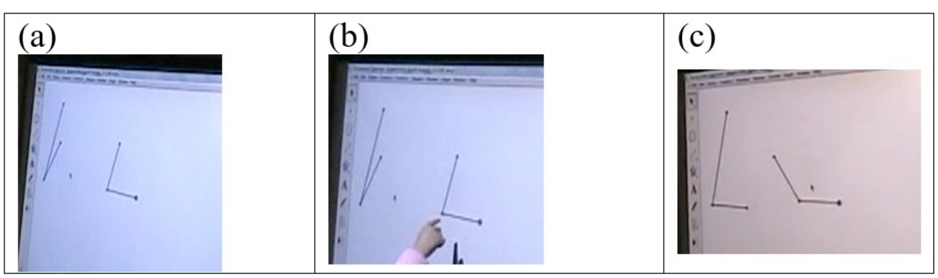


Figure 2: (a) Two angles presented for comparison; (b) Maya pointing at the bigger angle; (c) New angles for comparison

Later, the teacher presented a new set of angles (figure 2c). Pat responded that the second angle was bigger. In Larry's justification for the bigger angle in [1358], "because it is wider", the use of word "wider" points to something that includes a larger amount or covers a larger range or area. This gives confirmation that they are not attending to the length of the arms while comparing sizes of the angles. From the above responses of Maya, Pat and Larry, it seems like they had grasped the idea of bigger or smaller angles, but this was not true for all the children. The next excerpt outlines one such episode.

The teacher drew the next pair of angles as in figure 3a and asked:

No.	Who said/did	What was said/what was done
1364	T	Gia, out of these two which one has the bigger angle? (*Figure 3a*)
1365	Gia	That one (*Gia points towards one of the angles*).
1366	T	Can you come and touch it? Come point to the one you said for me. Okay.
1367	Gia	(*Gia touches the second angle as shown figure 3b*)
1368	T	Why does that one have a bigger angle?
1369	Gia	Because it has long lines.
1370	T	What does it mean? What does an angle mean?
1371	T	Does an angle mean how big a line is?
1372	Sss	No.
1373	Larry	Then how come people say angle of a line?
1374	T	That's a good question. That is a very good question. If angle does not mean the length of the lines, when people refer to the angle of a line, what do they mean? That's a really good question. What do you think they mean?

Then the teacher displayed the car sketch, the benchmark angles sketch, and the arms sketch one by one on the screen and asked the children what they did previously in all those sketches. To which the children responded, "turn". The teacher gave the following example.

1398	T	Hey, watch what happens here. Right now, I have a zero angle (*figure 3c*). Look at it. Really important. Look. Pat.
		Right now, I have a zero angle. (*Dragging the second line away, figure 3d)* What did I do with that second line?
1399	S	Make it turn
1400	S	Turned it
1401	T	I turned it (*gesturing as in figure 3e*). I made an angle.

1402	Larry	Angle is the amount of turn

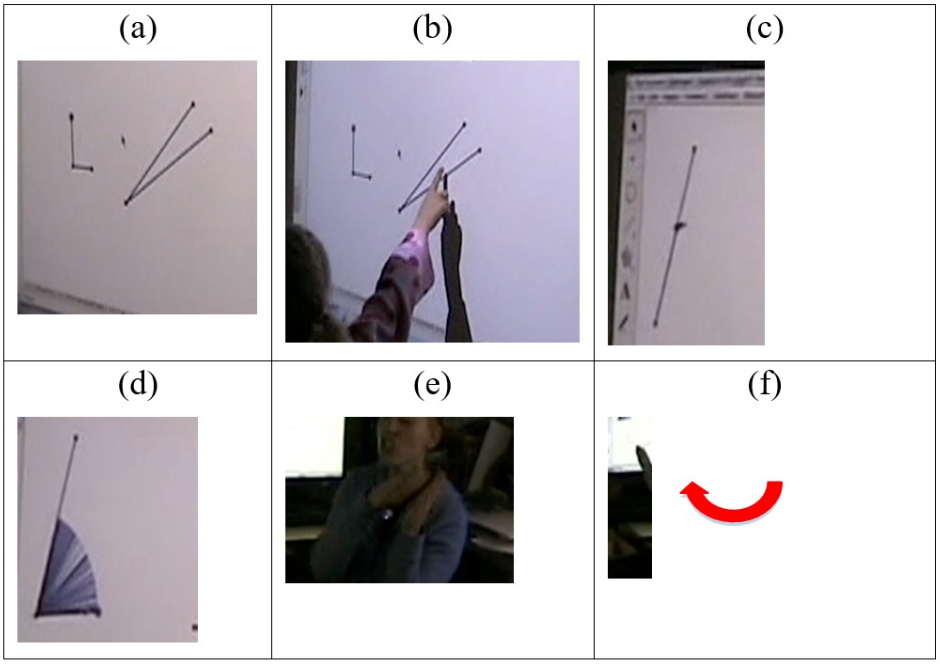


Figure 3: (a) Two angles presented for comparison; (b) Gia pointing at the bigger angle; (c) Zero angle position of two lines; (d) Angle created on dragging by the teacher; (e) & (f) turning gesture by the teacher to show the angle

In this exchange, Gia began by identifying the second angle as the bigger angle (as shown in figure 3b). Her utterance in turn [1369], "Because it has long lines," suggests that she was associating the size of angle with length of the line segments. This resonates with the findings of research literature where children are shown to have misconceptions about the angle sizes being related to length of the sides (Stavy & Tirosh, 2000). When the teacher questioned her in turn [1371], "Does an angle mean how big a line is?" most of the children responded with "No" [1372]. Their response "no" does not assure that they actually understand that the angle is independent of the length of the lines. Larry's statement, "Then how come people say angle of a line?" raises an interesting question. His utterance [1373] suggests that the confusion arises in part from the everyday language that we use for the angles.

To overcome this confusion, the teacher brought the children's attention to the work they had been doing with the sketches. She asked them what they had done in all the sketches, to which the children responded, "turn". Then the teacher used this opportunity to associate the static geometric representation of the angle with the dynamic conception. She put both arms of the angle together, so that they appeared as one (figure 3c) and were at zero angle position. The teacher dragged the line out tracing the path covered by it. The traces acted as a visual mediator. The children's utterances "make it turn", "turned it" in [1399] and [1400] respectively shows that they were noticing the turning of the line. But this does not mean that they associated it with the angles. The teacher associated the turning with creation of an angle in [1401] along with her turning gestures (figures 3e, f), which emphasized the association of turning with angle. Larry's statement in [1402], "Angle is the amount of turn" suggests his realization about the angle, which is his first explicit narrative about the definition of an angle that was later endorsed by the teacher.

Since, the teacher dragged one of the line segments out in figure 3d, she further wanted to make sure that the children noticed the particular kind of movement that was needed to create an angle. Below is the description of one such attempt by the teacher.

No.	Who said/did	What was said/what was done
1426	T	Okay with these two lines here, I am going to close them up. Close them up (*figure 4a*). Okay now I am going to turn the green one. Turn, turn, turn, so I make an angle (*figure 4b*). Now I am going to turn this one, so I make an angle (*figure 4c*). Which one has the bigger angle? Joyce, which one has the bigger angle?
		Peter which one has the bigger angle?
1427	Peter	The green one (*pointing at the screen*)
1428	T	The green one has the bigger angle, because why?
1429	Peter	Because you stretched it out
1430	T	Because I stretched it out. How did I stretch it out? Did I stretch it out this way by making the lines longer (*dragging the points on green angles making the green lines longer, figure 4d*)?
1431	Sss	No.
1432	T	How did I stretch it, because that's going to be important, Joyce?
1433	Larry	You turned the dot
1434	T	I turned the dot (*gesturing the turn, figures 4e, f*), so that's the important part.

The teacher made both angles in front of the children by turning the lines and asked about the bigger angle. Peter responded that the green one was bigger. His utterance [1429], "Because you stretched it out" was a result of visualizing the teacher's actions of dragging the lines away from each other. The teacher further asked if she stretched it out to make the lines longer, to which most of the children responded with "no". Larry said, "You turned the dot" [1433]. The use of word "turned" by Larry shows that he has associated the angle with the turning. The usage of the verbs stretched vs. turned is quite interesting. The verb stretching out could be associated with stretching in two ways i.e. making the lines longer (as in figure 4d) or making the space between the lines wider (i.e. turning). In contrast, the verb turning only refers to the changing angle between the lines (i.e. making it wider). Thus the teacher's later questions were aimed at making sure that the children were not confusing the word "stretching out" with the increasing length of the lines. Thus, the teacher's discursive move aimed to make sure that all the interlocutors were using the word "stretched it out" in the same sense.

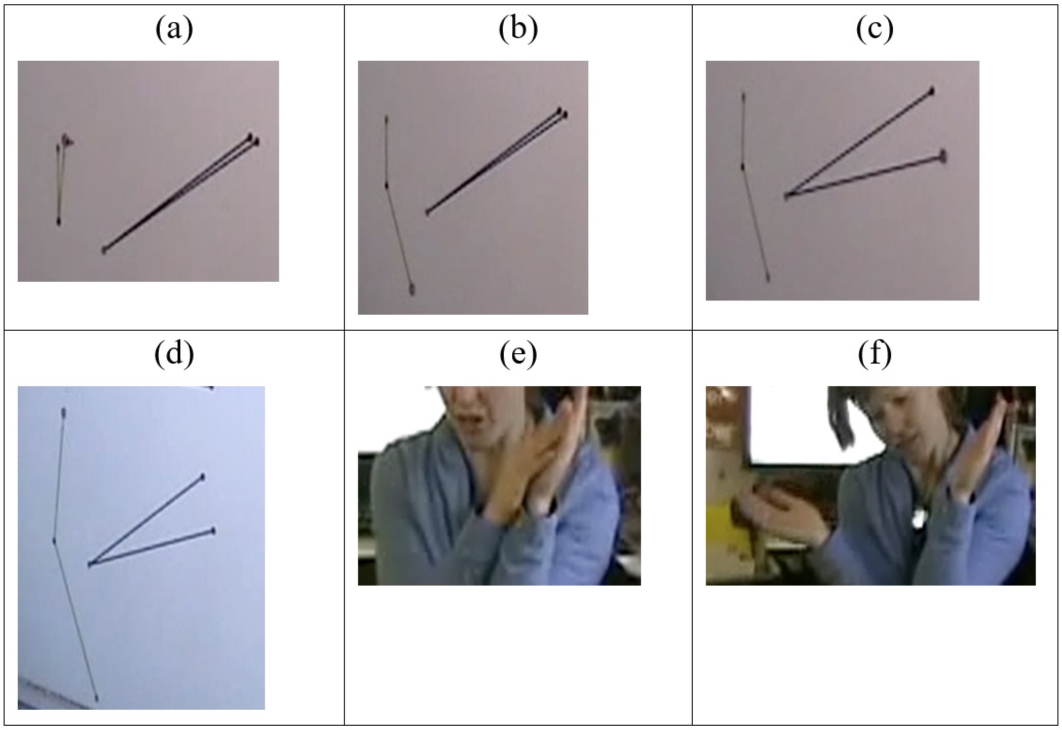


Figure 4: (a) Closing the lines of both angles together; (b) Making the first angle by turning lines out; (c) Making the second angle; (d) Stretching the green lines of first angle to make it longer; (e, f) the teacher's gesture for turning the dot

Thus, in the above episode, the teacher used the potential of the drag mode to help the children discover an important property of angles, i.e. change in the length of the arms does not change the size of angle. As an expert interlocutor, she made sure that all children were using the words stretched out in the same sense. Also, the teacher made use of different types of angles (not just prototypes with one horizontal line or with two arms of same lengths).

CONCLUSION

The discussion of above excerpts shows that DGE could be helpful in working through some of the common discursive conflicts about angles that have been reported in the research literature. For instance, the children were able to conceptualise turning in terms of angle and were able to connect static angles to turns. Larry's realization "angle is the amount of turn" emerged explicitly through the DGE-based instruction as he could visualise the amount of turn involved from one arm to another on the screen. The realization that angle measure is independent of the length of the lines is occasioned as the drag mode of DGE helped the children notice that the change in length of the arms of an angle does not change the size of the angle. Thus, the DGE approach was helpful in focusing children's attention on the quantity of turn rather than on the length of the line segments. This gives evidence that angle instruction in DGE could be helpful in avoiding discursive conflicts involved in thinking that the 'longer rays means greater angle' as reported in the literature (Stavy & Tirosh, 2000; Clausen-May, 2005). This study provides one way of establishing a relationship between angle-as-a-turn and angle-as-a-shape conceptions.

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WHAT MATHEMATICS TEACHERS SEEK WHEN APPROACHING PROFESSIONAL LEARNING THROUGH SOCIAL MEDIA

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Mathematics teachers are using social media to approach professional learning. This sort of activity is often carried out by calling on the digital community to respond to various questions related to the practice of teaching mathematics. This paper presents an analysis of a collection of such queries using a taxonomy of teacher wants. A revised taxonomy is emerged from the data that provides insight into the possible desires mathematics teachers have when using social media for professional learning. Results indicate that the social media space is used most prominently to seek pragmatic solutions for teaching, but that willingness to rethink more significant aspects of practice is not common. These results reveal the limitations of social media as a place for professional learning and inform professional development initiatives.

INTRODUCTION

Although teacher professional development is essential for enhancing the quality of teaching and learning in schools (Borko, 2004), there is growing concern about the sustainability and strength of common approaches to its implementation (Lerman & Zehetmeier, 2008). The robustness of a professional development initiative is dependent on ensuring both teachers and facilitators adopt a stance of inquiry, activities reflect and are driven by teacher needs and interests, and community building and networking are at the core (Lerman & Zehetmeier, 2008). However, initiatives are commonly limited to sparse one-time professional development workshops facilitated in face-to-face synchronous settings, which are not typically supportive of ongoing professional growth (Ball, 2002) and are generally driven by facilitator perceptions of what teachers need rather than by what teachers really want (Liljedahl, 2014).

In contrast to centrally organized, and sometimes compulsory, professional development initiatives, teachers from across North America are participating in decentralized, virtual, and autonomous professional communities. One such community involves mathematics teachers who regularly use Twitter and blog pages to asynchronously communicate their musings and practices, and have come to be identified as the Math Twitter Blogosphere (MTBoS) (Larsen, 2016). This unprompted, unfunded, and unevaluated teacher community is a rich phenomenon of interest that is largely unstudied and deserving of attention.

One frequent manner of interaction in the MTBoS is to post questions pertaining to mathematics teaching that are directed at the community. These questions have the potential to reveal desires of mathematics teachers irrespective of what a facilitator would want them to care about within a constrained professional development setting. As such, the study presented in this paper is driven by the question – what do mathematics teachers want when they approach professional learning via engagement in the MTBoS?

THEORETICAL FRAMEWORK

With an aim to identify the possible mathematics teachers desires with which they approach a professional development setting, the taxonomy of teacher wants developed by Liljedahl (2014) serves as an analytical tool to approach the questions that mathematics teachers pose to the MTBoS community. Liljedahl uses data from a variety of professional development settings, albeit not digital ones, to emerge categories of teacher desires when attending professional development. The scope of changes teachers are willing to make progressively widen throughout the categories, while the level of anxiety teachers hold towards making these changes diminishes.

The first of these categories is that of *resistance*, where the teacher does not want to be there, and their participation is directed by them not wanting to be there. They are in search of evidence, justification, and pragmatism, and are found to make comments such as ‘I already do that’ and ‘that will never work’. The next category is *do not disturb,* where teachers are there because they want to improve their practice, but their overly specific statements about what they seek indicate they yearn for “small self-contained strategies, lessons, activities, or resources that they can either use as a replacement of something they already do or cleanly insert into their teaching without affecting other aspects of their practice” (Liljedahl, 2014, p. 114). Examples of such statements include ‘I want something for the first 10 minutes of class’ or ‘I’d like a new way to introduce integers’. Liljedahl notes that teachers often act in this way either out of an anxiety about changing what they have already developed or because they genuinely believe small changes are a good way with which to approach change.

Liljedahl’s third category is that of *willing to reorganize*, where teachers are “willing to significantly reorganize their teaching and resources to accommodate the necessary changes” (p. 115). They may want to make specific improvements in their teaching, such as how they teach ‘problem solving’. Liljedahl notes that they are open to consequences related to the desired improvements and “are not hampered by anxiety around invalidating existing resources or undoing things learned” (p. 115). The fourth category is an extension of this and is referred to as *willing to rethink*. In this case, the desires for improvement are broader in scope and “often welcome a complete rethinking of significant portions of a teaching practice” (p. 115). This category also includes teachers wanting comprehensive guidance on how to implement new approaches to teaching they haven’t previously encountered. Examples include teachers who seek to learn about ‘approaches to numeracy’ or ‘ideas for differentiated assessment’. Liljedahl comments that teachers in this category “have a rough idea of what it is they want and are willing to rethink their teaching in order to accommodate new ideas" (p. 116). They also don't seem to hold anxieties towards changing things they have previously developed.

The last two categories include teachers who are willing to completely revamp their practice and show a more consistent and dedicated willingness to change. The first of these is *out with the old*, where teachers are explicitly seeking to reject significant components of their teaching practice and want to replace these with new ideas. They know their past ways did not work and they seek the facilitator's guidance on how to bridge the gap between theoretical ideas and the pragmatics of implementation. The last of the categories is that of *inquiry*, where teachers are seeking to develop their ideas about teaching in a reflective and reflexive manner. They do not seem to come with an agenda but are open to inquiry about practice.

Liljedahl (2014) further notes that the taxonomy can be conceived of as a pseudo-hierarchy in that although the categories seem to increase in terms of openness to change, it does not mean that teachers fall neatly into these categories all the time. In fact, teachers throughout the longitudinal study moved either progressively through the categories, became more open within one session, or regressed into more specific and pragmatic wants sporadically. Liljedahl adds that this movement may be related to the fact that as the facilitator, he often worked to 'upsell' teachers into higher categories of wants, which may not always be the case in other settings. However, Liljedahl emphasizes that teachers ultimately have the autonomy to take up new ideas or to implement change in classrooms.

The study presented in this paper considers how the taxonomy proposed by Liljedahl (2014) plays out within the MTBoS, a professional development setting housed on social media, where there is no facilitator and teachers have the ultimate autonomy. To this end, Liljedahl's taxonomy serves as an *a priori* analytical tool for the initial analysis of mathematics teachers' questions directed to the MTBoS community.

METHOD

The MTBoS is a vibrant community of mathematics teachers who connect via social media using Twitter and blogs. There are hundreds of posts made daily by users for various purposes, some of which include the #MTBoS hashtag and some of which don't. With over 500 self-identified MTBoS, many of whom post daily, and with newcomers joining continuously, the potential dataset is large to say the least. As such, a specific subset of posts was selected as the data for this paper based on the aim of exploring mathematics teachers' desires when approaching professional learning in this space. Namely, all posts made within an arbitrarily chosen two-week period between August 1, 2017 and August 15, 2017 that included a question mark symbol and the #MTBoS hashtag were identified. From this search, only posts pertaining to the practices of a mathematics teacher were selected, resulting in a set of 180 posts made mostly by mathematics teachers. Within these 180 posts, most users were unique, and user identities were not considered in the analysis other than that they were verified to be mathematics educators. These 180 posts form the data set for this study.

These data were then analyzed using a framework of analytic induction (Patton, 2002). "Analytic induction, in contrast to grounded theory, begins with an analyst's deduced propositions or theory-derived hypotheses and is a procedure for verifying theories and propositions based on qualitative data" (Taylor & Bogdan, 1984, p. 127 cited in Patton, 2002, p. 454). In this case, the *a priori* theory used to initially code the data was the taxonomy of wants developed by Liljedahl (2014) from teachers approaching a variety of in-person professional development settings. However, analytic induction allows for emerging new themes through recursive coding if they do not fit the initial theory.

Since this professional development setting explored in this paper is somewhat different in nature than that of what Liljedahl (2014) studied, some differences were expected to arise. Namely, upon a cursory look at the data, it was evident that the last two categories of the taxonomy could not be identified in the online space because there wasn't enough continuity of observation to determine if one had rejected past practices or if one was engaging in inquiry. As such, the data were initially coded according to the categories of *resistance, do not disturb, willing to reorganize,* and *willing to rethink*, but upon. Throughout the coding process, data that did not fit these initial categories were coded iteratively with more fitting categories, emerging new categories where necessary. Additionally, with the intention of understanding the content of the questions posed to the MTBoS, the data were also coded for topics that teachers sought, and these topics were categorized into themes.

RESULTS AND ANALYSIS

In what follows, I present the resulting taxonomy as it emerged from the data. This taxonomy reveals the various potential desires teachers have when approaching professional learning in a social media context. I also present a set of themes that indicates the topics teachers asked about within this dataset.

A taxonomy of teacher wants in the MTBoS

After the first pass of coding according to Liljedahl's (2014) taxonomy, it became evident that the *resistance* and *do not disturb* categories were more fine-grained within the social media setting than within the settings Liljedahl described. This is due to the autonomous nature of the social media setting, and that participant presence is not mandated or required. Nonetheless, the essence of these categories was still visible and was replaced with *seeking affirmation*, *seeking connection*, and *seeking pragmatism*. Further, a new category, referred to as *willing to entertain*, was created and came before the *willing to reorganize* and *willing to rethink* categories. In what follows, each of these six categories are outlined and exemplified.

Seeking affirmation

Self-affirmation is pervasively associated with social media use in general and is a way to explain why users spend so much time on social media sites. Social media can provide users with a sense of self-worth and improve one's self-efficacy as posts are 'liked' and valued within their network. In the context of this study, 30 out of the 180 posts were categorized as *seeking affirmation* because they seemed interested only in receiving affirmation for their ideas without any more interest in changing or developing them. In a way, these showed forms of resistance, but in a more positive way, and revealed that teachers want to be affirmed for what they are doing. Examples in this category included statements such as 'here's a Google form I use for students before they retake an assessment, any thoughts?', 'what do you think of my modified version of chutes and ladders?', or 'don't we fail to make the distinction about how many variables are in y = mx+b?' Although the statements don't directly come across as that of resisting change, upon follow-up, users either actively invalidated any alternative propositions to their own, or they touted what they already do and why they think it's appropriate. Although any social media post may be viewed as *seeking affirmation*, the posts categorized here solely sought this purpose.

Seeking connection

Some teachers actively pursued connection with others teaching the same topics or having the same experiences. There were 10 such cases in the data. Examples in this category included questions such as 'who is teaching calculus this year?', 'anyone have this sort of behavioural event happen in their classroom?', 'would anyone like to get the #Alg2Chat up and running this year?', or even more broadly, 'what do you like about teaching math?' While it is difficult to place these desires in terms of their willingness to change practice, it is evident they want to find people with whom they share a redundancy, which may be either for self-affirmation, for pragmatic resource exchanges, or for rethinking practice. While the purpose isn't entirely clear, teachers want to connect, and social media is one way to achieve this.

Seeking pragmatism

Seeking pragmatism includes posts that request very specific resources that can be cleanly inserted into existing practice, much like that of Liljedahl's (2014) *do not disturb* category. This category was renamed, however, because it was not possible to determine an anxiety towards changing existing practice, but instead, it was evident that these teachers sought very specific resources that made their lives easier without any further desire to rethink or question the purposes or rationale behind what they were using. The desire for pragmatic exchanges of resources was the most popular category in this dataset, with 61 cases. Examples in this category included questions such as 'I'm looking for resources that utilize technology for a unit on probability', 'I'm in search of a number line for my classroom, anyone have one they love?', 'anyone have a warmup sheet for doing Est 180, WYR, WODB, etc?', 'anyone have a pacing guide for Algebra I?', and 'what's the best thing for cleaning whiteboards?' It's not only that the requests were specific in nature, but that there was no further indication of an intention to change existing practice.

Willing to entertain

Willing to entertain includes statements that indicate a curiosity to learn about new ideas, but that have no indication of desire to implement these ideas or to change existing practice. The emergence of this category reveals that teachers can be curious about what is out there without necessarily committing to any sort of implementation. This was also a popular category with 40 such instances. Examples in this category included statements such as ‘I’m dreaming here – if you were given $ to spend on math tools for students, what would it be?’, ‘got meaningful high school algebra projects?’, ‘if you only had 10 minutes to do math with a group of high school students, what would you do?’, ‘pros and cons between GeoGebra and Desmos for high school geometry?’, ‘any ideas for math activities for the eclipse?’, or ‘ideas for good learning environments in secondary rooms?’ These ranged from more specific to less specific, and didn’t seem to have any implied agenda, but rather, a genuine curiosity to see what is out there in terms of the possibilities other teachers have thought of or tried.

Willing to reorganize

As an extension to the *willing to entertain* category, *willing to reorganize* included those searching for ways to change specific components of practice slightly broader in nature than that of the *seeking pragmatism* category, while indicating in some way the attempts they have already made. In short, Liljedahl’s (2014) definition of this category was maintained, and 15 of the 180 posts were categorized in this way. Examples included ‘anyone have a good set of rules, jobs, or norms for group work? I’d like to set the stage this year and don’t want to forget anything’, ‘I’m hoping to get some thoughts about standards-based grading in precalculus – want to start this year’, ‘I’m thinking about giving homework once per week on Monday due Friday, lagging from the week before – anyone do this or have a reaction?’, ‘any math teachers planning on bringing Charlottesville Curriculum into their classrooms this fall? How?’, and ‘anyone have information, resources, or advice for using the workshop model in high school math?’ In general, posts in this category revealed that the teachers can be willing to reorganize parts of their practice to accommodate new ideas, but often want specific ways to do this in order to be successful.

Willing to rethink

Some posts, however, did not have the flavour of specificity associated with most of the above categories. *Willing to rethink* includes broader questions about more significant components of practice and is compatible with Liljedahl’s (2014) definition. There were 23 posts of this kind, and included questions such as ‘please help me reflect on what we prioritize – what’s crucial for your students to take away?’, ‘what if my students could only spend time in class to learn everything they needed this year with no homework?’, ‘where can I find some good reading about vnps? I purchased boards and want to understand more deeply’, ‘what if a group isn’t working cooperatively – maybe a fish bowl strategy with debrief? Other ideas?’, and ‘how much time do you spend on classroom culture at the beginning of the year?’ Teachers writing posts in this category seemed to approach the MTBoS as a discussion space in which to hash out ideas pertaining to changing significant elements of practice and making the things that are typically non-negotiable, negotiable. Social media provides the space for this to happen but is not always utilized in this way.

Topics of interest in the MTBoS

In addition to determining the desires implicated by the queries made to the MTBoS community, the topics of these queries were also coded and iteratively grouped into the following five themes: *tools, class structure, curriculum,* and *class culture.* Identifying the topics that teachers approach the MTBoS with also informs the broad question of what teachers seek when approaching professional learning in the MTBoS. In what follows, these themes are briefly outlined.

Tools

A good portion of the queries (35/180) were requests for specific physical or technological tools used in mathematics teaching. Most of these fell under the *seeking pragmatism* category and pertained to investigating technological applications, virtual manipulatives, physical manipulatives, online homework systems, funding, and housekeeping tools such as whiteboard cleaners and label makers.

Class Structure

Another good portion of the queries (36/180) pertained to class structure and was relatively balanced between falling into the *seeking pragmatism*, *willing to reorganize*, and *willing to rethink* categories. These included questions about assessment, classroom organization, homework, and new models of teaching.

Curriculum

The majority of queries (71/180) were related to curriculum and thinking through the curriculum. Most of these fell into the categories of *seeking pragmatism* and *willing to entertain*, as well as *seeking affirmation*, and were associated with teachers who were *seeking connection*. Some questions were mathematical in nature, such as 'if 0.9999… = 1, then aren't x<1 and x<=1 equivalent?' Others were organizational in nature, such as 'does systems not directly follow graphing and solving linear equations?' And others were investigative in nature, such as 'I'm genuinely curious about what makes this student's misconception so powerful'. Curricular topics included mostly algebra and geometry, as well as social justice and problem solving.

Class Culture

Another significant category of query topics (38/180) was related to class culture. These were found across all categories of what teachers seek, and included many posts about setting classroom norms, about groupwork, and what to do on the first day. They also included questions about encouraging growth mindsets and student discourse.

Overall, it is interesting to note that most of the *willing to rethink* posts pertained to *class structure* and *class culture*, all of the *seeking connection* posts pertained to *curriculum*, and most of the *seeking pragmatism* posts pertained to *tools*. This shows that there is more to re-negotiate within *class structure* and *class culture*, that *curriculum* is a common source of redundancy that teachers can connect through, and that it is common for teachers to seek pragmatic advice for using various tools.

CONCLUSION

This paper presents a revised taxonomy of teacher wants in an asynchronous and public professional learning context unmandated by external bodies or facilitators. Namely, this taxonomy includes *seeking affirmation*, *seeking connection*, *seeking pragmatism*, *willing to entertain*, *willing to reorganize*, and *willing to rethink*. While there seems to be a hierarchy to this taxonomy, it should be considered as more of a pseudo-hierarchy, as in Liljedahl (2014), because the unit of analysis is a Twitter post unattributed to any individual participant. Therefore, it is more accurately seen as a *palette of possible desires* teachers may have when approaching professional learning.

Five topic themes also emerged from the data set: *tools*, *class structure*, *curriculum*, and *class culture*. The cross-analysis of these with the *palette of possible desires* reveals that mathematics teachers are interested in rethinking classroom structure and culture, and that there is space to negotiate these aspects of practice. However, this is not the most common way teachers engage in using social media for professional growth. Most prominently, mathematics teachers seek pragmatic resource exchanges pertaining to virtual or physical tools, ideas for class structure, and approaches to curriculum. They are often open to entertaining new ideas without necessarily planning to implement them, and they seek affirmation and connection with other like-minded teachers. Finding affirmation, connection, and new ideas to entertain may be a way to feel a sense of belonging, which is an important aspect of building a culture of inquiry.

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MATHEMATICAL LEARNING DISABILITIES AND INTERTEXTUALITY

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The study of mathematical learning disabilities (MLD), while relatively new, has received contributions from diverse fields such as psychology and education. The purpose of this paper is to analyze an early 20th century text on special abilities and disabilities to show how the discourses of psychology and education merge to constitute the burgeoning study of MLD and its participants. Norman Fairclough's version of critical discourse analysis will be applied, particularly his notions of intertextuality and discursive change. Findings suggest the sample text reflects the medico-scientific voice of psychological discourse, yet the book as a whole contains elements characteristic of educational discourse.

INTRODUCTION

While the history of mathematical learning disabilities (MLD) has been relatively short, it draws upon a diverse field of studies. Each field has contributed its own methods and perspectives to the study of MLD. One early contribution from the field of developmental psychology is Augusta F. Bronner's book *The Psychology of Special Abilities and Disabilities* (1917). This book contains early case studies of delinquent youth with defects in specific mental processes affecting the ability to do work in the school subjects including arithmetic. While this book is aimed primarily at other psychologists interested in delinquent or disabled youth, it is of interest to educators as it deals with the learning of school subjects and the treatment of students with specific learning disabilities.

This paper explores the ways in which the fields of psychology and education and their differing discourses come together within Bronner's book, first by a close analysis of a specific extract and then relating that analysis to the entire book as a whole. This analysis is significant as it shows the relationship between different disciplines as they compete over a similar discursive space and how the participants are interpellated by such a struggle. Fairclough (1992) uses the term "intertextuality" to describe this combining of text: "Intertextuality is basically the property texts have of being full snatches of other texts, which may be explicitly demarcated or merged in, and which the text may assimilate, contradict, ironically echo, and so forth" (p. 84). The multidisciplinary field of MLD did not appear ready-made, but was the result of a "negotiation" of sometimes contradictory perspectives and methodologies between disciplines. This paper attempts to shed some light on this negotiation through an analysis of discourses. There are three research questions: How can one describe the discourse style used in Bronner's text? How are the participants in Bronner's text (e.g.

psychologists, the youth with specific arithmetic disabilities, educators) constituted? In what ways does Bronner's text mix different discourse types and what are the larger ideological implications of such a mix for mathematics learning and MLD?

THEORETICAL AND METHODOLOGICAL FRAMEWORK

Fairclough's (1992) CDA is a useful framework for analyzing discourse as social practice and, in particular, the discursive practices used in texts. The main objective of Fairclough's approach to language analysis is to study social and cultural change. Shifts in language use play a central role in the understanding of changes in social phenomena. Fairclough's CDA synthesizes two different senses of discourse—the social-theoretical sense (such as Foucault's) and the "text-and-interaction" sense—and forms a three-dimensional model in the following way:

> Any discursive "event" (i.e. any instance of discourse) is seen as being simultaneously a piece of text, an instance of discursive practice, and an instance of social practice. The "text" dimension attends to language analysis of texts. The "discursive practice" dimension, like "interaction" in the "text-and-interaction" view of discourse, specifies the nature of the processes of text production and interpretation, for example which types of discourse (including "discourses" in the more social-theoretical sense) are drawn upon and how they are combined. The "social practice" dimension attends to issues of concern in social analysis such as the institutional and organizational circumstances of the discursive event and how that shapes the nature of the discursive practice, and the constitutive/constructive effects of discourse referred to above. (p. 4)

This synthesis of the socially and linguistically oriented views of discourse is what Fairclough calls a "social theory of discourse." His multi-dimensional approach emphasizes the importance of text and language analysis such as systemic functional linguistics in discourse analysis and has developed an explicit and operational approach for researchers to analyze discourses. There are two key focal points of any analysis in Fairclough's CDA: the order of discourse (the totality of discursive practices of an institution, and the relationships between them) and the communicative event (an instance of language use). Due to the small scope of this paper, the focus will primarily be on the communicative event (the text), and Fairclough's three-dimensional model of discourse (as text, discursive practice, and social practice) will be applied to it.

The analysis of these three dimensions for the chosen excerpt will be conducted separately (although it is not necessary to do so). The analysis of discourse as text will involve a focus on the linguistic features of the text (linguistic analysis). This will involve an analysis of vocabulary, syntax, and the grammar of sentences. The analysis of discourse as discursive practice concerns how authors draw on existing discourses and genres to create a new text, that is, through intertextuality. Fairclough's (1992) concept of intertextuality is key to understanding his notion of discursive change:

> The concept of intertextuality sees texts historically as transforming the past—existing conventions and prior texts—into the present. This may happen in relatively conventional

> and normative ways: discourse types tend to turn particular ways of drawing upon conventions and texts into routines, and to naturalize them. However, this may happen creatively, with new configurations of elements of orders of discourse, and new modes of manifest intertextuality [where specific other texts are overtly drawn upon]. (p. 85)

So a key question to ask is: Are discourse types (genres and discourses) used conventionally or creatively? Conventional discourse practice involves a normative use of discourse types and helps to reproduce the relationships in the order of discourse. On the other hand, creative discourse practice often mixes together a number of genres and discourses and helps to restructure the boundaries of the order of discourse. The last dimension to be considered when analyzing the sample is discourse as social practice: What is the relation between this discourse practice and the larger social structure to which it belongs (is it conventional and normative or creative and innovative)? Does it transform or reproduce existing social practices?

TEXTUAL ANALYSIS

The following text I have selected for analysis is from the book *The Psychology of Special Abilities and Disabilities* (as mentioned above) by Augusta F. Bronner, Ph.D. and published in 1917. Bronner lived from 1881 to 1966 and was the assistant director at the Juvenile Psychopathic Institute in Chicago at the time of publication. From her biography in *The Encyclopedia of Clinical Psychology*, Woodward and Hills (2015) state that Bronner "recommended that psychologists focus on the backgrounds of youth offenders rather than on their heredity. She emphasized nurture over nature, including home environment, education, companionship, recreation, and interests" (p. 1). For her dissertation, she worked under famed educational psychologist Edward L. Thorndike, the author of *The Psychology of Arithmetic* (1922) and co-author of *The Psychology of Algebra* (1923). Bronner published many works with William Healy whom she credits in the preface to her book and with whom she would later marry. With Healy, she emphasized how each individual should be treated as a special case: "This idea also guided their attempt to keep children in their homes, in a private school, or, failing that, in a foster home; institutionalization was the last resort" (Woodward & Hills, 2015, pp. 1-2). It is also of interest to note that Healy's book *The Individual Delinquent* (1915) contains one of the earliest accounts of someone with a specific disability in arithmetic.

Bronner's book *The Psychology of Special Abilities and Disabilities* argues in part that while much attention by educational psychologists has been paid to those with general low intelligence (e.g. the feeble-minded) little has been written about those individuals with specific disabilities (who otherwise exhibit normal or above normal general intelligence) and individuals with particular abilities (who exhibit below normal in general capacities). Her book attempts to fill this gap in the psychological literature and is based on numerous case studies conducted on delinquent youth. My focus in this paper will be on the sections of her book that focus on individuals with specific defects in arithmetic and the excerpt I have selected for close analysis is from Chapter V:

Special Defects in Number Work. I chose this specific paragraph because it describes the process that one goes through while diagnosing an individual with a specific defect in arithmetic ability. The paragraph also captures a particular voice that is prevalent in large proportions of the book and which I analyze below. The numbering of sentences in the excerpt is my addition as I refer to them frequently during the analysis.

> If now, in our study of individual problem cases, we find a child who is greatly retarded in number work, who seems to be incapable of normal advancement in this subject, it becomes necessary to make such an intensive investigation by means of psychological tests that we shall be able, if possible, to determine wherein the difficulty lies [1]. If we know the psychological processes involved in the learning of arithmetic, we ought to test these various mental functions in the individual in order to find which are normal and which are not [2]. Since memory for number and the ability to form arbitrary associations are elements in the learning of number work on the mechanical side, we must find whether these processes function normally in each individual case [3]. Furthermore, the more fundamental problems must likewise be answered, namely, whether the child has any concept of number and, if so, whether he has been able to form those abstractions which are necessary in the performance of the usual school tasks [4]. Perhaps he is able to solve problems when using concrete material and yet not able to perform correctly abstract work [5]. (Bronner, 1917, p. 55)

As an entryway into this paragraph, I will begin by analyzing aspects of text cohesion. In particular, I begin to look at the ways in which the clauses of the text relate to one another (connectives): "This will provide a way into looking at the sort of argumentation that is used, and the sort of standards of rationality [the sample] presupposes" (Fairclough, 1992, p. 171). Note the first three sentences of the paragraph have similar sentence structures. Schematically, the sentences can be represented as: If now/If/Since [clause 1], it becomes necessary/we ought/we must [clause 2]. This structure gives the paragraph a sense of procedure and obligation much like a doctor diagnosing and treating a patient. Even a portion of the fourth sentence can be rewritten to fit the pattern: "[If] the child has any concept of number [we ought to test] whether he has been able to form those abstractions which are necessary in the performance of the usual school tasks." The type of rationalization and argumentation in the paragraph seems typical of the medico-scientific voice of a psychologist and the use of the modals "necessary," "ought," and "must," add to this sense of authority of purpose.

Indeed, who is (are) the one(s) with the authoritative voice and who are the other participants referred to in this text? The use of "our study" and the pronoun "we" in numerous instances ("we find," "we shall," "we know," "we ought," and "we must") suggest not only the author (Bronner) but also the readers of the text. The "we" likely refers to other psychologists who study and test "individual problem cases" or researchers interested in the study of children with disabilities. Teachers, parents, other educators and administrators would also find this book of interest and may form part of the audience as it deals with the "learning of number work" and "the performance of the usual school tasks." But the "we" includes this group in a more generic sense of the word since they are not usually directly involved with the psychological testing of the mental processes of children but refer them to clinical psychologists if they detect problems in a child. Thus the "we" is ambivalent in the sense that, on the surface, it seems to include all readers, but the mention of specialized vocabulary and processes suggests a more exclusive group. In other words, the argumentation of the paragraph suggests the audience is those people with "insider" knowledge involved with the diagnosis and treatment of children with mental deficits.

The other participants mentioned in this piece of discursive practice are the "individual problem cases" or "child who is greatly retarded in number work." This individual or child is referred to with the pronoun "he" in the last two sentences. Since the cases described in the succeeding pages following this paragraph involve both genders, "he" is used generically and includes both genders, a very typical usage at the time. Regardless, the child with disabilities is constituted quite negatively in this paragraph. He or she is described as a "problem," "greatly retarded," "incapable," having "difficulty," possessing mental functions that are not normal, and not capable of performing certain tasks correctly. The transitivity of the sentences also suggests that the child is rarely the agent and remains a passive participant. It is the psychologist who is doing the finding, testing and investigating. It is the mental functions and processes of the child that is being probed but the child him or herself is not acting. The use of nominalizations such as "psychological processes involved in the learning of arithmetic" and "memory for number and the ability to form arbitrary associations" reinforce this as these nominalizations hide the agent, that is, the child who is doing the learning and the memorization.

The last two sentences, however, do describe the child as an agent of mental processes: "the child has any concept of number," "he has been able to form those abstractions," and "he is able to solve problems when using concrete materials." But these are merely hypothetical cognitive conditions of the child to which the psychologist must make further investigations and the child is not acting out of his or her own volition. Indeed, the fourth sentence can be (admittedly somewhat crudely) rewritten using nominalizations in order to hide the agent completely: "Furthermore, the more fundamental problems must likewise be answered, namely, whether there is any conception of number and, if so, the ability to form those abstractions which are necessary in the performance of the usual school tasks." Fairclough (1992) points out that "medical and other scientific and technical language favours nominalization, but it can be abstract, threatening and mystifying for 'lay' people" (p. 179).

There are other features of this paragraph worth pointing out that reinforce what has already been stated above. Firstly, the theme is the initial part of the clause and "can give insight into assumptions and strategies which may at no point be made explicit" (Fairclough, 1992, p. 184). The first four sentences begin with "if," "if," "since," and "furthermore." These logical connectives add to the rational argumentation being presented. Moreover, "we" is the theme of secondary clauses in the first three sentences, emphasizing the psychologists as the primary actors. Secondly, the use of modal words and phrases such as “seems,” “if possible,” and “perhaps” do not suggest a lack of affinity for the proposition to follow, but “is arguably for rhetorical reasons, motivated by the projection of an approved cautious and circumspect subjectivity and ethos for ‘the scholar’” (Fairclough, 1992, p. 162). Finally, the use of specialized words and phrases typical of discourse used by psychologists (such as “individual case,” “psychological tests,” “psychological processes,” “mental functions,” “normal,” “normally,” and “arbitrary associations”) reiterate the point that this text is not intended for the “lay” person.

TEXT AS DISCURSIVE AND SOCIAL PRACTICE

Analysis of other parts of the book suggests a consistency with the scholarly and technical style noted in the above extract. For instance, in Chapter I: The Problem, after quoting Thorndike on the notion of individuality (that there exists no “typical mind”), Bronner (1917) states:

> Experimental studies of different mental processes have led to the conclusion that, in all of their abilities, the majority of individuals cluster about an average; the greater the divergence from the average, the smaller the number of individuals found. One practical corollary of this general truth is, that while most people can adjust themselves satisfactorily to ordinary situations, there are some so far removed from the average that they are ill-adjusted under these same circumstances. (p. 2)

In this quotation, Bronner is referring to norm referencing and the bell curve. Standardized testing used in psychological diagnoses was widely used during Bronner’s time and she was a strong advocate of it. But the psychological voice of the passage changes when we consider the rest of the paragraph that follows:

> It is to these persons, numerically in the minority, yet forming a class socially very significant, that injustice is done in the present state of affairs. It is they who are often misunderstood, neglected, allowed to remain with their best possibilities undeveloped. It is for them, the individuals with particular abilities and disabilities, we would bespeak intelligent consideration. Among educators the most discerning thinkers have recognized this group as one meriting special consideration. (p. 2)

The voice used in this quotation is not that of a neutral scientist but that of a strong advocate for individuals whose needs have not been properly met and for individuals with particular abilities and disabilities who have not been educated in a way that takes into consideration their specific characteristics. Notice how the rhetorical force of her remarks is made through the repetition of the words “It is” at the beginning of the first three sentences. The sentences are written-as-if-spoken. She also uses strong words to describe those on opposite ends of the bell curve who have been treated unjustly: they are “misunderstood,” “neglected,” with their “best possibilities undeveloped.” She also appeals to ethos in the sentence: “Among educators the most *discerning thinkers* have recognized this group as one meriting special consideration.” So this passage contrasts with the earlier passages based in reason and procedure by its appeal to ethics and social justice. In the former, the child is the passive object to the psychologists’ investigations, whereas, in the latter, the child with special dis/abilities, although remaining passive, is nevertheless championed by the author as victims of educational neglect. There are other instances throughout the book where Bronner critiques a school system that ignores the special dis/abilities of students and continues to put square pegs into round holes as the saying goes. She even argues that much child delinquency could have been avoided if the specific needs of students were attended to, as the lack of success in school often leads to truancy.

But all of Bronner's assertions regarding education are based on the clinical evidence she acquires through mental testing. This book creatively combines information gleaned from tests such as the Binet-Simon Test for intelligence, so-called Puzzle Boxes, Construction Tests, Cross Line Tests and Knox tests and others (all of these tests are detailed in Chapter II of the book titled Methods of Diagnosis) with interviews to create a coherent case study. Consider the case of Mary L. 10 years, 9 months (Case 9). These are the results of her second testing conducted eight months after her first set of testing. All these "Records of Psychological Examination" are subordinated to an Appendix but the detailed case studies themselves are integrated into the text itself in the relevant chapters.

> **Binet grade**: through 10 years and 3 of the 12-year tests. **Construction Test II**: 2' 33", 17 moves. **Cross Line Test I**: correct first trial…. **Auditory Memory Span**: 5 numerals correct. **School Work**: Arithmetic (written): adds four 3-place numerals correctly. Cannot subtract, for example, says 50 – 42 = 10; says 7 – 7 = 7. (Oral) Some combinations of multiplication table correct and others' failures, *e.g.*, 4 X 3 = 12; 4 X 8 = 32, but 4 X 6 = 22. Fails to give correct answer to, 25¢ – 8¢ = ?; 25¢ – 4¢ = ?; says 10¢ – 6¢ = 4¢. With actual change cannot add 50¢ and 25¢, nor solve 50¢ – 42¢. Given a quarter and asked to return the change after 18¢ is spent, shows much difficulty in mental representation of the problem. Counts the 18¢ in change and then tries to find what is needed to make up the 25¢, but since it cannot be done with the change before her, fails to solve the problem. (Bronner, 1917, pp. 238-9)

In Chapter V: Special Defects in Number Work, Bronner details seven case studies based on her various assessments including Mary's. Consider the following extract from her case study:

> She could not succeed with so simple a sum as subtracting eighteen from twenty-five; it made no difference whether this was given her orally, as a written problem, or with actual money. She said that 'taking seven from seven leaves seven.' It was evident that the common sense which she used in other situations had never been called into play in number work, otherwise she would surely not have made so stupid a remark…. Mary was easily able to do much higher work in other school studies and in consequence she was wasting much of her time in school. (pp. 66-67).

This example of manifest intertextuality has Bronner taking results from a psychological exam and making strong educational claims about Mary. While this type of information is no doubt useful (e.g. whether a child is an auditory or visual learner, whether the child has any concept of number), there is also the danger of reducing mathematics to a set of finite mental processes without considering the wide variety of aspects involved with mathematics beyond merely arithmetic. Is the student better at geometry or algebra? What about word problems? Even with the knowledge provided by psychologists, it can sometimes be unclear or a challenge for how a teacher can use this information in the classroom to help a student. There still remains a separation between what goes on in the actual classroom environment and the clinical testing conducted by people such as Bronner. Nevertheless, this is a typical example of the mixing of genres of the psychological/medical testing evidence and the resulting educational recommendations/critiques through "intertextual chains" (Fairclough, 1992).

CONCLUSION

This paper has shown how the techniques of psychologists can enter into the field of education in innovative, and sometimes natural ways through the notion of intertextuality. This process has implications for mathematics learning and also for the construction of social identities such as those of psychologists, educators and the children labeled with special disabilities in arithmetic. This paper, however, has yet to address what the implications are for the larger order of discourse, or how does ideology fit in with this analysis. Fairclough's (1992) theory of discourse ties into a larger social theory that involves hegemony: "A conception of hegemonic struggle in terms of the articulation, disarticulation and rearticulation of elements is in harmony with what [Fairclough] said earlier about discourse: the dialectical view of the relationship between discursive structures and events" (p. 93). One way of thinking about the "struggle" between psychological and educational discourses within the field of MLD is through the notion of the "fragmentation" of discursive norms and conventions making local orders of discourse more permeable (Fairclough, 1992). This paper will benefit from the analyses of other discursive events from different points in history and with differing genres to further explore this "struggle."

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STUDENTS' USE OF EXTRA-MATHEMATICAL KNOWLEDGE DURING THEIR PROBLEM-SOLVING PROCESS

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When students work on mathematical tasks situated in reality, they need to connect their intra-mathematical skills with their extra-mathematical knowledge (EMK) to produce a realistic and reasonable solution. As such, EMK plays a crucial role in the quality of the generated solutions of such tasks. This study finds that the quality of students' generated solutions depends not only on students' existing EMK, but also on their abilities to identify, acquire, and apply these EMK.

INTRODUCTION

In a 1992 article, Dave Hewitt discusses the concept of "train spotting" – to solve problems by looking for patterns and paying attention to the numbers involved in the problem, but shy away from the richness of the problem. This results in students learning something mathematical about the patterns in the original situation but lose sight of the original situation. Hewitt suggests that what students seem to lack is not a fluency in their mathematical knowledge but a connection between their mathematical solution and the original situation, and the ability to recognize and use their knowledge related to the original situation to solve everyday problems. This disconnection between their solutions and the original problem also contributes to students' experience of the gap between reality and the world of mathematics.

A widely cited example that has been used to demonstrate this disconnection between students' mathematical solution and the original situation is the bus problem (Carpenter et al., 1983):

> An army bus holds 36 soldiers. If 1128 soldiers are being bussed to their training site, how many buses are needed?

Results of the study indicate that roughly 70% of students correctly carried out the division that is required to produce a mathematical solution. However, more than half of these students gave the answer of 31 buses or 31 buses remainder 12, and less than a quarter of these students gave the answer of 32 buses. The results of this study, among others (Greer, 1997; Verschaffel et al., 1994), show that while students often are capable of producing a mathematical solution, they tend not to interpret their mathematical solution from a real-world perspective. More importantly, these studies point out that the production of a realistic and reasonable solution rests on both students' intra-mathematical skills (Blum & Borromeo Ferri, 2009) and their extra-mathematical knowledge (Borromeo Ferri, 2006), or EMK, the ability to consider the situation and to validate the solution from a real-world perspective.

EMK is a term Borromeo Ferri (2006) uses in her modelling cycle to describe any knowledge or experiences that originate from outside of modelers' mathematical experiences. The modelling cycle begins with a real situation in reality. As modelers interpret the real situation, they build a *mental representation of the situation* (MRS), make assumptions, and draw on their EMK to build a real model. Next, modelers apply EMK to mathematize the real model into a mathematical one, utilize their mathematical skills to produce mathematical results, use their EMK once again to interpret these results in terms of the context of the problem, and validate these results by comparing them to the original situation. The modelling process concludes if modelers decide the real results are acceptable. If not, modelers re-enter the modelling cycle and make adjustments and modifications to their work.

EMK is used throughout Borromeo Ferri's (2006) modelling cycle and plays a crucial role in connecting problems situated in reality and the mathematics used to solve such problems. EMK may have little or no clear connections to mathematics but enables modelers to consider problem situations from a real-world perspective (Blum & Borromeo Ferri, 2009; Borromeo Ferri, 2006). Without the application of EMK, modelers would likely produce solutions that are purely mathematical and embody what Maaβ (2006) refers to reality-distant modelers.

This study intends to investigate a situation in which students have the required intra-mathematical skills but may not process sufficient EMK to solve a problem situation. In particular, this study looks at students' behaviour during their process of solving a mathematical task situated in reality, including the actions and strategies students use to deal with their lack of EMK, and the actions and strategies which contribute to students' success or failure in producing a realistic and reasonable solution to the task.

PARTICIPANTS AND METHODS

The students in this study worked on a genre of tasks called *Numeracy Tasks*. These tasks have been designed specifically to meet the numeracy goals of the local curriculum and are crafted around "an aggregate of skills, knowledge, beliefs, dispositions, habits of mind, communication abilities, and problem solving skills that people need in order to engage effectively in quantitative situations arising in life and work" (Steen, 2001, p. 7). The particular task that the students were asked to work on is called the *Design the New School* task (see Figure 1). Although this can broadly be considered a planning task (Liljedahl, 2010), it is, more specifically, a spatial planning task.

Data for the research presented here was collected in a grade 8 (age 12-13, n=13) and a grade 9 (age 13-14, n=26) class in a high school in western Canada while students worked on a numeracy task. Although it is not possible to know if the grade 8's had seen numeracy tasks in their previous years, this was their first numeracy task in the grade 8 school year. Conversely, the grade 9's had experiences with numeracy tasks in grade 8.

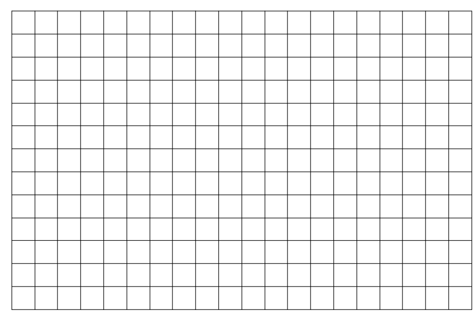
Students worked on the *Design the New School* task in randomly assigned groups of 2-4 during a 75-minute class. There were no instructions provided other than what can be seen in Figure 1. While the students worked, the teacher (the author) circulated naturally through the room and engaged in conversations with the students – sometimes prompted by her and sometimes prompted by the students. These conversations were audio recorded and transcribed. At the same time, photographs of students' work were taken, and students' finished work was collected. These, coupled with field notes summarizing the interactions as well as observed student activities, are used to build cases for each group of students. Each case is a narrative of students' task experience punctuated by significant moments of activity and emotive expression. These cases constitute the data.

Given that natural and unscripted nature of the teacher movement through the room, not all of the cases are equally well-documented. Regardless, each of these cases were analyzed separately through the lens of modelling using Borromeo Ferri's (2006) modelling cycle, with a focus on the EMK involved during students' solution process.

DESIGNING THE NEW SCHOOL

Your city is getting a new 11000m^2 middle school. It is going to be built on a lot (200m×130m) just outside of town. Besides the school, there will also be an all-weather soccer field (100m×75m), two tennis courts (each 15m×27.5m), and a 30 car parking lot on the grounds. The following requirements must be met:

- all fields, courts, buildings, and parking lots must be no closer than 12.5m to any of the property lines.
- any leftover property will be used as green space – grass, trees, shrubs.
- good use of green space is an important part of making the school grounds attractive.



To help you with your design and layout you have been provided with a scaled map of the property (every square is 10m×10m). Present your final design on a copy of this map. Label all structures and shade the green space.

Figure 1: Designing the New School task

In this paper, I focus on students' work on the parking lot and the EMK involved in their parking lot designs. At the time of the study, none of the participating students had reached the legal driving age, but all participating students have been driven around town by family members and others. As such, it could be assumed that they had no driving experiences, but experiences as a passenger in a vehicle.

DATA AND ANALYSIS

In this section, I first briefly describe some of the students' work on the parking lot, including their design process and their submitted work. The work presented here represents the full range of parking lot designs participating students submitted.

Group B: Becky and Bianca first attempted to generate a solution by multiplying 2, their estimated measurement of a vehicle, by 30, and stated the result, 60, as a solution to the problem. As they discussed their difficulties with their teacher, the teacher guided Becky and Bianca to create a layout of the parking lot and to determine the dimensions of each parking space and the driveway. Unfortunately, Becky and Bianca were not able to develop a deep understanding of what was being discussed and ended up using the lines on the grid provided as guidelines to develop their parking lot design. They created a parking lot that has 3 rows of 10 parking spaces (5m by 10m each) per row, connected by two 10m wide driveways (see Figure 2a).

Group G: Gabby, Gloria, and Gwen related the task with underground parking lots found at local shopping malls and decided to locate theirs in the basement of their school building and used the lines on the grid provided to generate a floorplan. They then discarded their work and assumed their parking lot to be 90m by 70m (same as their school building). As they discussed their work with their teacher, they realized that they overestimated the area required, and re-designed their parking lot layout with the teacher's help. During the process, the teacher guided the group through the process of making estimations and assumptions about the dimensions of parking spaces and driveways. Afterwards, the group adjusted the assumed dimensions and eventually created a parking lot that has 2 rows of 8 parking spaces (4m by 5.5m each) per row on either side and 2 rows of 7 parking spaces per row that run along the centre. These 4 rows of parking spaces are connected by two 5.5m wide driveways (see Figure 2b).

Group A: Amy and Angela recognized they had insufficient EMK to solve the problem and measured the width and the length of a randomly chosen vehicle parked in the staff parking lot (1.5m by 2.5m). They used this information to create a design of their parking lot: 1 row of 30 parking spaces (2m by 4m each) connected by a 1m wide driveway. They demanded and received minimal help from their teacher (see Figure 2c).

Group F: Fergus, Fiona, and Florence realized quickly they had insufficient EMK to solve the problem and measured the length and the width of a parking space and later on the width of the driveway at the staff parking lot. They constructed a parking lot using the information they gathered. Also, the layout of their design mirrors that of the staff parking lot: 2 rows of 15 parking spaces (2.51m by 4.92m each) per row on either side connected by a 5.75m wide driveway that runs along the centre. Similar to group A, this group demanded and received minimal help from their teacher (see Figure 2d).

Figure 2a: Becky and Bianca's parking lot design (Group B).

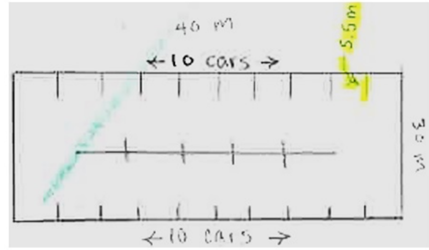


Figure 2b: Gabby, Gloria, and Gwen's parking lot design (Group G).

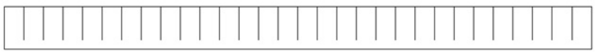


Figure 2c: A representation of Amy and Angela's parking lot design (Group A). Amy and Angela verbally discussed their parking lot in detail but drew only an outline of their parking lot in their submitted solution.

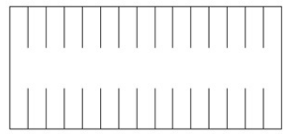


Figure 2d: A representation of Fergus, Fiona, and Florence's parking lot design (Group F). The group drew an outline of their parking lot in their submitted work and separately drew a few parking spaces to illustrate their calculations.

Figure 2: Parking lot designs from students in groups B, G, A, and F

In comparing the four groups of students, group B (Becky and Bianca) had the least understanding of parking lots (see Figure 2a). They failed to see any realistic aspect of the parking lot in their first attempt to solve the problem. Their unfamiliarity with parking lots led them to focus heavily on the mathematics, during which they extracted numerical values from the question, made measurement estimations without reflections, and manipulated these values to arrive at a solution. With help from the teacher, they eventually created a layout of the parking lot. However, Becky and Bianca were unable to develop the necessary understanding to build a reasonable design. Simply put, Becky and Bianca did not know what EMK was required and therefore were not able to determine and acquire the relevant EMK to build their solution. Furthermore, they were only able to apply very little of what was discussed with their teacher to their work. As such, while the design layout of their parking lot may seem realistic, the dimensions of their design were not reasonable.

Group G's (Gabby, Gloria, and Gwen) relation of the parking lot in the problem to underground parking lots commonly found at the shopping malls in town shows a form of their EMK (see Figure 2b). Unfortunately, they were limited by their existing EMK and were not able to extend their EMK to determine the dimensions of their parking lot until they discussed their work with the teacher. In the process, they did not acquire additional EMK independently. Rather, they relied on the teacher to provide them with the necessary help to develop a solution. It could be said that Gabby, Gloria, and Gwen were not certain what EMK was required to solve the problem but were able to use what was given to them to develop a solution.

In comparison to groups B and G, groups A and F approached the problem from a relatively real-world perspective. Both groups recognized their deficiencies and sought additional EMK on their own. Group A (Amy and Angela) applied their EMK and recognized that the dimensions of a parked vehicle, therefore a parking space, played a crucial role in their design, but did not recognize the importance of the width of the driveway (see Figure 2c). It is unlikely that they ignored the width of the driveway because they did not reflect on the problem situation. Rather, they lacked EMK and did not understand the importance of the driveway's width in their design.

Finally, group F (Fergus, Fiona, and Florence) recognized their deficiencies AND acquired additional EMK to help them develop a realistic solution independently (see Figure 2d). They used the staff parking lot as their guideline for both measurements and layout and developed the most realistic and reasonable design amongst the participating students.

As I look closely at students' submitted solution, I notice that their success in producing a realistic and reasonable solution lies beyond their intra-mathematical skills (Blum & Borromeo Ferri, 2009) and their existing EMK (Borromeo Ferri, 2006). Rather, their success is also dependent on: their ability to identify what EMK is required to solve the task, their ability to acquire such EMK, and their ability to apply their existing and/or acquired EMK to solve the task. The following table summarizes students' abilities to identify, to locate, and to apply EMK.

	Group B	Group G	Group A	Group F
Identification of the EMK required	N	N	Y[1]	Y
Application of existing EMK	N	Y	Y	Y
Independently acquire the EMK	N	N	Y	Y
Application of acquired EMK	Y[2]	Y	Y	Y

Table 1: Students' abilities to identify, acquire, and apply EMK to build a realistic solution

The table above points to the intimate relationship between students' abilities surrounding EMK and the quality of their submitted work: the higher students' abilities in these areas, the more realistic their solution. Table 1 also points to a common phenomenon: students will not always have all these aforementioned abilities. For example, both groups B and G were unable to independently identify and acquire the necessary EMK to solve the problem. However, this does not mean that

[1] Amy and Angela recognized they needed to measure the dimensions of a parked vehicle and extended their measurements to the dimensions of a parking space. They did not recognize the need to measure the width of a driveway.

[2] Becky and Bianca created the layout of their parking lot after receiving help from their teacher. However, their creation was disproportional despite the help they received.

these students failed to produce a solution or to approach the problem from some degree of real world perspective. In such cases, students applied strategies to cope with their limited EMK. Data show that students made assumptions about the situation and used the tools available to develop a possible solution: groups B and G designed a parking lot using the lines on the grid provided as markers for parking spaces and driveways; group G assumed at some point the parking lot had the same dimensions as their school building; group A assumed that a 1m wide driveway is sufficient; and group F's parking lot design mirrors that of the staff parking lot.

The assumptions students made and the approach they took were not necessarily the results of their interpretation of the question from only a mathematical perspective, nor the result of their lack of reflection on the situation. Rather, these assumptions and approaches were likely the result of their insufficient EMK – they did not have all the knowledge required and they did not know what knowledge was required to build a realistic solution. This further demonstrates the crucial role of EMK in solving tasks that are situated in reality.

However, this is not to say that students would be unable to produce realistic solutions if they had initial insufficient EMK. The data presented here demonstrate some students' ability to first identify the EMK they needed and then acquire it: group A first identified a parked vehicle and then later on a parking space as the determining factor of the dimensions they needed; group F first identified a parking space and then the driveway as crucial factors in their design. Both groups A and F identified the staff parking as a readily available source of information.

While students' abilities to acquire and to apply EMK play a crucial role in their solution, the quality of their solution still hinges predominately on their existing EMK. As all participating students were under the legal driving age, it would be fair to claim that they did not have very deep understanding of or experiences with parking lots. As such, their parking lot designs focused only on a few specific features such as parking spaces and driveways and neglected other features such as the entrance/exit and a driveway from the parking lot to the outside of the school grounds. While it is entirely possible to identify these features, their lack of experiences might have hindered them from forming such deep connections with the problem situation.

Finally, all submitted designs are very generic and represent typical parking lots found in the city. This again, is probably a result of students' EMK. Students' EMK is dependent on their lived experiences. As such, students' EMK varies as their lived experiences do. For example, recent development in technology has produced parking garage elevators and carousels. These designs minimize the area required for parked vehicles by vertically stacking these parked vehicles and replace the driveway with a mechanical system that retrieves the vehicles for the drivers. These designs are common in regions in the world where space is limited but are not seen locally. I suspect that if the *Design the New School* task is implemented in other regions around the world, such parking garage elevators and carousels may be common in students' submitted work.

CONCLUSION

Data demonstrate the crucial role EMK plays in students' ability in producing reasonable solutions and in the quality of their solutions when working with mathematical tasks that are situated in reality. Under ideal circumstances, students have the required EMK, recognize the need for this EMK, and apply the EMK effectively to build their solutions. Unfortunately, this is often not the case. In situations where students may not have the EMK needed, their ability to identify, acquire, and apply the EMK required are the initial steps to generating a successful solution.

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JUSTIFICATIONS SIGNAL SYSTEMIC CONTRADICTIONS AS CONDITIONS CHANGE WITHIN THE SECONDARY SCHOOL MATHEMATICS SYSTEM

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Engestrom's Activity Theory is a powerful structure through which to describe the activity of teaching mathematics. A view into the diverse actions and goals among the co-creators of this shared systemic object(ive) are outlined through Leont'ev's Activity Theory as actions are influenced directly by the conditions in which the broader system is operating. Each part of the broader system both influences and reacts to changes in conditions. As used in this paper, the theory allows insight into consistencies and contradictions as the co-creators of the object(-ive) adjust to changing conditions imposed by changes in the British Columba education system.

INTRODUCTION

Mason (2002) stated "most frequently there is some form of disturbance which starts things off" (p. 10), an observance which directs the focus of this research. The disturbance at the centre of this work is a change in systemic conditions related to a revised mathematics curriculum in British Columbia. This analysis explores the complexity of curricular enactment and explores signals as indicative of contradictions introduced into the stable education system. This is accomplished through the layering of Leont'ev's Activity Theory (linking goals, conditions and activity of the individuals) with Engestrom's Activity Theory (offering a framework for the broader system).

Three assumptions are foundational to this research: 1) A revised curriculum is implemented as the instrument of change within the education system; 2) The curriculum, classroom teacher, and curricular resources are co-creators of the *implemented* curriculum; and 3) Each participant within the activity system acts with its own goals and actions which are closely tied to the conditions within the system.

META FRAMEWORK: ENGESTROM'S ACTIVITY THEORY

The education curriculum in British Columbia has not seen significant change over the past 40 years (Brochmann, 1989; O'Shea, 2003) a length of time which suggests habituated conditions and activities for the broader system. As one contributor to the broader system introduces a change to systemic conditions this deeply habituated patterning becomes challenged at points where the changes contradict habits. Engestrom's Activity Theory (EAT) is the meta-frame which offers a clear structure which effectively outlines this complex system.

In this paper the curriculum revision is recognized as the *instrument* of change within the activity of teaching mathematics. Figure 1 offers the activity system as outlined by Engestrom. The co-relationships within the activity system are represented by double arrows within the figure. Most interesting within the system is the recognition of contradictions that pinpoint locations where the influence of the singular motive seems to direct the overall *object*(ive).

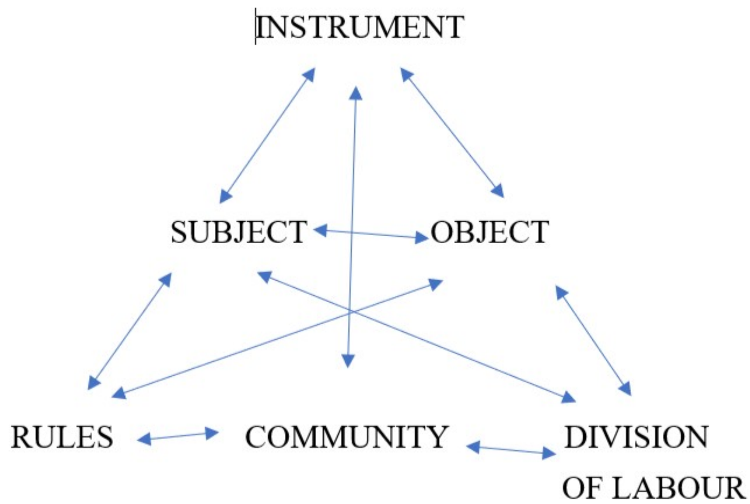

Figure 1: The co-participants of the activity of teaching secondary mathematics as outlined by Engestrom

The framework of EAT is a foundation bringing into focus the multiple pathways along which contradictions may be introduced into the system. The presence of contradictions within a deeply rooted system are signaled through comments from the subjects pointing to what is missing from the revised instrument thus preventing effective curricular implementation. These comments signal interactions suggesting possible divergence of the systems *object*(ive). The classroom teacher is the living entry point through which these changes within the system interact with the outside; Thus, situating them as the *subject* of the system and of this research.

Teacher as Subject

The secondary school mathematics teacher is one co-participant within the Activity System influenced by the *instrument* of systemic change: curricular revision. The initial interaction between the mathematics teacher and the curricular revision is taken to be the activity of reading the text. This precedes the transformation of curriculum documents into what is *implemented* within the mathematics classroom. Teachers appear to be signaling profound changes within the *division of labour* of *object*(ive) creation. This suggests teachers grant significant power to the role resources play in defining the *object*(ive). As such, the revised curriculum could be argued to professionalize teachers as the system shifts the balance of control away from resources and places greater responsibility for attainment of the *object*(tive) onto the teacher.

Resources as Division of Labour

This research recognizes the *division of labour* as dominated by textbooks, workbooks, and standardized examinations. In a system where the balance of influence on the *object*(ive) is skewed towards *division of labour* the role of the other co-creators of the system, such as mathematics teachers, must be questioned. A system which has connected curriculum and approved resources over 40 years has incentivized a powerful relationship between *Instrument-Object-Division of Labour* (see figure 2) thus de-emphasizing the important contributions of other participants within the system.

Resources attached to *division of labour* are available for public consumption thus introducing the argument the systems *object*(ive) is external to the teacher. This suggests the role of the teacher is not necessarily related to the professional knowledge they hold about the *object*(ive) rather how well they understand the delivery of the object as defined through the pre-packaged resources. In this scenario the systemic *object(-ive)* is accessible by all who have the resource. The training of the teacher becomes unnecessary.

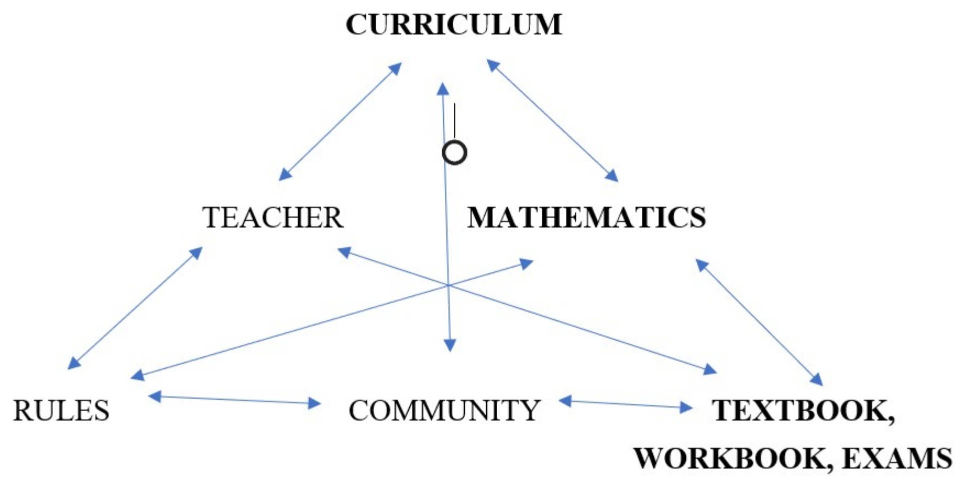

Figure 2: Notice how the diagram emphasizes the direct influence the *instrument* and the *division of labour* has on the *object*.

This paper emerges from the view that mathematics educational systems singular goal of content attainment has relied on the common curricular resources. Interconnections within the complex system are well-represented in the meta-theory of Engstrom's Activity Theory (EAT). This framework aligns with the reality of teaching being a complex cultural activity (Steigler & Heibert, 1999).

LEONT'EV ACTIVITY THEORY: A LENS TO OBSERVE THE EFFECT OF CHANGING CONDITIONS ON EACH PARTICIPANT WITHIN A SYSTEM

Activity systems are driven by communal motives that are often difficult to articulate for individual participants (Engestrom, 2000, p. 690). This nicely couples with Leont'ev's suggestion that conditions established within a system penetrate the goals and motives of each individual co-creator of the broader system.

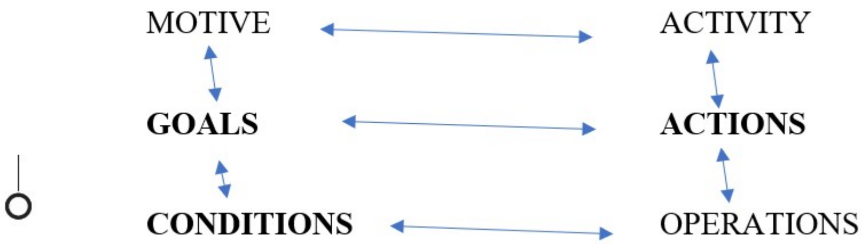

Figure 3: The Activity System of the systems co-participants as outlined by Leont'ev

If the goals (and therefore motives) of one single co-creator of a system dominate the *object(-ive)* then influence is drawn away from the other participants. It now becomes interesting to explore how, as the conditions outlined by the curricular revision change how the system of each participant is influenced. For this closer look at the individual, Leont'ev's Activity Theory is a powerful frame.

The Activity (Engstrom's *Object*(-ive)) of the Broader System as viewed through Leont'ev

The *activity* of the system involving curricular revision is the dissemination of mathematical knowledge. The revision is the *action* the curriculum imposes on the system. The *goal* of a curriculum is the maintenance of standards province-wide. The curricular revision is operating under the conditions of deep changes in the needs of students. The needs of students have evolved beyond the current structure based upon the industrial revolution. These new condition, external to the K-12 education system have forced education to operate with revised goals.

Conditions, goals, and actions of resources and examinations

The activity of disseminating mathematical knowledge dominates the purpose of the curriculum resource. The shift within the curriculum revision is the removal of approved resources from the teaching system with defines mathematics - the system to which British Columbia teachers are legally required to participate.

The presence of resources seems to have developed into the anchor of a goal upholding the promotion of consistency and standards in the reproduction of mathematical knowledge. The goal is knowledge of mathematics, but one must note the activities of the resource are restricted to written mathematics (Pimm, 1989). The resource operates as a control within the system to ensure all students are learning a standard mathematics. Bussolini (2010) recognized approved curricular resources have been operating as "an ensemble (set) of strategies of relations of force which condition certain types of knowledge and is conditioned by them" (p. 92). These have de-professionalized the teacher with respect to the necessary knowledge they require for the teaching of mathematics.

Research indicates that “for some teachers and students, the narrowed and normally linguistic version of mathematics as language is at least most familiar, if not all that is known” (Nolan & Graham, 2014, p. 604). This idea aligns with revelations from interviews such as with Victoria:

> “Basically for a lot of teachers they can go through the book um and then if they were in a subject in a grade that had a FSA or something um FSA or uh or provincial exam then you would also look at that as part of it um I remember one teacher from the north once and I think I said this in one of the meetings saying after seeing the exam twice now I know exactly what to teach so for her having seen the exam she knew what was going to be on the exam so that defined what she was gonna actually do in the classroom” (Victoria, March 2017).

The past decade has seen *approved resources* integrated into the activity of teaching mathematics. The resources teachers have grown to depend upon offer fascinating insight into what the role of teachers in knowing how to teach mathematics had become.

> “Publishers haven’t created a lot of resources like McGraws created one um but there’s been nothing new from Pearson I think Nelson made some slight changes so as teachers use [resources] as defining [content...] it kinda narrows the focus in on in on those type of things right like how far do we go […] what is a meeting expectation what is an exceeding expectation” (Victoria, March 2017).

Standardized examinations with a focus on mathematics content were also granted a significant role in defining the mathematics curriculum for the mathematics teacher.

> “Basically for a lot of teachers they can go through the book […] or provincial exam then […] having seen the exam […] knew […] what she was gonna actually do in the classroom” (Victoria, March 2017).

The reactions of teachers to the shift in the *conditions* in which the *activity* of teaching had been defined made me wonder is whether the heavy focus on resources for the past twenty years has resulted in the de-professionalization of the job of the mathematics teacher (Brochmann, 1989; O’Shea, 2003). Teaching based upon workbooks and textbooks, as well as teaching to an examination may have shifted responsibility for curricular implementation from the mathematics teacher.

The initial reactions of the teachers contain requests for specific guidance.

> “[…] we need assessment in specific subject areas. This could eventually bring us back to the excellent system of provincial exams we had a decade ago. We would once again enjoy a level curricular playing field in our secondary schools and the objective measure of achievement in each school would act to limit the inflation of marks that has been

occurring since the provincial exams were phased out. Bring on the inevitable return swing of the pendulum!" (BCAMT Listserve, May 11, 2017).

DISCUSSION

As a reorganization is introduced into the complex system, maintenance of the shared *object* necessarily becomes the focus. An object, previously defined by one single approved resource, becomes the responsibility of thousands of teachers province-wide. The tension becomes whether teachers are trusted to have the knowledge and ability to deliver a "standard" curriculum province-wide.

This suggests the goal teachers seek in reaching out for resources reflects a desire to maintain the *division of labour* which controlled the previous system. This is reflected in assessment requests from teachers:

> I am looking for information on numeracy assessments (K-12).
>
> Does your district use a district-wide numeracy assessment? If so, what assessment do you use and when do you administer it? And what happens to the information (student performance data) gained from the assessment?
>
> Also, if you use the Vancouver Island Net Diagnostic Math Assessment, how do you use it and does it align with the redesigned curriculum? (Tom, April, 2017)

The textual resources hold a place of curricular authority for mathematics teachers and without these resources teachers have lost an understanding of where to seek direction. The question of great concern is whether greater influence of the *subject* (with inherent diversity) can still lead to communal *objectives*?

The justifications provoked by the current curriculum revision suggested teachers recognize the contradiction introduced by the lack of resources. The focus on content reflects what written resources have reinforced through the past three British Columbia curricula. When implementing curricular change, it is essential to recognize both what is and what is not incentivized within the system.

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TENSIONS IN IMPLEMENTING MATHEMATICS JOURNALING: AN ACTIVITY THEORY APPROACH

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Research suggests there is a strong connection between mathematical writing and mathematical learning. As a result, many educators are implementing journaling in their mathematics classroom, which can be a challenging process. This paper identifies the tensions faced by an individual teacher implementing journal writing for the first time and interprets those tensions through the lens of activity theory. The results suggest that pinpointing the areas of tension within an activity system may provide a means of mitigating the challenges.

INTRODUCTION AND THEORETICAL FRAMEWORK

It is easy to understand why teachers are interested in implementing journaling in their mathematics classrooms. One of the primary benefits is its strong connection to mathematical learning (Morgan, 1998). There is the suggestion that the act of writing can help students synthesize new ideas and make meaning between old and new concepts (Hamdan, 2005), and that it can also foster positive mathematical beliefs (Sanders, 2009). For teachers, the written reflections can assist in assessing students' mathematical understanding and "give the teacher insights into areas of confusion or misunderstanding" (Sanders, p. 437).

However, there are challenges in journaling. The biggest barrier is how students initially respond to journal writing, as this is typically a new type of assignment for students in a mathematics class. Students tend to perceive journal writing about mathematics as something outside the norm, declaring that it "should not be part of mathematics class" (Williams & Wynne, 2000, p. 134). Additionally, it can be difficult for students to express mathematical ideas in writing, as Morgan (1998) suggests "it has largely been assumed that students will learn to write through experience and that they will develop appropriate forms of language 'naturally'" (p. 2).

Tension may result as teachers attempt to maximize the benefits while minimizing the challenges. Endemic to the teaching profession, tension encompasses the inner turmoil teachers experience when faced with contradictory alternatives for which there are no clear answers (Berry, 2007). At any given moment, a teacher may experience tension with students, with the task, with content, with assessment—tension that ebbs and flows throughout the course of a lesson, a day, or a lifetime of practice. This leads to the possibility of viewing tension as a web or series of interconnections. For example, a teacher says she is experiencing tension with assessing mathematical understanding. Is the tension 'what' to assess? How to assess? Is it because parents prefer summative assessment? Is it because students have test anxiety? Is it a combination of these? All of these? If these tensions are thought of as an alarm panel, which are the sections that light up and demand attention? Which sections are unlit? With these thoughts around tensions percolating in my mind, I came across Cole and Engeström's (1993) activity theory. It appealed to me instantly. Here was a way of understanding tensions as interactions in, and on, a dynamic, active system. Using activity theory, I could localize the tensions within an activity while considering their global effect on the activity itself. My goal then is to identify the tensions experienced by a teacher as he implements journal writing in his secondary mathematics classroom.

Activity Theory

Activity theory is built on the assumption that, at the level of the individual, all intentional human actions are goal-directed and tool mediated (Venkat & Adler, 2008). To expand from individual mediated action to the level of collective activity, Cole and Engeström (1993) introduced activity systems, which they define as communities engaged in activities which share common goals. They suggest these systems exist within socio-cultural settings like a classroom or school and can be seen as "natural units of analysis for the study of human behaviour" (Cole & Engeström, 1993, p. 9).

As illustrated in Figure 1, an activity system comprises six elements. The subject is the person, or group of people whose perspective is the focus of the analysis and the object is the overall goal of the system. Tools include anything used to mediate the activity, while rules are the explicit and implicit rules and norms that guide and restrict the activity. The community is the person, or people, who comprise the social context in which the subject belongs, and division of labour regards the roles within.

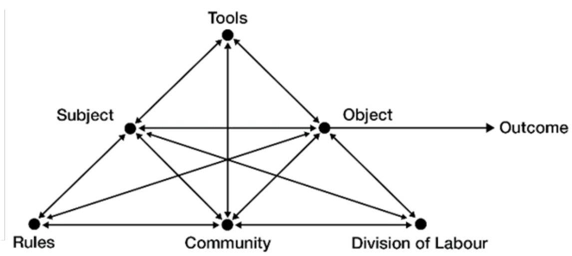

Figure 1. Activity System (Cole & Engeström, 1993).

Gedera (2015) notes that these six elements in an activity system "act as mediators and the relationships between these elements are constantly mediated" (p. 55). This suggests activity systems are dynamic and changeable. Cole and Engeström (1993) add that "activity systems are best viewed as complex formations in which equilibrium is an exception and tensions, disturbances, and local innovation are the rule and the engine of change" (p. 9). This means that the interconnections between the elements, and ways in which they influence and are being influenced, suggest a key area for exploration in that they "draw attention to those points where contradictions or tensions exists" (Jaworski & Goodchild, 2006, p. 55). These tensions can occur within an element of the activity system, between elements of the activity system or between connected activity systems and are considered essential for both understanding the motivation for particular actions and the overall evolution of a system more generally (Cole & Engeström, 1993).

Tensions also prove useful as a way for teachers to describe their own experiences of practice (Berry, 2007). To develop their classroom practice, it may be helpful for teachers to recognize and define these tensions (Lampert, 1985). "In the process of renaming what they know through their experience, the teachers critically reflect on—and thus begin to renegotiate—their ideas about teaching and learning" (Freeman, 1993, p. 488). To achieve this, the tensions first need to be identified at both the global level of the activity in its totality and at the local level of its constituent elements. This leads to my research question: What are the tensions the teacher experienced in the journaling implementation and where are they located within his activity system?

METHODOLOGICAL CONSIDERATIONS

Studies on tensions often rely on semi-structured interviews and self-studies as the primary means for subjects to identify and reflect on the tensions they experience (e.g., Sparrow & Frid, 2001; Berry, 2007). Inherent in revealing and understanding tensions then, is the element of reflection that originates from the subject. This study therefore, utilizes data from the subject's reflection of the activity rather than the researcher's direct observation of the activity itself. The rationale for this methodological decision is that it is necessary for tensions to stem from the subject's reflections rather than their observable actions, and furthermore, to make use of the tension, the subject must come to recognize and name their own tensions. It is in reflecting on the activity that the subject moves closer to understanding the tensions they experience within their activity system.

This paper in particular is a small scale qualitative study that seeks to prove the existence of a phenomena rather than its prevalence. It involves one participant, Dan, a secondary mathematics teacher. One of many teachers interviewed as part of a larger study regarding tensions in teaching, Dan was chosen for this study because of a specific experience he shared regarding implementing journal writing. Data used for analysis was obtained during a one-hour semi-structured interview that was recorded and transcribed in its entirety, along with written responses to follow-up questions. Using activity theory as both the theoretical lens and the analytical tool, the data was used in two ways. First, it was used to create a descriptive narrative of the subject, Dan, as he portrays himself as a teacher. This narrative was shared with Dan, to validate its accuracy, and subsequently used to outline his activity system. Secondly, the data was scrutinized to identify tensions. To begin, this was done by searching the transcript for evidence of tensions. In particular, evidence of utterances with negative emotional components such as "I was most disappointed with…" or utterances that conveyed doubt or uncertainty such as "I didn't know what to do." The list of identified tensions was then narrowed to include only those that stemmed from, or were related to, Dan's implementation of journal writing. Dan's own activity system, developed from the descriptive narrative, was then used as the frame for locating which specific elements contributed to, or were impacted by, the tensions he experienced.

Context of the Study

The next section contains two parts. First is a narrative describing Dan, which will be later used to outline his activity system. This is followed by a short description of the journaling Dan implemented in his mathematics classroom. Note that, from here on, all italicized words contained within quotation marks are Dan's.

Dan always planned to teach high school mathematics. He trained as an elementary generalist however, as he found it difficult to fulfill the senior mathematics requirements for secondary concentration. His plan was to use his elementary generalist degree as a stepping stone, thinking *"if I go in this way, maybe I can kind of go in the backdoor and get into secondary math"*. And, after two years of temporary contracts and substitute teaching, Dan was employed as a secondary mathematics teacher. He found this was a dubious success. Quite bluntly he states, *"I hated it. I hated secondary teaching; I did not fit in that culture"*. Calling himself *"philosophically misaligned"*, he spent his first few years figuring out where he fitted in as a teacher. He credits a strong bond with a teacher-mentor and his elementary generalist training with helping him establish his teacher identity. Despite initially viewing his entry into the elementary generalist program as a means to an end, he said that the elementary style of teaching and learning actually appealed to him. Sharing the adage that *"elementary teachers teach kids, secondary teachers teach content"*, his belief is that *"I can't do anything if I don't have a relationship"*.

Dan describes himself as *"kind of an outlier, typically out in front of things"*. He holds a Master's in secondary mathematics education. He also served as president for his province's mathematics teachers' association. His first role with his current district was the district's math consultant, a position he held for five years before going back to teaching secondary mathematics four years ago. Upon returning to the classroom, Dan noted a lack of engagement in his students: Lack of engagement with himself, and lack of engagement with mathematics. He wanted to create a culture in his classroom where his students were thinking mathematically, not simply repeating back words and steps he had provided. To that end, Dan began implementing changes in his practice. He began giving students opportunities to work collaboratively on problem solving at alternative working spaces. He began offering retests and flexible assessments. He also began working on alternative assessment practices that better described qualitatively what his students knew and could do.

Overall, Dan experienced varying success with the changes he implemented. What piqued the interest of this researcher was his mention of the tensions he experienced regarding one change in particular—his requirement that his students write journals about their problem-solving experiences. Dan had started off the current school year convinced that mathematical writing would be a beneficial experience for the students, helping them create personal connections to their mathematical learning. Instead he found the opposite; the change he wanted to implement was damaging his relationship with his students and their relationship with mathematics.

The journal writing Dan introduced was a weekly activity that his two classes of grade 10 students were expected to complete as homework. It involved having the students write up their problem-solving processes for a task they had completed collaboratively in class. In both his grade 10 classes, the students' reaction to journaling was immediate and overwhelmingly negative. Noting that he had not anticipated the anxiety they would experience, Dan discontinued journal writing after three entries.

ANALYSIS AND DISCUSSION

In the following analysis, activity theory is used to outline the five elements (object, tools, rules, community, division of labour) in Dan's activity system, in which he has the role of ***subject***. His own activity system is then used as a frame for identifying and interpreting the tensions he experienced. In all six tensions were identified, but due to space limitations, only three will be presented for analysis.

Dan's Activity System

Essential to Dan's activity system are his dual desires to develop meaningful relationships with his students, and for his students to develop meaningful relationships with mathematics. His ***object***, then, is in establishing what he called an *"ethic of care"* with his students, while figuring out how to *"push them forward mathematically"*, in this instance, through writing to learn. The ***tools*** Dan uses are pedagogical in nature. His approach to teaching combines tools of whole group, small group, partner, and individual. He attempts to engage students' mathematical thinking through tools such as journaling and collaborative problem solving. He also uses homework and note-taking, and offers flexible assessments. There are certain ***rules*** within which Dan's practice occurs. The content he teaches is guided by a provincial curriculum and there are school-wide assessment practices such as final exams. There are also well-established school-wide norms that dictate expectations for teachers, students, and classrooms. The ***community*** contained within Dan's immediate activity system comprise his students, parents, teachers, and administration. Lying farther out are the wider educational, professional, and social communities which Dan inhabits. The ***division of labour*** for each establishes expectations of the roles for the community, essentially who is to do what. For example, as the teacher, Dan has the authority to choose and assign homework; his students are expected to do the homework and hand it in.

Tensions between subject and tool.

Dan felt journaling would be an effective way of getting students *"to write for the learning of mathematics, to explain their thinking in written form"*. Initially stating that he likes the idea of journaling, he quickly amended his statement, *"No, that's not true. I don't like the idea of it. I think it's absolutely essential and critical."* His tensions with the tool lay in how to implement it effectively, not its efficacy. He believes he may have rushed the students into the process of journaling too quickly, not allowing them the necessary time or space to adjust and accommodate to this new tool. His students strongly resisted, to the point where he says, *"I was losing relationships with kids over this."* Dan reflects, *"I should have started in a more traditional way and eased into it."* This response is in keeping with Winograd (1996), who suggests that people deal with tensions regarding a tool in two ways; they either find ways to "work around" the tensions or they blame themselves. Dan, it seems, falls in the latter category. He places the blame for the tool failure on his own perceived shortcomings.

Interestingly, Dan never mentioned questioning the tool itself. His desire to have students writing to learn in mathematics led him to introduce a standard journaling technique of having the students write to him about their problem-solving processes. Noting that writing in mathematics is "best accomplished in contexts where there is an authentic need to communicate" (p. 15), Phillips and Crespo (1996) suggest that most mathematics writing activities are contrived and have the teacher as the intended audience. This has a detrimental effect on the writing. It is possible that changing the style of writing and/or changing the intended audience may have managed the tension Dan experienced.

However, for now, this is an ongoing tension that Dan is still facing. Despite having discontinued journaling this year, he continues to reflect on the experience as he wants to try again next year, with a different group of students. *"I'm committed to it"*, he says, *"it's really, really important."*

Tensions between rules and community.

Dan noted that journaling broke the rules regarding what his students believe about mathematics saying, *"I've been challenged by kids around their expectations of what math is and what math class is."* His students adhere very strictly to traditional notions of mathematics classrooms, reinforced by the community in which Dan works. He suggested that his students have figured out a way to *"survive math class, which is, you're going to give notes, you're going to give me homework, I'm going to study, get a tutor or whatever."* Dan said they were looking for *"give me the ten questions I need to know and I'll practice doing them"*, and instead, he was asking them to write. Morgan (1998) suggests this disengagement may occur if the students interpret journaling as an incidental extra activity unrelated to the learning of mathematics as they have come to expect.

This is an ongoing source of tension for Dan, who is trying to change how students *"see and do"* mathematics. He's frustrated by the limited notion of math that leads to students questioning *"when are we going to do some math here?"* whenever he tries anything 'untraditional', such as journaling. He goes on to say, *"I can't engage for them so I need them to buy-in to this."*

Tensions between subject and object.

Perhaps one of the most significant tensions Dan faced was maintaining his relationship with his students while trying to engage them in thinking mathematically. Noting that his students were resistant to being pushed to write mathematically, he worried that he was going *"too far outside their comfort level"*. He acknowledges the importance of fostering mathematical thinking and conceptual understanding yet recognizing, *"at the same time I can't be successful unless I have their trust. I can't be* *successful unless I have them with me and they see me as an advocate for them and not a barrier for them."*

In discontinuing the journaling activity, Dan was able to decrease the tension he was experiencing with his students, but it was at the expense of his corresponding object of engaging his students' mathematical thinking. This suggests that, although Dan contends that both are equally desirable, his true object is maintaining his teacher-student relationship. It is possible that this activity caused Dan to rearrange the priority of his goals: he may continue to value mathematical thinking but, in this instance, he gave higher priority to maintaining relationships. The strong emotional response from his students and himself, Leont'ev (2009) would argue, is necessary for establishing this hierarchy of objects, or what he calls motives. A change in the object hierarchy necessitates changes in the rest of the activity system, and the result may be a different outcome. In this way, tensions fuel the evolution of Dan's activity system.

Although discontinuing journaling managed the immediate tension surrounding relationships with his students, it remains an enduring tension of which Dan is keenly aware, *"and so I have to manage the tension between moving in a pedagogical direction that I think is best for their learning but at the same time that won't cost me the relationships I have with them or then I've lost them entirely."*

CONCLUSION

Imagine Dan's activity system as an alarm panel, with the connections between its components only lighting up between when tensions are experienced—his would have been a flashing array of warning signs (see Figure 2). He experienced tensions in six aspects of his activity system and the ultimate outcome was that he quit journaling. This was unexpected and disappointing for Dan. In reflecting on the experience, he shares:

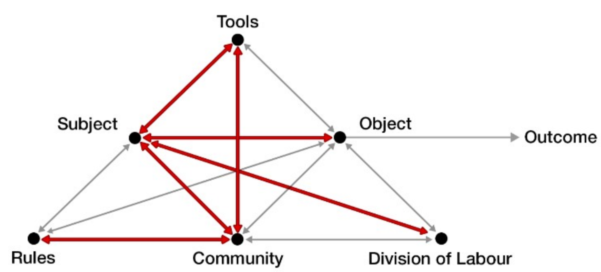

Figure 2. Dan's Activity System.

> "When I think back on this, it didn't occur to me... it wasn't a possibility that this wasn't going to work. So when it started to go sideways, I didn't know what to do. I didn't realize, so I continued to push forward with it. And then the pushback happened with the kids."

Initially, Dan spoke globally of tension in the activity of journaling. However, by focusing on the individual elements it is possible to see where the tensions impacted locally. And, as Dan is determined to try journaling again, identifying the local tensions that arise throughout his activity system could offer a means of reflection as he thinks through his next implementation. This fits with Lampert's (1985) view of a teacher as a "dilemma manager who accepts conflict as endemic and even useful to her work rather than seeing it as a burden that needs to be eliminated" (p. 192).

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ARE THEY GETTING ANY BETTER AT MATH?

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Conversations with stakeholders about students' improvements in mathematics invariably focus on student grades and work habits. Further, research into improvements in mathematical performance focus almost exclusively on the acquisition of mathematical content and improvement in test scores. This narrow focus makes assumptions about what it means to know, do, and improve mathematically. By exploring the micro and macroculture of mathematics classrooms and analyzing interview data with two secondary school mathematics teachers, I explore what it means to get better at math inside an inquiry tradition.

Are they getting any better?

I was having a conversation recently with a colleague of mine about her assessment practice. At the time, she was teaching English Language Arts, and we were discussing the practice of including marks from student writing from early in the year as part of the students' overall grade. It was the practice in her department to average student writing grades from September to June. We both had questions about this practice. Shouldn't the student's grade be determined exclusively on how the student was writing at the end of the course? After all, if the student's writing had improved significantly over the year, what was the point of including early writing marks? We both agreed that one of the main purposes of English Language Arts was certainly to help students improve their writing, among other competencies, and that, while the student should receive feedback throughout the course, evaluation seemed to make the most sense at the end.

> "Well what do you do in Math?", she asked me. "Do you average over the year, or just give them a mark at the end?"
>
> "We almost always average," I said. "It wouldn't make sense to just give them a mark at the end, because our grading is usually based on discrete units."
>
> "Well, how do you know if they're getting any better at math then?" she asked.

I can't remember exactly how I answered at the time. It's likely I said something about Math being different that English, and that we measure achievement by content acquisition, or something like that, but the question had been asked, and I knew as I answered it, I wasn't satisfied. "How do you know if they're getting better at math?" I had no idea. How could this be? How was it possible that I wasn't able to tell if students were getting better in the way she and I were talking about with respect to her English students. Certainly, my students were getting better at math throughout the year, weren't they? Ah, I thought, I've got tests! Yes, that's it. Look at what they can do that they couldn't do before! They can (mostly): factor polynomials, reduce radicals, graph linear equations. They couldn't do that stuff in grade 9, and now they could after grade 10. There, they'd gotten better at math. But somehow, the satisfaction of my answer rang hollow. Was that it? Was improving at math just about learning more content? Certainly, they must be getting better at something else? But what? And how did I know? I decided to examine my students' test and quiz scores.

After a thorough look at my gradebook for a year, it became clear from the data, and the method by which it was collected (i.e. quiz and test scores), that no reasonable inferences could be made as to whether or not my students had gotten any better at math. None of the data sets I chosen to examine showed improvement in scores over time; in fact, none of the students from any of my classes last year showed a consistent improvement over time. Even if the data had shown trends towards improved scores over time, it would be impossible to infer whether a student was getting better at math, or whether that student was simply getting better at writing the tests.

The method of data collection, however, has more profound implications. This data was used to measure, in large part, the success of students in my class. Parents, students, their next years' teachers, and other stakeholders, look to this data as, in many cases, the only indication of mathematical competence and proficiency. The data, however, is only part of the story. What I evaluate is linked to what I value and what I believe to be important.

Mathematical Culture

According to Bauersfeld (1993), mathematical activity depends on social and cultural processes. The classroom itself is certainly a dynamic system, in which social and cultural norms are introduced and reinforced by the teacher.

> "[T]he understanding of learning and teaching mathematics ... support[s] a model of participating in a culture rather than a model of transmitting knowledge. Participating in the processes of a mathematics classroom is participating in a culture of using math or better: a culture of mathematizing." (Blauersfeld, 1993, p. 4)

Mathematizing, in this sense defines, for each classroom, what it means to know and do mathematics. In their work on sociomathematical norms, Yackel and Cobb (1996) identify intrinsic aspects of a classroom's microculture, defined by teachers' and students' activity. They argue that these classroom normative understandings are modified by the ongoing interactions of students and teachers, and are unique to specific classrooms. While Yackel and Cobb (1996) look specifically at those interactions that sustain a culture of inquiry and problem solving, I contend that sociomathematical norms are present in classrooms with a focus on content acquisition also. Further, it is the activities of students and teachers that define both what mathematical engagement is for a particular classroom, and what mathematics is for students in that classroom culture.

Following the work of Richards (1991), Cobb, Wood, Yackel, & McNeal (1992) identify two classroom traditions: the *school mathematics* tradition, and the *inquiry mathematics* tradition. While it is tempting to view each tradition as theoretically dualistic, with the *school tradition* exemplified by direct teaching to passive receivers of knowledge, and the *inquiry tradition* as a dynamic environment where students make and test conjectures, solve problems, and actively construct meaning, Cobb et al. (1992) argue that this is an oversimplification. In both traditions, the meaning of mathematical activity is based on mutually agreed upon classroom interactions that constitute acceptable discourse in each tradition. The fundamental difference, however, lies in what is accepted as meaningful mathematical engagement in the respective traditions. The interpretation of what is meaningful can be connected to Skemp's (1976) discussion of instrumental and relational understanding, with respect to what constitutes normative activity in each classroom. In the school mathematics tradition, students view meaningful activities as those in which they can demonstrate correct procedures, while in an inquiry mathematics tradition, students see activities as meaningful when they facilitate the construction of personal meaning (Cobb et al., 1992).

Inside of the framework of sociomathematical norms, what do other teachers think it mean to get better at math and what implications does a focus on content acquisition have on what it means to learn and to do mathematics?

METHOD

This small scale qualitative study involves two secondary high school mathematics teachers, Jack and Diane, chosen because of their involvement in their school district's professional growth initiative to help integrate more problem solving into teacher practice. The data used for the analysis was gathered during a one-hour semi-structured interview with each teacher individually. The data was recorded and transcribed in its entirety. The data was subsequently used to identify themes in the teachers' descriptions of what it means to be good at, and improve in, mathematics. The selected transcripts below, while not comprehensive of theme, are provided as examples of the overall themes described.

RESULTS AND DISCUSSION

While several themes emerged from the two interviews, I will focus on two which explore the teachers' thoughts about what it means to get better at math. Firstly, that the teachers' view on what it means to be good at math depends largely on quick acquisition of new skills and a strong foundation of fundamental skills. Secondly, that students generally do not get better at math in the year students spend in the teachers' class or, over their high school experience.

What it Means to be "Good" at Math

One of the themes that emerged from the interviews was that content acquisition and the good work habits required to succeed at acquiring new content seemed to be the primary indicators of student they considered "good" at math. They also identified student success with mastery of foundational skills, getting the right answer quickly, and without difficulty as other indicators. While both teachers identified characteristics of adaptive reasoning in their descriptions of students who they considered to be good at math, students picking up new content quickly was seen as a major indicator of competence.

> Diane: Well, usually they can come up with answers quickly, but also you can give them a problem that's slightly different and they can still change their way of thinking and still find an answer to the problem.
>
> I give daily self-assessment quizzes. The beginning of class, we'll spend 10 minutes doing that. That gives me a change to walk around and see who knows their stuff and who doesn't know their stuff. If I'm walking around and somebody's answered all those questions and they've got without looking back at their notes or talking to the person next to them get hints then I know they've got their stuff.
>
> Yeah. I guess, are they good at all math? Who knows. Some kids might be really good at Algebra but as soon as you get to graphing, they don't get it anymore. That usually, if they're the ones that are feeling confident and can go through and do questions like that very efficiently than they're usually the ones that will pick things up really quickly and will be good at math.

> Jack: I think, often times, it's organization. You see them keeping good notes, they're completing homework, those kinds of things.
>
> And I guess that coincides with, like I say, a bit of a quick appropriation of those basic fundamental things. Maybe they already exist, or maybe they just picked them up quickly. And I guess just a strong background knowledge, that you get a sense of, somehow, I don't know if that's through seeing particular work, or if it's just a sense you get from somebody, based on the way they seem to be processing.

As mentioned above, the evaluation data of content acquisition for students is viewed as a measure of both success and proficiency. What we measure has an implicit and explicit value attached to it, since, in my classroom and in Jack and Diane's classrooms, the extent to which students can demonstrate their content acquisition is the most significant measure of their success. This is not to say that the instructional focus in our classroom was procedural; in fact, our beliefs are quite the opposite. We spent a great deal of time guiding students towards conceptual understanding by engaging them in discussions, activities, and explorations designed to deepen their appreciation and conception of the content. At some point, however, this focus narrowed, and the measure of their success became what they could do – what questions they could answer, how they could demonstrate their ability to perform skills, and how they could apply those skills to familiar questions on a test. I contend that this narrowing of focus defines for students what it means to know and to do mathematics. For our students, then, in spite of our efforts to focus on conceptual understanding and inquiry, the measure of their success was the degree to which they learned how to do the kind of mathematics that they could reproduce on a paper and pencil test, meaning that to improve in mathematics, one must learn to get better at performing on tests and quizzes.

The difference between engaging in the process of doing math and engaging in the process of learning math content is significant. The question of what Mathematics is, much like the validity and efficacy of paper and pencil testing, is beyond the scope here; however, it seems that if I am interested in trying to determine whether my students are getting any better at math, the question necessarily comes down to distinguishing between the doing of math and the learning of math as content. Both are a kind of "doing" of math or *mathing,* i.e. considering math as a verb, or the act of engaging mathematically. I recognize that these are not mutually exclusive, but it is in the *mathing* that the processes involved in doing mathematics lie. I contend that what constitutes mathing in a classroom defines for most students what it means to know and do mathematics, and, perhaps, even what Mathematics is.

Both the microculture of sociomathematical norms and the macroculture of classroom traditions are negotiated by teachers and students (Yackel & Cobb, 1996; Cobb et al., 1992) through an interactionist framework. This aligns strongly with my own experience. Classroom culture is a blend of my own philosophical, ideological, and epistemological perspectives, and those of my students. A profound tension exists, then, between an instructional approach that favours a tradition of inquiry and an evaluation plan that is grounded in a school mathematics tradition. How does the view that being good at math means picking up new content quickly influence the focus on an inquiry approach?

Do Students Get Better at Math Over Time?

When asked about how common it was, in their experience to have students improve over time, both teachers report that it is uncommon. When improvements do occur, the teachers attribute the improvement to maturation and better work habits.

Diane:	Yeah, I think because it gets more difficult. If you struggle in grade-8, you're still going to struggle, I think, in grade-12. I don't think you're going to get better. I feel like, yeah, you struggle more as you go ahead.
Interviewer:	The trend is generally that you're not going to get better overtime?
Diane:	Yeah. I'd say you got to maintain it or ...
Interviewer:	Or it drops?
Diane:	... or drop a little bit.
Interviewer:	You pointed to work habits being one thing that might change overtime to help someone get better over time?
Diane:	Mm-hmm (affirmative).
Interviewer:	What else do you think might?

Diane: Again, maturity might help, but I would feel that would help more with the work habits, and then that might see an improvement.

Jack: I would say it's not very common, yeah, not very common. I'd say that students can start in positions where they're doing poorly, in the sense that they might not pass a course, and then something happens. There's particular incentive or disincentive, that sparks something, and then habits change test marks are going up, those types of things. I've seen that happen before, but it wouldn't be the kind of thing where I would say this student was ... math didn't come naturally to them at the beginning of the year, and at the end of the year, it did come naturally to them. That transition doesn't seem to happen very much within the course of the year.

Finally, when asked how they knew if students had improved over the year they spent in the classroom, both teachers suggested that content acquisition was a way to determine improvement but recognized that improvement over time was difficult to infer based on changing content. Both teachers had challenges distinguishing between getting better as students and getting better at math and disagreed about whether their students' getting better at math was an important consideration in their practice.

Interviewer: Let's just imagine it's the end of the year and you've got your kids in front of you. How can you tell that they've gotten any better at math in the year that they've spent with you?

Diane: Good question. I don't know. Marks. Have they've gotten better at it? It's hard to see that because even with their marks because again, different units are different. If you were to give them a test from the beginning of the year at the end of the year and see if they do better on it, well, it's a different unit, it's been a while since we did it. I'm not sure if they're going to do better. I don't know if you'd gotten better at math.

Some of the kids, I think you wouldn't notice in general if they've just become better math students, that maybe they've gotten better at doing their math over of the course of the year, but I don't know. It's a good question.

Interviewer: Is that an important question, do you think? Is that something we should be concerned about at all or think about at all?

Diane: I don't know. I don't feel like it, no. Are you better at math now? Again, I feel like they've got a larger set of skills now so they should be better just in general. Their math skillset is larger than they should just be better in math by being there for a year and learning new things. I guess it's, what is better in math mean? That's a good question. Are you better problem solvers? You've got a bigger skillset so maybe you are? Maybe that's just it. Maybe at the end of the year, you are a little bit better in math because you've gotten a larger skillset.

Interviewer: When you say, "Skillset" because I think there's a lot of different versions of what skills say. When you say, "A larger skillset," tell me what you mean by that?

Diane: I guess I just mean you've learned more concepts, mathematical concepts. In grade-11 you learn about quadratic functions or you don't know about those in grade-10. You know more math.

Interviewer: Content?

Diane: Content, yeah. Thanks.

Jack: I think when we do assessments, or year-end evaluations, we're essentially saying we had some things that you were supposed to know, and this is the degree to which you appear to know those things. And I don't know that there's necessarily a requirement for them to have gotten better at math. Perhaps they did through the process, but it's an interesting question to me, because it doesn't seem to be, at least, the major thing that we're accessing directly. It's more topic based, and maybe that will change with the new curriculum, with things like that. But it seems mostly that it's a content-based thing, so I don't know. It's not a question that I've really ever asked myself, but seems to be one that should be asked.

The theme of improvement offers insight into the school mathematics tradition of content acquisition. As part of this tradition, Mathematics is seen as a subject of increasing difficulty of content and it is accepted, if not acceptable, that students will generally not improve over time. If they do improve, it is because the student has matured and is better able to acquire content. In the school mathematics tradition, acquiring content (and demonstrating this acquisition in written form) is a proxy for improvement. As Diane points out, since you've learned more concepts, you know more math.

Does acquiring more content count as getting better at math? Is it sufficient? What seems to be missing from the teachers' comments is the nature or quality of their students' mathematical engagement; are their students getting better at *mathing?* To determine this teacher's need to attend to and measure the processes students use to engage in mathematical inquiry, and not just measure their content acquisition.

This presents an alternative way of examining the mathematics classroom – by separating the way in which students engage with the content, and the content itself – describing *mathing* separately from the content that is the context for that mathing. If, as the research suggests, the mathing is a function of the micro and macrocultures negotiated in the classroom, then it follows that this process is relatively stable throughout the course, and defines what it means to *math* in that classroom. The content, on the other hand, is ever changing. As almost all math teachers know, the content is the clock; teachers and students both need to keep pace to be in certain chapters at particular times in the year. For example, if I don't finish trigonometry by

Christmas, I'm behind. Further, content is almost always unidirectional. Apart from review for mid-terms and finals, once we have finished a chapter, we almost never go back to it to. So, given that we have an ever-changing part of the course, i.e. the content, and a consistent part, that is, the way students *math*, if I want to know if they're getting any better, why do we focus almost exclusively on that which is always changing? If I want to know if students are getting better at doing mathematics, and not just getting better at acquiring mathematical content, I need to find a way to help improve the mathematical processes that students use to math. Essentially, the normative culture of the classroom, the stable and consistent environment of doing mathematics, is what I need to help students improve, not exclusively the variable content topics.

CONCLUSION

Ultimately, in reflecting on my colleague's question, how do I know if my students are getting any better at math, I can only conclude that, as long as I am only helping students improve in acquiring the content, my students' attention will be necessarily focused on the ever-changing part of the course, and not on the kind of *mathing* which is representative of the inquiry tradition I believe to be important. By re-visioning my goals for what it means to be good at math and what it means for get better at math, I can shift my attention to the processes they engage in inside an inquiry tradition.

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DOES THE GEOMETRIC SOFTWARE WE USE INFLUENCE GEOMETRIC THINKING?

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The study reported here is dedicated to research about our own way of thinking mathematically. It investigates the possibility of the existence of different such ways through using different tools and tries to explain how thinking mathematically could be realized by physical reactions. In doing so, the exposition will centre on a specific geometrical problem in two different environments, paper/pencil and the Geometer's Sketchpad. We identify dragging trajectories and other physical reactions in GSP as well as a solution based on theories in paper/pencil environment, consequently resulting in different visions to the problem (ways for attacking the problem). It is also suggested that a geometric software can prompt users to pay attention to certain features of a geometric problem that are ignored in a paper/pencil environment.

THEORETICAL FRAMEWORK

In our research, the theory of Instrumental Genesis (Verillon & Rabardel, 1995) will be used as our framework, the process through which an artifact becomes an instrument for a user. In fact, in this theory, an instrument is considered as a mediator between actor and activity and this meaningful relationship between them consists of two components: artifact and mental use scheme developed by the user. Regarding the mental schemes (the psychological component of Instrumental genesis) there are two main categories; usage schemes and instrumented action schemes (Trouche, 2004). Usage schemes are related to managing the artifact and instrumented action schemes directed towards the activity. The process of instrumental genesis of an artifact is also triggered by two directions, one towards the artifact which is named instrumentalization and the other towards the user which is named instrumentation (Trouche, 2004). The former is the process through which the affordances and constraints of an artifact deeply condition the action of a user in order to solve a given problem, and the latter is a process of personalization of the artifact. Also, in order to recognize the structure of our mental reaction to seeing a problem, we shall try to find a pattern of a solution by using dragging modalities (Arzarello et al., 2002; Hatterman, 2010) and other physical reactions which can act as a lens to organize and interpret dragging explorations in GSP.

RESEARCH QUESTION

"Is it true that the tools can influence our thoughts?"

The problem I am going to analyze in this paper is about the influence of the tool we use on the way we think. In fact, through the process of instrumental genesis of different tools we will be equipped with different ways of attacking a problem, and they can be considered as a new lens - at least in the case of geometry and problem solving. In other words, there is a tool (or at least we can make a new one) with the potential to shed light on the invisible parts in some specific environment (like paper/pencil). Or more importantly, during the process of instrumental genesis, what new utilization schemes are developed? How are these different in two environments? In doing so, we consider two different tools: paper/pencil and the Geometer's Sketchpad (Jackiw, 2012), and try to identify the influence of the utilization of drag modes in the Geometer's Sketchpad (GSP) on the way we think as well as the types of dragging modes which are concerned with (or result in) making conjecture and thinking mathematically.

TASK AND METHODOLOGY

I conducted this research study based on my own way of thinking while I was doing a task in a GSP. I utilized the drag modes in GSP. It was my first experiment doing a task in GSP, and I had no preparation session for this software except some basic constructions. In my investigation, I organized the activity through solving this problem: Construct a circle with radius 4cm that is tangent to two circles of radii 3cm and 4cm, and the distance between their centres is 7cm. I had some preliminary knowledge about circles and related theorems, though no experience with this type of questions in recent years. For data collection, I interrupted the experiment every one or two minutes, and wrote down my thought and what I was doing.

RESULTS

My experiment consists of three phases and a total of six episodes: first I tried to solve the question by using paper and pencil, then simulated my proving in GSP, and eventually I came to realize that the proving would not work and consequently I should solve the problem in another way.

In Episode 1, in more familiar environment, I solved the problem using paper and pencil and found two different answers. To provide this solution, I just used elementary circle properties.

In Episode 2 (Transferring a deductive structure to GSP**)**, as I have never worked with GSP, so this is the first stage of my instrumental genesis and my discovery is limited to the given problem.

In Episode 3, during this step (the psychological component of instrumental genesis) I have been able to appropriate GSP and find its potentialities relative to the question.

In Episode 4, I tried to solve the problem by simulating my solution in GSP. I constructed a line segment $\overline{AB}$ of length 7cm. In order to construct two circles centered at A and B with radii 2 and 3 cm, I should find two points C and D on AB such that AC=2 and BD=3 and then draw circles with centres A and B passing through the points C and D respectively. Based on my solution, it is also needed to construct two other circles centered at A and B and radii 6cm and 7cm respectively, as follows:

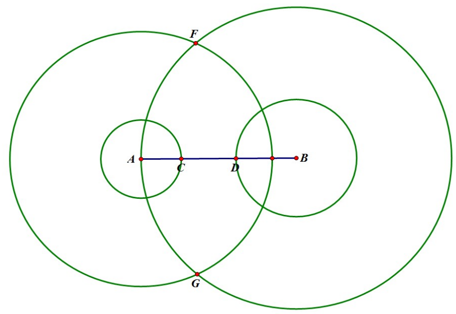

Figure 1

Each circle centered at F and G with the radius 4 is the desired answer for given question, see figure 2. Therefore, I found two answers to the question.

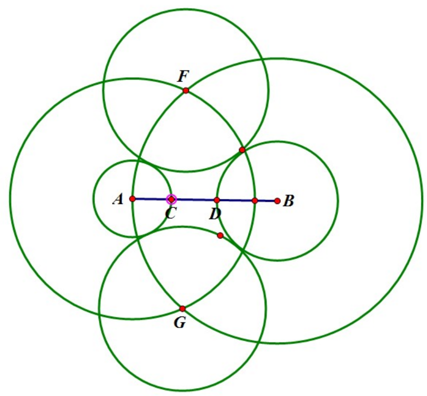

Figure 2

Many circles are confusing and it is difficult to continue and drag final circles. Therefore, I need to start from scratch.

Solve the problem without using background knowledge

In Episode 5, I construct a line segment $\overline{AA'}$ of length 7 cm (bound dragging) and then two circles centered at A and $A'$and radii 2cm and 3cm respectively. Then to find a circle tangent to both of them, I place a point, say D, on the sketch outside of the circles as follows:

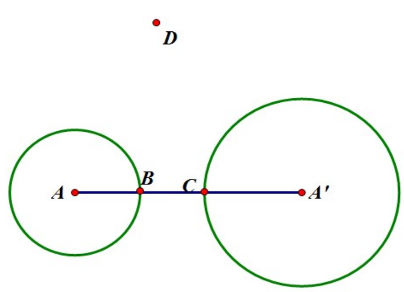

Figure 3

Now I construct a circle centered at D with radius 4 cm (Bound dragging-Adjust dragging) and try to drag the new circle while it is tangent to both supposed circles. However, I do not have any control on the measure of the radius, by dragging the centre point or the determined point (the point E in figure 4) the radius changes (Dragging test to check). So, it is needed to identify draggable objects concerning the new circle and check their functions as well (Bound dragging, Degree of freedom test). When I drag the arc of the circle, it moves without any change in radius but in the case of dragging the centre or the determined point on the circle the radius changes (Bound dragging and Function test).Then, I construct a new circle centered at D with radius 4cm by dragging the arc of the circle to find the desired centre point (Bound dragging, Adjust dragging).

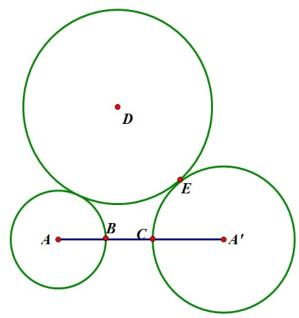

Figure 4

In Episode 6, it is the time to check my answer, in other words I should show that the new circle is tangent to the assumed circles. I just rely on some preliminary knowledge about tangent circles. And at this point I use this rule: Two given circles are tangent if the distance between centre points is the sum of the radii. In figure 4, I have $\overline{A'D}$ = 7 = 3 + 4 and $\overline{AD}$ = 6 = 2 + 4, therefore the answer is correct. In fact, I already knew that the answer is correct because parallel to my discovery I checked for some easy and useful properties of tangent circles, for example, A', E and D must lie on a straight line and while I was dragging, I considered this rule. Now, I am looking for the other possible answers for the question. By dragging the third circle around one of the two supposed circles (Bound dragging, Exploratory dragging) it reveals that there exist two other answers for the question:

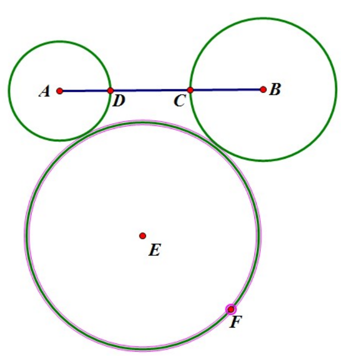

Figure 5

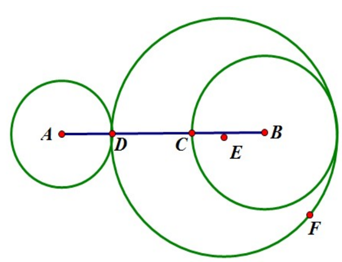

Figure 6

SYNTHESIS AND DISCUSSION

What I did	Dragging and Cognitive Modalities
Episode 1	
Using the definition of tangent circles, I found out that there are two answers for the question. In fact, if we consider the centres of three circles as vertices of a triangle, the side lengths should be 7, 7 and 6.	In this step, I used only paper/pencil to draw the related geometrical figure. By drawing two circles with radii 2cm and 3cm which the distance between the centres is 7 cm. I did not take into account internal tangents! This step took me less than 5 minutes.
Episode 2	
I am just trying to construct circles and segments. After spending about 5 minutes I found out how to construct a circle, a point and a segment labels.	Fitting GSP to my hand and this stage is devoted to discovery and selection of the relevant functions to the given problem. To find draggable points and their functions (*Exploratory dragging or Function test*). A component of Instrumentalization.
Episode 3	
I am about how to draw a segment with a specific length, and a circle with a specific radius. I draw a segment and find its length by using Measure | Length.	*Bound dragging*. Adapt the length of a segment and the radius of a circle (*Adjust dragging*). A part of instrumentation, directed to mental use scheme. About 15 minutes.
Episode 4	
I ready to display the details of given question in GSP and simulate my solution. To this end, I am following the sketch of my proof. Construct a segment with length 7 cm and two circles centered at each end points with radii 2 and 6 cm, as well as 3 and 7cm respectively. Then each circle of radius 4 centered at each of the intersection points of two circles of radii 6 and 7, is the answer of the question. So, till now the problem has two answers. What about	As I already solved the problem and have the conjecture, this step is a validation and my actions show descending control (*Dragging test* to validate). I use adapting segment and all circles to specific amounts (*Dragging to adjust*). Ignoring the potentials of GSP. About five minutes.

the other answer? Is there any other tangent circle? As I am puzzled with many circles I cannot check for the other possibilities.	
Episode 5	
I try to solve the problem without considering my solution in the first Episode. So, I construct a segment of length 7cm and then two circles centered at its endpoints of radii 2 and 3cm. Then I place a point outside them and try to construct a circle which is tangent to both of them. Now, I need to recognize draggable and non-draggable objects of the new circle as well as their functions. Then I try to find the answer by dragging the arc of the new circle.	To construct a segment and two circles with specific measures (*Bound dragging* and *Adjust dragging*). Place a point outside circles and construct a circle centered at it with radius 4cm (*Bound dragging, Adjust dragging*). To recognize draggable objects (*Bound dragging, Degree of freedom test*) and find out their functions (*Bound dragging, Function test*). Any movement in the determined point on the circle (point E, in figure4) and centre point will change the radius of the circle (*Dragging test, Degree of freedom test*). More than 5 minutes.
Episode 6	
Now it is needed to check the answer. In figure 4 we have $\overline{A'D}$ = 7 and $\overline{AD}$ = 6 so the new circle is tangent to the both. To find all possible answers, I drag the third circle around one of the two supposed circles.	I do not know any dragging mode to validate my conjecture, I just use some preliminary knowledge about tangent circles (*Dragging test*).

In the given task, since “pencil” is an instrument and can be considered as an extension of our body, I started the experiment with paper and pencil. Maybe thinking in this environment is more familiar than that of GSP.

As a result of doing the experiment in both environments independently, I realized two different types of thinking. The first one has more focus on mathematical theories and the second one has no specified structure until now.

As it is clear from the Result section, the schemes of the solutions were different. In fact, in GSP I focused more on the exact measures and the location of the circles. In other words, in such environment, this question is just an example and to solve it we may not need any theorem.

Also, the given question has three different answers, two outer tangent circles and one inner tangent, in paper/pencil environment I didn't not consider the inner tangent circles, but in GSP, by moving a circle around one of the supposed circles, it is much more likely to find it.

It seems individuals who gained mastery over a geometric software and those who just use paper/pencil see the geometry problem differently, because tools can impact our thinking and can lock us into certain patterns. Therefore, they make differences in the possible ways attacking the problem. As a result, different tools can result in different types of thinking.

This is the point that a software can facilitate understanding real mathematics. It can encourage us to think mathematically concerning at least to the given question. Thinking mathematically, in this case, is not only why the constructed circles are tangent to the two assumed circles, but also is there any other circle with this property? What are the all possible answers to the question?

Regarding my reactions during the task, I used "dragging test" and "degree of freedom test" to make a conjecture, and all the other ones were constructive modes which used for instrumentalization. In the table, I tried to list used dragging modes in order, but there were some other reactions which could reveal what was happening in my mind as well as some other effective factors such as: remove or delete an item; return from a path; priority of actions (their orders) in solving a problem; the contribution and importance of each reaction in solution.

CONCLUSION

In fact, our way of thinking, at least in the case of geometric problems, is strongly affected by the constraints and potentialities of the tool. For example, solving a problem in a paper/pencil environment is constrained in at least two ways: having trouble in drawing an exact picture of the given problem and the constraint of moving. Therefore, to solve a geometric problem using paper and pencil, we need to know the Theorems and why they are true. In this environment, the first step after understanding the question is to recall related definitions and theorems. While in GSP we can construct the problem in its real measures and it will play the role of example and theories stand up on a different level that we do not need them anymore. As a matter of fact, in the first environment it is an abstract problem whereas in the second one we have a more concrete form. Considering this experiment as a whole picture, I started with the discovery in the GSP and ended with a correct answer through the questions: Is there any other tangent circle? How many tangent circles exist? These types of questions are able to open the door to mathematical theories, that is why in the second phase there are ascending and descending modalities and it is easily observed moving from perceptions towards a more theoretical frame and back. My opinion though is that there is something greater than ascending and descending modalities. It may help to think more abstract. In fact, if there were the "third" type of tangent circle-which does not exist- I definitely was able to find it.

The following is the thought process studied in an article on "abstract thinking" and "concrete thinking" written by Ylvisaker et al. (2006). While it was published by an organization related to brain injuries, it has a section that is applicable for teaching abstract thinking.

In the following I recite some lines of it:

> Moving towards abstract thinking can be organized around the following thought processes:
>
> - searches for explanations, e.g., Are there other ways to think about this?
> - ways to evaluate, e.g., How can we decide if this is a good thing or not?
> - ways to draw inferences, e.g., If this is true, then what else must be true?

Therefore, there are some ways to move from concrete thinking to abstract one and improve learning mathematics in a computer learning environment and it seems that the possibilities a tool can suggest should be developed via rigorous argumentation.

Also, different situations require different types of thinking and each of which requires a different thinking strategy, so we should ensure that we use the best tool for the problem.

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EVIDENTIAL VS NON-EVIDENTIAL BELIEFS IN THE CASE OF ORAL ASSESSMENTS

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One of the most striking differences between the Canadian educational system and most European educational systems is the importance given to oral examinations, particularly in mathematics courses. In this paper, seven mathematics professors share their views on positive and negative aspects of oral and written assessments in mathematics. Four of the seven professors were born and educated in Bosnia, Poland, Romania, and Ukraine, and they are currently teaching in Canada. The other three professors were born and educated in Canada, the United States, and Germany, and they are all currently teaching in Germany. The results in this study show that non-evidential beliefs can affect views on oral assessments in mathematics.

INTRODUCTION

For many years the primary method of assessment in mathematics classroom has seemed to be strictly based on closed book written examinations. The USA in particular appears to be dominated by closed book examinations (Gold, Keith, & Marion, 1999; Nelson, 2011). Iannone and Simpson (2011) note that the majority of mathematics students in the UK seem to be assessed predominately using high stakes, closed book examinations at the end of almost every module. Joughin (1998) argues that the structure of the assessments today is either closed and formal, with little interaction between student and assessor(s), or open, with less structure and the opportunity for dialogue between student and assessor(s). When it comes to different types of oral assessment, according to Joughin (2010), they can be categorized into three forms: presentation on a prepared topic (individual or in groups); interrogation (covering everything from short-form question-and-answer to the doctoral viva); and application (where candidates apply their knowledge live in a simulated situation, e.g., having trainee doctors undertaking live diagnoses with an actor-patient). Although oral assessment is used in many areas, there is very little literature examining the use of oral assessments. Hounsell, Falchikov, Hounsell, Klampfleitner, Huxham, Thompson, and Blair (2007) note in their comprehensive review of the literature on innovative assessment that less than 2% of the papers address oral assessments. They reviewed the recent UK literature on 'innovative assessment' and of 317 papers considered, only 31 dealt with 'non-written assessments.' Within this category, only 13% addressed the use of oral examinations. There is a perception that oral assessments may make students more anxious than other forms of assessments. Hounsell et al. (2007) note that, "It is not clear whether oral assessments are scarier or just more novel" (p. 34). Also, Huxham, Campbell, and Westwood (2012) note that oral assessment anxiety may be primarily related to its unfamiliarity. Joughin (2007) notes, in his study of student experiences of oral presentations, that greater anxiety about oral compared with written assessments was associated with a richer conception of the oral task as requiring deeper understanding and the need to explain to others. On the other hand, Joughin (2007) points out that many students described the oral presentations as being more demanding than the written assignments, more personal, requiring deeper understanding, and leading to better learning. Today, there are many countries that still maintain an oral assessment as an important part of their assessment diet, such as Hungary, Italy, and the Czech Republic (Stray, 2001). Germany is also one of them.

Oral Examination in Mathematics

In most of the cases, students would have to take a written exam first, and then after passing the written exam, they would go to the next stage, which would be taking an oral exam. During the oral exam, students would have access to a blackboard, paper, and pen. The exam would be conducted by the course instructor, and each oral exam session could last anywhere from 30 minutes to 1 hour. Occasionally during the oral exam three or four students would be invited at the same time. The instructor would have prepared in advance a set of cards with questions of approximately equal difficulty, so a student would step in, randomly draw a card from the set of cards, and then he/she would take a scrap paper and go back to his/her desk and start working on the chosen question. After some time working on the question, each student, one by one, would go up to the board and present his/her answer to the instructor. In addition, the teaching assistant would be in the same room, monitoring students and taking the protocol. During the oral exams, usually students would be able to receive some help if needed and would receive a grade immediately following the exam. A typical card would have one theoretical question (for example 'prove the fundamental theorem of calculus') and one exercise (for example 'calculate integral': $\int arcsin^2 x \;\, dx$).

THEORETICAL FRAMEWORK

Green (1971) introduced two types of beliefs: *evidential* and *non-evidential*, defining them as follows:

> "When beliefs are held without regard to evidence, or contrary to evidence, or apart from good reasons or the canons for testing reasons and evidence, then I shall say they are held *non-evidentially*. It follows immediately that beliefs held non-evidentially cannot be modified by introducing evidence or reasons. They cannot be changed by rational criticism. When beliefs, however, are held on the basis of evidence or reasons, they can be rationally criticized and therefore can be modified in the light of further evidence or better reasons. I shall say that beliefs held in that way are held *evidentially"* (p. 48).

After having numerous discussions with mathematics professors in Canada as well as in the United States, I realized that oral examinations in mathematics courses at the university level are not present at all even though there is a number of research that indicate that oral assessments have a positive impact on students' learning of mathematics (Boedigheimer, Ghrist, Peterson, & Kallemyn, 2015; Fan & Yeo, 2007; Iannone & Simpson, 2012; Iannone & Simpson, 2015; Nelson, 2011; Nor & Shahrill, 2014; Odafe, 2006). Teachers' views "can provide significant insight into what teachers value and the relative importance they assign to different aspects of mathematics or the teaching of mathematics" (Wilson & Cooney, 2002, p. 131). In this paper, the following research question is investigated: *What are the mathematics professors' views on positive and negative aspects of oral and written assessments in mathematics?*

METHODOLOGY

The research design for this study is descriptive/qualitative. Seven participants were interviewed using open-ended questions to gather information about their personal experiences and perspectives on using written and oral assessments in the mathematics classroom. These participants were selected based on the following criteria: each participant has been exposed to oral assessment either as a student, teacher, and/or professor. In terms of recruitment, I used a methodology of snowballing, wherein I started with mathematicians whom I knew professionally, and then asked them to recommend others in the mathematics department or elsewhere, whom they suspected may have a history of experiencing or using oral assessment. Seven mathematics professors were selected for interviews: Melissa, Elisabeth, Van, Nora, Dave, James, and Jane. Melissa, Elisabeth, Van, and Nora, who were born and educated in Poland, Romania, Bosnia, and Ukraine, respectively, are currently teaching at a Canadian university while Dave, James, and Jane, who were born and educated in Canada, Germany, and the United States, respectively, are currently teaching at a university in Germany. With respect to familiarity with oral assessment, Van, Melissa, Nora, and Elisabeth had been previously exposed to oral examination in mathematics prior to moving to Canada while Dave and Jane, who were educated in Canada and the United States, had never been exposed to oral examination in mathematics prior to moving to Germany. James was born and educated in Germany, and thus, he has had a lot of exposure to oral assessment in mathematics. The audio recordings of interviews were transcribed and transcriptions were used for data analysis.

RESULTS

Based on the participants' responses, there are three major issues pertaining to oral assessment: fairness, time, and anxiety. Therefore, these three aspects of the results will be discussed in this section.

Fairness

When it came to the question of fairness, the views were divided between the following: oral exams can be perceived as less fair than written, and that there is no ideal assessment. The following comments exemplify this:

> "It may look like… in written exam everybody writes the same questions, right? But in the oral, there are different topics, different sections, so … at least when I was a student, somebody could draw a card with a topic from chapter 2 and somebody with chapter 13 and maybe that person didn't go that far as to study chapter 13" *(Elisabeth)*.

> “There is still debatable fairness because even if I have four TAs marking the same question, believe me, if I remark later, disparity, 4% or 5%. So, nothing is bulletproof… you cannot make guarantee that all will be extremely fairly assessed” *(Nora).*

The participants who had been previously exposed to oral assessments in mathematics, Melissa, Elisabeth, Van, Nora, and James, believe that there is a written record/proof of students’ work during the oral exams as each student would have a scrap paper with their work on it that would be collected at the end of the exam by the examiner. This type of belief can be considered as *evidential belief*. The following comment exemplifies this:

> “The room was quite big so they would be sitting at some distance. There was completely no chance to cheat. You’re sitting 4 or 5 meters away from another person. You were not allowed to come with any bags or anything. So, it was out of question. You come, the instructor would give you sheet of paper. You cannot bring anything with you. Pen, pencil. That’s it” *(Nora).*

On the other hand, the participants who had never been exposed to oral assessments in mathematics, Dave and Jane, do not believe that there is a written record/proof of students’ work during the oral exams. Their belief is probably based on their lack of experience with oral exams. Therefore, this type of belief can be considered as *non-evidential belief*. The following comment supports this:

> “When you have a written exam and there’s this record of like completely detailed record about what happened on the exam, then the student has some sort of form of recourse if they feel they weren’t, you know, graded correctly. And it’s there, it’s written and whereas, with the oral exam you’re kind of just taking notes, you know, maybe there’s a second person in the room who’s taking notes, but it’s kind of a sketch of what’s going on” *(Jane).*

Time

When it came to the second issue pertaining to oral assessment, issue of lacking time to perform oral exams, it was interesting to see that the majority of participants believe that because of large class sizes in their mathematics courses, it is difficult for them to find the time to perform oral exams. This can be also considered as a *non-evidential belief* for two reasons. The first reason has to do with the fact that those participants who are currently teaching in Canada spend a large amount of time assessing students in their mathematics courses, using different forms of assessments. Therefore, if there is a belief that the lack of time can be an issue for conducting oral exams, then how is it possible to find the time to assess students very frequently? The following comment exemplifies this:

> “So sometimes in Foundations of Analytical and Quantitative Reasoning course for example somebody looking from the outside may say ‘oh you're assessing them so much’ because we have 10 written quizzes, 10 homework assignments, many LONG CAPA quizzes. Every week there is a LONG CAPA quiz and two midterms and the final exam. So in the end the final grade is out of 30, 34 grades or something like that, so someone from the outside will say ‘what are you doing? Why are you assessing them so much?’” (*Elisabeth).*

The second reason is due to the large class sizes in mathematics courses. The question that I raised here was: Does class size really matter for conducting oral exams? The reason I asked this question was because when I was comparing participants' average math class sizes that they are currently teaching with average math class sizes during their undergraduate studies and/or their prior teaching, I realized that the numbers were quite similar. These numbers are shown in Table 1.

Participant	Average class size (Past)	Average class size (Present)		
	Lecture	Lecture A	Lecture B	Lecture C
Dave	-	80-150		
Elisabeth	120	30-35		
Jane	-	3		
Melissa	240	200-300	80-150	30-40
Nora	100-125	100-500		
Van	120	100-500	25	60

Table 1. Past and present average class size

Anxiety

In terms of anxiety, the views were divided between the following: the level of anxiety would be higher in oral than written exams, and that it would be hard to determine which type of exam could cause more or less anxiety among students. The following comments exemplify this:

> "So, there is a quite bit of pressure to perform in a short period of time and that's not easy… I believe with oral exam there is a higher level of anxiety" *(Van)*.

> "When I was a younger, I used to like the oral examination because I was very spontaneous and I liked to show off as a kid … but then in the university, I got to be a little bit shy to get together with people who are really mathematicians… I wasn't very good at oral examination. I was very shy and if I'm put now to grade people by oral examination that will be really hard for me especially if I have all the other people watching…" *(Elisabeth)*.

> "There was anxiety during the exams definitely, but – and whether it was more before written or oral, well … on average it was probably more anxiety but not significantly because there were students who preferred oral examinations, they felt that they could demonstrate their knowledge better… but overall my experiences from oral exams were positive" *(Melissa)*.

> … "it depends on the type of personality, I would say. I cannot generalize here. But from what I have seen around me, oral exam has its own anxiety, written exam has its own anxiety… in the written exam, the anxiety is that you can get something which you completely don't know. You studied everything and you didn't study this much, right? And you get exactly the questions, which are related to that two chapters, which you

> missed. What can you do? Nothing. Right? When it's an oral exam, there is anxiety because you talk eye to eye… in my country, it was normal" *(Nora)*.

Even though there might be a general assumption that the level of anxiety in oral exams is higher than in written exams, not all participants agreed on that. When Nora was asked about her experience with students taking written exams, she responded:

> "I have seen young men, not girls, young men who were sitting and shaking like that in the written exam… we had 700 students in the gym writing. I thought one girl would need to go to the emergency. One very good student was literally losing her mind, because there were so many people sitting around her, in completely unfamiliar settings in the gym where she has never been before… she couldn't perform. She got her much lower grade than she was actually able to get if there was a chance to talk."

DISCUSSION AND CONCLUSION

The question that could be asked here: What does it mean for any form of assessment to be considered as fair? Is fairness of assessment related to its objectivity? Does objectivity in mathematics assessment exist? Romagnano (2001) believes that all assessments of students' mathematical understanding are subjective, and that objectivity does not exist. Also, Romagnano (2001) thinks that a conclusion about a student's knowledge would require the teacher's judgment, and, therefore, "No "objective" assessment occurs; subjective—that is, human—knowledge, beliefs, judgments, and decisions are unavoidable parts of any assessment scheme" (p. 36). Human judgment about mental constructs is introduced when test designers decide "what items to include on the test, the wording and content of the items, the determination of the 'correct' answer, . . . how the test is administered, and the uses of the results" (FairTest: The National Center for Fair and Open Testing). On the other hand, there was no strong evidence here that showed if the number of students could be an important factor for conducting oral exams in mathematics. It seemed that despite the large classes, oral exams still played an important part in assessing students in mathematics for those who strongly favoured them. For instance, despite having only three students in her class, Jane was someone who strongly disfavoured oral exams, even though she had such a small class, which would be perfectly doable for conducting oral exams, she was strictly relying on written exams solely. Thus, this showed that the way Jane viewed mathematics assessments strongly affected how her mathematics classes were assessed regardless of their sizes. Moreover, Jane's non-evidential beliefs about oral assessments could not be modified even if she was provided with the evidence or reasons. Lastly, when it came to the question of anxiety, what needed to be asked here is: Is there a way to measure the level of anxiety in oral and written exams? How can we know which type of exam can cause more anxiety than the others? Is the level of anxiety determined by the personality of the exam-taker rather than the type of exam?

Overall, if we acknowledge that each student learns differently then having a common approach to assessment would be inadequate. Educators accept the need for differentiated instruction in order to deal with the individuality and variability of students, and thus, they also need to accept the need for differentiated assessment to represent the learning of the fractured student collective (Liljedahl, 2010). Also, it is very important for me to mention that in this paper I am not trying to depreciate written assessment, but merely to argue for a balanced diet of the most appropriate assessment methods for the students. I hope that the ideas and examples that I was able to present in this paper will encourage many mathematics educators to continue or to begin using oral assessment in their mathematics courses as well as to help promote discussion with their colleagues and students on this matter.

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