MEDS-C 2018

PROCEEDINGS OF THE 13TH ANNUAL MATHEMATICS EDUCATION DOCTORAL STUDENTS CONFERENCE

November 10, 2018

SIMON FRASER UNIVERSITY | FACULTY OF EDUCATION

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CONTRIBUTIONS

MEDS-C 2018 was organized by members of the Mathematics Education Doctoral Program. The conference would not have been possible without the following contributions:

Conference Coordinators: Sandy Bakos and Leslie Glen

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Program Coordinator: Arezou Valadkhani

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Lunch Coordinator: Judy Larsen

Snack Coordinators: Sam Riley and Max Sterelyukhin

Technology Support: Robert Sidley

Timers: Canan Güneş and Wai Keung Lau

PLENARY SPEAKER

Kitty Yan

WHAT'S THE STORY? IDENTIFYING KEY IDEA(S) IN PROOF IN UNDERGRADUATE MATHEMATICS CLASSROOMS

The mathematics education literature reveals an ongoing interest in fostering students' ability to construct and reconstruct proofs. One promising tool is the concept of "key idea". This study investigates how and how well undergraduate mathematics students identify key ideas in a proof and how they use them in reconstructing it. Drawn from an online survey and students' work, the findings show that while most of the students reported that they consciously identified key ideas in proofs, they varied widely in their interpretation of the concept itself. When asked to identify key ideas in the proof, though the majority of the students came close to capturing a key idea of the proof, very few were able to point to an idea that helped them both understand the proof and reconstruct it. The findings suggest that mathematics instructors need to extend considerable support to students by drawing their attention to features of proofs that are candidates for key ideas.

ABSTRACTS

Sandy Bakos

FINGERING IT OUT MULTIPLICATIVELY

This paper examines the design of an iPad touchscreen application that provides children with the opportunity for direct mediation through fingers and gestures. Explorations of a pair of third-grade students with a digital technology (TouchTimes) is described, in which engagement with number in a multiplicative sense draws on a singular interaction between the eyes and the hands. Using a theoretical perspective informed both by tool use and by embodiment in mathematical thinking and learning, I seek to gain insight into the affordances of the TouchTimes app in the development of multiplicative awareness in young children, with a specific focus on the multimodal nature of their mathematical interactions.

Jason Forde

THE (IMPLICIT) MATHEMATICAL WORLDVIEW OF RICHARD SKEMP: AN ILLUSTRATIVE EXAMPLE

In order to further advance a perspective in which mathematics is reconceived as the science of material assemblage, this paper deconstructs the implicit mathematical worldview of Richard Skemp, using it as an illustrative example of the ways in which problematic classical dualisms still impact modern thinking in the space of mathematics education. Inconsistencies within Skemp's articulations about the nature of mathematics are identified and critiqued from a non-classical viewpoint that extends forward from a radical enactivist perspective and aligns with a new materialist approach to embodied cognition. Alternative interpretations are given that more closely attend to the fundamental role that mathematics plays in both the structure and restructuring of all matter.

Leslie Glen

STUDENT-CENTRED COMMUNITY COLLEGE MATHEMATICS: AN RME EXPERIMENT

Because students at the tertiary level benefit from a student-centred, rather than a teacher-centred classroom approach as much as those in primary or secondary school, the theory of Realistic Mathematics Education (RME) was used to trial a modified Teaching Experiment in a remedial community college class where students are expected to learn the algebra of linear equations in two variables. The experiment is described and student responses to both team and individual assessments are analysed. The results strongly indicate that the theory of RME and a modified Teaching Experiment approach can be a successful combination in improving *community college student uptake of the concepts in a unit covering linear equations in two variables.*

Canan Güneş

LINGUISTIC CONFLICTS IN TEACHING POLYGONS IN A BILINGUAL MATHEMATICS CLASSROOM

This case study investigates language related conflict situations which occurred between students' expected and actual responses to mathematical tasks about polygons in a bilingual classroom. The data were created from a video record of a mathematics classroom in Hong Kong. Data analysis was based on Schoenfeld's framework "Teaching for Robust Understanding" and was focused both on the nature of the conflicts and on the teacher's reactions to these conflict situations. The results show that language conflicts occurred during the delivery of mathematical content and during the assessment of students' mathematics knowledge. The teacher used linguistic strategies, gestures, and vocabulary exercises when the conflicts arose.

Victoria Guyevskey

CONSTRUCTION IN DGE: LEARNING REFLECTIONAL SYMMETRY THROUGH SPATIAL PROGRAMMING

The goal of this paper is to gain insight into the construction process in a Dynamic Geometry Environment (DGE) and to see how, through modelling with geometric primitives, students come to understand the abstraction embedded in the concept of reflectional symmetry. Closely following a team of two upper elementary students as they construct a Leonardo da Vinci (or mirror-writing) machine in Web Sketchpad using geometric primitives, I describe and analyse several computational thinking (CT) practices that have emerged during the construction process. I then show how these practices supported the development of spatial reasoning skills and learning of geometry.

Judy Larsen

SOURCES OF COMMUNITY COHERENCE IN A SOCIAL MEDIA NETWORK OF MATHEMATICS EDUCATORS

An unprompted, unfunded, and unmandated mathematics teacher social media community is thriving and is touted by members as one of the best forms of professional development they have experienced. However, newcomers often find it confusing and difficult to navigate due to the frequency and mass of content shared. Although the space can seem chaotic, order emerges, and is informing of mathematics teacher needs, interests, and issues. This paper explores how order emerges in this community and considers the implications it has on the space of possibility it offers.

Wai Keung Lau

ACTUAL INFINITY AND POTENTIAL INFINITY: A CASE OF INCONSISTENCY

Despite the notion of infinity having been studied since ancient Greece, results are still controversial among academic circles, including the schools of philosophy, theology, logic, and mathematics. This paper provides a brief synopsis of the notion of infinity, including actual infinity and potential infinity, which is motivated by two views. First, high school students have cognitive conflicts when they are comparing different sizes of infinite sets. Second, actual infinity (e.g. Platonic or Cantorian) and potential infinity (e.g. Aristotelian or Kroneckerian) seem different. However, in reality, they are often treated as identical. This paper attempts to use Dubinsky's APOS theory and Sfard's dual nature of mathematical conceptions to indicate that cognitive conflict can improve students' intuitive thinking skill and argues that actual infinity can coexist pragmatically with potential infinity.

Minnie Liu

REDUCING REALITY AND REDUCING COMPLEXITY

When students work on modelling tasks, they simplify and idealize the situation to generate a real model to represent the situation. This study investigates the strategies students employ during the simplification process and finds two categories of strategies: reducing reality and reducing complexity.

Sam Riley

HISTORICAL CONTEXT IN MATHEMATICAL TEXTBOOKS

This paper analyses a linear algebra textbook to determine the reasons behind inclusion of historical sidenotes. Viewed through the lenses of Constructivism and Situated Cognition, the data is coded for either a mathematical purpose or a humanizing purpose. These codes are expanded on to explain how the sidenotes could specifically be used by a student to either situate the mathematics in history, construct the mathematics themselves, or to invite the student to do mathematics.

Sheree Rodney

EMBODIED CURIOSITY IN THE MATHEMATICS CLASSROOM USING TOUCH-SCREEN TECHNOLOGY

In this paper, I use data collected through video recordings from K-2 children aged between five and eight years old to discuss how touch-screen technology TouchCounts and its unique capabilities provide an outlet for students' bodily movement. In doing so, I draw on a self-generated theoretical construct called Embodied Curiosity (EC), *which has its roots in embodied cognition, to show that human bodies are essential components of shaping the mind and that students experience mathematical understandings through their bodies. I argue that human curious behaviour translates into bodily movements due to human and non-human agency, which leads to possibilities of constructing mathematical meanings.*

Annette Rouleau

INSIDE TEACHER TENSIONS: EXAMINING THEIR CONNECTION TO EMOTIONS, MOTIVES, AND GOALS

This paper examines tensions faced by mathematics teachers and their effect on teachers' actions using constructs from activity theory. Findings suggest that emotionally laden tensions can reveal motives, and impact teachers' goals by altering, prioritizing, or strengthening them. Therefore, in the relationship between emotions, motives and goals, tensions can be understood as drivers of teachers' actions.

Robert Sidley

ARE THEY GETTING ANY BETTER AT MATH?

While the goal of improving how students do mathematics is fundamental to the endeavour of mathematics educators, how and in what ways students improve over time is unclear. This study examines a Calculus 12 lesson on differentiation strategies to identify how students mathed and, through Variation Theory, contrasts the likelihood of improvement given the opportunities afforded students who worked alone compared to those who collaborated on white boards to work through practice questions.

Max Sterelyukhin

EXPERIENCING LEARNING MATHEMATICS AND REFLECTION: CALCULUS 12 PARTICIPANTS' STUDY

This study focuses on the assessment strategy that was designed in the 2017-2018 academic year in two Calculus 12 classes. Students' affect was at the centre of the research questions thus clinical interviews were used to create data on the relationship with mathematics as well as personal reflections on the learning of mathematics in the given year and overall in students' experience in school. Grounded Theory guided the research approach as themes began to emerge following with the analysis and conclusion of usefulness of these types of questions for students to reflect upon as the results were surprising and pleasing from the mathematics educator point of view.

Arezou Valadkhani

AN EXTENSION OF TOULMIN’S SCHEME TO DISCUSS THE WAY A MATHEMATICIAN DETERMINES A CONDITIONAL STATEMENT IN DIFFERENT CONTEXTS

The aim of this paper is twofold. Firstly, to explore how a successful mathematician determines conditional statements in different situations, namely mathematics, logic, and everyday context. Secondly, to extend Toulmin’s argumentation model in order to prepare the created data to be analysed.

Milica Videnovic

TENSIONS BETWEEN THE VIEWS ON WRITTEN AND ORAL ASSESSMENTS IN MATHEMATICS, AND MATHEMATICS ASSESSMENT PRACTICE

In this paper, seven mathematics professors share their views and experiences with teaching and studying mathematics in oral and non-oral assessment cultures. These participants come from Bosnia, Poland, Romania, Ukraine, Canada, the United States, and Germany. The results show that schooling and teaching experience as well as the lack of schooling and teaching experience with oral assessments in mathematics, institutionalized mathematics assessment norms, and socio-cultural assessment norms can influence views on oral assessment in mathematics.

FINGERING IT OUT MULTIPLICATIVELY

Sandy Bakos

Simon Fraser University

This paper examines the design of an iPad touchscreen application that provides children with the opportunity for direct mediation through fingers and gestures. Explorations of a pair of third-grade students with a digital technology (TouchTimes) is described, in which engagement with number in a multiplicative sense draws on a singular interaction between the eyes and the hands. Using a theoretical perspective informed both by tool use and by embodiment in mathematical thinking and learning, I seek to gain insight into the affordances of the TouchTimes app in the development of multiplicative awareness in young children, with a specific focus on the multimodal nature of their mathematical interactions.

INTRODUCTION

Multiplicative reasoning is the ability to work flexibly and efficiently with "the concepts, strategies and representations of multiplication (and division) as they occur in a wide range of contexts" (Siemon, Breed, & Virgona, 2005, p. 2), concepts that include direct and indirect proportion. Such reasoning involves learners viewing situations of comparison in a multiplicative sense rather than an additive one. As students progress to larger whole numbers and to decimals, fractions, percentages, ratios and proportions, multiplicative reasoning becomes key to many mathematical situations found in middle school (Brown, Küchemann, & Hodgen, 2010).

In the primary grades (K–3), however, repeated addition is commonly used for introducing multiplication and becomes firmly entrenched as the dominant perception of multiplicative situations (Askew, 2018). This can and does become problematic when students begin to engage with mathematics that requires a direct capacity to think multiplicatively (e.g. Siemon et al., 2005). Consequently, it appears that multiplication may prove a crucial turning point in student learning, possibly a turnstile for future mathematical competence. Rather than rely exclusively on repeated addition, approaches need to be developed and implemented earlier that highlight the function aspect of multiplicative reasoning that is so critical to success with mathematics.

Since Vergnaud (1983) wrote about the conceptual field of multiplicative structures, considerable attention has been given to the comparison of quantities using multiplicative thinking. Extensive research has documented difficulties students have with employing it in middle school and beyond, which Brown et al. (2010) claim has not improved since the 1970s. Limited student experience with different multiplicative situations is also proposed as a contributing factor to this problem (e.g. Downton & Sullivan, 2017). Furthermore, Askew (2018) contends that the lack of development of

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multiplicative reasoning in the primary grades is "a consequence of predominant approaches to teaching multiplication limiting access to opportunities through which thinking functionally can emerge" (p. 1).

Multiplicative reasoning and digital technology

Digital technology is providing new resources and means that show promise in supporting the mathematical learning of young children. The multitouch affordances of *TouchCounts* (TC) enable children to produce and transform objects through direct contact on an iPad screen. In their analysis of video excerpts of young children working with TC, Sinclair and Pimm (2015) describe new counting and early arithmetic opportunities enabled by the visible, audible and tangible design of this app. Digital technology allows children to create the "visual images of composite unit structures in multiplicative situations" (p. 306), that Downton and Sullivan (2017) argue are fundamental to the development of multiplicative reasoning.

In an extension of TC to include multiplication (provisionally named *TouchTimes* (TT)), children will be able to identify certain aspects of multiplication with distinct handedness, thus enabling new gestural experiences that provide direct feedback through both symbolic and visual representations. This paper describes an exploratory study of the affordances of TT for young children's multiplicative thinking, with a particular focus on the multimodal and joint nature of the mathematical activity of two girls working together on a single iPad.

Brief description of *TouchTimes*

The iPad application, *TouchTimes*, initially displays a blank screen split in half by a vertical bar. A user can place and hold fingers on one side of the screen to create coloured discs, called "pips". Each finger that maintains continuous contact with the left side (LS) produces a different coloured pip (Figure 1a) and the corresponding number symbol appears on top of the screen. When the user taps her finger(s) on the right side (RS) of the screen, a unit of coloured discs appears. These units, called "pods", are comprised of the coloured pips that correspond to the pips being created by the user's fingers on the LS at that moment (Figure 1b). The shape of the unit reflects the configuration of the fingers on the LS. As each tap creates a new pod, *TouchTimes* displays the number sentence that corresponds with the pips and pods created by the user. When a finger is taken off a pod, the pod remains on the screen, becoming slightly smaller, so that users can create many pods. As long as at least one finger remains in contact with the LS, the pip(s) are maintained within the pods, but when all fingers are removed, the pods disappear ("multiplying by 0"). Contact with the screen can be made one finger at a time or several fingers simultaneously.

TT 'takes care of the multiplying', both in terms of making sure that the pods are reflective of the number of pips on the LS, and in terms of ensuring that the equation on the screen corresponds to the pips and pods that have been created by the user. Embodying a model similar to Figure 1c (Boulet, 1998, p. 13), TT is designed to be a gesture-based, multimodal environment for multiplication that is multiplicative rather than additive. While the latter is privileged in approaches that rely on repeated addition, the former involves the co-ordination of two quantities. One way to conceptualise this action is to see the LS touches as the number of pips which are unitised into pods, with the pods then unitised into the product, as in the Davydovian approach (see Boulet, ibid.). In this view, 3 x 4 = 12 is read as the multiplicand times the multiplier equals the product, which reverses the typical North American approach (3)(4x). It is also possible to see the pods as groups of pips, which can be understood in terms of repeated addition. However, the simultaneity of the two-handed touching retains less of the temporal, sequential sense of repeated addition.

Figure 1: (a) Creating 3 pips (b) Creating 4 pods (c) Multiplicative model

THEORETICAL FRAMEWORK

The theoretical orientation of this study draws upon theories of embodiment and the relation between bodily movement and mathematical meaning-making (see Nemirovsky et al., 2013; Radford, 2009). I take the monist ontological position found in inclusive materialism (de Freitas & Sinclair, 2014) on the nature of body and mind, which does not subordinate sensorimotor actions to thinking, but instead recognizes that new ways of moving one's body *are* new ways of thinking. Given this orientation, and the gesture-rich design of TT, I am particularly interested in the *structured acts of gesturing* that arise through the use of TT. The epistemic and communicative nature of gestures has been well documented in the literature (see Sinclair & de Freitas, 2014) and warrants attention to gestures as particularly relevant structured acts of moving. Since multitouch environments enable children to work together, jointly structured acts of moving will also be a focus. Therefore, I will be investigating how a pair of students' interactions with TT prompts new gestures and how these new structured acts of moving are related to multiplicative thinking.

METHODS

The data for this paper comes from an exploratory conversation conducted as part of an iterative design experiment aimed at refining the TT prototype and developing appropriate tasks for use with grade two and three children. Two girls, whom I refer to as Jacy and Kyra, were chosen by their classroom teacher as a pair of students who had not yet used TT (three pairs of children had already participated). This interaction occurred in an elementary school in a culturally diverse and affluent neighbourhood in British Columbia, where the interviewer worked for approximately 30 minutes with the pair, in a setting separate from their classroom and teacher. A video-recording of the interaction was created, and the drawings produced by the girls were retained. The children were initially given time to become familiar with TT through independent exploration prior to any specific requests from the interviewer. Given that this was the pair's first encounter with TT, it provided an opportunity to observe how the girls made sense of the app and if their interactions with TT would lead to an ability to identify certain multiplicative aspects with distinct handedness.

After a period of about seven minutes for free exploration, the interviewer began to use prompts to focus the attention of the children on certain features of their creations on the iPad screen. Using TT, in conjunction with the significant presence of an adult, the aim of this research was to explore the influence of using gesture-based means for conceptualising, visualising, experimenting with and communicating about multiplicative relationships with students in primary grades (K–3).

Data analysis

In order to account for the multimodal, distributed nature of the phenomenon seen in the video-recordings, an *orchestral transcription* was produced, of ten-second increments involving three separate, but interacting, modalities: voices, hands and the iPad screen itself (Figure 2). The top three rows were designated for the voices of each child and the interviewer, thus providing a way to sequence the speaking visually in a manner that would effectively display overlapping voices. The section beneath the voices contains descriptions of what the children's hands were doing on the left and right sides of the iPad screen, in time to the speaking above.

Voices	Jacy		Laughs.............................
	Kyra	Wait, now I'll make a three. And then… laughs	They're dancing! Laughs
	Interviewer		
Hands	Left Screen	3 finger simultaneous touch (RH)	Middle finger repetitive taps (RH)
	Right Screen	Pointer finger touch (RH)	Pointer finger touch (LH)
iPad Screen	Top of Screen	3 3x1=3	2x1 2x2 3x2 2x2 3x2 2x2
			 
	Left Screen	3 pips	2 pips 3 pips 2 pips 3 pips 2 pips
	Right Screen	1 pod	2 pods

Figure 2: Orchestral transcription

When the importance of the iPad screen itself became apparent, additional rows were included to describe what could be seen on the top, left and right sides of the screen, supplemented by screen shots. The orchestral transcription enabled patterns within the structured acts of moving to be more easily identified throughout the interaction.

RESULTS

From the orchestral transcript, two intervals were chosen that illuminate some of the gestures made by the girls that are relevant to multiplicative thinking. Prior to presenting these two intervals, however, I will describe the pair's initial interactions with TT, which began with the researcher giving the pair permission to "play a little bit". Each girl placed and removed her pointer finger on and off the screen in random motions. After approximately fifteen seconds, Jacy placed her thumb on the screen in addition to her pointer finger, creating two pips. Kyra touched the screen next, creating a pod containing two pips (Figure 3a). Jacy then began to use both hands. It is shortly thereafter that Jacy said, "How do you get those mini ones?", thus marking a transition from random tapping to more intentional actions on the iPad screen.

Disappearing pips and 'dancing' pods

After five minutes of exploration, Jacy instructed Kyra to "Wait, just press a lot. Press your whole hand". Kyra responded by placing all five RH fingers simultaneously on the LS and waited while Jacy created pods on the RS one at a time with her index finger. Kyra abruptly removed her hand from the screen, causing both girls to laugh. At this point, Jacy began directing the creation of pips through the placement and removal of Kyra's fingers on the LS. After Kyra returned her five fingers to the iPad, her pinky and thumb were physically lifted from the screen by her partner. Possibly confused, Kyra briefly removed all fingers, then placed them onto the iPad once more. Jacy tried again, telling Kyra to "Wait. Put those [indicating Kyra's RH] and then take up your pinky and thumb". Kyra placed three fingers on the LS, while Jacy created one pod on the RS. This clearly was not what Jacy was after, and she further instructed Kyra to "Press your pinky and thumb away. No wait. Put your pinky and thumb down [Jacy physically pressed Kyra's pinky and thumb down (Figure 3b)], and now take them away". The pod that Jacy was 'holding' on the RS changed from a composition of three pips to five pips and back to three pips as Kyra alternated rhythmically, which in turn made the pod look like it was swinging back and forth (Figure 3c). Each time Kyra's pinky and thumb touched the screen, the resulting pips changed colour. After Jacy created a second pod, Kyra used her free hand to point to the pods and declare with a laugh that "They're dancing!".

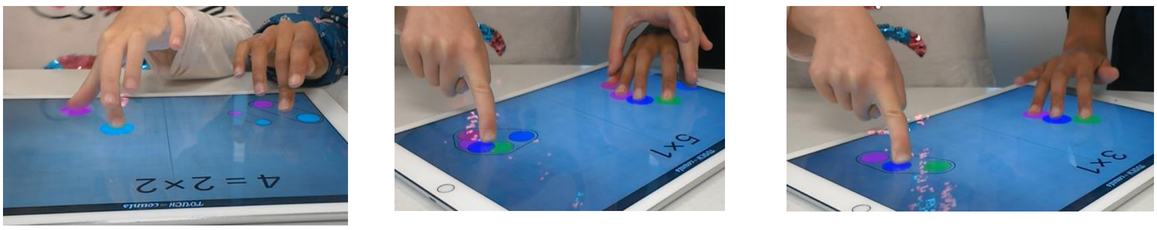

Figure 3: (a) Two fingers (b) Pinky and thumb down (c) Pinky and thumb up

Throughout this brief episode, Jacy became interested in exploring how the creation and deletion of pips affected the shape and colour of the pods. This complex interplay between the girls and TT involved a co-ordination of two pairs of hands, resulting in a *joint holding and repetitive-tapping gesture*. The appeal of the 'dancing' pod seemed to draw the girls' attention to the relation between the RS and LS finger touches, which is significant in terms of co-ordinating two quantities: the multiplicand (three or five pips) and the multiplier (one pod).

Moving backwards

After seventeen minutes, the researcher asked how the four, in the product (2 x 2 = 4) at the top of the screen (Figure 4a), could be made into a 12. Kyra said, “Okay, just let go Jacy” and nudged Jacy’s hands off the screen, which reset TT. Maintaining control of the iPad, Jacy counted and placed her fingers, "One, two, three, four” on the screen. On “four”, however, two fingers made contact with the screen rather than one, resulting in five pips. Kyra pointed out, “No, that’s more than four”, and tried to lift Jacy’s pinky finger from the screen. After removing all fingers, Jacy placed four fingers sequentially on the LS, while using her pointer finger on the RS to create a single 4-pod, stating, “I created *a* four”. Kyra echoed, “You created a four”. Jacy reset TT, created four pips by simultaneously placing four fingers on the LS, and then each girl created a pod, with Kyra quickly adding a third and final pod on the RS.

Again, resetting the app by removing her fingers, Jacy instructed Kyra to put four on the screen, which Kyra did with a simultaneous four-finger gesture. Using her pointer finger on the RS, pods were sequentially created by Jacy, while the equations flashed across the top of the screen…4 x 1= 4, 4 x 2 = 8, 4 x 3 = 2, 4 x 4 = 16. Jacy did not appear to notice these numbers until she asked, “What number are we going to?” 4 x 5 = 20, 4 x 6 = 24, 4 x 7 = 28. When the researcher responded, “Twelve”, Kyra said, “Four, eight, twelve”. Not seeming to hear this, Jacy looked at the equation, appearing unsure how to proceed. The researcher suggested, “You can put them in the trash if you want”, and Kyra counted her fingers in what seemed to be four, eight, twelve. Meanwhile, Jacy dragged pods to the trash (Figure 4b), while commenting on the equations at the top of the screen, “Twenty, sixteen and then it will be twelve”. Kyra stated, “Yes, four times three is twelve”. Kyra again pointed to and counted three of her fingers (Figure 4c), while confirming, “Yeah look…four, eight, twelve”.

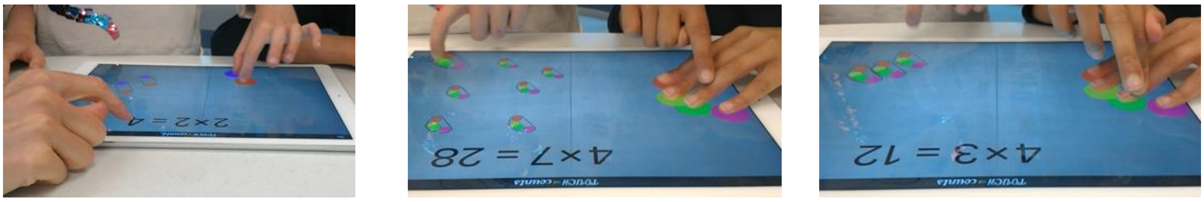

Figure 4: (a) Pointing to the product; (b) Pods to the trash; (b) 4, 8, 12

The girls had transitioned away from single touches and were now making a *simultaneous touch* gesture, as in the four-finger gesture described above. The interviewer’s question was meant to prompt the girls to notice the effect of increasing pods on the product displayed at the top of the screen. However, rather than creating the desired product of 12 by creating additional 2-pods (Figure 4a), which were on the screen at the time of the interviewer’s question, the girls started over, changing the number of pips, which is reflected by the composition of the pods. A product of 12 can be created in multiple ways, however right from the outset, when counting her fingers, Jacy seemed oriented towards creating four pips, thus thinking of 12 as a multiple of four (though it is possible she was counting to four because of the 4 on the screen).

After Kyra confirmed that Jacy had made four pips, each girl added a pod on the RS. When Kyra created a third and final pod, the pair had a product of 12, though Jacy did not seem to be aware of this. Instead, she directed Kyra's hand placement, which induced a *joint skip-counting* gesture in which Kyra was responsible for holding pips, while Jacy's sequential touches created pods. Unlike the skip-counting that occurs in many classrooms, where children count intransitively or read off a number line, the skip-counting in TT requires a co-ordination of the pips and pods. In the case of the skip-counting by four, each new pod created a new 4-pip unit and Jacy's comment, "I created *a* four", rather than "I created four" indicates some awareness of this.

When Jacy realized that she had a product of 28 instead of 12, she removed the extra 4-pods by dragging them to the trash. This self-correction is evidence of her attempt to co-ordinate the *joint skip-counting* gesture (tangible expression) with the products (symbolic expression). Kyra, who was responsible for maintaining the four pips on the LS and did not create the pods that stand for the unit, perhaps needed to physically create her own units, which she did by skip-counting on her fingers. As a result of the co-ordination of two quantities required in the *joint skip-counting* gesture, the skip-counting in TT is more multiplicative than additive.

DISCUSSION

The intent of the TT design is for learners to notice the relation between the number and colour of the pips, and the shape and content of the pods, as this is the basis for the multiplicative operation. Although numerous pips and pods were created, it was not until the *joint holding and repetitive-tapping* gesture that the girls seemed to co-ordinate this relation. Indeed, it seemed to be the shifting pod shape, and then the changing pip colour that initially drew their attention to this relation. The *joint holding and repetitive-tapping* gesture, which arose from manipulating the screen in exploratory ways, became a gesture for expressing the relation between pips and pods. The girls could make visible the effect of changing the unit through tapping on and releasing pips, which produced the changing size, shape and colour of the pods. When there were multiple pods, the *joint holding and repetitive-tapping* gesture used to create pips produced a multiplicative effect in which each tap also produced a new pip in *every* pod *simultaneously*.

The *joint skip-counting* gesture was more multiplicative than additive, although a gestural shift from tapping one finger at a time, to tapping several fingers simultaneously to create pods would provide an even stronger multiplicative effect. Since the girls rarely created multiple pods simultaneously, such a gesture may involve a more difficult co-ordination and need to be prompted by a particular task.

CONCLUSION

Designed to support the development of multiplicative thinking, TT provides young learners with ways of thinking about multiplication that are not solely dependent upon repeated addition. In the 30-minute episode reported above, two intervals in which the girls created and sustained a particular structured way of moving their hands were described as being relevant to the development of multiplicative thinking. The two primary gestures discussed were the *joint holding and repetitive-tapping* and the *joint skip-counting* gestures. The former prompted and enabled the girls to attend to the relation between the number/colour of pips and the pods, and thus the co-ordination of the two quantities. The latter enabled the girls to produce multiples of a number determined by the pip count. Attention to the symbolic expression available on the screen was particularly salient in the second interval. In future experiments, the insights gathered here will be used to design tasks that can effectively prompt and support similar types of gestures, and to link these gestures to other TT-based actions.

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THE (IMPLICIT) MATHEMATICAL WORLDVIEW OF RICHARD SKEMP: AN ILLUSTRATIVE EXAMPLE

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In order to further advance a perspective in which mathematics is reconceived as the science of material assemblage, this paper deconstructs the implicit mathematical worldview of Richard Skemp, using it as an illustrative example of the ways in which problematic classical dualisms still impact modern thinking in the space of mathematics education. Inconsistencies within Skemp's articulations about the nature of mathematics are identified and critiqued from a non-classical viewpoint that extends forward from a radical enactivist perspective and aligns with a new materialist approach to embodied cognition. Alternative interpretations are given that more closely attend to the fundamental role that mathematics plays in both the structure and restructuring of all matter.

INTRODUCTION AND THEORETICAL ORIENTATION

While numerous perspectives concerning the nature of mathematics have been proposed within the discourse of our field, I highlight here an especially salient one set out within the work of Richard Skemp. Skemp's articulations are of particular interest because of the manner in which they present a tentative and somewhat unconventional perspective on the nature of mathematics whilst simultaneously being bound to problematic (classical) dualisms. Certain internal inconsistencies inherent to Skemp's mathematical worldview ultimately highlight foundational characteristics of mathematics that are worth attending to more closely. To be clear, Skemp's worldview is a singular example that I leverage in order to illustrate how classical modes of thought still pervade modern thinking within our field. Considering the prominence of Skempian thought in the more recent history of mathematics education, it also presents an opportunity to demonstrate the value of implementing an alternative perspective. Thus, it is through the upcoming discussion that I not only identify inconsistencies embedded within Skemp's notion of what mathematics is, but also aim to rectify those inconsistencies through an interpretation aligned with the sense in which I see mathematics as *the science of material assemblage*.

The discussion to follow advances Campbell's (2001, 2003) radical enactivist view of mathematics as *the science of organisation*, as well as the approach to embodied cognition espoused by de Freitas and Sinclair (2013), both of which call for significant changes to the ways we characterize the unified, embodied experience of mathematical structures and processes.

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RESEARCH METHODOLOGY

Though this paper does not employ a formal critical discourse analysis, it does engage with and address curiosities identified through a close reading of selected excerpts from Skemp's seminal work *The Psychology of Learning Mathematics*. In effect, the primary strategy has been to attend as closely as possible to the *implicit*, and to use a more contemporary analytical lens to revisit Skemp's assertions (essentially reading between the lines, as it were). As opposed to focusing on the lineage of ideas underlying the identified curiosities, I instead take Skemp's implicit mathematical worldview as it is given and explore the implications associated with the articulated perspective(s). Where possible, efforts are made to provide a new interpretation with the capacity to overcome inconsistencies/deficiencies found within Skemp's worldview.

I purposefully emphasize the value of more recent theories of embodied cognition and their accompanying connotations about the nature of mathematics; however, the intent is not to discard or detract from Skemp's work in any way. Rather, the aim is to use Skemp's collective assertions and conjectures about the nature of mathematics as a valuable starting point upon which to ground, and with which to inform, the discussions to follow. While the notion of mathematics as *the science of material assemblage* is crucial to my overall discussion, a concise summary of what this entails shall not be offered immediately. Rather, the notion shall be elaborated upon throughout this paper, with Skemp's implicit mathematical worldview acting as a conceptual foil. Summative comments will be provided toward the end of the piece.

ANALYSIS AND DISCUSSION

There are various instances throughout the expanded American edition of his seminal work, *The Psychology of Learning Mathematics*, in which Skemp (1987) orbits around the nature of mathematics without speaking to it explicitly. Mathematical concept formation, for instance, is addressed within Chapter 2. Discussions of mathematical symbolism and structure appear in Chapter 5, with the communication of surface structure and deep structure of mathematics treated much later in Chapter 14. With all of these subthemes helping to illuminate Skemp's mathematical worldview, his notion of what mathematics actually *is* takes shape as the book progresses, with less incisive remarks appearing in formative chapters and more telling comments in ensuing ones. As an example of the former, Skemp (1987) alludes to the nature of mathematics through *non-example* when he makes the following simple distinction: "The automatic performance of routine tasks must be clearly distinguished from the mechanical manipulation of meaningless symbols, which is not mathematics" (p. 62). The subsequent discussion revolves around the meaning making that accompanies mathematical work, and while the point Skemp makes here is not especially revelatory when taken at face value (i.e. it is one that the vast majority of mathematics educators are likely to agree upon), it is his quiet assertion of what is *not mathematics* that poses something of a problem.

When reinterpreted through a view of mathematics as *the science of organization* (Campbell, 2001, 2003), and as *the science of material assemblage* more specifically, even the "mechanical manipulation of meaningless symbols" *ought* to be considered mathematics (or mathematical). That is, in terms of the organization and reorganization of material processes and structures, even basic symbol manipulation *unaccompanied by understanding* is inherently mathematical activity. To be fair, Skemp frames his own discussion around the psychology of *learning* mathematics, and his primary interest is in the role that work with mathematical symbols plays in developing *understanding* of mathematical concepts. Thus, the distinction he makes between the automatic and the mechanical (the former being attributed to knowing/thinking mathematicians, and the latter to machines that are unaware of the processes they enact) is a convenient foreshadowing of the way in which he eventually adopts and applies Stieg Mellin-Olsen's notions of relational and instrumental understanding; but it ultimately conflates the *understanding* of mathematics with the *nature* of mathematics.

In establishing his view of instrumental understanding, Skemp (1987) offers that: "I would until recently not have regarded [it] as understanding at all. It is what I have in the past described as 'rules without reasons' […]" (p. 153). Through use of the qualifying phrase "until recently", he acknowledges the change to his own preconception, essentially conceding recognition that instrumental understanding is still a *form* of understanding, if only one that is less meaningful than the relational (i.e. less *full* of meaning). In light of this, I would suggest that Skemp's earlier statement about what is *not mathematics* requires a similar amendment. Granted, the "mechanical manipulation of meaningless symbols" may not be particularly *meaningful* mathematics, but it is mathematics nonetheless. Its grounding in processes of material organization and reorganization assures this, and making such an amendment would bring Skemp's earlier remark about the nature of mathematics more in line with his later assertions about mathematical understanding. Indeed, reading between the lines of Skemp's writing bears out the interesting entanglement between the material and the meaningful, both of which are closely linked to (and possibly emergent from) the mathematical.

Skemp's Type 1 and Type 2 Theories

> A type 1 theory is an abstract, general, and well-tested mental model of regularities in the physical world. It embodies what are sometimes called laws of nature, and to qualify for this description it must have explanatory and predictive power.
>
> (Skemp, 1987, pp. 129-130)

> A type 2 theory is a model of regularities in the ways in which type 1 theories are constructed […] It is a mental model of the mental-model-building process.
>
> (Skemp, 1987, p. 130)

Following Skemp's (1987) discussions of type 1 and type 2 theories in Chapter 10, it is in Chapter 11 that he probes more deeply into the heart of his mathematical worldview by posing to readers, and possibly himself, a crucial underlying question: "What kind of theory is mathematics?" (p. 142). It is notable that Skemp ultimately concedes that mathematics does not fit into *either* of the proposed categories, and the same query is reiterated elsewhere. Though the question of theory is not answered in any conclusive sense by Skemp, his motivating educational concern *is* made clear: "[H]ow can we usefully think about teaching it [mathematics] if we do not know what kind of subject it is that we are trying to teach?" (p. 149). Even within the readership comprised of mathematics educators and teacher-researchers, I imagine that much of Skemp's intended audience might initially be taken aback by the posing of this question, if only momentarily; for a nondescript yet somehow taken-as-shared sense of what mathematics is seems fairly commonplace/ubiquitous and the question itself is rarely raised by practicing mathematics educators or teacher-researchers. Skemp further notes that theories about the teaching and learning of mathematics, as "theories about how we construct mathematical theories" (ibid.), should be distinguished from theories about mathematics itself. This is consistent with his working definitions of type 1 and type 2 theories given above, but it more importantly makes clear Skemp's conviction that a taken-as-shared sensibility about the nature of mathematics is not sufficient. This is a conviction that I share. As I see it, this broader question from Skemp does align with some of my own reasons for perturbing common notions of what mathematics is; however, there are also ways in which Skemp's view diverges significantly from my own, or rather mine from his.

Just as I invoke Wigner's (1960) reference to the unreasonable effectiveness of mathematics in the natural sciences, Skemp (1987) draws upon a similar query attributed to Einstein: "How can it be that mathematics, as a product of human thought independent of experience, is so admirably adapted to the objects of reality?" (p. 149). Though the heart of this question may be likened to that of Wigner's, I find both the *disembodiment* and *depersonalization* of mathematics within to be concerning, and I would submit that neither of these is tenable. Moreover, Skemp himself also refers to mathematics as "an activity of our intelligence" (p. 142). Though there is some value in this second claim, I amend it and the Einstein quote quite strongly by noting that mathematics should not be seen as an activity of our intelligence *alone*. This is to say that mathematics is as much of the body as it is the mind; for both are integral to the material assemblage that underlies our embodied and unified *experience* of mathematical structure.

de Freitas and Sinclair (2013) provide a powerful counterpoint to the Einsteinian and Skempian perspectives expressed above when they elaborate their notion of "the body in and of mathematics" (p. 454), and clarify that this new materialist approach to embodied cognition is one that "helps us rethink the body in/of mathematics so that embodiment entails mathematical concepts and artifacts, as well as human learners" (p. 468). The view they forward is one in which mathematics is neither divorced from experience, nor entirely driven by the intellect. Rather mathematics is not only imbued within, but closely *entangled* with the material self. "Instead of seeing [mathematical] concepts as entirely discursive or abstracted and dislocated from an inert matter, we have argued that activity should be studied for the way that new leaner-concept assemblages emerge" (ibid.). Thus, by entangling the mathematical with the material, de Freitas and Sinclair shift the emphasis away from the purely/exclusively intellectual, negating the senses of disembodiment and depersonalization that characterize the Einsteinian and Skempian perspectives.

Indeed, much of the present document is devoted to justifying a perspective in which mind and body are taken to be ontologically *indistinct*. To speak of mathematics only as a function of intellect and as independent of experience reverts us back to the classical dualisms that have already proven so problematic. To be fair, Skemp acknowledges that he does not *fully* answer Einstein's question; so I admit that there is a possibility he may not be entirely committed to these dualisms and that the overarching issue may be more related to, or grounded in, the dualism-preserving Cartesian phrasings that permeate the English language.

In close proximity to the excerpts discussed above, Skemp also claims that mathematics is not, itself, one of the natural sciences. While this could be considered true from a traditional perspective that maintains strict boundaries between the disciplines of physics, chemistry, biology, astronomy, geology, et al., it is *inadequate* within my characterization of mathematics as *the science of material assemblage*. It not only deprives mathematics of a primacy that I aim to emphasize, but also elides any sense of mathematics being *natural*, of being *innate* to the structure (and restructuring) of all matter. In contrast, from the perspective I am working to develop, it would actually be much more appropriate to conceive of mathematics as *the most foundational* natural science. If only to acknowledge the decidedly Pythagorean roots underlying this sensibility, I will even suggest that *reinstating* it as such would not be at all unreasonable. In fact, this alternative view of mathematics as the most foundational natural science (i.e. the science of material assemblage) rightly restores mathematics to a position of prominence that supersedes even the philosophy-physics of the atomists Democritus and Leucippus, who many consider to have laid the foundations for modern Western science.

Interestingly, Skemp (1987) does offer that mathematics:

> can be regarded as a conceptual kit of great generality and versatility, so valuable to anyone who wants to construct a scientific theory as to be almost indispensable. Did I say "almost"? Francis Bacon wrote: "For many parts of nature can neither be invented with sufficient subtlety, nor demonstrated with sufficient perspicuity nor accommodated into use with sufficient dexterity without the aid and intervention of mathematics." Likewise, Jeans: "All the pictures which science now draws of nature and which alone seem capable of according with observational fact are mathematical pictures."
>
> (Skemp, 1987, pp. 149 & 150)

Unlike Skemp's allusion to the almost-indispensable conceptual kit of great generality, the more evocative excerpts from Bacon and Jeans suggest that mathematics is indeed much more fundamental than the natural sciences of which we normally speak, and the above excerpt consequently sets out a slightly inconsistent pairing. Bacon and Jeans would seem to be pointing toward the primary status of mathematics, or its significance/immanence as the underlying structure of material reality, whereas Skemp's characterization of it as a conceptual kit is more suggestive of a convenient toolset that simply happens to be incredibly efficacious. This is to say that it implies a *coincidental utility* much more than it does a *fundamental status*. Again, it is notable that Skemp admits that mathematics does not fit the criteria for either the type 1 or type 2 theories outlined earlier in his work. As he says of type 1 theories: "Collectively, they form the natural sciences" (p. 130). Via his conceptual kit analogy and the quotes from Bacon and Jeans, Skemp also appears to be saying that the natural sciences are built upon mathematical foundations, yet mathematics is not itself a type 1 theory, nor is it characterized by Skemp as a *collection* of type 1 theories. I ultimately choose to favor the voices of Bacon and Jeans, which Skemp also gives more prominence than his own in the cited passage. Nevertheless, Skemp's overall message here is somewhat mixed.

Revisiting the Question of Theory

So, as does Skemp (1987), let us also return to the inciting question: "What kind of theory is mathematics?" (p. 142). In spite of the fact that the previous inconsistencies are not addressed by Skemp himself, his exploration of this query continues. He ventures into a space to which his choice of language does not seem entirely suited, yet it is worth quoting him at length.

> […] I suggest that we regard mathematics as a theory of a unique kind, having all the characteristics of a type 1 theory except mode 1 testing. It is the mental stuff of which type 1 theories are made; or to put this differently, it is pure form […] I have said on many occasions that I regard mathematics as a particularly pure and concentrated example of the functioning of human intelligence. This suggests that it is a kind of essence […].
>
> It is still hard to say why this should be. I still have not answered Einstein's question, so I offer the following as a beginning. Here is another quotation, this time from Galileo. "Where our senses fail us, reason must step in." With the help of our senses, we perceive regularities of our physical environment. These regularities are embodied in what I have called *primary concepts*. Next, by the use of our intelligence, we find regularities among these regularities – we form secondary concepts. In mathematics, we repeat this process to form more abstract concepts, representing regularities of great generality, and relations between these. All this time we are getting further away from what is accessible to our senses; yet, paradoxically, we seem to be getting closer to the essential nature of the universe.
>
> (Skemp, 1987, p. 150)

As with his earlier description of mathematics as an activity of our intelligence, Skemp still adheres to an exclusively mental characterization, belying the classical underpinnings of his perspective, and a bias toward (if not a compete commitment to) Cartesian dualism. That said, his interpretational extension of Galileo in the final paragraph above reworks the perspective by speaking to a more circulatory exchange between sense and intellect, and a more complementary interplay between inner and outer experience. The repetition of the process that he characterizes is reminiscent of the continuous circulation back and forth between the Husserlian natural and phenomenological attitudes, the double-embodiment encapsulated by Merleau Ponty's (1962) ontological notion of "flesh", and even the entangled state of being/becoming/knowing that I align with the enactive monist approach to cognition.

In my reading of Skemp, it is the curious combination of his classically dualistic statements about the exclusively mental character of mathematics, and the subsequent assertions regarding abstraction and generality that suggest he might have one foot planted on fairly classical Cartesian ground, with the other inching toward terrain where the ontological distinctions already addressed hold less sway.

At least with respect to his articulations about the nature of mathematics and its relation to the material world, I sense a sort of indecision on Skemp's part. In his discussions of abstraction (1987), he clearly states that 'more abstract' means 'more removed from experience of the outside world', which fits in with the everyday meaning of the word 'abstract' (p. 14). He also notes that mathematical concepts are "far more abstract than those of everyday life", such that the "communication of mathematical concepts is therefore much more difficult" (ibid.). At a certain level, his concession that mathematics is neither a type 1 theory, nor a type 2 theory, but a *theory of a unique kind*, the *stuff of which type 1 theories are made*, and *pure form* raises even more questions than his inquiry into what mathematics is. While it does seem reasonable that mathematical abstraction initially draws us further from the everyday experience of the outside world (i.e. into the idealized spaces of the inner world), I would also suggest that the subsequent capacity to generalize from abstraction and to use generalizations as the bases for *further* abstractions (possibly under different contexts) ultimately brings us right back to the outer world again, and I hesitate to break apart these two processes, preferring instead to envision something cyclic or circulatory and potentially even self-perpetuating.

For Skemp, it is paradoxical that the simultaneous move away from sensory experience in the outer world into the abstracted inner space of the intellect should draw us closer to the essence of the universe (I have paraphrased heavily here). However, from a perspective that reconceives mathematics as the science of material assemblage, there is really no need to consider this circumstance as paradoxical at all; for just as mind and body are already ontologically unified, so too are inner and outer experience. From a classically-grounded perspective, there may be a certain irony accompanying the sense that higher order abstraction simultaneously distances us from the outer world whilst bringing us closer to our experience of the inner world; but from the conjoined enactivist/new materialist perspective that underlies my broader mathematical worldview, that irony dissipates completely. Even reconceptualising Skemp's notion of *schema* such that it is not driven solely by intellect should not be overly problematic, particularly in light of the manner in which scholars like de Freitas and Sinclair re-imbue the cognitive *with* the material.

Essentially, Skemp's paradox is only paradoxical when one's mathematical worldview is constrained by classical modes of thought. Even Einstein's question of how mathematics can be so admirably adapted to the objects of reality can be revisited and revised in a similar way. In fact, I completely co-opt Einstein's phrasing by asserting that mathematics need *not* actually be "adapted to the objects of reality"; for it is already innate to those very same objects (i.e. it is immanent in matter). It is the viewpoint that *divorces* mathematics from the material in which the ontological problems rest. As *the science of material assemblage*, mathematics embodies the essential framework upon which material reality is built, from which its objects emerge, and according to which they (co-)evolve. This is to say that mathematics embodies the very principles according to which matter organizes and reorganizes itself, and in that sense, the unreasonable effectiveness of mathematics spoken to by Wigner is *not* unreasonable at all. In this view, mathematics not only underlies the material structure of reality, but also encapsulates the conditions and constraints through which the dynamic processes of material assemblage are manifested.

CLOSING REMARKS

I acknowledge that the space of this paper is insufficient for a full explication of the sense in which I envision mathematics as the science of material assemblage; however, it is hoped that this abbreviated deconstruction of excerpts from Skemp's seminal work has prompted the reader to consider the nature of mathematics somewhat differently. Additional work will be necessary in order to more fully/clearly demonstrate how mathematics relates to the entanglement of the material and the meaningful.

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STUDENT-CENTRED COMMUNITY COLLEGE MATHEMATICS: AN RME EXPERIMENT

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Because students at the tertiary level benefit from a student-centred, rather than a teacher-centred classroom approach as much as those in primary or secondary school, the theory of Realistic Mathematics Education (RME) was used to trial a modified Teaching Experiment in a remedial community college class where students are expected to learn the algebra of linear equations in two variables. The experiment is described and student responses to both team and individual assessments are analysed. The results strongly indicate that the theory of RME and a modified Teaching Experiment approach can be a successful combination in improving community college student uptake of the concepts in a unit covering linear equations in two variables.

INTRODUCTION

This paper examines some of the outcomes of teaching the algebra of linear equations in two variables to community college remedial algebra students using a modified Teaching Experiment approach. The concepts addressed here are first seen in a secondary school elementary algebra course, but the audience receiving instruction in the study reported by this paper were community college students enrolled in a "remedial" or "pre-college" course. Community colleges have endured a reputation as subordinate educational institutions. As a result, there is little research about community college teaching and learning. One of the aims of this paper is to redress that deficiency; another is to examine the uptake of the concepts being taught when the audience is community college students, and the approach is not the conventionally accepted lecture model, but a student-centred approach.

THEORETICAL PERSPECTIVE

Realistic Mathematics Education (RME) is the theory that rather than teach an abstract concept in a manner that emphasizes mechanical manipulation, the concept should be embedded in a "realistic" problem so as to make its solution, and the emergent mathematics, meaningful (Cobb, Zhao, & Visnovska, 2008). The core tenets of the RME theory were adapted and extended in 2000 by van den Heuvel-Panhuizen to six principles; it is from these that RME can be identified in a contemporary setting:

- the *activity* principle: "students are treated as active participants in the learning process [and] ...mathematics is best learned by doing mathematics" (van den Heuvel-Panhuizen & Drijvers, 2014, pp. 522-523)

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- the *reality* principle: "the importance [of] students' ability to apply mathematics in solving 'real-life' problems…." (van den Heuvel-Panhuizen & Drijvers, 2014, p. 523)
- the *level* principle: "students pass various levels of understanding" (van den Heuvel-Panhuizen & Drijvers, 2014, p. 523; see also Treffers, 1987)
- the *inter-twinement* principle: "mathematical content domains…are not considered as isolated curriculum chapters but as heavily integrated" (van den Heuvel-Panhuizen & Drijvers, 2014, p. 523)
- the *interaction* (or *interactivity*) principle: "learning mathematics is not only an individual activity but also a social activity" (van den Heuvel-Panhuizen & Drijvers, 2014, p. 523)
- the *guidance* principle: students have "a 'guided' opportunity to 're-invent' mathematics" (van den Heuvel-Panhuizen, 2000, p. 9)

It is important to clarify the meaning of "realistic" in the sense in which it was originally meant in the RME context. In English, the word "realistic" will likely be interpreted to mean "real", "actual", or "factual". In RME, however, "realistic" is more accurately interpreted as "imaginable".

> "The Dutch translation of 'to imagine' is 'zich REALISEren.' It is this emphasis on making something real in your mind, that gave RME its name. For the problems presented to the students, this means that the context can be one from the real world, but this is not always necessary. The fantasy world of fairy tales and even the formal world of mathematics can provide suitable contexts for a problem, as long as they are real in the student's mind" (van den Heuvel-Panhuizen, 2000, p. 4, emphasis in original).

Realistic Mathematics Education, then, is a theory that allows for the exploration of the mathematical elements of a real or realistic situation.

Also utilized in this study, the Student-Centered Learning (SCL) approach ensures that students' learning is prioritized over the teacher's needs or desires. A broad spectrum of methods is available to the instructor wishing to use SCL.

THE STUDY

"Elementary algebra" curriculum in Washington state community colleges includes linear equations in two variables. Students enrolled in such a course are expected to acquire the concepts of Cartesian coordinates, solutions to linear equations in two variables, slope, intercepts, and linear graphs. This course, like most college courses, is conventionally lecture-based.

Using a modified Teaching Experiment approach (which teaches to a whole class rather than a small group, and for which iterations were implemented from one whole class to another rather than from one session to another with the same small group), all of the concepts in this study were taught by presenting a scenario in which a distal goal is described. Students worked in teams to accomplish tasks and reach proximal goals, each of which aligned with a course objective. Each task was embedded within the scenario, and while many of the chapters of the story are fantastical, each is sufficiently believable to be considered realistic by RME's definition of that term. In the interest of brevity, only the concepts of Cartesian coordinates, slope, and slope-intercept form of an equation are discussed here.

At the start of the unit, a story is told about a missing artefact requiring students' expertise to locate. Teams are given maps of a country with about a dozen cities; the map is overlaid with a grid of indeterminate unit, and a few cities fall neatly at intersections of grid lines. The land mass is not identifiable, so any activities requiring measurements to be taken, for example, cannot be researched on the Internet. Several activities were developed, and tasks designed and delivered around the same distal goal of finding the missing item.

PARTICIPANTS

The study was implemented with a class of 23 students representing diverse populations in terms of sex, age and ethnicity. Some students entered the course with previous exposure to linear structures; however, the curriculum calls for a rigorous treatment of the subject, and as such, the unit of interest in this paper addressed the concepts of Cartesian coordinates, slope, intercepts, points on a line, graphical interpretation of linear equations, and writing linear equations.

The aim of the study was to determine whether students would take up conventional concepts if presented as embedded in a narrative. A series of activities was designed that were intended to capture students' imagination while requiring them to utilize concepts about which they had read before arriving in class. Upon finishing each activity, teams were asked to document their analysis in a "log entry".

Cartesian Coordinates: A local "convention" used in the fantasy location was explained: the capital city is known as the "origin" of commerce, labelled with coordinates $(0,0)$ and other locations on the map can be identified by referencing their distance east and north from there. Teams were given a chart containing the coordinates of a few cities and towns on the map and asked to fill in the coordinates for the others.

Slope: Referred to in the narrative as the "directional command", the fundamental idea of the average rate of change was explained in words rather than through a formula. The description of a programmable drone, too primitive to accept anything but a single value as its directional command, was meant to encourage students to see that while there are two components to the direction of flight, they work together as a single ratio of values.

Slope is a concept that many students struggle with, and this difficulty is extremely well documented (e.g. Barr, 1980; Bell & Janvier, 1981; Leinhardt, Zaslavsky, & Stein, 1990; Orton, 1984; Simon & Blume, 1994). The hope in introducing slope via a realistic scenario was that students would see the slope as a relationship first and a numerical value second, but also that the connection between these two modes of expressions would become apparent.

Slope-Intercept Form of Equation: The slope-intercept form of the equation of a line is a linear combination of slope and y-intercept. Teams were required to find the particular combination of the two values they need in order to create an equation. Students were expected to have read an introduction to the concept in the textbook, so in addition to the terminology used in the narrative, they had seen the nomenclature used in the conventional setting. This preparation meant that they already knew the form in which the equation they were asked for should be, and they were determining the combination they needed based on work they had done to find the slope and the y-intercept.

RESULTS AND ANALYSIS

In addition to finding that the approach described here more actively engaged students than does lecture, it also appears to have held up under conventional assessment. This is not to say that the modality described by this report guarantees that all students will learn better and remember longer, nor that there are no students who learn better by listening passively to a lecture, but the vast majority of students who end up in pre-college algebra at community college have already been let down and put off by the traditional academic model. For these students, delivering the same content the same way is unlikely to result in a different outcome. If students have any chance of progressing, something needs to change in order for them to learn algebra in *this* course when they could not learn it before. While multiple reasonable alternatives exist, and others are being investigated, Realistic Mathematics Education is one which has been tried and found to be a strong influence for positive change, not only in the primary classroom, but also in secondary and, as shown by this study, tertiary classrooms as well.

Team Activities

To convey the concepts required of the curriculum, students read from the textbook before coming to class and worked in teams to complete activities that used the concepts about which they had read. The expectation was that students would collaborate to answer the questions presented within the activities, but submissions were graded individually. Students in the study were required to submit their work individually; but, as they were not being assessed individually (at this point), it is not surprising that within any team, the responses were often identical. If a student was not present, he could still submit the required worksheet, but he would either need to work out the solution on his own or ask his team for help.

What follows is an analysis of the activity in which teams in the study were required to establish the equation of a line from two points on that line, although in keeping with the narrative, the instructions were given in that spirit. The activity in its entirety required teams to find the “directional command” (the slope) and the “longitudinal crossing point" (the y-intercept) and then to put these values into their correct positions in the slope-intercept form of the equation of a line.

Finding the Slope: Teams were easily able to interpret "directional command" to mean "slope", and most used the coordinates of the points they had recorded in the Cartesian Coordinates activity (not described here due to lack of space) and the slope formula to find the desired value of $\frac{32}{19} \approx 1.6842$. All teams but one gave an answer consistent with the value $\frac{32}{19}$. One team gave the fraction $\frac{15.9}{9.5}$, having been more exacting in establishing their coordinate locations in the "coordinates" activity. These responses were deemed correct given that students were measuring with very coarse tools, and the goal was to capture the idea of slope rather than accuracy of measurement.

Another team did not parse the information in the narrative correctly and ended up with a slope of 1.2. This error is based on their having selected the wrong destination for their calculations, but the calculations they did do were correct for the purpose of finding the slope.

Finding the y-Intercept: Determined algebraically, the y-intercept in the scenario had a value of $\frac{27}{19}$, or approximately 1.4211. Only one team arrived at this value; they did so by using the slope-intercept form of a line ($y = mx + b$), replacing m with 1.68 and x and y with -5 and -7 respectively (from a location set by the narrative); solving the resulting equation for b returned the correct y-intercept.

In general, during guided instruction following on from self-exposure and group discussion and negotiation, teams found the slope easily, the vast majority of them through the use of a formula (two found the slope by counting). On the other hand, only a few teams found the y-intercept by using a formulaic approach; several found the intercept by observation.

The Equation of the Line: Having determined the slope and the y-intercept, most teams were able to use these values to construct the equation of the line. A few used some form of formula which resulted in re-computation of a value or values that they had already found. Re-computation is common during exam conditions, but during the activity, students were working not only in a casual environment, but collaboratively, and with no restrictions on the resources they might use to help them answer the questions they were asked. That students can correctly find a slope given two points, but then re-compute the same slope when asked to write the equation of a line suggests that they have not made the connection between the slope of a line and the role that it plays within the equation of a line. It may be that these students are so used to finding the equation of a line by purely mechanical means that on being asked to do so, their only way forward is to work through the steps "find the slope", "plug in x and y", "solve for b", "write out the equation". That they have already found "m" eludes them because it does not fit into the process that they have memorized for this task.

Summary of Team Activity

Although some teams did not correctly interpret the instructions given, all but one individual student submission showed correct interpretation of the directional command as "slope"; more importantly, all of these students correctly analysed and computed the slope, insofar as its meaning was identified. Several also exhibited competence in finding the y-intercept, although some did so with incorrect replacement values. Finally, the vast majority correctly used the slope and y-intercept to create the equation of the line that went through the two given points, even if some of these had to recalculate the slope in order to do so. In comparison to lecture-based courses taught by this author, the collaborative and narrative-based approach to delivering this concept was considerably more successful in producing student work that correctly used the information they were given to accomplish the requested task. Because of the nature of the collaborative group, it is impossible to say whether each individual student successfully grasped the concepts from the activities, or whether some of them simply copied out the work of their team mates. This behaviour was not forbidden, and in fact the group activity was designed to allow it; it is not unusual for understanding to follow mechanics, (Skemp, 1987) and the collaborative environment was intended to ensure that every student would be able to submit work with confidence. To determine whether individuals had achieved understanding, the students were also assessed individually.

Individual Assessment

At the end of the series of activities on linear equations in two variables, students were assessed by a conventional exam to determine their uptake of the concepts in that unit. Assessment is a common method of determining a student's ability to perform the mechanics of the concepts in question, but in this case another reason was to determine whether students had successfully learned the concepts under the RME delivery modality. While the activities were team endeavours and individual grades were based on the efforts of the team, the exams were individual assessments and grades were based on individual performance. The questions addressing coordinates and the equation of a line included in the examination are analysed below:

Question 1: …plot the points $(\mathbf{0}, \mathbf{0})$ and $(-\mathbf{5}, \mathbf{3}.\mathbf{5})$ and sketch the line that goes through these two points.

All students were able to plot a Cartesian point one of whose coordinates was a negative integer and the other of which was a non-integer value. Plotting a Cartesian point is a simple task for the expert, but for the novice, this task holds several traps. This task is usually mastered by most students, but a large minority do not conquer it early on. That 100% of students in this study successfully answered this question is noteworthy.

Question 4: Write the equation of the line that goes through the points $(-5, 3.5)$ and $(4.5, 9)$.

Eight students arrived at the correct solution to this question by finding the slope of the line between the two points (or writing the ratio of differences), replacing the "m" in the equation of a line, replacing x and y with the values from one of the given points, solving for b and rewriting the equation. An additional six would likely have returned the correct response had it not been for an error in rounding, arithmetic or algebraic manipulation. 14 out of 24 students, therefore, responded essentially correctly, a greater ratio than is normally realised in this unit.

Summary of Individual Assessment

It is impossible to know whether the students using analysis in this class are doing so because they are more engaged or whether it is because they are required to pre-read the text; certainly, the combination of pre-reading and RME appears to be very effective. Anecdotally, many students in courses such as this one, when unprepared for class, simply do not attend. Students in this class did occasionally miss a session, and a few did so chronically, but not at the rates common in classes at this level. This class was also no exception to the well-known attrition problem, but attrition occurred at a much lower rate than usual.

DISCUSSION AND CONCLUSION

The question of Student-Centered learning (SCL) is not new; it has been around since at least 1951 (Rachman, 1987), although it can be argued that it became a "modern" phenomenon in the early 1970s (e.g. Foster, 1970; Clasen & Bowman, 1974). That SCL is more effective than lecture has become difficult to dispute, given the ongoing and increasing research into its benefits (e.g. Wilson, Sztajn, Edgington, & Myers, 2015; Dondlinger, McLeod, & Vasinda, 2016; Osmanoglu & Dincer, 2018). One area that would benefit from additional research is in determining which approaches work best for which audiences. In particular, there is a great deal of room for more research, and for specifics, such as whether RME, designed for use in elementary classrooms, can be effective in other environments. This paper demonstrates that RME is a valid approach for the remedial college classroom. It is critical that more college and university instructors become informed of the benefits of SCL. Certainly, as more instructors use such methods, their students who themselves become educators are more likely to employ alternatives to lecture, but the field is resistant to this type of change. It is hoped that this paper helps to convince the reader that SCL is an approach that benefits students, but also that it need not mean reinventing the wheel. If the RME and Teaching Experiment combination can be successfully modified for use in the college classroom, then there must be many other such theories and approaches that can be similarly modified.

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LINGUISTIC CONFLICTS IN TEACHING POLYGONS IN A BILINGUAL MATHEMATICS CLASSROOM

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This case study investigates language related conflict situations which occurred between students' expected and actual responses to mathematical tasks about polygons in a bilingual classroom. The data were created from a video record of a mathematics classroom in Hong Kong. Data analysis was based on Schoenfeld's framework "Teaching for Robust Understanding" and was focused both on the nature of the conflicts and on the teacher's reactions to these conflict situations. The results show that language conflicts occurred during the delivery of mathematical content and during the assessment of students' mathematics knowledge. The teacher used linguistic strategies, gestures, and vocabulary exercises when the conflicts arose.

INTRODUCTION

Linguistically diverse mathematics classrooms involve participants, either learners or teachers, who are potentially able to draw on more than one language in their engagement in mathematical activities. Speaking more than one language involves cognitive advantages, but it also brings compound challenges in learning mathematics (Truxaw & Rojas, 2014).

Bilingual classrooms have two kinds of problems in learning mathematics: Type A and Type B problems (Berry, 1985). Type A problems typically occur when the language of instruction (e.g. English) is not the student's mother tongue, whereas Type B problems result from the "distance" between the cognitive structures natural to the student and implicit in his mother tongue and culture, and those assumed by the teacher. The severity of both problems correlates positively with the student's lack of fluency in the classroom language.

Truxaw and Rojas (2014) found that learning mathematics in another language is challenging due to both cognitive and affective factors. It is exhausting to understand academic instruction in a second language. Public participation in classrooms might decrease because asking or answering meaningful questions is intimidating for bilingual students. Students might get confused with unfamiliar representations and contexts.

This study focuses on the language related confusions that influence teaching mathematics in a bilingual classroom. Specifically, I answer the following research questions:

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1. When does language involve in conflicts between students' expected and actual responses to mathematical tasks in a bilingual mathematics classroom?
2. How does a mathematics teacher respond to language conflicts in a bilingual mathematics classroom?

THEORETICAL FRAMEWORK

Successful mathematical classrooms which enhance student learning consist of five dimensions: mathematics; cognitive demand; access; formative assessment; and agency, ownership, and identity (Schoenfeld, 2017). According to this framework, students should engage with mathematical content which represents best current disciplinary understanding. Classroom interactions should create and maintain intellectual challenges for students to develop cognitively. Mathematical activities should invite and support engagement of all students in the classroom. Subsequent instruction should respond to students' ideas by building on productive beginnings and addressing misunderstandings. Students should have opportunities to build on each other's ideas to develop agency and ownership over the content.

In each dimension, language acts either as a tool to share ideas or as a mediator to build mathematical knowledge. The mathematical content appears in the classrooms via at least one of oral, written, or symbolic language. Therefore, language plays a role in students' access to the mathematical content. In addition to information transfer, language mediates verbal interactions in the classroom, which are necessary to create cognitive demand and agency and to assess student thinking.

This framework suggests that success in learning mathematics is based on success in using language. The importance of language in each dimension can explain why mathematical instruction with a second language is challenging.

METHODS

In this case study, I created my data from a classroom video taken from The Third International Mathematics and Science Study (TIMSS) 1999 Video Study. This study investigated eighth-grade science and mathematics lessons from seven countries, three of which were English-speaking countries.

Hong Kong was the only bilingual country among the English-speaking countries within the study. The official languages of Hong Kong are both English and Cantonese. English fluency is 53% while Cantonese fluency is 94% (Mair, 2017). Therefore, I chose a video from Hong Kong. It was the first session in a sequence of four lessons working towards the more advanced concepts of polygons. It was in English and lasted for 34 minutes. It was reported that there were 41 students in the class.

The TIMMS video is provided with additional resources. One of the resources is the lesson graph which separated the lesson into nine episodes based on the type of classroom activity. According to this graph, there are five public classworks and four private classworks. In public classwork, the teacher invites students to the board to answer some questions and makes explanations based on students' drawings. In private classwork, students make some drawings and solve some problems in their notebooks individually. I focused only on the public classworks because interaction between the teacher and students occurred only in these episodes.

DATA ANALYSIS

I conducted a video analysis by following Powell, Francisco, and Maher's (2003) analytical model. They claimed that there is no proper model for video analysis in mathematics education. Therefore, they devised this model to study the development of mathematical thinking. It consists of seven phases: (1) Viewing the video data attentively, (2) describing the video data, (3) identifying critical events, (4) transcribing, (5) coding, (6) constructing a storyline, (7) composing a narrative.

I tried to get familiar with the video content at first. During my viewing, I audio recorded my comments about some moments which were interesting to me. Some of these moments were related to classroom routines, such as the students' way of greeting the teacher and seating arrangement. Others were related to language use in the classroom. After I watched the video, I described the video content with time-coded notes about transitions of situations and activities. According to my research questions, I identified the critical events which included moments where language constituted a problem. I transcribed both audio and visual data separately, but combined them in the same transcript to perform synchronous coding. The visual data were indicated both with parenthesis and smaller font size in the transcript (see Figure 1).

I= I=I= I= I wa:nt you to draw--an= another type of the Polygon

(Camera focused only on the student who drew a convex pentagon)

U::h= okay= thank you (.)

(Teacher gazed at the student's drawing. Teacher raised his both hands on shoulder level with palms towards him and moved his palms approximately about 90 degrees toward the student until they were parallel to the ground while he was smiling and nodding)

Figure 1: The combined transcript of audio and visual data

TIMMS video is supplied with the transcript of the lesson. TIMMS transcription includes the time stamps of utterances and punctuation marks. Punctuation marks are used not only to follow writing rules, but also to represent repetition of words and long breaks between utterances. I extracted the relevant parts of the TIMMS transcription into a word document. I discarded the existing punctuation marks and modified the transcription based on Jefferson (2004) by adding relevant punctuation marks to indicate above characteristics of a speech. Moreover, I discarded time stamps and added line numbers to help readers to follow the analysis.

I read the transcript line by line and typed my codes next to the relevant line. First, I coded teachers' actions according to Schoenfeld's (2017) framework, to analyze which dimensions involved language conflicts. Then I color coded teachers' gestures together with the verbal data. The same color means that the teacher utterance and the gesture happened at the same time. Finally, I colored each kind of punctuation mark differently to code sounds.

RESULTS

In this study, linguistic conflict refers to the moments where students' interpretations of a mathematical expression do not match the teacher's interpretation. Below, I introduce an incident which involved linguistic conflict and then provide the teacher's reactions after the incident.

Moments of Conflict

The linguistic conflict resulted from the mismatch of interpretation of the expression "another type", as in the following example:

1	T:	I= I=I= I= I wa:nt you to draw--an= another type of the polygon
2		(Camera focused only on the student who drew a convex pentagon)
3	T:	U::h= okay= thank you (.)
4		(Teacher gazed at the student's drawing. Teacher raised his both hands on
5		shoulder level with palms towards him and moved his palms
6		approximately about 90 degrees toward the student until they were parallel
7		to the ground while he was smiling and nodding)

Before this incident, one student drew a convex hexagon on the board. Then the teacher asked another student to draw another type of polygon (line 1). The student's drawing of a convex pentagon (line 2) indicates that student is aware that she should draw something different from the convex hexagon.

Convex pentagon and convex hexagon seem to be different types of polygons for the student. She changed the number of sides of a polygon to change its type. Thus, the characteristic which separates polygons for the student seems to be the number of sides of a polygon.

When the teacher said "another type", he intended to direct students' attention to interior angles of a polygon not to the number of sides. This intent is evident from the description of the episode and from the teacher's response to the student's drawing. According to the lesson graph, this episode is a revision of convex and concave polygons. Therefore, the teacher most likely expected the student to draw a concave polygon, which is not the same type as a convex polygon.

The teacher's reaction after the student's drawing of the convex pentagon (line 2) indicates surprise. When the student drew the pentagon (line 2), the teacher gazed at the pentagon (line 4) while he was saying "uh" in a prolonged manner (line 3). The prolongation might have given the teacher some time to think what to say next because this surprise might have created uncertainty in the teacher in terms of how to proceed with the classroom. It seems that the teacher did not expect the student to draw a convex pentagon. In other words, he did not expect the student to change the number of sides of a polygon to make it another type. Instead, he might have taken for granted that the student would draw a concave polygon.

In addition, the teacher's body language also signaled a mismatch between the student's response and the teacher's expectation. The teacher made a particular gesture when students did not answer his questions correctly (lines 4, 5, 6, & 7). The teacher made the same gesture in this lesson when students could not recall the name of a pentagon:

8 T: Do you=do you know the name of thi:s uh polygo::n (Teacher bounced his
9 fist below the pentagon with a sound "tik tik" and kept his hands on the
10 board until the pause)
11 T: You forget it (He suddenly raised his hand to his shoulder level with palm
12 showing upward parallel to the ground, smiling)
13 Okay=it is a pen:tagon—It is a pentagon

The word "forget" (line 11) refers to being unable to remember something. Therefore, it indicates that the teacher assumed that the students knew the name of that certain kind of polygon and he expected them to answer him correctly. However, the students' inability to answer him surprised the teacher. The teacher's utterance "you forget it" (line 11) and his gesture (lines 11 &12) happened at the same time. Since this moment of surprised is accompanied with a gesture, it might be asserted that this gesture reflects surprise which result from the mismatch between the students' responses (or lack thereof) and the teacher's expectation.

The phrase "another type" (line 1) seems to represent different meanings for the teacher and the student. This mismatch prevented the teacher from assessing the student's knowledge about the difference between convex and concave polygons because the meaning of the word "another" for the student did not resonate with the teacher's intention.

The phrase "another type" indicates a categorization based on a specific characteristic. However, the teacher used the expression without explicitly specifying the relevant property to categorize. The reasons to do so might be several but the result of the incomplete use of the mathematical expression is related to Schoenfeld's dimension of mathematical content. This utterance exposed students to deficient use of mathematical terminology.

This linguistic conflict occurred while one student was participating in a mathematical task. In other words, linguistic conflict corresponded to the development of student agency in the classroom. Since the student's mathematical interpretation was not aligned with the teacher's and it was not reinforced, this mismatch might influence the student's agency negatively.

The Teacher's Strategies for Language Problems

The teacher devised linguistic strategies for language problems. After the student drew a convex pentagon (line 2), the teacher asked the same question with different wording (line 14):

14 T: I (.) I want you to draw an::other ty:pe of the polygon=another property
15 (.)Okay (Teacher raised his hand to shoulder level with palm showing
16 students. He moved his hand from shoulder level to belly level quickly
17 palm downward as if he throwed a ball towards the ground. His hand went
18 up and then pushed an imaginary point in the air. The palm which was the
19 towards students was shaken right and left quickly)

The teacher used the phrases "another type" and "another property" almost in a row (line 14). This successive utterance indicates that the teacher tried to indicate the same concept with different wording. This shows that the teacher likely thought that the student did not answer the question correctly, because she did not interpret the meaning of "type" correctly.

The teacher used gestures all throughout the lesson. Apart from the above example, the teacher used at most two different gestures between two pauses during the lesson. However, when he referred to the same concept with a new word, he used four different gestures between two pauses. This increase in the way of gesturing might indicate that gestures were denser when he tried to solve linguistic conflicts. However, it is not clear how these gestures were related to the words they accompanied.

The teacher resorted to vocabulary exercises when students could not recall "pentagon" a second time. The main vocabulary exercises are spelling the word and reading it aloud with the students altogether at the same time. The teacher might have used them to strengthen the memorization of the word "pentagon". The teacher associated students' failure to answer his question with an inability to remember the term. Instead of asking "how is it called", he asked about the name of the polygon by saying "Do you still remember how to call this".

DISCUSSION

In this video, the conflicts related to language occurred both in formative assessment, agency, and mathematics dimension within Schoenfeld's TRU framework. The difference between the teacher's and students' interpretation of the word "type" prevented the teacher from making a valid assessment of student knowledge. The teacher's deficient use of the word "type" does not represent best current disciplinary understanding of mathematics.

The teacher's strategies to cope with language problems in the mathematics lesson reflects the literature. As in Kasmer and Billings (2017), the teacher devised mostly linguistic strategies for language-related problems. This suggests that the teacher does not associate communication difficulties either with mathematical understanding or with cultural issues (Adler, 1995; Gorgorió & Planas, 2001). Different from Kasmer and Billings (2017), this study shows that instead of translating math terms into the students' mother tongue, the teacher resorted to other English words when there seemed to be misunderstanding.

This study suggests that, Schoenfeld's (2017) framework could be modified in two ways. First it could be enlarged to include "language use" as another dimension. The language dimension could refer to the extent of language use which minimizes possibilities of miscommunication. Second, since language is both a tool and a mediator for success in each dimension, it might be added as a prerequisite dimension. This would change the nature of the framework. Schoenfeld suggested it in a non-hierarchical manner. However, adding the language dimension as a prerequisite would make the framework hierarchical.

It is difficult to separate the sources of these conflicting situations from language problems in learning mathematics to language problems in learning at linguistically diverse classrooms. However, this paper aimed to document conflicts without investigating the reasons. Future work can focus on this separation.

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CONSTRUCTION IN DGE: LEARNING REFLECTIONAL SYMMETRY THROUGH SPATIAL PROGRAMMING

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The goal of this paper is to gain insight into the construction process in a Dynamic Geometry Environment (DGE) and to see how, through modelling with geometric primitives, students come to understand the abstraction embedded in the concept of reflectional symmetry. Closely following a team of two upper elementary students as they construct a Leonardo da Vinci (or mirror-writing) machine in Web Sketchpad using geometric primitives, I describe and analyse several computational thinking (CT) practices that have emerged during the construction process. I then show how these practices supported the development of spatial reasoning skills and learning of geometry.

INTRODUCTION AND THEORETICAL BACKGROUND

Hoyles and Noss (2015) suggest, "It is impossible to be a citizen of the 21st century and not have some idea of what it means to write a computer program, of what it means to build a mathematical model". Kotsopoulos et al. (2017) underscore the benefits of integrating computational thinking (CT) and mathematics:

> When children write code, they come to (1) understand in a tangible way the abstractions that lie at the heart of mathematics, (2) dynamically model mathematics concepts and relationships, and (3) gain confidence in their own ability and agency as mathematics learners. (p. 1)

Computational thinking and mathematics

Wing (2006) defined CT as "an approach to solving problems, designing systems, and understanding human behaviour that draws on concepts fundamental to computing" (p. 34). Among the plethora of CT definitions available today that are non-specific to mathematics (e.g., Weintrop et al., 2016), the frameworks of Hoyles and Noss (2015) as well as Brennan and Resnick (2012) were found helpful in analysing DGE-based programming.

Hoyles and Noss (2015) identified that all CT attributes fall into either practices or concepts, and defined CT as entailing abstraction, algorithmic thinking, decomposition, and pattern recognition. Similarly, but with a slightly different focus, Brennan and Resnick (2012) defined CT as involving three key dimensions: computational thinking concepts, computational thinking practices, and computational thinking perspectives. Within practices, Brennan and Resnick observed four main sets: being incremental and iterative, testing and debugging, reusing and remixing, and

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abstracting and modularizing. It is this group of practices that is of main interest to me, due to its potential to help explain the programming process live.

Spatial programming

Jackiw and Finzer (1993) studied the potential of DGE as a problem-solving domain, which involved exploration of the process of expressing geometric relationships visually and by demonstration. They developed a notion of "spatial programming", which they defined as "visual identity between a program and its output" (p. 295) and found that in DGE, there was no distinction between the geometric content domain and the spatial programming domain; students using it encountered programming as the central activity. "The distinction between programmer and user disappears; the two coalesce into one – the student" (p. 294).

Jackiw and Finzer (1993) argue that:

> Constructing a sketch in GSP [The Geometer's Sketchpad] is programming, in the straightforward sense of building a functional system which maps input to output. The unconstrained elements of the sketch […] constitute the program's inputs or parameters. The relationships between parts of the sketch […] correspond to a program's production statements. In GSP's case, the semantics of the production language are governed by traditional Euclidean constructions. (p. 295)

They further state that a program's structure and its output were isomorphic, and that by manipulating the program's inputs, the student generated further output, meaning that manipulating is performed in the same domain as constructing the initial sketch.

Sinclair and Patterson (2018) argued that Sketchpad can be an effective programming language in the context of complex high school tasks. After analysing finished DGE sketches created by high-school students in Belgium, they conclude that many CT practices associated with the use of propositional programming languages were also featured in the more spatial and temporal register of the geometric 'language' of DGEs. While their analysis was focused on already-made sketches in high school settings, I am interested in observing the construction process live in hopes that it will provide additional insight into the phenomenon of using DGEs as CT tools that support learning of geometry. My research question thus is: What kind of programming might be involved in an elementary school construction task and how does this programming support learning of geometry?

METHOD

The project described in this paper took place in a Grade 6/7 classroom in the spring in a high-density, affluent neighbourhood elementary school in British Columbia, Canada. Students participating in the project had been exploring geometry with DGE using iPads for approximately one hour per week since the beginning of the school year. Two researchers, of whom I was one, led the project. We employed a team-teaching model, with one of us always being engaged with either the whole class, a small group, or individual students. Isometric transformations were selected as the content backdrop for these explorations, as they are part of the provincial curriculum for this grade level (though are usually presented through coordinates). From the topic of transformations, a sub-topic of reflection was chosen as a starting point, with the hope that the concept of reflectional symmetry could offer a low-floor entry into the world of transformations.

The first lesson was introductory and "unplugged": students verbally created definitions of transformations based on knowledge from previous years. The second lesson involved introduction of the concepts of symmetry and reflection. The third and fourth lessons revolved around students working with reflectional symmetry. Following this, the students were told Leonardo da Vinci's story, were shown a picture of the mirror-writing machine that he designed to encode his writings, and then they were invited to create such a machine using a Basic Geometric Tools websketch: http://www.sfu.ca/content/dam/sfu/geometry4yl/sketchpadfiles/BasicGeometryTools/index.html.The researchers suggested that they start by placing a point (for the "pen") and a segment (for the "mirror") as a possible first step. The Continuous Symmetry sketch was the "black box", which code students had to uncover.

In this paper, I examine the work of one pair of students, Danny and Dexter, as they created their construction of a mirror-writing machine. They were good friends and since the beginning of the year, both were very keen on working with Web Sketchpad. The boys worked together for approximately fifteen minutes during each of two sessions that were one week apart. At the end of the first session they reached a partial solution, and by the end of the second session, they were finally able to carry out a workable procedure to their own and the researchers' satisfaction, creating a machine "that can write stuff backwards". This teamwork was documented via audio-recording and screen capture tool, and later transcribed and analysed for the presence of prominent computational thinking practices identified by Hoyles and Noss (2015), as well as Brennan and Resnick (2012).

MIRROR MACHINES

Along with myself (the classroom teacher), the boys worked in a small office adjacent to the main classroom, so as to make clear audio-recording possible. I was mainly observing, but occasionally offered suggestions and scaffolds, especially during the "being stuck" phases. I will now briefly summarise thirty minutes of work, while highlighting two pivotal points.

The boys had to do multiple restarts during the two sessions: they tried to use two intersecting circles, one circle with a diameter, and two perpendicular lines, but had not been successful in creating a mirror-writing machine, either getting the translation as a result, or no transformation at all.

At one point, the boys decided to start from scratch. Dexter constructed two parallel lines, each running through the two centres of the circles, one small and one large. He added tracers to the centres of two smaller circles and attempted writing with the right tracer, but no transformation happened. Danny took over and added tracers to the intersections of the larger circles. He dragged the pen around, but his writing was now rotated twice, with the angle of rotation being 60 degrees. The boys silently watched the action (see Figure 1).

Dexter: Try making an L.

Danny: No, it does the same thing.

Dexter: No, look at this one. [Dexter took over and tried to draw an L.] See, this one.

Danny: Yeah, but it's not backwards, it's just a different angle.

Dexter: Oh yeah, but it's close. It's closer.

Danny: No, it's not.

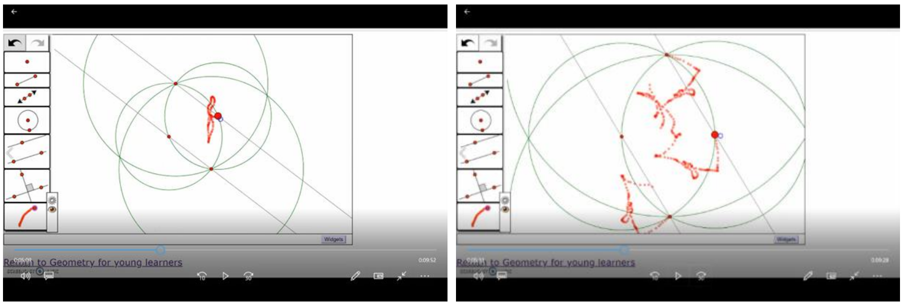

Figure 1: Rotations of letter "L".

The boys could not yet see that if they added another 60-degree rotation increment, their L would be at 180-degree rotation, which would make it upside down, and it would be an upside-down mirror-writing machine. Danny saw the same shape even after a 120-degree rotation, but Dexter thought it was different enough to be considered "closer" to mirror writing. It is not too early to mention that this was the first significant step towards reflectional symmetry: the boys began using their spatial reasoning skills and contemplating what degree of an angle constitutes a reflection.

The boys came back to the segment-point starter, but instead of either a perpendicular line or a circle, they constructed two loosely symmetrical right triangles. That attempt failed to produce any transformation. Then another attempt to start with a segment-point bundle followed, but this time with a line perpendicular to the vertical segment, which ran through the free-standing point. After that, another vertical line was added, which was parallel to the one already there. This did not produce the desired result, so the boys started anew.

They constructed a circle and a segment for the diameter and added tracers to the endpoints of that segment. When the circle dilated, the tracers drew a diagonal line in opposite direction.

Danny: Yeah, see, if you spread the circle [Dexter expanded the circle and the pen drew a horizontal line that was now reflected.] see, it goes different ways. That's it.

Dexter then moved the entire circle without dilating it, so the tracers were now translating rather than reflecting; he shrunk the circle and the line was reflected again. Dexter saw the limitations of Danny's construction, but Danny was so excited they had the solution, that he did not even notice how the reflection turned back into a translation until later, when he had a chance to watch the video of it (see Figure 2). It seems helpful to say now that it was the second major turning point, when the boys realised that moving in the opposite direction was not enough, and the relation of the two circles was needed. They here demonstrated further development of their spatial reasoning skills.

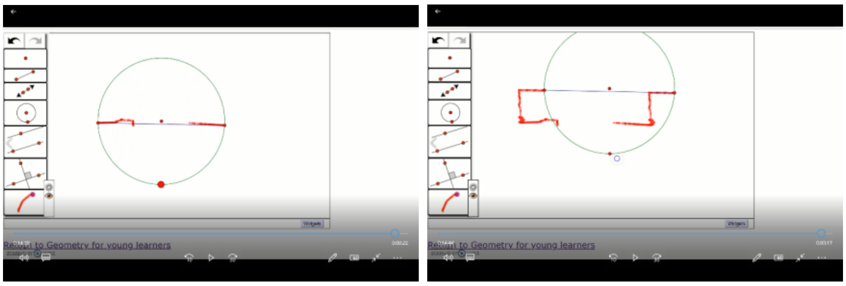

Figure 2: "That's it!" …Or is it?

The second session was more productive. Dexter interlocked two circles in a horizontal orientation, then oriented the structure vertically and wrote "Hi", but it translated again. Dexter then reluctantly constructed a segment and a point suggested earlier by the researchers as a way to jumpstart the construction. Danny centered the circle on the endpoint of the segment and connected the circumference to the arbitrary point. After adding the second circle and tracers, Danny first dragged the lower endpoint of the segment to cause only the left point to leave a trace. Then he dragged the left intersection point and the writing finally reflected on the other side.

Dexter: Oh my gosh, we did it! We created a machine that writes stuff backwards!

ANALYSIS AND DISCUSSION

Upcoming analysis of the data described above will demonstrate that CT practices such as being incremental and iterative, testing and debugging/decomposition, and abstraction were all part of the DGE construction process to varying degrees. Debugging/decomposition practices were especially prominent.

Being incremental and iterative

Brennan and Resnick (2012) emphasized that students move forward non-sequentially and in increments. This definitely holds true for the project being described here as neither Dexter nor Danny had a well-devised plan as to how to proceed – to the point of being reluctant to make use of the starting hint suggested by the researchers. I have chosen to refer to their experience as "computational wandering", as they generated many unplanned, seemingly random designs before arriving at the correct solution. Often, they would repeat the same buggy procedure over and over again (like their persistent attempts to use the two interlocked circles). Other times, they would come very close to the correct solution only to abandon it for something less promising, because they were not yet able to see the relationships at hand. For example, at one point, Danny saw the potential of one of the two parallel lines to serve as a line of symmetry, but he was not yet clear on how to do it and abandoned it half-way, letting Dexter to take over. Overall, the journey to the solution could be described as two steps forward and one step back (or sometimes sideways). Having to debug paid off as the experiences were getting richer, and ultimately scaffolded the boys to the solution.

Testing and debugging/decomposition

According to Brennan and Resnick (2012), testing and debugging practices emerge when there is a breakdown in code. Most testing and debugging practices they described were developed through trial and error, transferred from other activities, or initiated by knowledgeable others. Sinclair and Patterson (2018) also found that breakdowns in the behaviour of the constructions were the primary source of debugging. Students had to determine why the relationships between objects were compromised under motion: “Such breakdowns can be seen as bugs if we consider them to be wrongly expressed relationships between the different objects in the sketch” (p. 69).

In the case of Danny and Dexter, testing and debugging was the most frequent practice observed. Both boys were relatively new to GSP, being in their first year of its usage. However, both boys had by now become comfortable interlocking two circles – circles intersecting each other’s centres – for various purposes (e.g., to construct equilateral triangles or perpendicular lines) and then if needed, iterating further circles centring them on the intersection of any two circles, which would afford further construction of congruent segments and regular polygons. Interestingly, debugging did not happen within this context. However, using less frequently explored widgets, like adding a perpendicular line or constructing a three-circles design, often resulted in soft constructions with variable angles, causing spatial relationships to be affected in undesirable ways.

Jackiw and Finzer (1993) wrote:

> In the programming of a sketch, a bug may be considered as an inaccurately or insufficiently expressed relationship between two or more objects. Bugs occur most frequently with novice GSP students, who readily arrive at a drawing of what they want, only to have the desired relationships between objects disappear when they drag a free node.” (p. 303)

Overall, the boys encountered a bug twelve times throughout the two 15-minute sessions, hoping to see mirror writing to occur, and instead seeing either a sole trace, translation or rotation rather than reflection. It prompted them to keep searching for a solution. This resonates with the practice of decomposition, described by Hoyles and Noss (2015) as solving a set of smaller problems prior to solving a problem – unless one solves how to get the two tracers moving in opposite directions, one cannot move on to producing mirror writing.

One of the benefits of constructing in DGE is access to immediate visual feedback: breakdown in code could be seen immediately as the boys dragged the design, which would not behave in the desired way. They would take a couple of seconds to ponder over it and start over, and instead of having to fix the alphanumeric code first, they would make adjustments right there in the sketch. That saved a lot of time and led to the correct design after twelve restarts during 30 minutes of work.

Abstraction

When Dexter and Danny were designing a procedure for the mirror-writing machine, there was abstraction involved in that they had to see the relationship between the writing pen and the movement and position of the circle, as well as how that circle related to the other circle or the line perpendicular to the line of symmetry. When the boys first encountered the finished model of the mirror-writing machine, it looked like two symmetrical points of different colour, and there were no geometric objects visible, so it was difficult – if not at that point impossible – for them to see the relationship between the objects that comprised the inner mechanism of the machine and the writing the pen was producing. In spite of being jumpstarted into action by being constrained by the available widgets of the web sketch, and knowing the suggested starting point, the procedure was not at all obvious to them. However, a couple of pieces of the puzzle were in place from the very beginning by mere presentation of the finished product of the mirror-writing machine, which did not reveal the mechanisms, but vividly demonstrated what the machine was capable of. This "black box" experience was very important in helping the boys understand the behaviour they had to model: they already knew that movement had to be symmetric, but it was not at all obvious to them what was the relation between the two points.

The relationship was becoming more and more palpable, however, as they observed the mirrored pen in action: it either remained static, or it would follow its own trajectory, or it would translate or rotate the shape produced by the main pen. From all this the boys painstakingly extrapolated the intricate and complex relationships between the objects involved: for example, that mirrored writing meant needing to have a point on the opposite side, that circles had to be connected but not locked, that the perpendicular line needed to move up and down, or that there had to be a line of symmetry that would anchor the entire machine. At first, they heavily relied on intuition. For example, at one point, Dexter referred to the perpendicular widget as "maybe we could use this somehow, 'cause it has the right angle tool", not being quite sure why he would want the right angle.

Their reasoning was becoming more grounded (e.g., "dot will have to go to the other side, since it will be backwards", or "if you spread the circle, writing goes in different ways"). Even though the boys did not rely on the use of CT or mathematical terminology, their ability to abstract was evident from their gestures and on-screen behaviour. This ability did not necessarily translate into full understanding of the relationships, but certain utterances indicated that many pieces of the puzzle were already in place. Consider, for example, how Dexter, after having written another translation remarked: "It doesn't work, because there are no circles moving up and down". Of course, the up-and-down movement is not enough to produce a string of letters, and the relational dilation of the two circles was required, but nevertheless, it was a solid step towards being able to abstract.

CONCLUSION

In this paper, I focused on the process of construction in DGE, hoping that it would provide additional insight into the phenomenon of using DGEs as CT tools that support spatial reasoning. I also carried out analysis of the CT live, and demonstrated what kinds of programming might be involved in an elementary school construction task. The process of debugging/decomposing was very salient in this example, yielding twelve attempts with various tools and various configurations. The boys were able to test out directly whether the construction worked, and they could get visual feedback of what they constructed. The available tools of the Web Sketchpad were providing hints for the boys of the kinds of things they could try just by being there and provoking curiosity, and they were responding to those more than to the researchers. In order to crack the "black box" code and get access to the hidden geometric relations of reflectional symmetry, the boys needed to develop the ability to abstract, and they managed to do it through extensive testing and debugging.

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SOURCES OF COMMUNITY COHERENCE IN A SOCIAL MEDIA NETWORK OF MATHEMATICS EDUCATORS

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An unprompted, unfunded, and unmandated mathematics teacher social media community is thriving and is touted by members as one of the best forms of professional development they have experienced. However, newcomers often find it confusing and difficult to navigate due to the frequency and mass of content shared. Although the space can seem chaotic, order emerges, and is informing of mathematics teacher needs, interests, and issues. This paper explores how order emerges in this community and considers the implications it has on the space of possibility it offers.

INTRODUCTION

Teacher professional development is essential for enhancing the quality of teaching and learning in schools (Borko, 2004), and the robustness of a professional development initiative is dependent on ensuring activities reflect and are driven by teacher needs and interests, and that community building and networking are at the core (Lerman & Zehetmeier, 2008). However, initiatives are commonly limited to sparse, one-time professional development workshops facilitated in face-to-face synchronous settings, which are not typically supportive of ongoing professional growth (Ball, 2002) and are generally driven by facilitator perceptions of what teachers need rather than by what teachers want (Liljedahl, 2014).

In contrast to these centrally organized, and sometimes compulsory, professional development initiatives, teachers from across North America are participating in decentralized, virtual, and autonomous professional communities. One such community involves mathematics teachers who regularly use Twitter and blog pages to asynchronously communicate their musings and practices, and have come to be identified as the Math Twitter Blogosphere (MTBoS) (Larsen, 2016). MTBoS participants often have very promising statements about the possibilities for professional growth they experience in the space and are found suggesting that MTBoS is a safe space to share ideas and to find others who help improve these ideas.

> Following this weird #MTBOS hashtag on twitter has changed my teaching practice in so many ways. (@MrOrr_geek, 6 Feb 2018)
>
> That's what the MTBoS is all about... Sharing ideas and having others improve it! (@mathymeg07, 10 Sep 2017)
>
> I love that Twitter can be a safe space for learning. (@mavenofmath, 2 Sep 2018)

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Although communication through Twitter is generally random and unprompted, the MTBoS seems to be treated as an established space, one that is determined by participation from members who contribute and continue to use the MTBoS hashtag. This rich phenomenon of mathematics teacher professional development is largely unstudied and deserving of attention. As such, the study presented in this paper is driven by the overarching question – How does the MTBoS continue to thrive and what can it teach us about mathematics teacher professional learning? And more specifically, how does order emerge in the MTBoS and what does this order teach us about mathematics teacher professional activity?

THEORETICAL FRAMEWORK

With an aim to understand the autonomous organism of the MTBoS, this study is guided by complexity theory (Davis & Simmt, 2003; Davis & Sumara, 2006). Complexity theory provides the tools to describe a system of individual agents who seem to generate emergent macro-behaviours. Complex systems do not merely exist, they also learn and adapt. In complexity theory, learning is expanding the space of the possible and is primarily concerned with "ensuring conditions for the emergence of the as-yet unimagined" (Davis & Sumara, 2006, p. 135). The goal of complexity theory is not to identify interpersonal collectivity, as do other social theories of learning, but rather to understand 'collective-knowing', where knowledge is not attributed to any one member, but sits atop the social network.

To this end, Davis and Simmt (2003) identify five interdependent conditions necessary for complex emergence; that is, for a complex system to learn and thrive. These conditions include *diversity*, *redundancy*, *neighbour interactions*, *decentralized control*, and *organized randomness*. Davis and Sumara (2006) further theorize these conditions into complementary pairs: specialization (tension between *diversity* and *redundancy*), trans-level learning (*neighbour interactions* through *decentralized control*), and enabling constraints (balancing *randomness* and *coherence*).

Specialization has to do with agent attributes. *Diversity* among agents allows for novel actions and possibilities, while *redundancy* allows for stability and coherence. Without *redundancy*, agents may not be able to communicate, but without *diversity*, agents may never have anything to communicate about. However, trans-level learning has to do with ideas. This means that *neighbour interactions* are ideational rather than social, even if they require a physical component such as oral or written expression to provide means for interaction. Similarly, *decentralized control* has to do with the emergent conceptual possibilities among ideas, and typically emerges a scale-free network that has the same structure at smaller scales as at larger scales (Davis & Sumara, 2006). Scale-free networks are prone to developing topological hubs with many links that are topologically central in the network (Mitchell, 2006). Finally, liberating constraints refer to the space of possibility emergent from systemic activity. That is, *randomness* is the unexplored space of possibility, and *coherence* is the explored space of possibility that allows the collective to maintain focus of purpose and identity (Davis & Sumara, 2006). These work together to occasion the system’s capacity for ‘collective-knowing’, which allows it to adapt to changing conditions (Davis & Sumara, 2006). As such, the purpose of this paper is to explore the ‘collective-knowing’ of the MTBoS by considering how *neighbour interactions* through *decentralized control* contribute to sources of *coherence* and *randomness*.

METHODS

To explore *neighbour interactions*, which are ideational interactions, it is necessary to consider what ideas are being communicated, and how these ideas are interacting. Twitter allows users to post publicly available ‘tweets’ up to a maximum of 280 characters along with weblinks, photos or videos, tagged users, and hyperlinked hashtags. When users post a tweet, they may be communicating several ideas, and by posting these ideas together they may be bringing about new meanings. For instance, the tweet in Figure 1 states, “I ‘love’ #playwithyourmath! Perfect activity for after a test!” (@hbolur7, 20 Sep 2018). This tweet presents the ideas of #playwithyourmath, engaging students in an activity after a test, and the MTBoS/iteachmath community. In doing so, it is positioning #playwithyourmath as something for students to work on at desks after a test. In this way, new links between ideas are created that link #playwithyourmath and post-test activities. This sort of analysis is pursued to develop an ideational network formed from a selection of tweets made in the community.

Figure 1: #playwithyourmath tweet

Since the #MTBoS community began developing as early as 2007 and has grown to more than 800 self-identified members who post multiple times a day, it is impossible to pursue analysis of the entire corpus of data due to its sheer mass. Instead, a smaller selection of tweets that include the hashtag #MTBoS is chosen for analysis. This selection is formed from taking every second tweet made with the hashtag #MTBoS from one randomly selected school day, Sept. 21st, 2018. It includes 129 tweets made by 107 unique users from all over North America and forms the dataset for this paper.

Each of these 129 tweets was coded according to the ideas, or *ideational artefacts*, that were evident in the tweet, which may have come from any part of the tweet’s text including hashtags and user handles, or from associated links and media in the tweet. Relationships between the ideational artefacts from each tweet were recorded in a spreadsheet equipped with NodeXL Pro software, which allows for building a network graph from the relationships. The software tabulates the codes from the relationships into a nodes tab, which lists all the ideational artefacts that were coded throughout the dataset. Each of these artefacts was in turn coded recursively into categories to emerge a typology of ideational artefacts.

It is important to note that the researcher has participated in the community for almost five years and takes the ethnographic stance of a participant observer (Jorgensen, 2015). The insider experience helped with the interpretation of some of the hashtags or nuanced community references in tweet content. Without becoming an insider, these observations would have been more difficult, if not impossible, to make. However, careful consideration was given to maintain an observational perspective in identifying the ideas that each tweet is communicating. All ideational artefact codes and relationships were code checked with a second pass before continuing analysis.

The ideational artefact codes and relationships were then used to create a network graph that showed the artefacts as nodes and the relationships as edges between nodes. Colours were used to indicate types of artefacts. However, since the graph became very dense with over 600 relationships and 250 nodes, a new graph was created by removing all relationships that included #MTBoS, the central node. This allowed for a network diagram that showed the *residual ideational network*, one not dependent on the community hashtag. The connectedness and density of this residual network revealed the robustness of the ideational network that thrives atop the social network.

To pursue further analysis of the ideational network, topological hubs were identified by looking at their node degrees as well as their topological centrality in the network diagram. The top two hubs for each ideational artefact category were chosen, and the tweets for each of these nodes were drawn from the data to map out how each tweet highlighted a set of nodes and relationships on the graph. Observations of how each ideational artefact hub attracted relationships were made to emerge themes about how sources of coherence emerge in the MTBoS and what space of possibility they offer.

RESULTS

The ideational artefacts identified for all the tweets in the dataset fell into eight categories: *math topics, people, practices, resources, teaching issues, teaching values, community values*, and *community identities*. *Math topics* ranged from place value to calculus and were most often the context of an activity a user was sharing about, but sometimes was the direct object that was being communicated about. *People* were only included if they were treated as resources, and not if they were part of the conversation. *Practices* ranged from quiz games to creative projects, and indicated approaches used in teaching. *Resources* included tools or sources that could be used in various teaching contexts. *Teaching issues* were things teachers were faced with, such as standardized testing, that they had to grapple with. *Teaching values* were things they valued such as generating student discussion or making math homework meaningful. *Community values* were values of the community such as not sharing answers to problems and informing practice through research. *Community identities* were things the community is known for or associated with, such as math jokes or chats like #edchat.

With #MTBoS, the network diagram (Figure 2) revealed a robust decentralized network with many cross-categorical connections, particularly between teaching values, practices, math topics, and resources. Without #MTBoS, the residual ideational network resulted in a robust major connected component with several topological hubs, mostly in the categories of *practices*, *resources*, and *teaching values* (Figure 3).

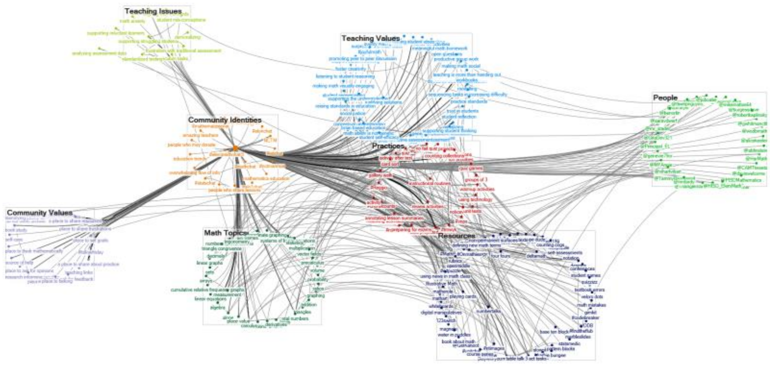

Figure 2: Network diagram of all data with #MTBoS

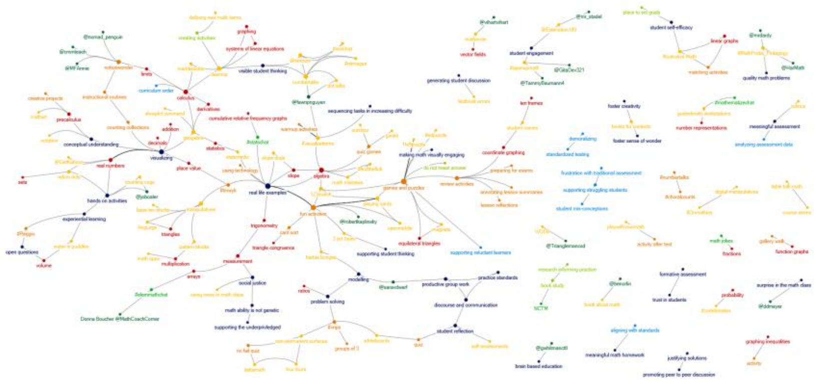

Figure 3: Residual ideational network without #MTBoS

The top two hubs were identified for each category by looking at the nodes with highest degree (number of relations to other nodes) for each category (Table 1).

Ideational Artefact Type	Top Ideational Artefact Hubs (degree)
Math Topics	algebra (6); calculus (6)
People	@fawnpnguyen (3); @saravdwerf (2)
Practices	using games & puzzles (10); using fun activities (8)
Resources	desmos (6); numbertalks (6)
Teaching issues	supporting struggling students (2); standardized testing (1)
Teaching values	Using real life examples (9); visualizing mathematics (7)
Community values	Creating activities (2); book study (2)
Community identities	#statschat (2); #elemmathchat (2)

Table 1: Ideational artefact hubs

Tweets related to each of these hubs were mapped onto the subgraphs that each hub generated. However, for purposes of brevity, only a select number of hubs are presented in this paper based on their representativeness of the themes that emerged from the complete analysis. Namely, hub mappings revealed that certain hubs attract more overlapping relationships, while others attract less. The hubs that attracted more overlapping relationships indicated more ideational coherence than those that had fewer overlapping relationships with more randomness. Exemplars of each of these cases are presented in what follows.

Randomly connected ideational hubs

Some hubs were formed by several tweets among which the only overlapping idea was the hub artefact itself. The math topic of calculus exemplifies this and is seen in Figure 4, where each colour represents a different tweet. The only idea in common with each tweet is that it had to do with calculus.

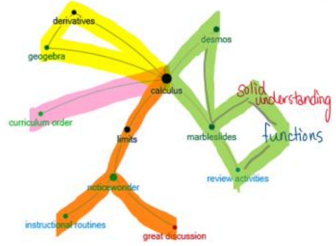

Figure 4: Ideational hub around calculus

One of these tweets states, "@Desmos Marbleslides are such a great way to check that Ss have a solid understanding of functions. Calculus Ss will be doing all of them to refresh their memory before we move on." (@Abaziou, 21 Sep 2018). This tweet is relating the *math topic* of calculus with the *resource* desmos, and more specifically, the *resource* called Marbleslides, which is an activity within desmos that prompts students to write equations that graph along given structures of stars so that a marble that slides along the graph runs into the stars. But it not only relates a *resource* to a *math topic*, but also *associates* it with the *practice* of using it as a review activity for the *math topic* of functions so that students build a 'solid understanding', which is a *teaching value*. The rich network of connections among various ideational categories makes the post very specific in how it relates a *resource*, a *teaching practice*, *math topics*, and a *teaching value*. However, none of these relationships repeat in other posts within this hub.

Another one of these tweets states, "How can we verify that the derivative of y=abs((cos^2x)(sinx)) does not exist at the points where absolute value turns graph around? @geogebra helps. Can we do it without?" (@mrdardy, 21 Sep 2018). This one relates the *math topic* of calculus to a different subtopic of *derivatives*, and relates it to a different *resource*, this time, geogebra. So, it is less specific in terms of the ideational categories it draws together: it *associated* a *math topic* with a *resource* but does not indicate a *teaching value* or *practice*. It is more mathematically oriented rather than teaching oriented. The nature of the tweet is also different than the first in that it is asking for input rather than sharing the result of using a resource. This shows the diversity that this hub attracts.

The other tweets in this hub contributed even more randomness, with one relating the *math topic* of calculus to the *teaching issue* of curriculum order by stating how the teacher organized their calculus course content. Another related the *math topic* of calculus with the *teaching practice* of using a *Notice Wonder* instructional routine, stating that it prompted great discussion, yet another *teaching value*. As such, although calculus seems to be a hub in this network, it is attracting more ideational randomness than ideational coherence.

Coherently connected ideational hubs

In contrast, a hub that attracted a lot of coherence among its relations was around the *person* @fawnpnguyen, who was referred to as a resource more than someone that conversation was directed to. @fawnpnguyen is known for developing the continuously growing *resource* of #visualpatterns, which has a website, and has become quite popular in the community. This is evident in the relationships around this hub, even though it only includes three tweets. All three of the tweets relate the *person* @fawnpnguyen with the *resource* #visualpatterns. Two of the tweets position #visualpatterns as a warmup activity, a *practice*, that targets the *math topic* of sequences. Another two tweets position #visualpatterns as something that allows students to solve in many ways, which is a *teaching value* (Figure 5).

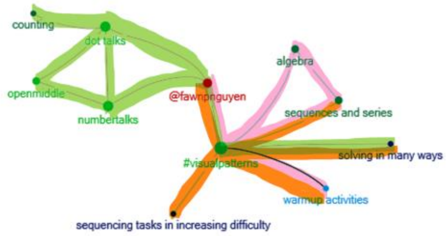

Figure 5: Ideational hub around @fawnpnguyen

In this way, the relationships that repeat relate the *person* @fawnpnguyen with the *resource* #visualpatterns, with the *practice* of using it as a warmup, with the *teaching value* of solving in many ways, for the *math topic* of sequences. This hub therefore emerges a sense of coherence in that there are many overlapping relations between nodes. The recurrences among overlapping relations are also specific enough to continue attracting these same relationships in future posts. That is, there is generative possibility in this hub, which is seen in the data as one of the posts provides a new example of a coloured square visual pattern that teachers can use as a warmup activity.

DISCUSSION AND CONCLUSIONS

Two important themes emerge out of this analysis: *recurrence* and *association*. The fact that ideas and combinations of ideas are recurring through various tweets from different users sets the stage for the possibility of coherence. However, through the analysis, it’s evident that *recurrence* is necessary but insufficient to occasion *coherence*. Rather, it is the *recurrence* of *associations* that are specific enough that creates a sense of *coherence*. The *randomness* within the *coherence* allows for generative activity that prompts further activity in the same direction.

Fundamentally, the social media space of the #MTBoS is defined by people making posts. However, these posts communicate ideas, and these ideas become linked through living together in a post, thus forming an ideational network. The nodes and connections in this network encounter *recurrence* over time as ideas and idea combinations are repeated by various agents in the space. Whether intentionally or unintentionally, agents are building an ideational network greater than the sum of any of their individual posts, a network that reveals the emergence of sources of *coherence* in the community.

These sources of *coherence* are reinforced when ideas are *associated* together repeatedly, and they begin to travel together. The implication here is that these sources of *coherence* can shape what mathematics educators can perceive as possible activity in this space. Sources of *coherence* limit the space of possibility, but can allow for nuanced differences to be made, which include adaptation of ideas into diverse teaching contexts. This begs the question of what other sources of *coherence* emerge in the MTBoS space, and how these sources of *coherence* direct agential activity, and in turn, adaptation to various teaching contexts.

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ACTUAL INFINITY AND POTENTIAL INFINITY: A CASE OF INCONSISTENCY

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Despite the notion of infinity having been studied since ancient Greece, results are still controversial among academic circles, including the schools of philosophy, theology, logic, and mathematics. This paper provides a brief synopsis of the notion of infinity, including actual infinity and potential infinity, which is motivated by two views. First, high school students have cognitive conflicts when they are comparing different sizes of infinite sets. Second, actual infinity (e.g. Platonic or Cantorian) and potential infinity (e.g. Aristotelian or Kroneckerian) seem different. However, in reality, they are often treated as identical. This paper attempts to use Dubinsky's APOS theory and Sfard's dual nature of mathematical conceptions to indicate that cognitive conflict can improve students' intuitive thinking skill and argues that actual infinity can coexist pragmatically with potential infinity.

HISTORY OF INFINITY AND COGNITIVE CONFLICT

The notion of infinity can be traced back to the pre-Socratic thinker Anaximander (~600 BC), who first used the word *apeiron* to represent the concept of infinite, boundless, and unlimited. Zeno's paradox of Achilles and the tortoise (c. 490–430 BC) is one of the famous arguments involving the notion of infinity and has drawn the attention of mathematicians and philosophers. In this paper, I will focus on the conceptions of infinity at the high school level, including the notion of cardinality but avoiding the philosophical debates around how those concepts are cognitively attainable or unattainable (Dubinsky et al., 2005a).

One of the cognitive conflicts regarding infinitely large (infinity) or infinitely small (infinitesimal) is that, whenever action is regarded as to 'infinity', we are involving both psychical and physical causal laws. For example, consider "the equation $0.999\ldots = 1$" (Dubinsky et al., 2005b). On one hand, $0.999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$ is psychically an infinite process without a last counting number. On the other hand, one may use the sum of geometric series as a mathematical 'proof' to show that "1" is physically reached or, at least, the equation is pragmatically correct. That is:

$$0.999\ldots = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots = 1$$

Note that the above equation does apply the rule $\lim_{n\to\infty} \left(\frac{1}{10}\right)^n = 0$. In other words, the actual ones, "1", is based on axiomatic arithmetic on an infinite quantity n which is

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very controversial, for to arithmeticize an abstract notion ∞ by the usual arithmetic is not immediately perceived. This is because $\lim_{n\to\infty}\left(\frac{1}{10}\right)^n = 0$ is neither a finite quantity nor an infinitely small quantity. Bishop George Berkeley (1685-1753) coined the phrase "ghosts of departed quantities" to describe infinitesimals such as these. More details about $0.999\ldots = 1$, as well as the common inconsistencies for students when they are comparing two different infinite sets will be discussed later in this paper.

BRIEF DESCRIPTIONS ABOUT INFINITY WITH EXAMPLES

Below are some informal descriptions that may be helpful for students to understand more clearly about infinity at the high school level.

A. Infinity: infinity is a mental concept describing something without any bound. For example:

    1) If the number of elements in the set of natural numbers is infinite, we denote this set as $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$, in mathematics convention, we use the symbol ∞ to denote infinity. Note that, ∞ is an abstract conceptual symbol. An infinite set may or may not be denumerable (countable).

B. Cardinality: the term 'cardinality' was coined by Georg Cantor (1845-1918) who first defines the arithmetic for cardinality (or cardinal number) to measure the quantity (size) of two, or more infinite sets (see Dubinsky et al., 2005a, p. 346, p. 355). Note that, application of 'cardinality' is not limited to infinite sets, it also works perfectly in finite sets. For example:

    2) Finite set $S_1 = \{1, 2, 3\}$ has a cardinality of 3, and finite set $S_2 = \{-3, -2, -1, 0, \mathbb{N}\}$ has a cardinality of 5. In this case, despite $\mathbb{N}$ is an infinite set, "$\mathbb{N}$ can be seen as a single object" (Dubinsky et al., 2005b, p. 261).

    3) The infinite set of all-natural numbers, $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$, has a cardinality of $\aleph_0$ (notation is coined by Cantor in his Cantorian set theory).

C. Property of cardinality: two sets have the same cardinality if there exists a one-to-one correspondence mapping (bijective mapping) between two sets. For example:

    4) $S_1 = \{1, 2, 3\}$ and $S_2 = \{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$ have the same cardinality of 3, because the mapping $\begin{pmatrix} 1 \leftrightarrow A \\ 2 \leftrightarrow B \\ 3 \leftrightarrow C \end{pmatrix}$ is bijective between S_1 and $\mathbf{S}_2$.

5) $\mathbb{N}_1 = \{1, 2, 3, 4, \cdots\}$ and $\mathbb{N}_2 = \{4, 8, 12, 16, \ldots\}$ have the same cardinality because there is a one-to-one correspondence mapping $f(n) = 4n$ form $\mathbb{N}_1$ to $\mathbb{N}_2$, or pictorially, $\begin{pmatrix} 1 \leftrightarrow 4 = 1 \times 4 \\ 2 \leftrightarrow 8 = 2 \times 4 \\ 3 \leftrightarrow 12 = 3 \times 4 \\ 4 \leftrightarrow 16 = 4 \times 4 \\ \vdots \\ n \leftrightarrow 4n \\ \vdots \end{pmatrix}$ is bijective between $\mathbb{N}_1$ and $\mathbb{N}_2$.

D. Potential infinity: potential infinity means a procedure that gets closer and closer to, but never quite reaches, an infinite end. In other words, the procedure is always on going. For example:

6) A sequence of positive integers 1, 2, 3, 4, ... means adding more and more numbers, we assume that this procedure will never be completed physically and mentally. This constitutes the concept of potential infinity.

7) Sequences like $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ contain infinitely many divisions, and we assume this process is never completed. In this case, the infinitely many divisions constitute a potential infinite division.

E. Actual infinity: actual infinity means an infinity that one truly reaches, its action is already done. For example:

8) We assume that the sequence of integers $1, 2, 3, 4, \cdots$ will never be completed physically and mentally. However, if we group it as a completed object $\mathbb{N} = \{1, 2, 3, 4, \cdots\}$, this completed entity, $\mathbb{N}$, does constitute the notion of actual infinity of $1, 2, 3, 4, \cdots$.

9) In mathematics, the geometric sequence $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$ becomes zero as steps of division 'tend' to infinity. Then zero is the actual outcome for this infinite sequence. Similarly, by using the formula that students use in secondary school, the sum of geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ converges to 1. In other words, despite that the infinite summation $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ is potentially never-ending; its summation has an actual infinity of 1.

BRIEF BACKGROUND ABOUT ACTUAL AND POTENTIAL INFINITY

Historically, Plato (~ 428-344 BC) claimed that no matter if it is in the real world or in the heaven of God, infinity is the actual infinity, while Aristotle (384-322 BC) denies all actual infinities, and postulates that actual infinity is impossible in both reality and in mathematics. More details about why Aristotle made a clear decision in favour of potential infinity can be found in Körner (1968). Thus, a Platonic viewpoint of infinity is actual infinity whilst an Aristotelian viewpoint is potential infinity. Indeed, debates on actual/potential infinity have been argued for more than two thousand years and seem to have no end. In addition, "their debate has dealt with and touched on many scientific disciplines, including philosophy, logic, the theoretic foundation of computer science, mathematics, and others" (Zhu, Lin, Gong, & Du, 2008, pp. 424-425).

Moreover, the development of calculus since the mid-17th century heated the debate intensely; both Newton's fluxions and Leibniz' infinitesimals were being criticized heavily by philosophers and mathematicians. Bishop George Berkeley (1685-1753) was one of the most representative opponents against Newton and Leibniz; on his *The Analyst or A Discourse Addressed to an Infidel Mathematician* (1734), one may notice that Berkeley's critiques were grounded on both philosophical and mathematical analysis, and his influential critiques lead the development of the concept of infinity.

CONCEPTUAL FRAMEWORK ON DUBINSKY'S APOS AND SFARD'S DUAL NATURE OF MATHEMATICAL CONCEPTIONS

There are many ideas about how to develop appropriate mathematical concepts for the students at elementary/high school level. To narrow down various good ideas and the amount of literature, this paper focuses on Dubinsky's (2001) APOS theory (Actions-Processes-Objects-Schemas) and Sfard's (1991) ideas regarding the dual nature of structural and operational conceptions on mathematics.

Dubinsky's APOS theory

Initially, "Dubinsky developed this framework as an adaptation of some of Piaget's ideas that are central to the study of advanced mathematical thinking" (Zazkis & Campbell, 1996, p. 544). Piaget focused on reflective abstraction in children's learning, whilst Dubinsky extended this mental mechanism to advanced mathematics (Dubinsky, 2001). Regarding actual/potential, Dubinsky explicitly connects action, process, and object (actual/potential infinity) to develop the thematization of schema (see Figure 1).

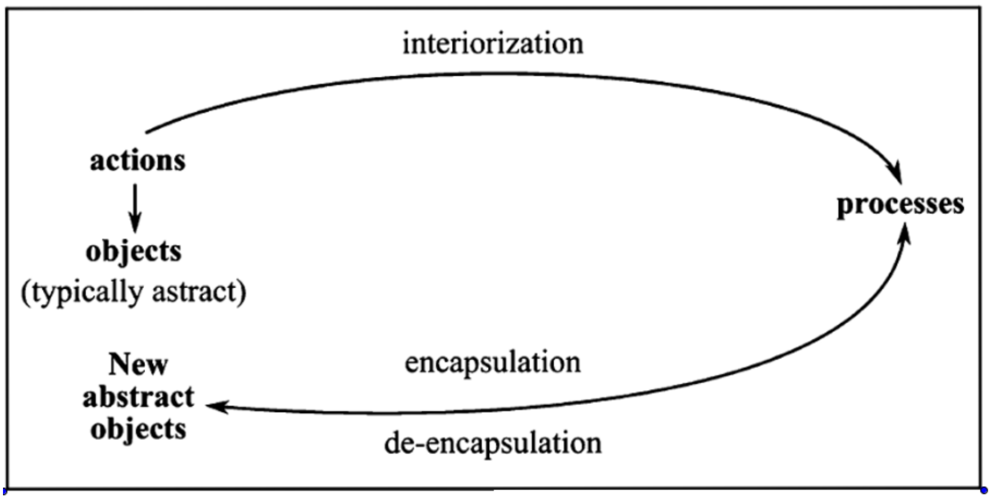

Figure 1: Diagram of Dubinsky's APOS Schema

To do so, for example, if one revisits "the equation $0.999... = 1$", the typical abstract object is potential (0.999…) vs. actual (the ones, "1"), and the action is a desire to understand this abstract object. Through mental interiorization, the action becomes a process, and one may then encapsulate the process into a new abstract object or a new higher cognitive stage. In this example, the process of conception of $0.999...$ may not directly produce the ones, "1"; however, if an individual can see the process as a totality, and then interiorize an action of evaluation on the series $0.999... = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \cdots$, then the encapsulation of the process is the transcendent object, "1" (Dubinsky et al., 2005b). In other words, 0.999… and 1 can coexist. Surely, how to achieve the interiorization and encapsulation mentally and effectively is the turning point of cognitive learning. Dubinsky's APOS theory does, in any case, provide a fruitful theoretical framework regarding learners' mental mechanism.

Sfard's dual nature of mathematical conceptions

APOS theory is not the only way to approach the abstract mathematical objects systematically. Sfard's (1991) dual nature of mathematical conceptions also works well in mathematics education and has some similarities with Dubinsky's action-process-object-schema. Sfard's structural (as objects) and operational (as processes) duality is accomplished through three steps, namely interiorization, condensation, and reification, and does provide "a theoretical framework for investigation the role of algorithms in mathematical thinking" (Sfard, 1991, p. 1). For example, as Sfard states, "[t]he term 'interiorization' is used here in much the same sense which was given to it by Piaget: we would say that a process has been interiorized if it 'can be carried out through [mental] representations'" (p. 18). Sfard's condensation stage means "a person becomes more and more capable of thinking about a given process as a whole" (p. 19). It is fairly similar to how Dubinsky encapsulates the process on the sequence 0.9, 0.99, 0.999, … as the transcendent object because "0.999… is considered as an object" (Dubinsky et al., 2005b, p. 261). The stage of reification is the point where an interiorization of higher-level concepts begins; one may compare this stage to Dubinsky's new abstract object (return to Figure 1). Note that Sfard's duality is not equivalent to a dichotomy, as Sfard states that "the terms 'operational' and 'structural' refer to inseparable, though dramatically different, facets of the same thing. Thus, we are dealing here with *duality* rather than *dichotomy*" (p. 9). Recall the case of $0.999... = 1$ again; there is no evidence that actual infinity deserves higher esteem than potential infinity and vice versa. Both are closely tied together with a special school of thought: for example, intuitivism or formalism. To decide which component is more significant is, in some sense, rooted in one's philosophical viewpoint and the purpose.

COMMON INCONSISTENCY BY COMPARING THE SIZE OF TWO INFINITE SETS

There are two main ways to compare the size of two sets:

A) part-whole consideration (part/whole relationship) and

B) one-to-one correspondence justification (bijective mapping)

Part-whole consideration states that a part is less than the whole, “a whole greater than its parts” (Sutherland, 2004, p. 180), while one-to-one correspondence justification states that two sets (finite or infinite) contain the same number of elements if there exists a one-to-one correspondence relation between them.

When dealing with finite sets, both part-whole consideration and one-to-one correspondence should work perfectly, and usually students have no cognitive conflicts once these two methods are introduced. But, when students are dealing with infinite sets, for instance, $\mathbb{N}_1 = \{1, 2, 3, 4, 5, \ ...\}$ and $\mathbb{N}_2 = \{4, 8, 12, 16, 20, ...\}$, students might apply the part-whole consideration and claim that the size (or the magnitude) of $\mathbb{N}_1$ is strictly larger than the size of $\mathbb{N}_2$ because $\mathbb{N}_2$ is part of the $\mathbb{N}_1$. We denote it by $|\mathbb{N}_1| > |\mathbb{N}_2|$.

Pictorially, $\mathbb{N}_1 =$ {1, 2, 3, **4**, 5, 6, 7, **8**, 9, 10, 11, **12**, 13, 14, 15, **16**, 17, 18, 19, **20**, …}

yields $\mathbb{N}_1$ contains the elements 4, 8, 12, 16, … etc. which is surely the infinite set $\mathbb{N}_2$. However, if one changes the above representation from horizontal form to vertical form as shown below,

$$\mathbb{N}_1 = \{1, 2, 3, 4, 5, \ldots\}$$

$$\mathbb{N}_2 = \{\mathbf{4}, \mathbf{8}, \mathbf{12}, \mathbf{16}, \mathbf{20}, \ldots\}$$

some students might notice that there is a one-to-one correspondence relation between $\mathbb{N}_1$and $\mathbb{N}_2$ because $4 = 1 \times 4$; $8 = 2 \times 4$; $12 = 3 \times 4$; $16 = 4 \times 4$, … etc.

i.e.,

$$\mathbb{N}_1 = \{1, 2, 3, 4, 5, \ldots\}$$

$$\mathbb{N}_2 = \{1{\times}4, 2{\times}4, 3{\times}4, 4{\times}4, 5{\times}4, \ldots\}$$

This bijective mapping $f(n) = 4n$ is indeed satisfying the criteria of one-to-one correspondence. Hence, set $\mathbb{N}_1$ and $\mathbb{N}_2$ should have the same size. In other words, the number of elements in $\mathbb{N}_1$ is equal to the number of elements in $\mathbb{N}_2$. If we denote it by $|\mathbb{N}_1| = |\mathbb{N}_2|$, it seems to contradict the previous result from the part-whole consideration $|\mathbb{N}_1| > |\mathbb{N}_2|$. This contradiction causes confusion and inconsistency for many high school students or even for university students.

Despite Cantorian set theory, many students do prefer the one-to-one correspondence relation rather than part-whole consideration, and one-to-one version is also considered to be very normative in mathematics; however, both are not undebated.

The purpose of this demonstration is to claim that students’ intuitive thinking can be enhanced by cognitive conflict, and that making critical thinking intuitive should be one of the important components in mathematics education.

SUBSEQUENT QUESTIONS

Notions of infinity such as actual infinity and potential infinity can belong to the disciplines of philosophy and mathematics or even psychology; therefore, it is not unusual that students have a hard time understanding their meaning, especially merely in terms of discursive language. I think the following questions are also worth thinking about:

1) Is ‘infinity’ definable by using a finite number of written words?
2) Is the truth behind infinity beyond human perception and hence unattainable for human beings?
3) How much do we know about infinity and infinitesimal?
4) How much do we need to know about infinity at the high school level?

CLOSING REMARKS

As Körner (1968) notes, the concept of actual infinity may be separated into three schools, namely the finitists (such as Aristotle, Gauss, and the intuitionists), the transfinitists (aligned with Cantor and his followers), and the methodological transfinitists (like Hilbert). All of them are by no means easy for beginners to comprehend. However, after experiencing discrepancy, cognitive conflict, and inconsistency by comparing the size of two infinite sets, students will gain intuitive thinking and critical thinking skills on the notion of infinity.

Both Dubinsky’s APOS and Sfard’s duality are indeed well-known theoretical frameworks for the studies on mental mechanism and conceptual understanding. Cantor indeed did fully understand the spirit of part-whole composition and one-to-one correspondence; his creation on ‘cardinality’ possibly passed through interiorization, condensation, and reification indirectly. However, how to turn theoretical frameworks into practice or into day-to-day curriculums are still critical issues.

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REDUCING REALITY AND REDUCING COMPLEXITY

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When students work on modelling tasks, they simplify and idealize the situation to generate a real model to represent the situation. This study investigates the strategies students employ during the simplification process and finds two categories of strategies: reducing reality and reducing complexity.

INTRODUCTION

Modelling tasks are problems situated in the real world. They require students to approach the problem from a real-world perspective and to use mathematics as a tool to produce a real solution that fits the original situation (Borromeo Ferri, 2006). The process by which students solve modelling problems can be described using modelling cycles (Figure 1). As students develop an understanding of the real situation and generate a mental representation of the situation (MRS), they simplify the situation and create a real model to represent the situation, mathematize the real model into a mathematical model, determine a mathematical solution and a real solution, and validate the real solution against the original real situation (Borromeo Ferri, 2006). The modelling process relies heavily on students' use of their extra-mathematical knowledge (EMK), which includes students' lived-experiences and their ability to consider the situation and to validate the generated solution from a real-world perspective.

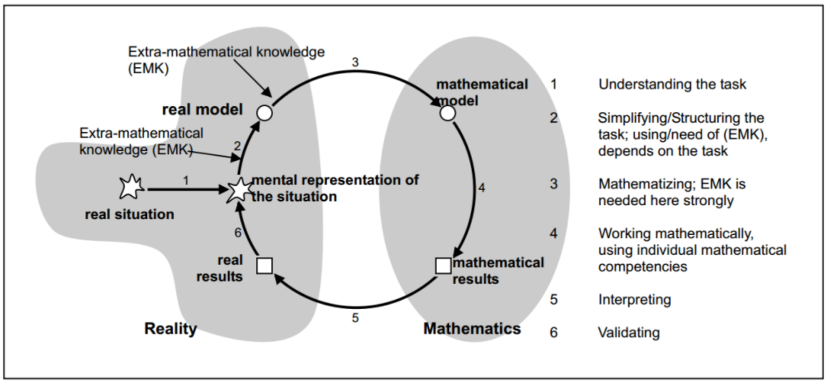

Figure 1: Modelling cycle proposed by Borromeo Ferri (2006)

In this paper, I investigate the strategies which students employed to simplify the real situation and make the situation accessible to them prior to building a real model.

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PARTICIPANTS AND METHODS

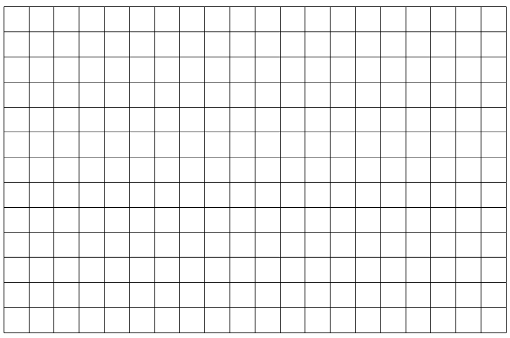
Data for the research presented here was collected in a grade 8 (age 12-13, n=13) and a grade 9 (age 13-14, n=26) class in a high school in western Canada while students worked on a modelling task. Although it is not possible to know if the grade 8's had seen these tasks in their previous years, this was their first modelling task in the grade 8 school year. Conversely, the grade 9's had experience with modelling tasks in grade 8.

Students worked on the *Design a New School* task in randomly assigned groups of 2-4 during a 75-minute class. There were no instructions provided other than what can be seen in Figure 2. While the students worked, the teacher (the author) circulated naturally through the room and engaged in conversations with the students – sometimes prompted by her and sometimes prompted by the students. These conversations were audio recorded and transcribed. At the same time, photographs of students' work were taken and students' finished work was collected. These, coupled with field notes summarizing the interactions as well as observed student activities, are used to build cases for each group of students. Each case is a narrative of students' task experience punctuated by significant moments of activity and emotive expression. These cases constitute the data.

Given the natural and unscripted nature of the teacher movement through the room, not all of the cases are equally well documented. Regardless, each of these cases were analysed separately through the lens of modelling using Borromeo Ferri's (2006) modelling cycle, with a focus on the strategies which students employed to simplify the task.

Design a New School

Your city is getting a new 11000m^2 middle school. It is going to be built on a lot (200m×130m) just outside of town. Besides the school, there will also be an all-weather soccer field (100m×75m), two tennis courts (each 15m×27.5m), and a 30 car parking lot on the grounds. The following requirements must be met:



- all fields, courts, buildings, and parking lots must be no closer than 12.5m to any of the property lines.
- any leftover property will be used as green space – grass, trees, shrubs.
- good use of green space is an important part of making the school grounds attractive.

To help you with your design and layout you have been provided with a scaled map of the property (every square is 10m×10m). Present your final design on a copy of this map. Label all structures and shade the green space.

Figure 2: The Design a New School task

DATA AND ANALYSIS

In the process of building a real model, students simplified, idealized and made assumptions about the situation. These actions are in line with those suggested by modelling literature (for example, see Borromeo Ferri, 2006, 2007). However, students' modelling behaviours suggest different intentions in their simplification strategies. One type of simplification strategy led to a reduction in reality and retained the messiness and complexity of the original problem. Students who applied this type of strategy intended to simplify the situation enough for them to approach the problem from a real-world perspective. I refer to these strategies as reducing reality. Another type of strategy led to a reduction in complexity and changed the nature of the problem. Students who applied this type of strategy intended to change the nature or the goals of the problem. I refer to these strategies as reducing complexity. In what follows I describe students' strategies to simplify and idealize the real situation and provide examples from four groups of students to illustrate these strategies.

Reducing reality

Reducing reality serves as a way for students to filter information and make decisions about the real situation during their modelling processes. This is similar to Borromeo Ferri's (2006) descriptions of students' modelling behaviours as they build a real model. I use the following example to illustrate the idea of reducing reality: when asked to determine the height of a pile of straw bales (5 straw bales at the bottom, 4 on the next level, then 3, 2, and 1 on top), students simplified straw bales into cylinders but took into account the weight of the straw bales towards the top of the pile pushing down on the bottom ones (Borromeo Ferri, 2007). In other words, students simplified the situation but did not reduce the messiness of reality. They remained in the realm of reality and their assumptions idealized the situation and removed some negotiable aspects of the situation so they could proceed to solve the problem and arrive at a solution which fits or can be used to describe the original situation. My analysis shows that there are a number of scenarios which students reduced the reality of the *Design a New School* task. These scenarios can be classified as a mathematical approach, convenient assumptions due to insufficient EMK, and modification of the instructions.

A mathematical approach

Students took mathematical approaches to simplify the question without removing themselves from the realm of reality. They focused on converting the measurements of various building structures from metres to number of squares. This is especially observed in students' treatment of the border, the soccer field, the tennis courts, and the school building. In these scenarios, students' mathematical interpretation of the problem minimizes the realistic considerations of these building structures, such as the placement of the tennis courts and the lengths and widths of their school building in comparison to school buildings found in reality. Their mathematical approach to these building structures allowed them to simplify the problem and to generate a solution that satisfies the instructions quickly. I consider this reducing reality because while students took mostly a mathematical approach, they satisfied the instructions and kept the messiness of the problem: some incorporated the tennis courts into the school building to reduce the ground space these buildings occupied, some discussed the possibility to create a 2-floor tennis court, and all students made use of vertical space and created a multi-floor school building.

Convenient assumptions: insufficient EMK

Another strategy which students applied to simplify the situation is by making convenient assumptions about the situation. Convenient assumptions have strong ties with students' insufficient EMK, where students overcome their insufficient EMK by making convenient assumptions rather than research on the situation. These convenient assumptions are made to address the messiness of the situation, but they do not remove the messiness of the situation.

An example of convenient assumptions due to insufficient EMK is noticed in group A-S' work on the parking lot, where Amy and Angela added an additional 50cm to the width of the vehicles to accommodate for drivers and passengers opening the doors. They did not recognize the possibility to use a parking space as their unit of measurement, but assumed that an additional 50cm to the width of their measured vehicle would suffice. They recognized the situation is messy and made what they assumed to be realistic assumptions to deal with the messiness of the situation.

Modifying the instructions

There are a few scenarios where students deliberately modified the instructions during their modelling processes. For example, the border rule was in place to force students to consider using vertical space. Without the border rule, students would have enough space for all buildings even if they were to build everything on ground level only. As group G-S worked on the task, they modified the border rule by reducing the distance between the property line and all buildings from 12.5m to 10m.

I consider their decision a form of reducing reality because it allowed them to simplify the question, to outline the usable space using the lines on the grid rather than estimating or measuring 1.25 squares from the edge, but it did not remove the messiness of the question – they were already thinking to build a multi-floor school building. Conversely, if they were to change the rule to fit all things on ground level it would have been a case of reducing complexity.

Reducing complexity

Another form of strategy which students employed to simplify the situation is reducing complexity, where they removed the founding criteria or the non-negotiable aspects of the original real situation to avoid answering the problem, or avoid reflecting deeply about the situation and to look for a possible answer to the question as quickly as possible by removing certain variables of the problem. For example, when asked to fairly split the bill between two people who made different amounts of purchase, students may decide to split the bill evenly but the person who had a bigger purchase buys the other lunch to compensate for the cost

Students' actions in the above-mentioned example changed the criteria or the non-negotiable aspects of the original situation to avoid answering the question. In this scenario, the students' decision allowed them to quickly find an answer to the problem while avoiding answering the original question – to split the bill fairly. While this is a possible solution to the problem, the students avoided the messiness of the question and they oversimplified the question. There are a number of scenarios in which students reduced the complexity of the situation. These scenarios can be classified as breaking rules and convenient assumptions to finish quickly.

Rules are meant to be broken

There are a few scenarios where students interpreted the instructions of the task to their liking to reduce the complexity of the problem. This is especially observed when students dealt with the school building and the parking lot. These scenarios are different from modification of rules as a form of reducing reality, as they remove the non-negotiable aspects of the situation and avoid answering the original question.

In particular, group B-S did not bother to calculate the floor area after a decision was made to build a 4-floor school building; group F-S removed the 11000m^2 school building constraint and claimed that their school building exceeds 11000m^2 without any verification; and group G-S did not like their 2-floor school building (55m×100m) and modified the dimensions to 90m×70m. In these scenarios, students deliberately disregarded what the question had asked them to do. They avoided the messiness of the question but produced an answer that fits the question based on their interpretation that the modifications of the instructions were acceptable.

Also, group G-S wanted to assign the entire basement of their school building (90m×70m) as their parking lot with an understanding that the area exceeds what a 30 car parking lot requires. They attempted to lift the 30 car constraint so their solution could be considered an acceptable solution.

Convenient assumptions: to finish quickly

Unlike making convenient assumptions due to insufficient EMK, these convenient assumptions are made to remove the messiness of the situation, especially towards the end of students' modelling processes. For example, all students except for group A-S assumed all remaining space as green space. Students' assumptions allowed them to satisfy the instructions and to produce a complete solution quickly and to avoid reflecting on how they could distribute the green space to make the school grounds attractive, which was an original goal of the original problem.

DISCUSSION

It is no surprise that students' actions to reduce reality are in line with modelling literature during the stages where students build a MRS and a real model to represent the real situation. Students' modelling behaviours suggest that students intended to simplify and idealize the problem so it becomes accessible to them while retaining the messiness of the problem.

As I reflect deeply on students' actions to reduce complexity, I notice that students in this study reduced the complexity of the problem for reasons more than making the problem accessible. These include the avoidance of work, students' insufficient EMK, and an attempt to keep themselves in flow.

Avoidance of work

Some scenarios in this study suggest that students actively avoided the complexity of the problem or the messiness by reducing complexity. In these scenarios, reducing complexity by making convenient assumptions provided students with an easy way out and allowed them to finish solving the task as soon as possible. Examples of students avoiding work include students removing the constraints such as the school building's floor area and assuming any leftover space as greenspace.

Beyond reach

In other scenarios, students reduced the complexity of the problem when the problem was beyond their reach. Some examples are groups G-S' work on the parking lot and their school building, and group F-S' work on their school building. Students in these scenarios likely reduced the complexity of the problem because they had insufficient EMK, or little understanding of the situation. As such, reducing the complexity of the problem allowed them to simplify the problem and to produce a solution.

Because students' actions have roots in their insufficient EMK, it is possible to change students' behaviour by providing them with guidance and the EMK required to solve the problem so that the problem is now within their reach. This is observed in group G-S' progress as they worked on the parking lot, where they went from designating the basement of their school building as their parking lot to a proper and reasonable parking lot design.

Flow theory and reducing complexity

Csíkszentmihályi (1990) uses the state of flow to encapsulate the essence of optimal experience, during which "people are so involved in an activity that nothing else seems to matter; the experience is so enjoyable that people will continue to do it even at great cost, for the sheer sake of doing it" (p.4). The state of flow is only created when there exists a balance between challenge and ability.

If students' ability exceeds the challenge offered by the activity students easily become bored. Conversely, if the challenge offered by the activity far exceeds students' ability then students are likely to feel anxious and become frustrated. The balance between challenge and skill could be represented in Figure 3.

Liljedahl (2018) extends Csíkszentmihályi's (1990) flow theory and finds that there exists a state of tolerance between flow and boredom, where students work on repetitive tasks but do not feel bored or quit, and a state of tolerance for the mundane between flow and frustration, where students find the challenge provided far exceeds their abilities but do not feel frustrated or quit.

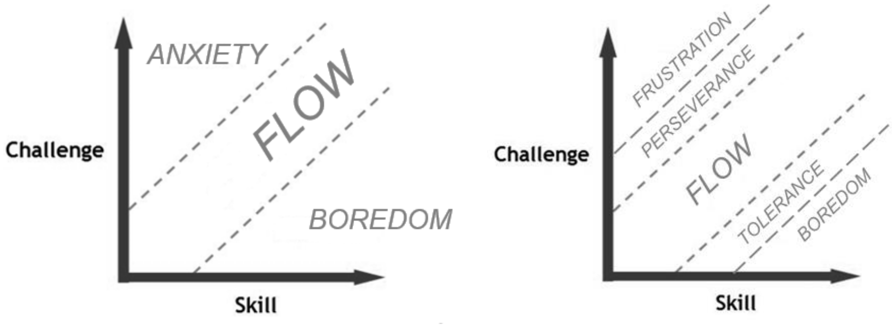


Figure 3: Csíkszentmihályi's (left) and Liljedahl's (right) modified graphical representation of the balance between challenge and skill

The states of tolerance and perseverance act "as buffers between flow and quitting by delaying the transition to boredom or frustration long enough for the imbalance between ability and challenge to be rebalanced. In the case of *tolerance*, this rebalancing was the result of an increase in complexity while in the case of *perseverance*, rebalancing could happen as a result of either a decrease in challenge or an increase in ability" (Liu & Liljedahl, in press).

Many students experienced an imbalance of the challenge found in the *Design a New School* task and their abilities to determine a reasonable solution. Some examples include group F-S' work on their school building and group G-S' work on their parking lot. Instead of increasing their abilities, these students aimed to recreate the balance of challenge and ability and to keep themselves within the band of flow by decreasing the challenge. In doing so, they reduced the complexity of the problem.

SUMMARY

Reality is messy. Students in this study purposefully simplified the situation or approached the situation in a specific way to achieve a goal. In some scenarios students aimed to simplify the situation and retain the realistic aspects of the situation. These assumptions removed some of the freedom found in the problem in order to devise a plan to solve the problem and resulted in the removal of some of the negotiable constraints of the problem. In other scenarios students made assumptions to reduce the complexity of the situation by removing the non-negotiable aspects of the situation to avoid work, to remove the messiness of reality, and to keep themselves in flow.

As I reflect on these strategies, especially reducing complexity, I do not think that reducing complexity is necessarily a strategy which we want students to completely avoid. Reducing complexity represents students' interpretation of the problem and the way which they determine a solution based on this interpretation. Reducing complexity provides students with a way out when they are stuck and a way to solve the problem.

However, reducing complexity allows them to avoid the original problem, and it may remove or change the intended purpose of the original problem.

Does reducing complexity make the solution wrong? Not necessarily. But the results of reducing complexity often have lots of room for improvement. In some cases, students over-simplified the situation and it removed students from the realm of reality. In other cases, reducing complexity provided students with a way to solve the problem. It is not necessarily the way I wanted them to solve the problem or what the problem intended for students to do. But it sheds light on students' insufficient EMK and possibly their misunderstanding of the situation. I do not think reducing complexity is avoidable, nor should it be avoided. Although it is not how I want students to approach and solve the problem, reducing complexity serves as a way for students to access the problem, and possibly a way for students to discuss their difficulties with me or with each other.

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HISTORICAL CONTEXT IN MATHEMATICAL TEXTBOOKS

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This paper analyses a linear algebra textbook to determine the reasons behind inclusion of historical sidenotes. Viewed through the lenses of Constructivism and Situated Cognition, the data is coded for either a mathematical purpose or a humanizing purpose. These codes are expanded on to explain how the sidenotes could specifically be used by a student to either situate the mathematics in history, construct the mathematics themselves, or to invite the student to do mathematics.

INTRODUCTION

There has been much research into using history in mathematics classrooms, and it is even part of the curriculum in certain parts of the world, such as the Nordic countries (Jankvist, 2010). While the purpose can be to impart the historical aspects of mathematics, more educational bodies use the history to apply meaning to the mathematics (Jankvist, 2009). This meaning can include introducing the contexts behind mathematical discoveries or the arguments over different approaches (Liu & Niess, 2006). In a roundup of studies about using history in mathematics education, Fauvel (1991) presented a list of reasons why using history in mathematics could be beneficial, which I summarize below:

- Following the history of mathematical progress can be a template for students' mathematical progression.
- Historical problem areas may point to problems students are still having today.
- Historical context can pique students' interest and provide motivation.
- Stories about the people behind mathematical breakthroughs can make mathematics more accessible.
- Mathematics comes from all cultures and all peoples; exploring that can build students' confidence that they too can do mathematics.

The majority of the studies have focused on the classrooms and how teachers incorporate the historical aspects; less have looked at the historical content in textbooks. Of those that did, it was found that the historical content in textbooks was largely superficial, and light on any mathematical purpose, which caused instructors and students to skip over it (Smestad, Jankvist, & Clark, 2014).

2018. In J. Forde & A. Rouleau (Eds.). *Proceedings of the 13th Annual Mathematics Education Doctoral Students Conference*, pp. 73-80. Burnaby, Canada: MEDS-C.

RESEARCH QUESTIONS

My research question started with the authors' purpose of including historical context in a current linear algebra textbook. Specifically, how does this context support the mathematical concept that it accompanies? After reviewing the data, I extended my query to include the humanizing aspects of historical stories: How do they help the students see themselves as mathematicians?

THEORETICAL FRAMEWORK

The theory used in this study draws from both Situated Cognition and the Constructivist viewpoint espoused by von Glasersfeld. In von Glasersfeld's (2002) description of Constructivism, he focuses on how students build knowledge, while highlighting their need to take ownership of the material and build off their current knowledge base. We can apply his theories to the study of historical context by suggesting that including this context can humanize mathematics, inviting students to see themselves 'doing math' while also building up the knowledge base of mathematics in the same manner that mathematics was formed. This recapitulation argument asserts that "to really learn and master mathematics, one's mind must go through the same stages that mathematics has gone through during its evolution" (Jankvist, 2009, p. 239).

"In situated cognition, one cannot separate the learning process from the situation in which the learning takes place" (Caffarella & Merriam, 2000, p. 59). This idea is most often discussed in the terms of apprenticeships or learning in a lab setting, but it has been argued that information is best understood if taught in a way that embraces the entirety of its history from its origins to its current use. Brown, Collins, and Duguid (1989) believe that to understand a new concept, students should understand the context of its creation, how it is used now, as well as the cultural aspects of both.

METHOD

To do this study, I analysed the first three chapters of a linear algebra textbook that is currently in use at Simon Fraser University in British Columbia, Canada. I reviewed the chapters on Vectors, Systems of Linear Equations, and Matrices and Matrix Algebra from *Contemporary Linear Algebra* (Anton & Busby, 2003). I chose these chapters as each introduces some content that should be completely new to the students and does not necessarily build off previous mathematical knowledge. I focused on the sidebars, information that was considered external to the main mathematical content and was embodied in a highlighted area in the margins of the text. These sidebars were a running series called *Linear Algebra in History*.

I looked into some common analysis types used with literature and decided on thematic analysis as laid out by Braun and Clarke (2006). Thematic analysis is a reflexive way of looking over data; the researcher can come into the data with some pre-established ideas, but will code the data according to those and other themes that arise. Each piece of data is looked at repeatedly to fit into any themes that do emerge, which means that each piece of data can end up with multiple codes.

DATA ANALYSIS

There were a total of 17 sidebars, but I found 42 instances where meaning could be derived from the discussions within that I wanted to code. I went into the data with the pre-determined themes of 'situating the data into historical context' and 'assisting students in constructing their own meaning', but quickly realized that the sidebars were performing many more functions than just those two.

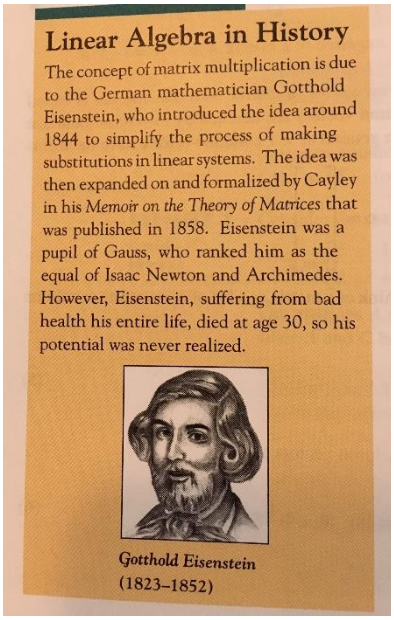


Figure 1: Sidebar from chapter on Matrices and Matrix Algebra (Anton & Busby, 2003, p. 82)

For example, I coded the content of Figure 1 (above) as both contextual, in that it explained why matrix multiplication came into existence, and constructivist, in that the authors have already discussed linear systems, so by including that information students could construct their own idea of why matrix multiplication would be necessary. But the sidebar continues with a discussion of mathematicians that were involved in the development of this concept. I used two different codes for this part. The fact that Cayley expanded on the idea of matrix multiplication means that, as brilliant as Eisenstein was, there were parts that he missed, just like there will be parts of mathematics that the students will miss. I called this 'Humanizing Mathematics'. The further information about Eisenstein, that he had bad health and died early, was humanizing him as a mathematician.

Overall, I ended up with two main themes, "Humanistic" and "Mathematical", with five subthemes. There were nine instances that just named a person or a book that a term or concept referred to. For the purpose of this study, I coded them as references, but did not analyse them further.

Humanizing Mathematics

The overall purpose of this category was to make mathematics and mathematicians more relatable to students. There were three ways that the authors aimed to achieve this goal. The authors were evenly split among telling stories that depicted mathematicians as people who had lives outside of their discoveries, which I have called 'Humanizing Mathematicians', and telling stories around the difficulties and foibles that surround the mathematics in new discoveries, which I have labelled 'Humanizing Mathematics'.

Under Humanizing Mathematicians, students learn that these historical figures had other interests, health issues, legal troubles, and other incidents that give the impression of a full life. As discussed by Fauvel (1991), it can seem to some students that mathematics can only be done by the greatest minds solely focused on the material; so by seeing that these people had full lives, students can imagine that they too can do mathematics.

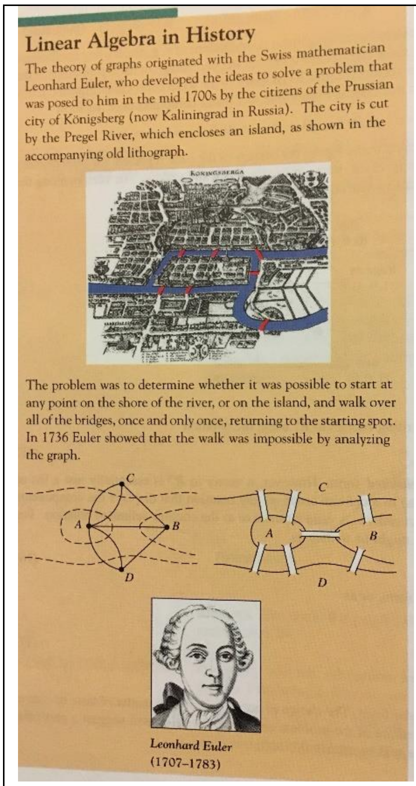
There were a number of instances in Humanizing Mathematics, where the authors took a mathematical discovery and talked about either: how another mathematician expanded it; how, at the beginning, a certain part was left out; or how it took a few people working together to come to a discovery. In these stories, the mathematics is shown to be something that even famous mathematicians struggle with and get wrong, so it would follow that students would also struggle. In all of these anecdotes, though, the person (or people) does succeed, which gives a hopeful pathway to the student on how to continue in mathematics. Mathematics is something that is often thought of by students as always being perfect, and having always been perfect; so these examples show that humans worked hard to create mathematics, with many bumps along the way. It shows students that what they are learning now is where we currently are at this point in the history of mathematics, and that there is still so much more mathematics for them, and for all of us, to learn.

The last theme under this heading had half the amount of the other two; I called this 'Diversifying Mathematics'. The purpose of this heading was to open up mathematics to groups that, in Western Education, are not generally thought of as mathematicians. The sidebars that were coded under this heading highlighted that the mathematicians in question were in a minority category; in these specific instances that included individuals who were Jewish, gay, female, or Asian. Highlighting the minority status of mathematicians invites students that share that status into the conversation. If a student can see others like them have succeeded, then they are more likely to believe that they can succeed in that endeavour.

Mathematical Purpose

Within the sidebars, almost half of the content included was mathematical in nature. Of that content, it was almost evenly split between giving context to the term or idea being discussed, which I called 'Situated Concept', and giving information that could help a student construct meaning for themselves, called 'Construct Meaning.' Often both were included at the same time.

Under ‘Situated Concept’, the vast majority situated it in the localized history of mathematics or the wider historical context. These examples ran the gamut from the example in Figure 1, where the idea of matrix multiplication is used to make substitutions in linear equations easier, to the one below in Figure 2, where Graph Theory is born from the placement of bridges in an old city.

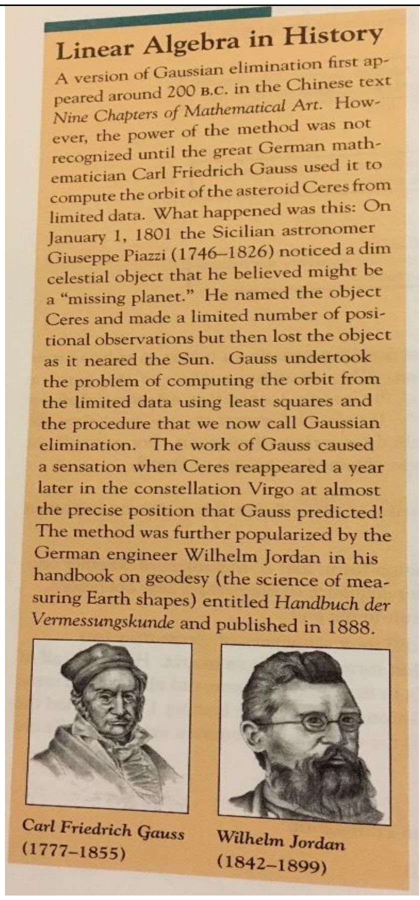


Linear Algebra in History

The theory of graphs originated with the Swiss mathematician Leonhard Euler, who developed the ideas to solve a problem that was posed to him in the mid 1700s by the citizens of the Prussian city of Königsberg (now Kaliningrad in Russia). The city is cut by the Pregel River, which encloses an island, as shown in the accompanying old lithograph.

The problem was to determine whether it was possible to start at any point on the shore of the river, or on the island, and walk over all of the bridges, once and only once, returning to the starting spot. In 1736 Euler showed that the walk was impossible by analyzing the graph.

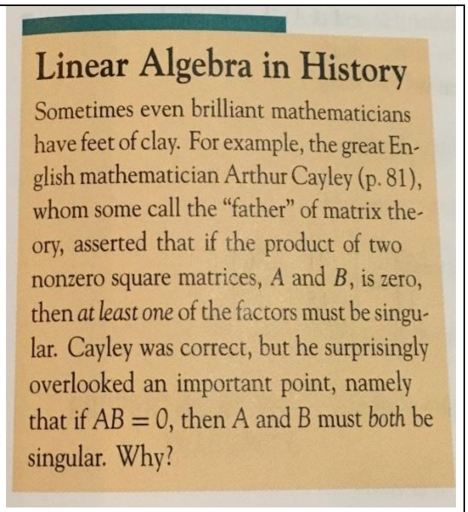
Figure 2: Sidebar from chapter on Vectors (Anton & Busby, 2003, p. 12)



Linear Algebra in History

A version of Gaussian elimination first appeared around 200 B.C. in the Chinese text *Nine Chapters of Mathematical Art*. However, the power of the method was not recognized until the great German mathematician Carl Friedrich Gauss used it to compute the orbit of the asteroid Ceres from limited data. What happened was this: On January 1, 1801 the Sicilian astronomer Giuseppe Piazzi (1746–1826) noticed a dim celestial object that he believed might be a “missing planet.” He named the object Ceres and made a limited number of positional observations but then lost the object as it neared the Sun. Gauss undertook the problem of computing the orbit from the limited data using least squares and the procedure that we now call Gaussian elimination. The work of Gauss caused a sensation when Ceres reappeared a year later in the constellation Virgo at almost the precise position that Gauss predicted! The method was further popularized by the German engineer Wilhelm Jordan in his handbook on geodesy (the science of measuring Earth shapes) entitled *Handbuch der Vermessungskunde* and published in 1888.

Figure 3: Sidebar from chapter on Systems of Linear Equations (Anton & Busby, 2003, p. 54)



Linear Algebra in History

Sometimes even brilliant mathematicians have feet of clay. For example, the great English mathematician Arthur Cayley (p. 81), whom some call the “father” of matrix theory, asserted that if the product of two nonzero square matrices, A and B, is zero, then *at least one* of the factors must be singular. Cayley was correct, but he surprisingly overlooked an important point, namely that if $AB = 0$, then A and B must *both* be singular. Why?

Figure 4: Sidebar from chapter on Matrices and Matrix Algebra (Anton & Busby, 2003, p. 101)

These sidebars delve deeper into the history of mathematics and the history of our world, in trying to show where a discovery came from and why it was needed. It helps students see that these discoveries were often made to solve problems, usually to make a mathematician’s life easier. So, while these examples give historical context to mathematical concepts, they also have a humanizing quality where students can see the trajectory of the mathematics they are learning.

Under those sidebars that allowed the students to ‘Construct Meaning’ for themselves, most did it in terms of situating the concept. As in Figure 1, while students can follow the progression of the mathematical discoveries, they also can see how it mirrors their own progression. By now, they have learned of linear systems and practiced substituting into those systems, so they can use this information to work out why matrix multiplication would simplify this process. In Figure 2, the same could be said. The sidebar gives the wider historical context behind graph theory; but, by including the diagram, it lets students attempt the same problem and start to understand the power behind this theory.

There were a few sidebars that only fit under ‘Situated Concept’ or ‘Construct Meaning’. They are above, labelled Figure 3 and Figure 4. In the first of these, the story behind the historical usefulness of Gaussian Elimination is shared, but does not give enough information applicable to the students’ background to where they could try to replicate the actions. In this case, the mathematical purpose of the sidebar is to just give a context for the new concept in history. Students have an understanding of space and could see how knowing the orbits of certain celestial beings could be useful; so, even without grasping how Gaussian Elimination helped in this instance, they can determine the overall usefulness of the method and the reason it was so important at the time.

The sidebar in Figure 4 does not situate the concept in history, other than to let students know that this subject has been a matter of study for thousands of years. However, it does use history to introduce an activity that they could use to further their own understanding. This also has the side benefit of ‘Humanizing Mathematics’, as even Cayley missed some things that the students can prove and understand. Of the 42 instances where I could code meaning from the sidebars, only 4 were of an historical activity, such as the one just described and the one in Figure 2; the rest were more anecdotal.

DISCUSSION

The textbook analysed did provide historical context for some of the new terms and concepts that students encountered. The ones that I reviewed were in the form of sidebars, therefore off to the side of the main content. In the guide in how to use the textbook, these historical perspectives are listed last of all the things in the text (see Anton & Busby, 2003, p. viii). We can assume, then, that while the authors saw a need for this type of content, they felt it was of lesser importance than the mathematics that they were presenting; the historical notes were more incidental than a focus. That said, the authors were careful to draw the eye to these sidebars by having them next to the concepts discussed and by using similar typesetting in the text and the sidebars for the word or phrase of interest.

Even given that, there were still seventeen sidenotes in the three chapters I reviewed. As seen in Chart 1 below, almost half of the information in these sidebars had a mathematical purpose, evenly split between situating the concept, explaining the background and why it was important historically, and using history to help the student construct their own understanding of the concept.

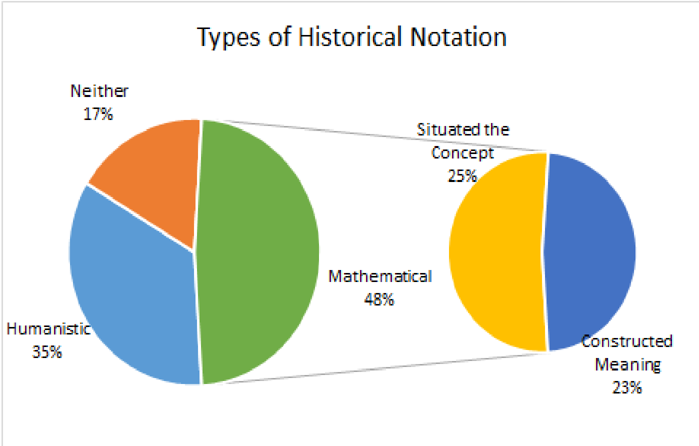


Chart 1: Types of Historical Notation

Those that situated the concept in history did not go into detail about the process of the mathematical discovery or the reason behind its origin; they instead just imparted a sentence or two that contained an original use, or a reason that the concept came into being. These stories would not be enough to allow the student to trace the concept from origin to today, but could spark an interest that would lead an interested student to do so. As stated earlier, a third category was referential, where a sidebar would just give the name of a person or book related to a concept's discovery. While, this did not give any information to a student to use in their mathematical studies, it could lead a student to outside research.

I would argue that the authors included these small snippets of the historical context along with references less as a tool that students can use to learn the mathematics within this book and more as a way to broaden the culture of mathematics. While it does not fit the general idea of Situated Cognition, it does provide students with a sense of the life and times in which these discoveries were made. These snippets, those that do not have a constructivist bent, give students a taste of the history behind mathematics, the historical uses of mathematics (even if the students cannot currently comprehend those uses), and the scope of humanity involved in mathematics. Even the reference category helped expand that scope, as students can see the different eras, languages, and people involved in the creation of these concepts.

The remainder of the concepts, those that help students construct the mathematics for themselves, and those that work to humanize mathematics, all belie the same mathematical purpose: giving students permission to try mathematics.

The humanizing aspect allows students to see that mathematics is not handed down from on-high, fully and perfectly formed. It is created by humans, often by many humans working together, and failing consistently, until they get it right. These humans have multifaceted lives, and they are representative of all of humanity; so any student should be able to see themselves in a mathematician and therefore as a person capable of doing mathematics. This humanizing aspect then gives the student permission to try mathematics while the constructivist scenarios and activities give them tools that not only build on their own past learning but also use their past learning to explain how a concept even came to be. Referring back to Figure 1, in that sidebar students learn of a mathematician who had a full, if tragic, life, and made a discovery that is explained in a way students can tie to their own past learning; then that discovery was furthered by others. Just in that sidebar, they can see the humanity of mathematics and that they also can participate in that human endeavour.

Overall, the authors seem to have chosen the information to include in the sidebars mostly for the above purposes. They wanted to invite students to do mathematics, give students tools to create their own meaning in these new concepts, broaden the subject of mathematics to be a part of the historical culture, and to give students key information so they can continue their own inquiry if they choose.

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EMBODIED CURIOSITY IN THE MATHEMATICS CLASSROOM USING TOUCH-SCREEN TECHNOLOGY

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In this paper, I use data collected through video recordings from K-2 children aged between five and eight years old to discuss how touch-screen technology TouchCounts and its unique capabilities provide an outlet for students' bodily movement. In doing so, I draw on a self-generated theoretical construct called Embodied Curiosity (EC), which has its roots in embodied cognition, to show that human bodies are essential components of shaping the mind and that students experience mathematical understandings through their bodies. I argue that human curious behaviour translates into bodily movements due to human and non-human agency, which leads to possibilities of constructing mathematical meanings.

INTRODUCTION

There has been growing concern about what is involved in the "doing" of mathematics. Equally as important, is the way in which students construct mathematical meanings using digital technology tools as mediators. Many scientists as early as the 1950s (Harlow, Harlow, & Meyer, 1950; Berlyne, 1960; Festinger, 1957; Harmon-Jones & Mills, 1999) suggested that many learning theories led us to search for different ways to answer questions and develop deeper understandings about how knowledge is sought and retained. It is my belief that curiosity (a desire for knowledge) may provide answers to questions about how children learn and develop mathematical meaning. Furthermore, technology has made the question of mathematics learning more different. Noss and Hoyles (1996) for instance, suggested that the computer can be used as a form of "webbing" comparable to the role of the World Wide Web, portraying mathematics learning as a process of networking and implying that children construct mathematical meaning when they draw on supportive knowledge. In this sense, Noss and Hoyles' interest does not lie in the computer itself, but rather "on what the computer makes possible for mathematical meaning-making" (p. 5). It is through these understandings that this paper evolved about the relevance of curiosity to mathematics education and more specifically on the role touch-screen technologies play in stimulating curiosity to influence how students develop mathematical meanings.

Curiosity

Curiosity is immeasurable, intangible, and elusive to define. However, authors such as Baxter and Switzky (2008) specified curiosity as a desire to know or a state of

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inquisitiveness, while Loewenstein (1994) positioned curiosity as a term evolving from the disciplines of psychology and philosophy and contended that “curiosity has been consistently recognized as a critical motive that influences human behaviour” (p. 75). As a result, curiosity emerged with a common understanding as being evoked in three distinct ways: by biological factors, from a desire to fill an information gap, and by an imbalance between something and an individual’s view of the world. Furthermore, Rowson et al. (2012), proposed a fourth type of curiosity which they called “tactile curiosity”. They established tactile curiosity in a view that human cognition is inherently embodied and that things are not only understood in sensory terms, but also in terms of how they can perform. Consequently, they believed that our perceptual worlds are not only embraced by objects that we inherently absorb, but of the possibilities which lead us to think and act in certain ways. This implies that embodiment is linked to curiosity, within which my interest lies.

THEORETICAL FRAMEWORK

Embodied Curiosity (EC), which guides this study, is self-generated. It is in keeping with the mind grounded in the details of sensorimotor embodiment. Drawing on Lakoff and Núñez’s (2000) and Lakoff and Johnson’s (1980) versions of embodied cognition (with conceptual knowledge assuming the relation to which human bodies operate in the environment) and Gol Tabaghi and Sinclair’s (2013) understanding that learning takes place when speech, body movement, gestures, and materials work together in a harmonious relation, I juxtaposed Pickering’s (1995) account of human and material agency. He suggested that there is a line drawn between actions of scientists (humans) and the actions of nature, which credits agency both to people and to things.

As a result, agency has to do with the influence of one thing onto another. Furthermore, Pickering proposed that in people’s desire to understand, they are led to do certain things, thus encountering resistance from various sources, including material objects (such as an iPad). Resistance usually hinders the smooth running of a process. In order for humans to accomplish a task, they must first make accommodations to overcome or circumvent such resistance. From a posthumanist point of view, human agency is not given precedence over any other form of agency (in this case material agency) and therefore curiosity arises from a human-material interaction. I see EC as the harmonious relationship bounded together by agency, among human curiosity, materials, and body movements. EC emphasizes curiosity emerging from the ways in which learners and technology tools interact with each other and how mathematical meanings are developed from body movement.

The term—embodied curiosity— is not novel, in fact, it was used by Fridman (2013) who, at the time, was seeking a place to store things discovered by her own curiosity. For Fridman, embodied curiosity meant changing ideas into action by infusing curiosity into her own life and acting on curiosity as often as possible. To the best of my knowledge, Fridman did not explore this concept any further and the term was left untheorized. My use of the term does not suggest a continuation of Fridman’s work, nor imply that my development of the theory is grounded in her work. Instead, I acknowledge the term "embodied curiosity" is not new.

Description of TouchCounts

TouchCounts (Sinclair & Jackiw, 2011) is a multimodal touch-screen application for the iPad that provides young children with means for creating and representing numerosity. Within this application are two micro-worlds called the enumerating world (for counting) and the operating world (for adding and subtracting). In the former world, learners use their fingers to tap the iPad screen, which produce numbered yellow discs (called herds), along with aural number names. When learners are working in the enumerating world (with gravity on) a horizontal line called the shelf appears on screen and the discs disappear when tapping is done below the shelf. Conversely, tapping above the shelf produces discs which are visible on the horizontal line (Figure 1a). In the latter world, tapping creates large circles (called herds) representing the cardinality of the set produced by the fingers (Figure 1b). However, the emphasis of this paper is on the operating world since the task and episodes involved students' experience with adding numbers.

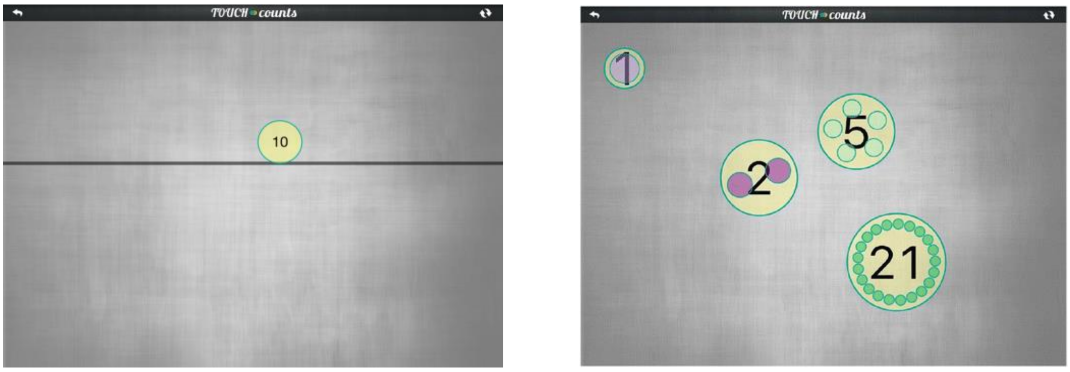

Figure 1: (a) tapping in the enumerating world (gravity on), (b) tapping in the operating world

METHODS

Data for this research responds to the extent to which embodied curiosity fosters the construction of mathematical meaning, and came from eight months of observing and video-recording interactions between children of ages ranging from five to eight. The participants were taken from an after-school day-care in Burnaby for K-2, where they engaged in a series of tasks while waiting on their parents for pick up. I worked alongside two other researchers (Nathalie and Sean) for approximately one hour each week (30 minutes per group) with children in groups of two to four. Students were randomly selected either from a list presented by one of the day-care educators or when they walked into the research site volunteering themselves to "play". The interaction in this study was taken from a session with one group of girls comprising students whom I named Chelly, Nadison, and Oliah. The focus of analysis was on how touch-screen technology may or may not influence children's embodied curiosity. The video-recording was then transcribed in different rounds focusing on various features of the episode. Firstly, I recorded the voices from humans and then "voice" from the material which is an affordance of the touch-screen application. My next phase was to pay attention to children's body movement and align them with human curious behaviour and the mediative role of the digital tool.

Data Analysis and Results

To give an interpretation for the synergy among material, human curious behaviour, and body movement, I used an episode taken from a video-recording involving one researcher (Sean) and a group of girls who were working on the task of "making thirteen". This episode was selected primarily because of the potential it possesses in demonstrating the way EC is operationalized. That is, embodied curiosity occurs when children transform their wonderings while using technology into body movements and their body movements into possibilities of developing mathematical meanings. I first observed how children's interaction with the technology tool and the mediative role of the technology stimulated their curiosity. Subsequently, I analysed the relationship between the ways they demonstrated curiosity (through body movements) and how mathematical meanings were developed as a consequence of their curiosity.

Episode One: Make Thirteen

Sean pushed the iPad to the girls individually from Chelly-Oliah-Nadison with the same five herds (2, 2, 7, 6, 1) and asked that they each make thirteen. Chelly and Oliah both attempted the task and were unsuccessful, pinching (7, 2, 2, 1) and (7, 2, 1, 2) respectively. The children gazed at the iPad screen (Figure 2a) as Oliah contemplated how she would perform the task of making thirteen; but this was not sufficient to determine whether or not the children were curious or what they were curious about. The episode below occurred during Nadison's turn and the analysis provided a clearer description of the ways in which the children were curious. I used transcript conventions of curly brackets to represent the iPad's action or voice and squared brackets to show body movements. The < and > signs were used to depict my understanding of students' feeling within the moment.

1 S: I want you to make thirteen

2 O: Oh yeah! it's gonna be the same thing [Beating fist on the table. In a rhythmic fashion while she utters "the same thing"]

3 C: Who cares! <with disappointment at her attempt>

4 O: She might get thirteen [Nadison using her index finger and thumb simultaneously to pinch together the disc, 7 and 6] (iPad says thirteen)

5 N: Oh! I did only two! I didn't even expect that! that was so-oo weird [Nadison opens her hand wide and gasp in disbelief]

Figure 2: (a) children backed away, (b) leaning forward, (c) Nadison shows surprise

Initially, the children demonstrated curious behaviour in the way they clustered together over the iPad while Nadison performed the task, pinching the herds (7, 6) together. They were fixated on the screen with a deep gaze, to see which herds she would choose to produce 13 (Figure 2b). I see this as the interplay between human and material agency as described by Pickering, where the affordances (including the visible herd, the aural number names which mapped to the visual representations of numbers, and the touch-screen interface) acted upon and influenced the children with a desire to see and hear what resulted. This is a sign of curiosity which may lead to further exploration of different ways in which the herds could be pinched together to achieve thirteen. This signifies the possibility that numbers can be composed and decomposed in different ways.

During the interplay between children and material, the material was granted a central role in the process, giving importance to what the iPad could do, as seen in turn 4 with the utterance "she might get thirteen", as if thirteen were a 'thing' to be handed out by the iPad. Likewise, Nadison was surprised that thirteen appeared on the screen after using only two herds (7 and 6) as oppose to four, which was previously used by the other girls. This was evident in the way she opens her hands wide (Figure 2c) and in her utterances "I did only two" and "I didn't even expect that!" at turn 5. In addition, the "weirdness" she spoke about also at turn 5 reveals that she might have been unaware that two numbers could be added to yield thirteen, or at least those two; 7 and 6.

Figure 3: (a) Oliah's decision, (b) Nadison's decision

It is also important to note that, in Figures 3a and 3b, eye-hand coordination played an important role in the decision related to which herds were used to make thirteen. Therefore, embodied curiosity occurs when children's eye-hand coordination, the visual and aural capabilities of the iPad, along with the children curious behaviour (as displayed by their fixation on the touch-screen interface and how they leaned forward spontaneously when a task is being performed) coordinate. This occurred in a synchronizing way which suggested that the children came to an understanding that numbers can be composed and decomposed, as well as the children have attached the meaning of addition to the pinching motion of their fingers. Hence, 7 and 6 makes 13.

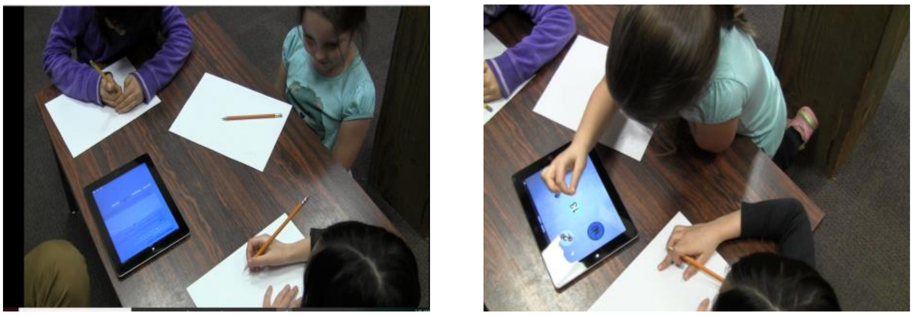

Figure 4: (a) Children reluctant, (b) Chelly revisits the iPad

(c) Chelly's fingers with meaning, (d) Adding with fingers and arrows

Shortly after, the children were asked to add together 7 and 5, this time, using paper and pencils. They drew the circular herds depicting 7 and 5 on the paper as they appeared on the iPad screen. That is, with small circles inside, representing the cardinality of the numbers. Chelly was reluctant to begin at first (Figure 4a), but later decided to revisit the iPad to concretize her understanding of "putting together" (Figure 4b). She experienced EC when her fingers directly moved from the iPad screen in Figure 4c with a new meaning of "putting together" these two numbers on the paper. In this sense, Chelly's fingers became the objects of addition, which she believed was transferable from the iPad to the paper. This implies that the interplay between the tool (iPad) and the children's fingers generated a possible understanding of finger-pinching as addition. This was further reiterated in their drawings on paper (Figure 4d) when Chelly also drew two fingers and used an arrow to show how they were added together.

DISCUSSION

The mathematical ideas which emerged from the data was significant in providing an explanation of how the relationship between the mediative role of the touch-screen technology and children's curious behaviour through their body movements influenced the generation of mathematical knowledge. For instance, in episode one, the desire to know which numbers will make thirteen was displayed in the way both human and non-human agencies interacted with each other. The visual display of numbered discs on the screen evoked students' interest (Figures 2c & 3a) and curiosity which further led them to lean their bodies closer to the device in anticipation of which numbered discs could be used to make thirteen. Furthermore, this coordinated relationship between technology tool, curiosity, and children's body movement led to an understanding that thirteen could be made in different ways and a knowledge that 7+6=13 was one possibility. In addition, the movement of the hands in a widened position followed by a gasping sound indicated that this curiosity has been satisfied with the potential of new "wonderings" emerging. Moreover, the emergence of the mathematical idea of the closure property in arithmetic, where two numbers of the same set (whole numbers) when added produced a member of the identical set was evident particularly when the task was transferred from touch-screen device to paper and pen. Though the children may not have articulated this, the idea of pinching the numbers together with two fingers generated a new way of looking at numbers.

CONCLUSION

The analysis provides supporting evidence that touch-screen technologies such as *TouchCounts* stimulate curiosity and offer an opportunity for young children to perform actions. The data also revealed that *TouchCounts* and its unique aural and visual features stimulated a strong desire for knowledge, whether it is by exploratory means or by attending to a specific piece of knowledge. Children's body movements such as leaning forward and backwards, movement of the hands, eye movements are all responsible for the way in which children construct mathematical meanings. *TouchCounts* provided visual, tactile, and aural cues which foster the stimulation of curiosity.

Using EC: the coordination of human curiosity, materials, and body movement as an analytical tool helped me to understand that the interplay between children and technology tools within the mathematics classroom plays an important role in stimulating children's inquisitiveness which leads to continuous exploration and development of mathematical meanings. In addition, this study also accentuates the significance of considering touch-screen technology as essential tools for stimulating human curiosity, and helps me to understand that mathematics teaching and learning should not be concerned solely with the nature of mathematics but also the nature of human beings.

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INSIDE TEACHER TENSIONS: EXAMINING THEIR CONNECTION TO EMOTIONS, MOTIVES, AND GOALS

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This paper examines tensions faced by mathematics teachers and their effect on teachers' actions using constructs from activity theory. Findings suggest that emotionally laden tensions can reveal motives, and impact teachers' goals by altering, prioritizing, or strengthening them. Therefore, in the relationship between emotions, motives and goals, tensions can be understood as drivers of teachers' actions.

INTRODUCTION

Reflective conversations with mathematics teachers invariably contain what I have come to call 'I…, buts':

> I want to implement problem solving in my classroom, but I don't know how.
>
> I know collaboration is a good thing, but some of my students work better alone.

For those working in professional development, these conflicts are familiar refrains. Some teachers want to make changes, but do not know how. Others are trying to make changes but are encountering challenges. Conflicts such as these are endemic to teaching and are commonly referred to as tensions (e.g., Berry, 2007; Carr, 1998; Jaworski, 1999; Mason, 1988). Studies in mathematics education have produced lists of tensions that impact mathematics teaching (e.g., Liljedahl, Andrà, Di Martino, & Rouleau, 2015; Mason, 1988; Page & Clark, 2010); however, there has been less focus on their mechanisms of action. Liljedahl et al. (2015) argue that "better understanding of these tensions would allow us, as mathematics education researchers, to better understand the intentions and actions of mathematics teachers — and to better respond to their needs in the crafting and delivery of professional development opportunities" (p. 200). My aim with this study is to further develop that "better understanding" of tensions. I move beyond the identification of tensions to examine more closely how it is tensions act, and how they are acted upon, by using constructs based in activity theory.

THEORETICAL UNDERPINNING

One of the findings from Liljedahl et al. (2015) was that tensions are tied to teachers' needs. For this reason, I turn to Leont'ev (2009) to understand the complex relationship between tensions and needs. Consider again, for example, the opening refrain:

> I want to implement problem solving in my classroom, but I don't know how.

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We see that underlying the tension (I don't know how) that causes the teacher to actively seek professional development is an unfulfilled goal (implementing problem solving) that is impeding an unspoken need (e.g., being seen as a good teacher or improving student learning). This is the essence of Leont'ev's (1974) notion of activity: "Behind the object, there always stands a need or a desire, to which [the activity] always answers" (p. 22). He adds that a motive is an object that meets a certain need, and that generally, motives go unperceived by the subject:

> The paradox lies in that motives are revealed to consciousness only objectively by means of analysis of activity and its dynamics. Subjectively, they appear only in their oblique expression, in the form of experiencing wishes, desires, or striving toward a goal. (Leont'ev, 2009, p. 171)

Goals, however, are conscious; we are typically aware of what it is we are striving to achieve and can pinpoint our aims. If visualizing activity as a hierarchy, we would see unconscious motives/needs driving the activity, with the activity being directed at the conscious goals and their related actions (see Figure 1). For Leont'ev activities are composed of actions, which are, in turn, composed of operations. These three levels correspond, respectively, to the motive, goals, and conditions. As indicated by the bi-directional arrows, all levels can move both up and down (e.g., goals can become motives, actions can become operations).

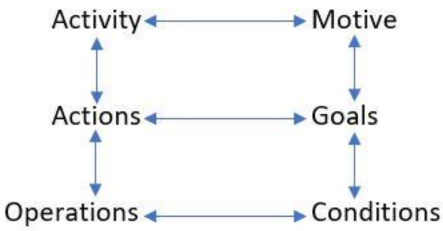

Figure 1. The activity hierarchy of Leont'ev (2009)

To understand the dynamic nature of the components within the activity hierarchy, Leont'ev (1974) described a situation in which the activity is learning to drive a car with a manual transmission. When first learning, shifting gears is a conscious action with the goal of smooth coordination of the clutch and gear-stick. Later, the action of shifting gears becomes operational as the learner no longer has to think "How do I move my hand or foot?"; these are now unconscious operations determined by the conditions (e.g., the speed of the car or the position of the gear-stick). Eventually, driving the car is no longer an activity in itself, it becomes an action in another activity such as getting to work (for which the unconscious need for status may be one possible motive).

For the purposes of this study, only the top two levels of Leont'ev's activity hierarchy will be considered, where activity can be seen as comprising actions related to associated goals. Engeström (2009) argues that tensions are essential for understanding the motivation behind particular goals and actions. For example, a subject can have conflicting needs, or there can be a conflict between a need to be satisfied and the actions that are allowed to be done in order to satisfy it. In both cases, the subject feels a tension that can drive the subject to action. Tensions, thus, belong to the subject's consciousness, while motives and needs can be subconscious — even if they cause the tension.

This impacts analysis as Kaptelinin and Nardi (2006) point out the difficulty in establishing motives or needs in human activity: "The link between what an individual is doing and what she is trying to attain through what she is doing is often difficult to establish" (p. 58). Noting that, for Leont'ev, emotions were signals of the actualization of a motive or need, Engeström (2009), suggests that "to gain access to motives, one must proceed along a "round-about way," by tracing emotionally marked experiences. In other words, the study of action-level emotional experiences is an avenue to an understanding of activity-level motives" (p. 7).

Emotions, then, play an important role in activity theory where they reflect relations between motives (needs) and the real or potential success of an activity. Referring to emotions as "internal signals", in that they arrive from lived experiences rather than intellectual reflection, Leont'ev (2009) states, "they appear as a result of actualization of a motive (need), and before a rational evaluation by the subject of his activity" (p.166). Leont'ev places emotion at the level of activity rather than at the level of actions or operations but suggests that both can be affected by the emotions experienced by the subject.

This suggests a potential connection between tensions, emotions, and motives, discernible at the action-goal level, and leads to my research question: How do emotionally laden tensions affect goals and motives?

METHOD

In this study, I adopt an exploratory and qualitative approach that focuses on documenting the presence of a phenomenon rather than quantifying its prevalence. Data for analysis was taken from a larger study involving six teachers whose teaching experience ranged from 5 to 16 years. The data used was obtained during semi-structured interviews that ranged from 40 to 60 minutes. The interviews were audio-recorded and then fully transcribed. The structure of the interview aimed at letting tensions and emotions emerge through a narrative rather than by direct questioning. For example, the teachers were asked to describe their school, their relationship with their colleagues, and with parents, without explicitly asking them to describe the tensions they lived. This allows for richer descriptive data of personal experiences that leading questions may inhibit. The transcripts were then scrutinized for utterances with emotional components such as "I wasn't happy…" and for utterances that conveyed doubt or uncertainty such as "I wasn't sure, but…". The identified emotions and tensions were then re-examined for their potential connections to goals and motives.

FINDINGS

In the following, I highlight three ways in which tensions impact goals. I characterize the instances with excerpts from the interviews and explicate their interconnections with emotions and motives.

Changing goals

Excerpts in which the mathematics teachers expressed tensions frequently began with some variation of "this never used to bother me, but…". This piqued my interest and these instances were explored to examine what had changed. For instance, Eric mentioned that he used to assign, collect, and mark homework. He would give zeros for unfinished work and felt it was a good use of both his own, and students', time. His implicit goals were gathering evidence of learning and work habits. As his actions worked cohesively towards achieving his goals, there was no apparent tension. This changed when he encountered a student who took no notes, handed in no homework, sat at the back of the class and yet engaged fully in the lesson:

> He'd sit at the back and say "No, you're wrong" or "I disagree" or "What about this?". And I loved it because there was this back and forth, and like this is good! So, I think zeros, forget that! And man did he bring something to the [class]. I loved it. So that really changed my philosophy on taking in homework. Because he just sat there, but he was into it. I thought this was great! A lot of the students, all they do is just hand me homework, I like this better.

I argue that the emotion Eric experienced indicates a perception of motive. He recognized that, beyond his goal of collecting data for assessment, was the deeper desire to engage his students. This became apparent when he further explained:

> I used to collect the homework and mark it and there was no engagement with the students.

In recognizing his motive, he marked it as his new goal — to engage with his students. This shift in goals is accompanied by a tension, as he is not yet certain how to proceed.

> I do know one thing — after 10 years of collecting homework and marking, I don't want to do that anymore. And so I'm trying to… I want to change. I'm trying to fix that, but I'm still struggling with that. So I'm going to make that my focus for the next year.

This is an ongoing tension for Eric, coloured by uncertainty, as he searches for actions that will help him achieve his new goal.

Prioritizing goals

There were several instances where tensions experienced by teachers caused them to rearrange the priority of their goals. Although the teachers valued both goals, the tensions they experienced made them realize that they had been favouring one goal at the expense of the other. We see this in Lacey who explained that, when she began teaching, she was fine with having her students learn mathematics by completing worksheets. She was teaching a combined class of grade one and two students and felt that assigning individual worksheets helped with her dual goals of managing the behavioural issues in her classroom and fostering student learning.

> I would give them worksheets and then get them to sit at the tables and work on it independently. And then I would teach the grade ones, like whole group.

These actions worked well for her until her participation in professional development brought her to the realization that her students were developing very little mathematical understanding. Behavioural issues were under control, but at the expense of student learning. This resulted in a tension for Lacey, as she explained:

> I knew now that it wasn't working. I wasn't happy with my math program, but I didn't know how to change it either.

In Lacey's later description of the recognition as a *"horrible feeling"*, we see a strong emotional response as she comes to an unwelcome realization: her need for classroom management superseded student learning. It also suggests that her unconscious motive may have been to be a good teacher. Unlike Eric, the tension did not cause Lacey to shift from one goal to a new goal. She continued to value her goal of classroom management but had given higher priority to finding complementary actions to aid her goal of developing her students' mathematical understanding.

Strengthening goals

I found instances where the ongoing tensions experienced by a teacher served to strengthen their resolve to achieve a goal. This was exemplified in an excerpt from Mia who views herself as a progressive teacher who wants students to problem solve and think mathematically. She further explained that she wants to teach in a way that makes learning mathematics *"an enjoyable experience for students and meaningful for kids"*—her professed goal. To assess their learning, she relies heavily on formative assessments that lead to mastery. She experienced tension when she was forced to measure her students' learning in standardized assessments:

> It was really frustrating in that I had this idea of how I wanted to teach and how I thought students should learn. And especially after what I would consider a successful unit or a successful lesson and then I would give them this formalized test that was the same as all these other classes and then it actually meant little, because if the average mark of the class wasn't 75%, then there was something wrong with my teaching or my marking. So, if my average was 78%, that probably meant that I was marking too easy and if my average was too low, I wasn't teaching them good enough. I found it really difficult, because I had this idea that if I taught my students well, then they would succeed at what they were learning and if they were above average that maybe meant that I had done something right, that I had taught them well. So that was just stifling.

Mia's strong emotions evidence the tension she experienced when her teaching style was threatened by the imposed assessment. Not wanting to lose sight of her goal, Mia's response was to push back:

> I did [pushback], and that kind of eventually settled and you find enough working terms, but it was an issue.

She went on to explain that she continued experimenting to find teaching methods that suited her students and did not let the threat of standardized testing interfere with her goal.

We see this same tension regarding her teaching style appear again for Mia when she recounts colleagues questioning her practice:

> So, they're like, this isn't right, they [students] need to know these steps. And I'd be like, why? They understand what they're doing and they're getting the right answer… like get over it.

Mia likened this to *"harassment"*, an emotional response that reflects the ongoing tension surrounding her pedagogical choices. However, her final comeback, "*like get over it"* indicates Mia's determination to continue with her actions of working towards her goal. Rather than weakening her goal, both instances of tension appear to solidify it. Leont'ev (2009) suggests that the emotion that accompanies goals is short-lived and does not bring awareness of underlying motive.

Therefore, I find a third instance referencing this tension enlightening. In this instance, Mia had begun preparing her students for their year-end standardized exams. However, this time Mia experienced tension when she found she had to defend her teaching style to her *students*:

> They were starting to stress out, "We have never seen this, what are you talking about? How are we supposed to do this? You never taught us anything". They started to get angrier and angrier as the exam got closer. And I don't know, like that really bothered me—in that okay, for my students, like for their sakes, they do have an exam at the end and I wanted them to be prepared for it. And even though I could see that they had done some amazing math, they never were aware of what they had done.

Here we see again Mia's emotional response. She explicitly mentioned feeling bothered, but implicitly there is also a sense of uncertainty and disappointment. As with the other instances, her teaching style played a role in her tension, but this time the results are different. Rather than solidifying her goal, this time the tension caused Mia to alter her goal and thus her actions:

> And then what happened the following year when I had them [same students] again, they started the year with some of that tension that was still there, even though it sort of had settled over summer. It was new curriculum, a new year. But now they were like, "Well, what's this going to be about?" And then I ended up teaching them very traditional. And, like on my scale of traditional too. So like, I taught them very traditional this year in comparison to last year. But as I said earlier, my traditional is still not a typical traditional.

The emotion and accompanying tension Mia experienced with her students caused her to reorient her goals. Mia may have expressed her need to teach in ways that make mathematics meaningful for her students, but I argue that in backing down, Mia revealed her true motive — building relationships with her students. And like Eric, recognition of her motive marks it as her new goal. Tensions with assessment and colleagues did not cause her to change her goal or actions, but tensions with her students did.

DISCUSSION AND CONCLUSION

For the teachers in this study, tensions arose that, I argue, made self-evident their motives, and subsequently required them to adjust their goals. According to Leont'ev (2009), the level of actions and goals is most readily understood, while the level of activity and its motive is less accessible to individuals. It is through the emotional colouring of their actions that an individual's motive for the activity is revealed. However, it is important to note that emotions are not a reason to act, they are a result of activity (Leont'ev, 2009). Therefore, I suggest that while emotions are effective in revealing tensions, it is the tensions themselves that are vital to the subsequent changes in goals and actions. It was only through experiencing tension that the teachers were motivated to change their actions. This resulted in the participants changing, reordering, or strengthening their goals. This leads to two interesting conjectures.

First, context creates tension. The participants in this study were content in pursing their goals until confronted with a new context. We see this in Eric who happily collected homework until he met with a student for whom homework had no purpose. Likewise, Mia held strong to her goal of student engagement despite pushback from colleagues, but this changed when she met resistance from her students. This suggests the tensions teachers feel subjectively are there for a reason; they are the objective result of goals clashing with a (new) context. The resulting emotionally laden tension makes apparent the teachers' motive, which, in Eric's case, becomes a new goal (see Figure 2a). In Mia's case, the goal was strengthened until it met yet another context (see Figure 2b).

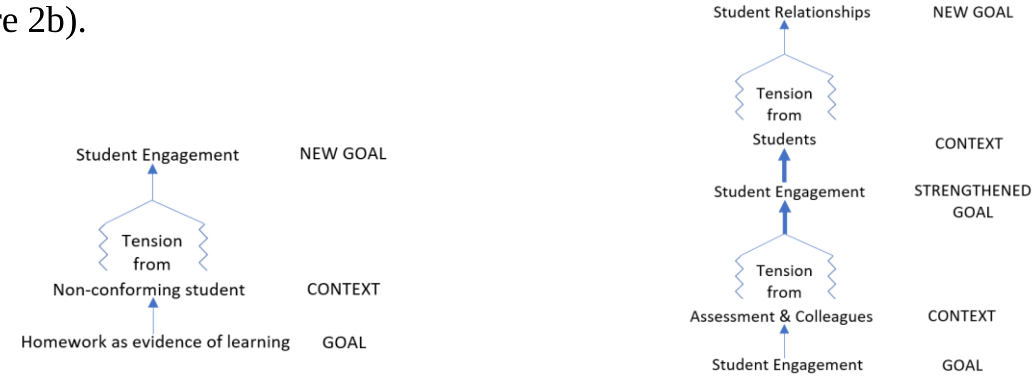

Figure 2. (a) Eric's Goal Transformation (b) Mia's Goal Transformation

Related to this is that tensions are also useful in delineating primary and secondary goals. This is evident in Lacey, who, through participation in professional development (new context), comes to the realization that she values student learning even more than classroom management. This suggests an image of tandem goals functioning in parallel until they hit a context that creates tension (see Figure 3). What emerges is a prioritization of goals where both are in play, but one is given higher priority.

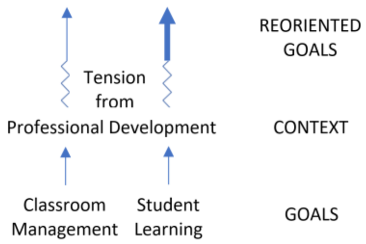

Figure 3. Lacey's Goal Transformation

In this dynamical framing of the relationship between emotions, motives, and goals, tensions can be understood as drivers of teachers' actions. I see tensions as complex collections of opposing forces between possible actions and contrasting motives. Tensions give rise to emotional responses that, in turn, make teachers conscious of their motives. Tensions have an emotional nature and, consequently, they act as signals; the teacher feels, through tensions, that the motives of her actions are contradictory. As such, tensions drive the teacher to action.

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ARE THEY GETTING ANY BETTER AT MATH?

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While the goal of improving how students do mathematics is fundamental to the endeavour of mathematics educators, how and in what ways students improve over time is unclear. This study examines a Calculus 12 lesson on differentiation strategies to identify how students mathed and, through Variation Theory, contrasts the likelihood of improvement given the opportunities afforded students who worked alone compared to those who collaborated on white boards to work through practice questions.

INTRODUCTION

Student improvement is certainly one of the most important goals of mathematics education. As important a goal as that is, however, what improvement means is not universally understood or agreed upon. Does improvement mean knowing more math, being able to solve novel problems, students explaining their thinking, representing their knowledge in multiple ways, collaborating with others, or doing better on tests? Furthermore, what do teachers do in the classroom to support or facilitate improvements in students' ability to do math? Teachers certainly focus on the acquisition of content. But does the acquisition of content knowledge in itself constitute improvement? What mathematical thinking, acts, or actions students get better at while they engage in learning mathematics is the broad question that guides the inquiry in this paper. What are students doing when engaged in the learning of mathematics? How do they *math* and how do the learning activities and classroom structures contribute to this improvement?

MATHEMATICAL THINKING AND DOING

While it is beyond the scope of this paper to unpack all the different ways of describing mathematical thinking and doing, in order to describe student mathing, some frame of reference is necessary. Learning to think mathematically involves, among other things, learning to value the processes of mathematization and abstraction, as well as knowing when to use them (Davis & Hersh, 1998; Mason, Burton, & Stacey, 2010; Schoenfeld, 2016). Schoenfeld (2016) also points out the importance of being able to mobilize mathematical content knowledge in the service of problem solving as being key to mathematical cognition. In *Mathematical Thinking*, Mason et al. (2010) describes three factors that influence how effective one's mathematical thinking is:

1. One's competence in the use of the processes of mathematical enquiry

2. One's confidence in handling emotional and psychological states and turning them to one's advantage
3. One's understanding of the content of mathematics and, if necessary, the area to which it is being applied.

Mathematical thinking, he argues, involves using natural processes in mathematical ways. Mason identifies several mathematical themes also: Doing and undoing involves performing a mathematical action or resolving a question, then reversing the action or resolution for further exploration; invariance allows for attending to what is changing and remaining the same in a mathematical situation; freedom and constraint is helpful when considering solutions, as Polya (2014) describes, "to find", or when constructing tasks. By adding or removing constraints, solutions become more or less specific and can layer towards general solutions.

While mathematical thinking cannot be observed directly, the activities students engage with serve to actualize their mathematical thinking into some sort of doing. How students should engage with activities, both to acquire and use their mathematical content knowledge, has been prescribed by the National Council of Teachers of Mathematics (NCTM) in their Process Standards document (National Council of Teachers of Mathematics, 2000). Problem Solving, Reasoning and Proof, Representations, Connections, and Communication, all serve to highlight the processes students should use to learn and use their mathematical content knowledge. These processes, in various versions, inform numerous mathematics curricula throughout North America. And while the NCTM Process Standards provide teachers with references and goals for student engagement, they do not describe what students are actually doing in response to mathematical activities.

According to Bauersfeld (1993), mathematical activity depends on social and cultural processes. The classroom itself is a dynamic system, in which social and cultural norms are introduced and reinforced by the teacher.

> [T]he understanding of learning and teaching mathematics ... support[s] a model of participating in a culture rather than a model of transmitting knowledge. Participating in the processes of a mathematics classroom is participating in a culture of using math or better: a culture of mathematizing.
>
> (Bauersfeld, 1993, p. 4)

Mathematizing, in this sense, defines, for each classroom, what it means to know and do mathematics. In their work on sociomathematical norms, Yackel and Cobb (1996) identify intrinsic aspects of a classroom's microculture, defined by teachers' and students' activity. They argue that these classroom normative understandings are modified by the ongoing interactions of students and teachers, and are unique to specific classrooms.

While Yackel and Cobb (1996) look specifically at those interactions that sustain a culture of inquiry and problem solving, I contend that sociomathematical norms are present in classrooms with a focus on content acquisition also. Further, it is the activities of students and teachers that define both what doing math is for a particular classroom, and what mathematics is for students in that classroom culture. While micro cultures and sociomathematical norms offer an explanation of how the mathematical activity for a particular classroom culture comes to be, they do not necessarily describe the actions of students in response to classroom activities.

What is needed is a way of describing the mathematical acts or actions of students as they engage in the mathematical activities of a particular classroom culture and determine if their way of engaging, their *mathing*, improves over time. To this end, the purpose of this paper is to answer the following questions:

1. In what observable ways do students engage in the mathematical activities in a mathematics lesson? In other words, how do they *math*?
2. Given the structure of the lesson and the ways students engage, what can be said about the likelihood of improvements in the ways students *math*?

The theoretical analysis for this study will be viewed through the lens of Vygotsky's sociocultural theory (Vygotsky, Hanfmann, Vakar, & Kozulin, 2012; Vygotsky & Cole, 1981). Unlike the incremental acquisition of content, improvements in student *mathing* will be viewed as developmental, involving a unity of psychological functions, and mediated by the activities of the classroom. In Vygotsky's view, learning is culturally situated and leads psychological development. To address the potential for improvement, I will use Variation Theory (Lam, 2013; Marton, 2006, 2015; Marton & Tsui, 2004) to describe the learning activities and the potential for students to discern the critical aspects of how they math.

METHOD, DATA, AND ANALYSIS

The data for this investigation was collected during a single class of Calculus 12 in a secondary school located in Southwest British Columbia. The class was videotaped, and the relevant episodes were identified and transcribed. The class itself was taught by the school's Mathematics Department Head, a fifth-year teacher with a Master's degree in Mathematics. The students, most of whom were taking Pre-Calculus 12 concurrently with Calculus 12, were in their third unit of study. Previous to this lesson, they studied Functions, Limits, and Derivatives. This lesson was the second in a unit on Differentiation Rules. Several days after videotaping the lesson, I informally discussed the lesson with the classroom teacher. During that conversation, she discussed her thoughts on teaching strategies and student activity. Some of what she shared in that conversation is included in my analysis.

Lesson Structure and Mathematical Activity

The mathematical content of this lesson on Differentiation Rules included the Constant Multiple Rule and the Power Rule for Differentiation. Forty-five minutes of the 70-minute class were devoted to the teacher showing her students how to implement both rules. From her position at the front of the class, pointing to a screen displaying her prepared notes for each of the differentiation rules she introduced, she began by showing the students two examples of how to use the rules. This was followed by an invitation to students to try a couple of prepared examples on their own.

This lesson can be described as having two main segments; the *content acquisition* segment of this class (the first 45 minutes), and the *practice* segment (the remaining 25 minutes). The *content acquisition* segment can be further divided into three sections; *teacher demonstration, "now you try" or emulation,* and *teacher consolidation.*

During the *content acquisition* segment and the *teacher demonstration* segment of the lesson, and for student who chose to do individual or group practice at desks, the register of actions associated with this mathematical activity was predictably narrow. The actions of students, in other words, the way they *mathed*, were in line with teacher expectations and align with what one would expect to see in a content driven, teacher centered lesson. Students **watched** and **listened** as the teacher explained how to apply each differentiation strategy to various examples, **responded** to the teacher's prompts during the *demonstration* segment, and **clarified** a procedural move the teacher made that was not clear. During the *"now you try"* segment, students attempted to **imitate** the teacher's strategy, **asked questions** about the correctness of their steps, **clarified** in what algebraic form the answer should be represented, and **adjusted** their work according to teacher feedback or correction. The students who worked individually using paper and pencil on the practice set of exercises, did so silently.

By contrast, the students who chose to work in groups of two, three, or four on the classroom's white boards had a broader register of activity. While still trying to **imitate** the teacher's demonstrated strategies, **responding** to teacher questioning and prompts, and **clarifying** with each other, they engaged in the problem set by using four additional observable actions, **directing**, **vicariously solving**, **negotiating**, and **looking at other groups' work**.

Directing

This describes the action of one student telling another student what to write. This occurred when the student with the pen had stalled in solving the problem. The director bridged the gap in procedure by telling the student with the pen what to write, and then stopped directing when the student with the pen was able to carry on independently. The director appeared to be **vicariously solving** during this action. In other words, the directing was immediate and not as a result of the student with the pen asking for help.

Vicariously Solving

This describes members of the group actively watching the student with the pen solve a problem, as if they themselves were solving the problem. It was exemplified by one or more group members directing when the student with the pen stalled, or by group members immediately pointing out an error.

Negotiating

When students disagreed about the next step in the procedure, they negotiated a resolution. Sometimes the issue was resolved internally; however, occasionally the issue was resolved by asking for help from another group nearby or by **looking at other groups' work**.

Looking at Other Groups' Work

When a group became stuck and no member of the group could direct, students often would look around the classroom at other students' white board work to see if they could resolve their impasse. While the entire solution to the problem they were currently having difficulty with may have been visible on another white board, students seemed to only be interested in attending to the specific part of the work they were stuck on.

The contrast between how students *mathed* during the practice segment of this lesson was dramatic. How the students grouped individually, together at tables, or in groups at white boards qualitatively changed the way they did mathematics. The nature of the white board activity facilitated a collaboration that was not evident when students worked side by side in groups at tables. While vertical non-permanent surfaces like white boards may help students engage in tasks by removing the anonymity of desk work (Liljedahl, 2016), they also appeared, in this study, to support collaboration, communication, and a sense of group community. As mathematics is seen as being more of a collaborative and social endeavor (Albers & Alexanderson, 2008; National Council of Teachers of Mathematics, 2000; Steen, 1988), there is a changing view of the role collaboration has on how students know and learn:

> ...for those engaged in the kinds of collaborative efforts discussed by Steen, membership in the mathematical community is without question an important part of their mathematical lives. However, there is an emerging epistemological argument suggesting that mathematical collaboration and communication have a much more important role than indicated by the quotes above. According to that argument, *membership in a community of mathematical practice is part of what constitutes mathematical thinking and knowing*.
>
> (Schoenfeld, 2016, p. 12)

Improvement

While a single lesson is too narrow a focus to determine if students improved the way they math, it does provide a context for examining the potential for improvement. Given the structure and interactions of this lesson and the many similar to it that students experience throughout their school mathematics experience, what is the likelihood that students will improve in the ways they do math? One way to examine this question is through Variation Theory (Marton, 2006, 2015; Marton & Tsui, 2004). For those who put forward Variation Theory, there are some conditions which are necessary for learning. If these conditions are not met, learning will not occur. Marton (2015) suggests that the difference between people who handle situations in more or less powerful ways is in how they "see" the situation. Being able to discern the critical aspects of a situation allows someone to respond more powerfully than someone who does not. In order to learn to discern the critical aspects, he argues, the learner must be exposed to contrast, or variation, in those critical aspects. To what extent are the students in this class exposed to contrast and variation in the critical aspects of the way they math? For the purposes of this analysis, I will focus on comparing students who worked individually on the practice set with students who worked collaboratively at the white boards; however, much of the discussion can be extended to all the students working individually during the *content acquisition* segment of the lesson.

In order for a student to improve at any of the actions they were engaged in during the lesson, they would need to be exposed first to contrast in that activity so they could discern a critical aspect. This opens up simultaneously a dimension of variation, and qualities of the aspect being discerned. For example, a student who is working individually is exposed to the difference between how she solves the problem and how her teacher solved the problem. Her goal is to emulate the teacher's strategy and successfully match her answer with the answer provided in the textbook. The potential for improvement is provided in this case by the contrast between her work and the target work of the teacher and textbook. By contrast, the students working on white boards are exposed to more opportunity to see differences in the way other students think about solving the problem, how others write their solutions, and how their thinking is the same or different from others. These added dimensions of variation provide greater opportunities to discern the critical aspects of the situation and learn to handle the situation in more powerful ways. For the student working alone, she cannot contrast her way of doing math with those of her peers. She has improved when her solutions more consistently match the textbook and more closely resemble her teacher's way of solving the problem. Working in isolation limits her ability to experience the variation in those critical aspects of doing mathematics that account for handling new situations in more powerful ways. It is possible, of course, for the student working individually to notice and discern the critical aspects necessary for powerful responses to mathematical situations; the purpose of this analysis is to describe the ways in which the learning environment provides the contrast and variation necessary for learning. It is also possible that the students working on white boards will not experience the variation they are exposed to. The point is that the necessary conditions for learning are present in that situation. While both the students working individually and the students working collaboratively may improve in how they math, because the students working collaboratively at white boards are exposed to significantly more contrast and variation in the critical aspects of working on the solution, the potential for improvement in the ways of doing mathematics that align with how mathematics thinking and doing have been described (Mason et al., 2010; Schoenfeld, 2016) is far greater.

CONCLUSION

I set out in this paper to identify the ways that students engage in the mathematical activities of the classroom, and to comment on the likelihood of students improving in the ways they *math*, given the structure of the lesson. When students were working individually, either while acquiring procedural content knowledge from the teacher or during the time allotted for practicing those procedures, the register of mathematical actions was narrow. Students **watched**, **responded** to teacher prompts, and occasionally **sought clarification** from the teacher about the procedure being demonstrated. When students collaborated in small groups at white boards to practice the procedure, however, the activity of students broadened to include several additional actions which qualitatively changed the way those students engaged in the activity. These changes in the way students *mathed* encourage argumentation, generalization, the mobilization of mathematical content knowledge, the development of a mathematical point-of-view through a community of learners, as well as supporting the construction of mathematical knowledge. This is not to say that the students working alone do not develop these same skills and attributes; however, the *mathing* which occurred at the white boards facilitated it.

Much of the same can be said of the likelihood of improvements in the way students did math. While students working alone may become aware of the dimensions of variation required to see their task in a way so that it creates the conditions necessary for learning, these conditions were not afforded them as part of the classroom experience. Students working at white boards were provided with the opportunity to see multiple dimensions of variation, including ways of thinking about and through the problem, ways of representing the solution, ways of negotiating meaning, and ways of becoming unstuck. Having these dimensions of variation opened up simultaneously, as a function of the way they were engaging in the mathematical activity, occasioned them the opportunity to improve.

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EXPERIENCING LEARNING MATHEMATICS AND REFLECTION: CALCULUS 12 PARTICIPANTS' STUDY

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This study focuses on the assessment strategy that was designed in the 2017-2018 academic year in two Calculus 12 classes. Students' affect was at the centre of the research questions thus clinical interviews were used to create data on the relationship with mathematics as well as personal reflections on the learning of mathematics in the given year and overall in students' experience in school. Grounded Theory guided the research approach as themes began to emerge following with the analysis and conclusion of usefulness of these types of questions for students to reflect upon as the results were surprising and pleasing from the mathematics educator point of view.

INTRODUCTION

Assessment in mathematics classrooms has been a very hot topic in the field lately, as it appears to be full of tension. There are powerful voices trying to influence the practice of teaching in classrooms and beyond, as there are decades-old discussions with unresolved problems in defining terms and explaining phenomena (Frey & Schmitt, 2007). There is a strong traditional pull of a system of tests and quizzes as historically practitioners have been exposed and graduated from such a system (Buhagiar, 2007; Romagnano, 2001). Furthermore, because of the strong traditional influence, there are instances of masking the old traditions in the innovative kind (Shepard, 2005). This is driven by the fact that assessment has been taken place primarily for the purpose of evaluation (McTighe & O'Connor, 2005). The other side of the argument is calling for stepping away from the evaluative nature and aligning a new purpose of assessment: "Classroom assessment and grading practices have the potential not only to measure and report learning, but also to promote it" (McTighe & O'Connor, 2005, p. 10). One part of the argument to change assessment has been feedback, as it is claimed to be most effective function for improving student learning (Guskey & Bailey, 2001; Wiggins, 1998). The above literature influenced an attempt to change assessment in 2017-2018 Calculus 12 class taught at an independent school in Lower Mainland of British Columbia, Canada.

THE STUDY

The main concern coming into the position of change was noticing of students' experiences in a math class. The following graphical representation was produced to illustrate these experiences:

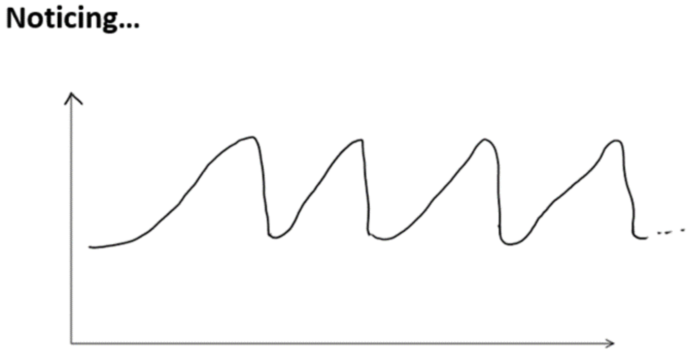

Figure 1: Noticing Illustration

In Figure 1 the horizontal axis is time and vertical axis is confidence, enjoyment, or positive feeling. The dips correspond to an assessment, usually a test given to students. This pattern, once thought about for a while, makes a scary contemplation with a simple calculation: 5 years of high school times an average of 8 tests per course = 40 such dips! If that pattern does teach learners something, it is definitely not a positive correlation with mathematics and their experience with it. With this in mind, the following assessment method was devised for the 2017-2018 year calling it the Check-Point System. The outline of the course was broken down into major topics and subtopics. Each subtopic became a trackable element for each student, which they could view at any point in time as a shared Google sheet. So, instead of a regular marks book, now every student had the profile with their continuous progress, as each of the subtopics was repeatedly assessed. Another option given to students was an interview at the end of each term. In this interview students could showcase that they know a certain subtopic better than their overall mark for it. Below is the screenshot of one such spreadsheet:

Description	CP 1	CP 2	CP 3	CP 4	Total:	T1 Mark: 88	T1 Mark before: 78	CP 5	CP 6	Mid Year: 86	CP 7	CP 8	T2 Total: 86
Topics:													
1.1: Linear and Quadratic Functions	3				3		3			3			4
1.2: Basic Classes of Functions	2.5	3.5			3		2.5			3			4
2.1: Limits, Rate of Change, Tangent Lines													
2.2: Limits, a Numerical and Graphical Approach		3	4		3.5		3			3.5			3.5
2.3: Basic Limit Laws		3.5	4		4		3.5			4			4
2.4: Limits and Continuity			3.5		3.5		3.5			3.5			3.5
2.5: Evaluating Limits Algebraically			4	3.5	3.5		4			3.5			3.5
2.6: Limits at Infinity			3	3.5	3.5		3			3.5			3.5
2.7: Intermediate Value Theorem			3	4	3.5		3			3.5			3.5
3.1: Definition of the Derivative				3.5				4		4			4
3.2: Derivative as a Function				3				4		4			4
3.3: Product Rule								3.5	3	3	3.5		3.5
3.4: Quotient Rule								3.5	3	3	3.5		3.5
3.5: Higher Derivatives													
3.6: Trigonometric Derivatives								4	2.5	3			3
3.7: Chain Rule								3	2.5	3	3.5		3
3.8: Implicit Differentiation									1		3		3
3.9: Related Rates											3		3
4.1: Linear Approximation											3	4	4
4.2: Extreme Values											4		

Figure 2: Sample Tracking Sheet

By the second half of the year it was curious to see what the students thought about this approach and their relationship with mathematics. Situated in the Grounded Theory, the data was given a chance to develop into self-emerging themes prompted by the questions outlined below via semi-formal interviews. In what follows, I outline the environment and participants, method and data, and discuss emerging themes.

ENVIRONMENT

Two Calculus 12 classes had the implementation of the Check-Point assessment practice. These classes totalled 28 students in Grades 11 and 12. The school in question offers three levels of calculus: Calculus BC, Calculus AP, and Calculus 12. Typically, the latter is chosen by students who want early exposure to calculus and are planning to take it in their university years. Flipped classroom approach together with the discussion-based learning were the primary vehicles of instruction and day-to-day structure of these classes. Students were expected to come to class prepared to have watched the videos and attempted a series of questions (Sterelyukhin, 2016).

PARTICIPANTS

Four students were selected from the cohort of the two classes to be interviewed.

(1) Ethan is a student who works hard in class. He was put into an accelerated program in Grade 8. In this program students complete Grade 8-10 mathematics curriculum in two years. The selection process for this includes the marks for the first three units of Grade 8 and teacher recommendation. The advantage of being in this cohort is staying one year ahead of their peers in a regular stream. This put Ethan in the position of already completing Pre-Calculus 12 last year. Ethan was selected because he was showing excellent results and participation in class.

(2) Nancy always found mathematics challenging and had many issues with the subject throughout her career at the school. One way to help herself that she developed over the years is to pay a very close attention to examples and then mimic her work based on those. Nancy was taking Pre-Calculus 12 concurrently with Calculus 12. Nancy was chosen because she was showing excellent results, and it was particularly interesting to enquire about such a turnaround in her success in a mathematics class.

(3) John has always shown great success in his mathematics classes throughout high school. He exhibited a natural aptitude and interest in mathematics. It looked like math came easy to him and he was able to construct meaning for himself to the level that allowed him to be very successful in every math course he took thus far. John has also come from the accelerated stream. John was selected because of his excellent results on the Check-Points and insights he was offering during class discussions.

(4) Sam came to the school in Grade 11. She was not exposed to flipped classroom and discussion-based learning before. She was in the accelerated program at a different school with the same outcome of finishing Grade 10 math in her Grade 9 year. Sam was chosen because of her good results, participation and in-depth conversations about

learning over the course of the year. In addition, prior to the interview, Sam had written a summary of her 13 years of learning math.

METHOD AND DATA

As outlined above, these four students were chosen to conduct a semi-formal interview about their learning experience with mathematics as a whole and particularly this year. As the interest was situated in student experiences with the new implementation of the course, a similar study by McGregor (2018) dealt with anxiety in middle-school mathematics classroom and a new approach to reduce it. The questions prepared ahead of time were as follows, adapted from McGregor:

(1) What does mathematics mean to you?

(2) What does learning of mathematics mean to you?

(3) What has changed for you this year in math? How do you feel you are different in and with math this year?

(4) How are you feeling about learning math this year?

(5) Describe feelings, emotions, associations that come to mind when you are in a math class. Try to reflect on the whole experience.

Interviews took part during the school day when students either had a spare block or lunch period. A quiet place was found without anybody listening in or distracting. All the interviews were recorded, totalling in over one hour of recording time. After the attempt to transcribe the entire collection of recordings and running into timing constraints, it was decided to listen to the interviews first to see if there were any emerging themes from what was heard. After listening to all the recordings, five themes were identified that emerged from careful listening and reflecting. Then, only the excerpts that corresponded to these themes were transcribed. The focus was in what the students were saying and not the aesthetics of speech, pauses, etc. Therefore, other aspects of the recordings were not coded. The following five themes emerged: **Math vs. English**, **Coming Back to Topics (Using Check-Points)**, **Social Aspect (Not getting it but the rest did)**, **Negative Experiences From the Past**, and **Enjoying Calculus This Year**.

Due to the constraints in the length of this paper, we only present two out of the five themes here. Below is the data created with the themes heading each set of transcriptions:

Theme 2: Coming Back to Topics (Using Check-Points)

(2) Nancy:

1 T: Do you have explanations for why particularly this year, particularly this is, do you feel any different, do you, like, what's...?

2 N: I think, well, for Calc I like how I can, you give us a second chance. A lot of the time in most of our courses I don't get a second chance, so I'm one of those people if I get something wrong, I want to prove myself I can get to do

better. I think it's just also the way how you teach now, how I like it. But I think if every other course was like that, it would really help me to improve, which is nice.

(3) John:

1 T: Anything else you want to mention that you have not from what we have been talking about?

2 J: Just any final statement?

3 T: Yeah.

4 J: Ahm, I think the Check-Points have been a big thing for me this year because, like I said earlier, I get time to finish it, but I think it is also nice to go over a concept multiple times, especially for learning purposes, I think it's great, cause there is a lot of times where you spend a whole month working on something and then you write one test on it and then it's just, it's gone, the concept does not re-appear until the final exam and then you, crap the bed on it, cause you have not seen it in forever and it's, makes it quite difficult, that I think that sort of approach which I think is nice about the Check-Point, because you see it a couple of times at least before the end of the year, so I think it's quite effective for the learning purposes, and also the marks.

(4) Sam:

1 S: Oh, also, another thing that I feel like I appreciated this year was the whole "Check-Point" method because I know that last year we talked about aiming for mastery, how can you preach that thing, but not actually do it, because when we do old system, you are not aiming for mastery, taking the test, done, that's it, learning something new and that's it. And with the Check-Point system you are learning something seeing oh, lol, ok I got this part wrong, I can aim for a "4" next time you try it again and again, until you get a "4". I like that you let us show that we can understand it.

Theme 5: Enjoying Calculus This Year

(1) Ethan:

1 T: Thinking about this year in particular, has anything change for you in terms of math this year, or has it been all kind of the same?

2 E: I don't know if it is necessarily cause like the accelerated program, not being in it now basically, being with my own peers, it definitely makes more sense this year, I'm able to understand it this year, the thing also is that cause having you as my teacher made it much more enjoyable, Pre-Calc 12 was my least favorite year of math. I seem to understand this stuff, and you can sort of picture it a bit more, compared to past years.

(2) Nancy:

1 N: I would say this year it's kinda the only year that I've actually really enjoyed math. So, which is really interesting I don't know, I find it interesting, so I do take Calc I guess, and Pre-Calc 12. And I take Chemistry, English, and Physics, but out of the like 5 courses I take, I enjoy going to math the most. It is interesting, it all kinda turned. No, I'm just saying that not because you are asking me, but over this year I kinda liked it better. I don't really know exactly what it means to me though, but I did really enjoy it.

2 N: I think... over the years I just kind of kept pushing and kept wanting to do better and then I've noticed if I look at my grade 11 marks to my grade 12 marks, I see like a huge jump in progress and so I do go to tutoring, but I actually go to tutoring for Pre-Calculus and I don't go for Calculus, which is also kinda crazy cause I'm doing better in Calc then in Pre-Calc so I find it like always my tutoring actually helping me and so now, where I am at in when I'm in math today I feel more confident, like, I can go to the board and not be nervous I used to find it very neur-wracking or talking even sometimes I still do if I don't understand a concept, but now if I get it I don't feel as nervous to like express my ideas.

(3) John:

1 T: Describe the feeling, emotions and associations that come to mind when you are in a math class. Try to reflect on your whole experience, not just this year, but if you were to think about your experience of learning mathematics as a student from grade 1 to now when somebody says "this is math now", what do you feel about it?

2 J: Ahm, I think... In the past I had a little of an embarrassed attitude towards it because I had so much pressure on myself with math in particular, I've always, math has always been, supposed to be, my strongest class, so the pressure was to perform and show that I was not less capable than my peers so I wanted to make sure I always was on top of, putting a little pressure in the back of my head, maybe I did not always follow through studying that I should have to maintain what I wanted, but I've always held math in great priority, compared, especially my other subjects, so I think I've always put a lot of pressure on myself in past years. And I think this year it's been easier knowing that I have a math course under my belt already, with an ok mark, that I can submit already to university, so that's good, but this year it has been a lot less intense feeling.

THEMES AND ANALYSIS

In this section we will elaborate on the observations from the data on two of the five themes. It was very pleasing that these five themes emerged so clearly from nearly all the students interviewed as the interviews were only about 15 minutes long on average. Furthermore, the themes did not directly follow from the questions that were asked.

First, we turn our attention to the Check-Point method and the opinions about it. Clearly, for the three students who decided to talk about it, Check-Points made a difference. Nancy, John and Sam all comment on the positive aspects of coming back to topics, being given another chance, and maximising the learning. Interestingly, that even though their motivations are quite different (Nancy and John are much focused on the marks and measure of achievement through that, and Sam is centered around understanding), the idea of revisiting, solidifying, getting rid of the “once and only once” moments has given rise to positive experience in a math class when talking about assessment and evaluation. From this feedback we are confident that our idea to make the assessment process a positive experience has succeeded and clearly is making a difference in not only students’ perception of mathematics class, but also in their learning and how they approach it. This echoes McTighe and O'Connor (2005), noted previously.

For the last theme of enjoying the class this year one can easily identify the element of less stress and anxiety. Students are telling us that they are more confident in their math learning, feeling that they know it well. They have evidence for such conclusions and are able to track it at all times. Also, knowing that there will be other opportunities to demonstrate their learning along the way decreases the value and the “now and only now” feeling when major assessments are happening. Assessment should create data of what a learner knows up to the moment of time when the assessment is taking place and feedback from it should prompt a learner to analyse where improvements are needed and to go ahead and make an appropriate change. This enables more opportunity for positive experiences in a math class, and thus the want to keep going, coming back and persevering are more likely to happen, increasing mastery and personal satisfaction from the learning process.

CONCLUSION

From what we have seen in the data, the study of personal relationship with mathematics and learning of mathematics appears very interesting to students. They are more than willing to share their experiences with the subject and are very honest when talking about their feelings and emotions. It is pleasing to see the themes emerge from such a small set of interviews, clearly indicating that there are a number of items to be investigated further. One factor seems to be prevailing from all of this: students need to be asked about their learning of mathematics, what they like, what they do not like, where there are positive moments and where there are negative ones. From a small sample of four students from a pool of 28 between the two classes it is evident that their relationship with the learning of mathematics is much more than just marks and tests.

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AN EXTENSION OF TOULMIN'S SCHEME TO DISCUSS THE WAY A MATHEMATICIAN DETERMINES A CONDITIONAL STATEMENT IN DIFFERENT CONTEXTS

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The aim of this paper is twofold. Firstly, to explore how a successful mathematician determines conditional statements in different situations, namely mathematics, logic, and everyday context. Secondly, to extend Toulmin's argumentation model in order to prepare the created data to be analysed.

INTRODUCTION

Regarding the role of formal logic and the necessity of learning rules of inference in proving and reasoning in mathematics, there are different views among researchers and mathematics educators. Some mathematics educators, for example Hanna and de Villiers (2008), and Selden and Selden (2009), argue that introducing formal logic to students does not necessarily build up their ability in proof construction because of the distinction between "proving" and "reasoning logically" as different mental activities. However, many other researchers, such as Epp (2003) and Hoyles and Küchemann (2002), have stressed the role of logic in students' understanding of mathematical proof, and stated that we need to have more focus on introducing logical relations, especially implications. The majority of research on cognitive paradigms supports that reasoning skills are highly dependent on context and it is not easy to apply a rule learnt on a topic in another topic. See, for instance, Lehman and Nisbett (1990), and Morris and Nisbett (1992).

There are many conditional statements that are conditional by logic rules, but do not make sense in either the realm of mathematics or colloquial language. In fact, at first glance, it seems there has to be some relation between the meanings of premises and conclusions. Lewis (1917) stressed that to prove/justify something, there has to be some relevance of content or meaning. Along with the work of Lewis (1917), and having in mind Baldwin's (2009) distinction between "semantics" and "situation" for the concept of variables in algebra, I attempt to consider similar triangles with corresponding vertices for logic, and distinguish two types of logical meaning: formal meaning of symbols in a logical context and interpretation of logical statements in content. Baldwin (2009) added a third component to "syntax" and "semantics" in the discourse of mathematics (specifically for algebra), namely "situation". He makes a distinction between "semantics" and "situation", and proposes that these are two sorts of mathematical meaning: meaning of symbols in a mathematical context and interpretation of mathematical statements in the physical world. He also considered

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"syntax", "semantics", and "situation" as three needed components for learning variables in algebra, claiming that these are essential elements of learning algebra.

RESEARCH QUESTION

Even though there are studies focused on the way individuals evaluate conditional statements, there is currently no adequate understanding of the ways individuals determine implications. These two problems are essentially different, as many mathematicians can evaluate conditionals correctly, but cannot recognize implications either situated in other contexts or with false assumptions, whereas the formal concept definition of a conditional is well understood.

The introductory part of the current work raises this problem that a symbolic expression in logic may be ambiguated by situating it in either mathematics or everyday language, and asks how people determine a conditional statement in different contexts. Specifically, as the final goal of this paper, the manner in which a successful mathematics student determines an implication will be investigated under the following questions:

How does a mathematician react to a conditional statement in different situations, namely classical logic, mathematics, and colloquial language? Do individuals with a good background in either formal logic or mathematics apply rules of inference to make a conclusion in daily life reasoning?

THEORETICAL FRAMEWORK

There might be some theories of reasoning that can account for the empirical data created during this research; but, since the intent of this study is not to compare the performance of different frameworks for a certain event, I choose the one that seems most suited to the questions and the created data. To analyse data that is a collection of arguments for making decisions about a situation, among all potential models, Philip Johnson-Laird's Mental Models theory of reasoning provides the best framework within which to analyse how the interviewee recognizes conditional statements.

Mental Models Theory of Reasoning

The Mental Models theory of reasoning proposed by Johnson-Laird (1983) is one of the famous theories of cognition and reasoning. This theory suspects that reasoning involves formal operations over logical forms, and that instead of following logical rules during reasoning, people reason over models in which such forms are true. Such models are constructed in individuals' minds, which they modify and reason from; these models are concrete representations of situations, rather than abstract things assumed in mental logic. Johnson-Laird (ibid.) stated that, when a new situation is encountered, the reasoner goes through the following three stages:

1) They look at the premises and create a mental model of the possible situation they find themselves in.

2) They form a non-trivial conclusion that is based upon the premises of their model.
3) They look for counterexamples to their model and conclusion. If they cannot find any, then they accept the conclusion.

Toulmin's Model of Argumentation

Toulmin's model is a method of reasoning introduced by Stephen Toulmin (1958) in his work on logic and argument. According to this model, an argument consists of at least three essential parts called the core of the argument (data, conclusion, and warrant), along with three additional, optional parts (backing, modality/qualifier, and rebuttal). This method can be used as a tool for developing, analysing, and categorizing arguments. Based on this model, the arguer starts by putting forward the data (D) and showing, via the warrant (W), that the conclusion (C) follows. If the warrant is not immediately obvious, then some justification or backing (B) for it is required. The qualifier (Q) gives an indication of the level of certainty contained in the argument (of course, in mathematics, arguments are traditionally seen as aiming to establish the full certainty of claims rather than a level of probability in them). The final part, the rebuttal (R), occurs when the conviction in the argument is non-absolute.

Applying Toulmin's Argumentation Scheme

Regarding Toulmin's model, though it adequately describes argumentation in the context of mathematics, in this work I found it more useful as a tool/method to organize some critical parts of interviews by somehow visualizing the transcription that helps to find patterns. But we cannot simply use Toulmin's model for disagreement arguments, or those with uncertainty and no conclusion, because an argument for something does not have the same structure as an argument against that thing. In a refutation scheme, there needs to be additional parts, because to refute something, there is always a reason or backing. So, we may need to consider a new box for the "source or rebuttals". Also, Toulmin's scheme can provide a method to separate and organize parts of arguments, but we are not limited to its original structure introduced in the previous part.

For example, in this paper, I picked the most important parts of the argument, considering the research questions, and filled the boxes with exactly what the interviewee said, with no changes in applied words (See Figures 1, 2, and 3 later in the paper). I also decided to include long pauses (at least 1.50 seconds) to Toulmin's model. My reason for such changes is that hesitation pauses anticipate sudden increases in information or uncertainty in the message being produced, and such pauses will tend to occur at points of highest uncertainty in spontaneously produced utterances (Osgood & Sebeok, 1954/1965). Also, research by Grosjean and Deschamps (1975) shows that the more complex the communicative task, the greater the number of pauses. Although I made a few changes in applying Toulmin's scheme, we can still add some other factors to have a better view of the argument, such as using symbols to indicate intonation and falling/raising voice pitches, and so on.

TASK AND METHODOLOGY

In the current study, to have a closer observation and more accurate response to the questions that this work is set out to answer, I conducted semi-structured clinical interviews (Cohen et al., 2000) as my method to create data. The interviews were structured in three phases. In the first phase, I examined the participant's information about implications in both colloquial and mathematical contexts. In the second phase, I saw how they determined conditionals in different situations. And finally, in the third phase, which began after about 10 minutes break, I selected some of the interviewee's responses to elucidate the underlying process, and asked them some general questions about the second task.

Phase 1, Discovery

This phase examined the interviewee's knowledge about conditionals in the three different realms of classical logic, mathematics, and everyday language.

Phase 2, Identification

In this phase, the following questions were designed in the realms of mathematics and everyday language; for some of them, there are no rational relationships between premises and conclusions. The following table classifies the designed tasks and represents the applied pattern, ensuring that they are consistent with the research questions, and that all possibilities are considered in almost the same weight.

	Colloquial	**Mathematics**
Related	If you pay 10$, you will have my pen. If you will be late for meeting, then you will be fired.	If a population consists of 40% men then 60% of the population must be women. If $(x - 2)(x + 1) = 0$, then $x = 2$ or $x = -1$. If ABC is a triangle, then $A + B + C = 180^o$.
Unrelated	If elephants could fly then you win the lottery. If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan*.	If $x + 1 = 0$, then $z = 5$. In triangle ABC we have $A + B + C = 180^o$, then the area of a *circle* is pi times the radius squared (πr^2). If $4X^2 - 5X – 6 = 0$, then $\sin \theta = 0.546$.

Unknown	If sun erupts from an active region called AR 2673, then there will be loops of plasma tens of times the size of the Earth.	If A is a Banach algebra, then for every Banach A-bimodule X, H (A, X') = 0. If cos θ = 0.81 , then $\theta = 2\pi \pm \frac{13\pi}{173}$. False

Table 1: Designed tasks for phase 2 of interview

Phase 3

In the final phase of the interview, the interviewee was asked to determine conditional statements in the context of math and colloquial language. However, nothing was presupposed and, based on their responses, I would decide about the next questions.

In order to see how the interviewee's knowledge assisted or misled him in doing such tasks, I picked participants from people with a good background in first order logic and solid knowledge in mathematical reasoning. Also, in order to find the pattern of the interviewee's thinking model while performing the task, I applied Toulmin's model as a tool to prepare data for analysis. So, it is essentially different from the applied method to create data, and not used as a framework; it only provides a mediation between transcription and data analysis.

DATA ANALYSIS

All thirteen statements in the second task are conditionals and the interviewee recognized them correctly, with the exception of five. After decoding the statements, it emerged that all five incorrect answers pertained to the second row of Table 1, where there are no rational relationships between premises and conclusions. The interviewee evaluated that three of these five statements, all of which are in the realm of mathematics, are not conditional. And for the two other statements in colloquial language (stated below), he preferred to not give any answer. It is also worth noticing that the whole interview for the second task lasted about 22 minutes, and that we spent approximately 7 minutes for the two following statements:

- If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan.*
- If elephants could fly then you win the lottery.

Although these two assertions are clearly in colloquial language, the interviewee explicitly looked at them from two different views, mathematics and daily language. The following is an excerpt from the first item:

Interviewee: If you buy a fresh fish tonight, then *Bill Gates meets Aamir Khan* [6 sec]. Yes. It doesn't make sense but it is a conditional ↓.

Interviewer: Can you say why?

Interviewee: If-then. If you buy something, <u>then</u> [stressed], something happens, though they are not very related but, errr, it is a conditional, though meaningless.

Interviewer: So, what is this?

Interviewee: Only an idiot would say that [laugh]. (6sec) In fact, mathematically, it's not a conditional but in daily language we take it as a conditional.

Interviewer: So, is it A conditional or not??

Interviewee: With regards to math or daily language? Actually in mathematics, it's not a conditional but for daily language it is ↓.

Interviewer: So?

Interviewee: With regards to math it's not, but as a sentence it is. Almost↓.

The following is a part of our discussion on the first above statement that is coded using Toulmin's model. The boxes through the main line stand for "conclusions", above the main line represent "rebuttals", and below the line indicate "warrants". Also, each rebuttal and warrant can be justified by a "backup". The number of conclusions and pauses shows the interviewee's uncertainty to decide about this statement.

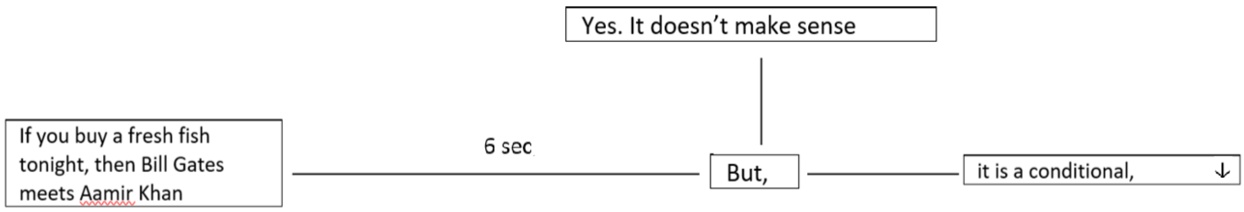

Figure 1

Then I asked him to elucidate his response, and he argued as follows:

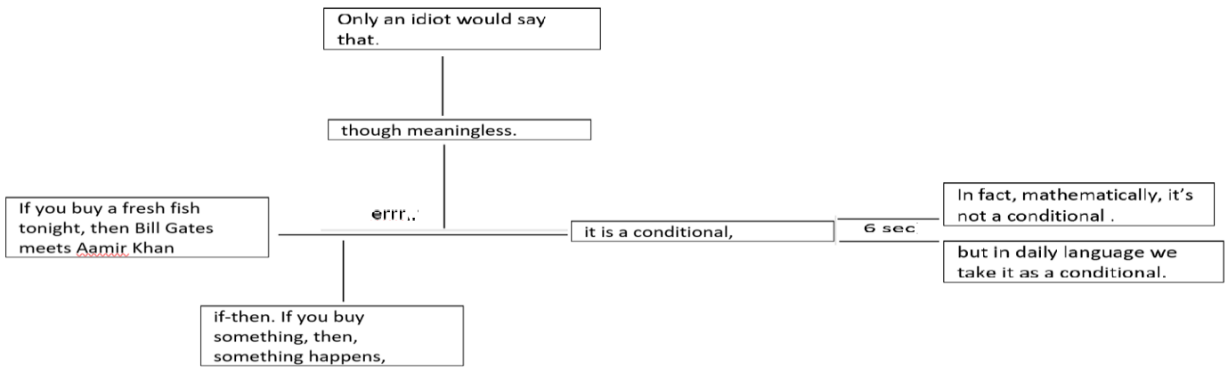

Figure 2

The following is an excerpt from the second selected statement coded by Toulmin's model:

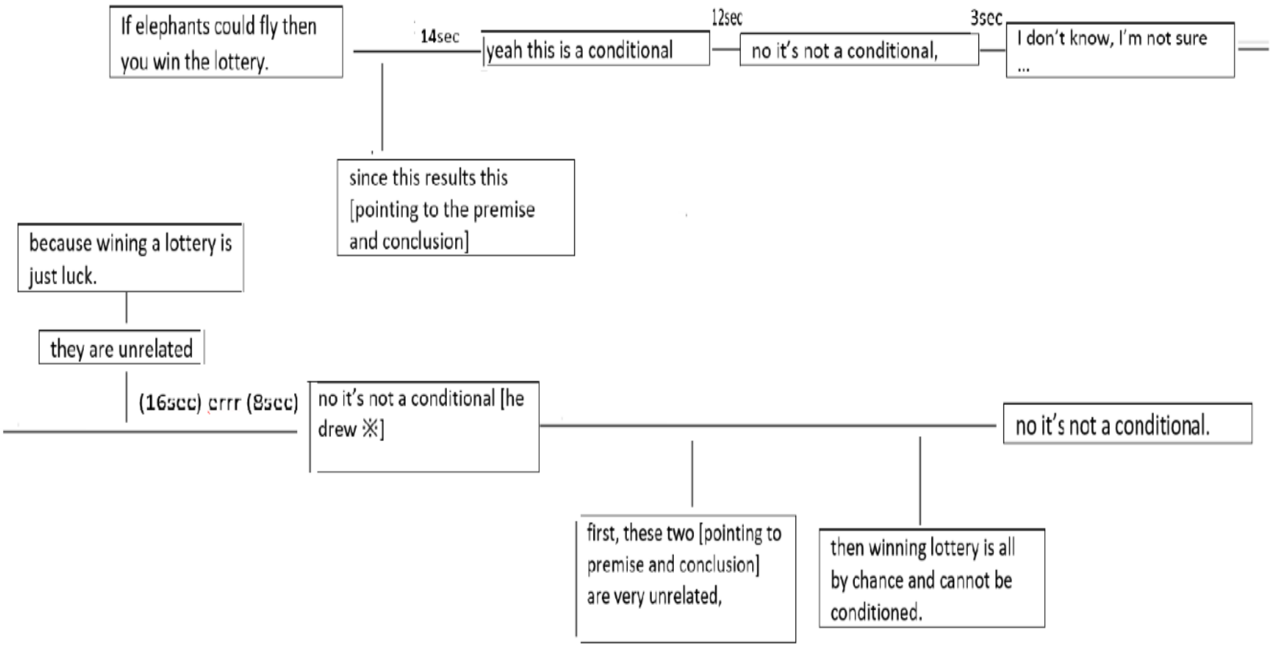

Figure 3

The interviewee's perception of a conditional may be shaped by his knowledge and pragmatics. They may add their information to the statement, then decide if it is a conditional or not. If there is a mental model for something, it may influence the way of arguing. Regarding this interviewee, to see each assertion is an implication, he has his own mental model. The mental model posed by the interviewee is indicated in the schemes coded using Toulmin's model. He first was looking for an *if-then pattern*, and then *relatedness* put the end on his decision. He argued that those with unrelated parts are not conditional statements, even if they are in *if-then* form. Long pauses and the frequency of different conclusions on the main line of the scheme may represent that the interviewee could not decide about the answer.

He also stressed that, in mathematics, the above statements are not conditional, but in colloquial language they might be. As it is clearly indicated in his model of argumentation, different situations of mathematics and colloquial language explicitly influence his decision. The interviewee gave more credit to situated meanings than semantics. In fact, he did not consider any of the assertions as statements in formal logic. And, in some cases, his mathematics view conflicts with daily language

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TENSIONS BETWEEN THE VIEWS ON WRITTEN AND ORAL ASSESSMENTS IN MATHEMATICS, AND MATHEMATICS ASSESSMENT PRACTICE

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In this paper, seven mathematics professors share their views and experiences with teaching and studying mathematics in oral and non-oral assessment cultures. These participants come from Bosnia, Poland, Romania, Ukraine, Canada, the United States, and Germany. The results show that schooling and teaching experience as well as the lack of schooling and teaching experience with oral assessments in mathematics, institutionalized mathematics assessment norms, and socio-cultural assessment norms can influence views on oral assessment in mathematics.

INTRODUCTION

A large amount of research on teachers' beliefs focuses on beliefs about mathematics, mathematics teaching, and mathematics learning (Beswick, 2007; Cross, 2009; Ernest, 1989; Handal, 2003; Liljedahl, 2009; Maasz & Schlöglmann, 2009; Philipp, 2007; Raymond, 1997; Stipek, Givvin, Salmon, & MacGyvers, 2001; Thompson, 1992; Žalská, 2012). However, in the most recent review of assessment in mathematics education, there was almost no research on students' and teachers' beliefs about assessment in mathematics (Suurtamm et al., 2016). Although there were numerous papers written about the inconsistency between teachers' beliefs and teachers' practice, I found no research studies on teachers' beliefs and teachers' practice from an oral assessment perspective. Most of this research showed that there is a difference between teachers' espoused beliefs and their actual classroom practice (c.f. Vacc & Bright, 1999; Wilson & Cooney, 2002). In addition, there was a large amount of research that indicated the disjunction between teachers' intentions of practice and their actual practice (Cooney, 1985; Karaağaç & Threlfall, 2004; Noyes, 2004; Skott, 2001). According to Skott (2015), beliefs are "results of substantial social experiences" (p. 19). Richardson (1996) describes three categories of experience that can influence beliefs about teaching: personal experiences, experiences with schooling and instruction, and experiences with formal knowledge. Also, Raymond (1997) describes different factors which can influence the teachers' beliefs about mathematics: the central reciprocal relationship between beliefs and practice, past school experiences, immediate classroom situations (students' abilities, attitudes, and behavior; time constraints; the mathematics topic at hand), personality traits of the teacher, and teachers' educational programs where they were trained. In this study, I looked at the relationship between mathematics professors' views on oral and written mathematics

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assessment and their mathematics assessment practice in oral and non-oral assessment cultures — where non-oral assessment culture is defined as a culture in which oral assessment in mathematics is not part of the system of education while an oral assessment culture is one where oral assessment is an important part of assessment practice in mathematics.

Oral Examination in Mathematics

In most of the cases, students would have to take a written exam first, and then after passing the written exam, they would go to the next stage, which would be taking an oral exam. During the oral exam, students would have access to a blackboard, paper, and pen. The exam would be conducted by the course instructor, and each oral exam session could last anywhere from 30 minutes to 1 hour. Occasionally during the oral exam, three or four students would be invited at the same time. The instructor would have prepared in advance a set of cards with questions of approximately equal difficulty, so a student would step in, randomly draw a card from the set of cards, and then, he/she would take scrap paper and go back to his/her desk and start working on the chosen question. After some time working on the question, each student, one by one, would go up to the board and present his/her answer to the instructor. In addition, the teaching assistant would be in the same room, monitoring students and taking the protocol. During the oral exams, usually students would be able to receive some help if needed and would receive a grade immediately following the exam. A typical card would have one theoretical question (for example, 'prove the fundamental theorem of calculus') and one exercise (for example, 'calculate the integral': $\int arcsin^2 x \ dx$).

Research Questions

The main research question of this study is: *"What factors influence mathematics professors' views on oral assessment in mathematics?"* More specifically, this primary question breaks into the set of following questions:

a) *What are mathematics professors' views on written assessment in mathematics?*

b) *What are mathematics professors' views on oral assessment in mathematics?*

c) *What are mathematics professors' views on their mathematics assessment practice?*

METHODOLOGY

The research design for this study is descriptive/qualitative. Seven participants were interviewed using open-ended questions to gather information about their personal experiences and perspectives on using written and oral assessments in mathematics classrooms. These participants were selected based on the following criteria: each participant had been exposed to oral assessment either as a student, teacher, and/or professor. In terms of recruitment, I used a methodology of snowballing, wherein I started with mathematicians whom I knew professionally, and then asked them to recommend others in the mathematics department or elsewhere, whom they suspected may have a history of experiencing or using oral assessment. Seven mathematics professors were selected for interviews, and their pseudonyms are: Melissa, Elisabeth, Van, Nora, Dave, James, and Jane. Melissa, Elisabeth, Van, and Nora, who were born and educated in Poland, Romania, Bosnia, and Ukraine, respectively, are currently teaching at a Canadian university, while Dave, James, and Jane, who were born and educated in Canada, Germany, and the United States, respectively, are currently teaching at a university in Germany. With respect to familiarity with oral assessment, Van, Melissa, Nora, and Elisabeth had been previously exposed to oral examinations in mathematics prior to moving to Canada while Dave and Jane, who were educated in Canada and the United States, had never been exposed to oral examinations in mathematics prior to moving to Germany. James was born and educated in Germany, and thus, he has had a lot of exposure to oral assessment in mathematics. The audio recordings of interviews were transcribed, and transcriptions were used for data analysis.

RESULTS

The participants' views on written and oral assessments in mathematics, and mathematics assessment practice in oral and non-oral assessment cultures are presented in Figure 1.

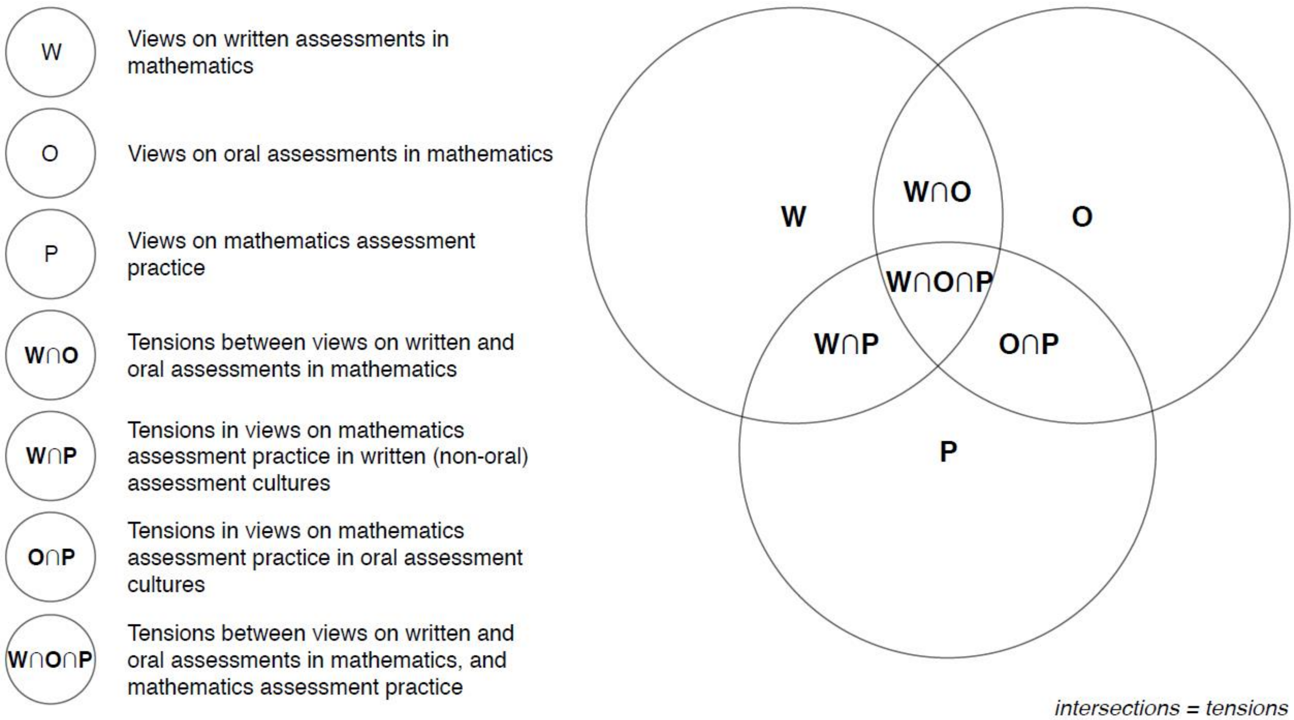

Figure 1: Mathematics professors' views on written and oral assessments in mathematics, and mathematics assessment practice in oral and non-oral assessment cultures

W = ***Views on written assessments in mathematics***

What are the mathematics professors' views on written assessment in mathematics?

When it comes to the participants' views on written assessments in mathematics, they can be divided between positive and negative views. Based on the positive views, written assessments in mathematics: allow the relation only between the student and the subject of mathematics that is being assessed; provide a written record of student's performance; allow the asking of deep-probing questions; provide an opportunity to answer questions in order of student's preference. On the other hand, based on the negative views, written assessments in mathematics: do not prevent plagiarism; do not provide an opportunity to redeem; cause students to feel anxious when they are taking exams in large class sizes.

O = ***Views on oral assessments in mathematics***

What are the mathematics professors' views on oral assessment in mathematics?

When it comes to the participants' views on oral assessments in mathematics, they can also be divided between positive and negative views. Based on the positive views, oral assessments in mathematics: are reactive to students' needs in terms of they provide an opportunity for discussion, follow up questions, and instant feedback; reaffirm/improve students' grades; better assess students' understanding and knowledge of the concept; prevent plagiarism; provide an opportunity for students to assess themselves by listening their classmates; can assess students' thinking; provide an opportunity to redeem; allow differentiated assessment; provide an opportunity to adapt the level of questions to each student's level of response. On the other hand, based on the negative views, oral assessments in mathematics: can make students feel intimidated or discriminated by the examiner; can make students feel anxious when they need to present the material; are less fair than written.

P = ***Views on mathematics assessment practice***

What are the mathematics professors' views on their mathematics assessment practice?

The participants' views on their mathematics assessment practice are mostly based on their prior schooling and teaching experience, school culture, and study program within the assessment culture. For participants who are teaching in non-oral assessment cultures, their current assessment practices in mathematics are not aligned with their personal views of mathematics assessment, due to constraints of the university context that exist within non-oral assessment cultures. On the other hand, for the participants who are teaching in oral assessment cultures, their current assessment practices in mathematics are aligned with their personal views of mathematics assessment, due to flexibilities of the university context that exist within oral assessment cultures.

W∩O = *Tensions between views on written and oral assessments in mathematics*

Based on the participants' responses, there are some tensions when it comes to views of the participants who were educated in oral assessment cultures versus the participants who were educated in non-oral assessment cultures. Therefore, five participants, who had been previously exposed to oral assessments in mathematics, agreed that written exams can mostly assess procedural knowledge and instrumental understanding while oral exams can better assess conceptual knowledge and relational understanding. The other two participants, who had never been previously exposed to oral assessments in mathematics through their prior schooling, agreed that written exams solely can assess both procedural knowledge and instrumental understanding, and conceptual knowledge and relational understanding. For these two participants, Jane and Dave, time is essential for choosing the most suitable mathematical questions for the exam. They both relate conceptual types of mathematical questions to the questions that would take more time to think about, and thus, they can be only answered through written exams. Accordingly, Jane and Dave consider procedural types of questions in mathematics to be questions that can be answered quickly, and thus, only these types of questions can be assessed orally. On the other hand, the other five participants think completely opposite to these two, so that conceptual mathematical questions can only be assessed through oral exams while procedural mathematical questions through written exams.

In addition, there is also a tension in participants' views when it comes to anxiety and fairness in oral and written assessments in mathematics. Based on their responses, it is still not quite clear which type of exam, oral or written, could cause more or less anxiety among students nor which of these two types of exams can be considered to be more or less fair in comparison to each other.

W∩P = *Tensions in views on mathematics assessment practice in written (non-oral) assessment cultures*

When it comes to tensions in the participants' mathematics assessment practice in written (non-oral) assessment cultures, these participants face many constraints within their assessment practice and teaching of mathematics, such as: issue of time to administer oral exams; students' expectations and behaviors; institutional and mathematics department beliefs; school cost; professors' teaching evaluations; the adopted mathematics curriculum and mathematics textbooks.

O∩P = *Tensions in views on mathematics assessment practice in oral assessment cultures*

When it comes to participants' mathematics assessment practice in oral assessment cultures, these participants are given the opportunity to teach and assess mathematics in correspondence with their own personal beliefs. On the other hand, the data shows that non-evidential beliefs (Green, 1971) can affect views on oral assessments in mathematics. Thus, based on the data, there are tensions pertaining to oral assessments in that there is no written record of students' work to be shown during an oral examination and a lack of time to administer oral exams. Five participants, who had been previously exposed to oral assessments in mathematics, agreed that there is a written record of students' work during the oral exams as each student would have a scrap of paper with their written work on it, which would be collected by the examiner. On the other hand, the other two participants, who had never been exposed to oral assessments in mathematics, both agreed that there is no written record of students' work during oral exams.

In terms of lacking time to administer oral exams, the participants believe that because of large class sizes in their mathematics courses, they would not be able to find the time to administer oral exams. But there is no strong evidence that shows if the number of students could be a main factor for administrating oral exams in mathematics based on the comparison of the participants' average mathematics class sizes that they are currently teaching and average mathematics class sizes during their undergraduate studies and prior teaching. These numbers are very similar.

W∩O∩P = *Tensions between views on written and oral assessments in mathematics, and mathematics assessment practice*

What factors influence mathematics professors' views on oral assessment in mathematics?

There are certain factors that influence the participants' views on oral assessments in mathematics. The key reasons for the participants' views of oral assessments in mathematics are based on their own prior schooling and teaching experience. Moreover, some participants face many constraints within their teaching institution and mathematics department, which affect their current mathematics assessment practice. Also, the participants' assessment practice and their views are certainly influenced by the social context of their current assessment culture. Therefore, the following factors influence mathematics professors' views on oral assessment in mathematics:

- Their schooling and teaching experience with oral assessments in mathematics (evidential views).
- Their lack of schooling and teaching experience with oral assessments in mathematics (non-evidential views).
- The institutionalized mathematics assessment norms (the adopted mathematics assessment practice by the teaching institution and mathematics department).
- The socio-cultural assessment norms (the adopted mathematics assessment practice in oral and non-oral assessment cultures).

DISCUSSION AND CONCLUSION

The participants' views of the nature of mathematics assessment and their impact on mathematics assessment practice within oral and non-oral assessment cultures, introduces some aspects of oral assessment from the mathematics professor's perspective which can potentially enlighten readers' views on written assessment limitations. This study has implications for at least five aspects of educational practice: the incorporation of oral assessment into mathematics curricula; the design of oral assessment items; the preparation of students and teachers for oral assessment in mathematics; the implementation of oral assessment in teacher mathematics education programs; and to serve as a guide to anyone who is about to experience the transition within their teaching of mathematics in moving from an oral assessment culture to a non-oral assessment culture and vice versa.

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