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December 8, 2012

8:40 – 8:55	**Welcome and Coffee**	
	EDB 7600	**EDB 7610**
9:00 – 9:35	**Chanakya Wijeratne** Paradoxes of Infinity – The Case of Ken	**Oi-Lam Ng** Gestures And Temporality: Children's Use Of Gestures On Spatial Transformation Tasks
9:35 – 10:10	**Gaya Jayakody** Interplay Between Concept Image & Concept Definition: Definition Of Continuity	**Simin Jolfaee** What Are We Sure About? What Do They Tell About Our Probabilistic Thinking?
10:10 – 10:20	**Break**	
10:20 – 10:55	**Darien Allan** Gaming: Explaining Student Behaviour?	**George Ekol** Examining Constructs Of Statistical Variability With A Semiotic Lens
11:00 – 12:00	**Plenary Speaker:** Lulu Healy	
12:00 – 12:15	**Plenary Q & A**	
12:15 – 1:15	**Lunch: In Situ**	
1:15 – 1:50	**Natasa Sirotic** Secondary Mathematics Teachers Tinkering about How to Teach Radical Equations	**Sean Chorney** Intra-Action Of Agents In A Geometric Activity
1:50 – 2:25	**Arda Cimen** Qualitative Learner Profiling: Using Basic Concepts Of Elementary Number Theory	**Minnie Liu** Model-Eliciting Activities And Numeracy Tasks: A Comparison
2:25 – 3:05	**Olga V. Shipulina** Bringing 'Reality' Into Calculus Classrooms: Mathematizing A Problem Simulated In Virtual Environment	**Harpreet Kaur** Effect Of Dynamic Geometry On Children's Performance In Angle Comparison Tasks
3:05 – 3:20	**Break**	
3:20 – 3:55	**Veda Roodal Persad** Mathematics As Desire: The Life Of André Weil	**Mina Sedaghat Jou** Numbers On Fingers
3:55 – 4:30	**Lyla Alsalim** Perspectives On The Saudi General Aptitude Test	**Kevin Wells** Teacher Judgements In The Classroom: What Is It We Attend To?

PLENARY SESSION

Mathematical Cognition and embodied experience: Learning from students with disabilities

Lulu Healy

This contribution explores the role of the body's senses in the constitution of mathematical practice. It examines the mathematics activities of learners with disabilities, with the idea being that by identifying the differences and similarities in the practices of those whose knowledge of the world is mediated through different sensory channels, we might not only become better able to respond to their particular needs, but also build more robust understandings of the relationships between experience and cognition more generally. To focus on connections between perceptual activities, material and semiotic resources and mathematical meanings, the discussion concentrates on the mathematical practices of learners who see with their hands or who speak with their hands. This discussion centres around two examples from our research with blind learners and deaf learners and, in particular, analyses the multiple roles played by their hands in mathematical activities.

ABSTRACTS

Gaming: Explaining student behaviour?

Darien Allan

The goal of this review is to explore the varying usage of the idea of gaming as student behaviour. This paper briefly describes the concept of game theory, and investigates the appearance of 'gaming' in educational and popular literature. Rather than the formal definition in game theory, a more holistic, general use of the term 'gaming' is proposed in order to describe and explain student behaviour.

Perspectives on the Saudi General Aptitude Test

Lyla Alsalim

The General Aptitude Test (GAT) is a standardized ***test designed*** to be used as an ***admissions*** tool by colleges and universities in Saudi Arabia. Since its establishment, the debate about the GAT has been growing. Different viewpoints and ***perspectives*** on the issue have resulted in conflicting opinions, some supporting the test and believing in its value and others opposing it. The purpose of this paper is to explore the perspectives of different stakeholders about issues related to the GAT in order to highlight any disparities and similarities in perceptions and how these differences and/or similarities affect the way in which the GAT is being applied. The paper highlights the key themes gathered from interviews.

Use Of Phenomenology Theory Perspective In The Analysis Of Professional Development Sessions: Teachers Engagement And Use Of Resources.

Melania Alvarez (No presentation)

This research paper considers how Phenomenology Theory can be used to explain the factors that affect teachers' engagement throughout a professional development experience and what are the factors that are more likely to contribute to sustained engagement in acquiring new knowledge and pedagogical skills.

Intra-Action of Agents in a Geometric Activity

Sean Chorney

In an attempt to understand the intra-actions and, consequent, co-constitution of student, mathematics, and digital tool, this research report adopts a post humanist stance decentring the gaze of observation from the student to the integration of each component listed above. Using the construct of agency as a method of understanding this relation between parts, this paper addresses the intra-acting agencies of an exploratory activity of grade 9 students engaging in geometrical activities both digitally and using paper and pencil.

Qualitative Learner Profiling: Using Basic Concepts of Elementary Number Theory

Arda Cimen

The objective of this study is to look in depth into personal factors affecting metacognitive monitoring and control in self-regulated study and restudy of basic concepts of elementary number theory. By incorporating a wide spectrum of observational methods such as behavioural and physiological, and self-reporting techniques and demographics of the participants, I aim gaining deeper insights into personal factors implicated in learners studying a mathematical text. My ultimate objective is to provide “learner profiles” with the help of these qualitative tools that can be used to better inform assessment and tailor instructional design in mathematics education.

Examining Constructs of Statistical Variability with a Semiotic Lens.

George Ekol

This paper aims at shedding light on how students represent notions of statistical variability in a dynamic computer-based learning environment. Taking a Vygotskian perspective, I explored students’ understanding of variability as expressed through spoken word, gestures, drawings and inscriptions. Participants interacted with dynamic mathematics sketches that I designed in order to make more explicit the notion of variability and analyzed their emerging understandings. Based on the analysis of the changes in their multimodal communication, I argue that the use of dynamic mathematics environments can help promote a more physical and temporal understanding of statistical variability.

Interplay Between Concept Image & Concept Definition: Definition Of Continuity

Gaya Jayakody

This study looks at the interplay between the concept image and concept definition when students are given a task that requires direct application of the definition of continuity of a function at a point. Data was collected from 37 first year university students. It was found that different students apply the definition to different levels, which varied from formal deductions (based on the application of the definition) to intuitive responses (based on rather loose and incomplete notions in their concept image).

What Are We Sure About? What Do They Tell About Our Probabilistic Thinking?

Simin Jolfaee

In this study the prospective teachers' understanding of 100% probabilities is studied through their examples of such events. Watson and Mason's Learner Generated Examples (LGE) theory is employed to justify the type of data used in this study and to emphasize the importance of examples in learning about different levels of the learners' probabilistic thinking.

Effect of Dynamic Geometry on Children's Performance in Angle Comparison Tasks

Harpreet Kaur

This paper examines the effect of the use of dynamic geometry environments on children's thinking about angle. Using a driving angle model in Sketchpad, kindergarten children were able to develop an understanding of angle as "turn," that is, of angle as describing an amount of turn. After the classroom lessons with dynamic sketches, students were interviewed on various angle comparison tasks. It emerged out that gestures and motion played an important role in their developing conceptions of angles as well as in their decision making on angle comparison tasks.

Model-Eliciting Activities and Numeracy Tasks: A Comparison

Minnie Liu

Literature has shown that models and model-eliciting activities are important aspects in the learning of mathematics and a powerful tool in promoting students' higher order thinking. On the other hand, numeracy and numeracy tasks are vaguely defined and are not fully recognized as useful tools to promote students' higher order thinking. In this paper, I examine the similarities and differences between model-eliciting activities and numeracy tasks, and examine the possibility to integrate numeracy tasks into the curriculum.

Gestures and Temporality: Children's Use of Gestures On Spatial Transformation Tasks

Oi-Lam Ng

This paper discusses findings from a task-based interview with 5 elementary school children working on a spatial transformation task. The paper focuses on children's gestural and verbal communication when engaging in the task. Findings suggest that children use gestures as multi-modal resources to communicate temporal relationships about spatial transformations. Although research has shed light on the use of gestures to represent functions deictically, iconically and metaphorically, this work has not addressed aspects of temporality and the dynamic nature of gestures. This paper raises questions for further research in the area of gestures and communication to address the temporal aspects of mathematics.

Mathematics as Desire: The Life of André Weil

Veda Roodal Persad

When mathematicians write about their involvement with mathematics, what lies beneath? What do such accounts tell us about the nature of the discipline and the attendant demands, costs, and rewards? Working from the autobiography of the French mathematician, André Weil (1906-1998), and using the Lacanian notion of desire, I examine the forces that shape and influence engagement with mathematics. I contend that, at an elemental level of human development, these forces turn on the notion of subjectivity and the forms of desire.

Numbers on Fingers

Mina Sedaghat Jou

This paper describes TouchCount application (designed for iPad) and its two Counting and Adding world. We also explore how a five-year child (Kindergarten) builds meaning through communicative-touch based activity involving talk, gesture and body engagement. The main goal of this paper is to show the impact of touch-based interactions on the development of children's perception and motor aspects of ordinality and cardinality of numbers. In this case study, we found a strong value of mathematics embodiment in emergent expertise in producing and transforming numbers, which can be supported with the Perceptuomotor integration approach.

Bringing 'Reality' into Calculus Classrooms: Mathematizing a Problem Simulated in a Virtual Environment

Olga V. Shipulina

The study explores how students, who had completed the AP calculus course, mathematized the optimal navigation real-life problem simulated in the Virtual Environment (VE). The particular research interest was to investigate the factors determining the ways of students' horizontal and vertical mathematizing, including the role of their empirical activity in VE and the role of intuition. It was found that empirical knowledge prevails over intuitions and that horizontal mathematizing is fully grounded on empirical activity.

Secondary Mathematics Teachers Tinkering about how to Teach Radical Equations

Natasa Sirotic

This report presents findings from a collaborative teaching experience on the topic of solving radical equations in a Grade 11 mathematics classroom. An in-service professional development process was employed in a K-12 suburban school over an extended period of time in which teachers created, implemented, and reflected upon their mathematics lessons in the traditions of "community of inquiry" and "lesson study". Teachers' discourse during the phases of planning for instruction and reflecting upon the teaching experience were analysed with respect to what teachers notice about students' mathematical thinking. Through the process, the teachers became attuned to critically examine their practice and how it affected what students are doing, thinking, and understanding.

Teacher Judgements in The Classroom: What is it We Attend To?

Kevin Wells

When meeting a group of students for the first time teachers can often make judgements, wittingly or not, about the students' ability. In this paper I will examine some possible clues teachers attend to which may be enabling them to make this judgement. In this instance I am considering the feedback a teacher receives from observing a group of students problem solving. I show that certain features of the dialogue, along with the body language of the students, can offer clues as to the level of understanding the students have regarding the material. Using tools of Conversation Analysis and an analysis of gesture, I show that certain features are recognizable amongst students that are successful in their problem solving, and suggest that the experienced teacher may develop a subconscious recognition of such traits.

Paradoxes of Infinity – The Case of Ken

Chanakya Wijeratne

Previous studies have shown that the normative solutions of the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation are difficult to understand even for learners with advanced mathematical background such as doctoral students in mathematics. This study examines whether this difficulty is due to the way they are set in everyday life experiences. Some variations of the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation and their abstract versions set in the set theoretic language without any reference to everyday life experiences were given to a doctoral student in mathematics. Data collected suggest that the abstract versions can help learners see beyond the metaphorical language of the paradoxes. The main contribution of this study is to reveal the possible negative effect of the metaphorical language of the paradoxes of infinity on the understanding of the learner.

GAMING: EXPLAINING STUDENT BEHAVIOUR?

Darien Allan

Simon Fraser University

Studenting is defined as what students do while in a learning situation. A subset of studenting behaviours, that I call gaming behaviours, subverts the intentions of the teacher. This paper introduces the notion of gaming as a subset of studenting, describes the behaviours categorized as gaming, and argues for the further study of these actions together with an investigation of the student goals that coincide with gaming behaviour.

Keywords: Studenting, gaming, goals, behaviour

INTRODUCTION

Anyone who has been a student knows that students often display behaviours that are not conducive to learning, and sometimes detrimental to learning. Some of these actions are intentional and conscious, while others arise from a non-conscious level. The classroom is rife with examples of non-learning types of behaviours. During group work, some students may sit back and wait for others to do the majority of the work. In a lesson, often students sit back and wait for the 'keeners' to answer the teacher's questions, or wait for the teacher to answer them. Students figure out how to get around policies and rules regarding homework, attendance, and tests, to name just a few, in order to minimize their efforts and either maximize achievement (reward) or minimize censure or discipline. Students learn very quickly how to 'get away' with doing less work and how the teacher's rules can be manipulated.

The following transcript is taken from an informal discussion with a group of senior high school mathematics students. They were asked discuss their behaviours regarding school and dealing with teachers.

N: You learn how to read the teacher, which is really bad because you learn like their weaknesses. Teachers do that to us too.

A: With some teachers there's no way out of it, so you prioritize that class, study for that first, and then, the teachers who are a little more lenient then you can probably take those ones off, if everything's coming together on one day, study for the one who's not going to let you off.

M: Also, what I've seen or heard of is to turn on the emotions; crying, or you come to the teacher really angry.

S: Or for teachers that aren't very emotional you can just be really technical about it I guess – like I had this, this, and this, and I really tried hard.

M: Sometimes, like genuine disappointment…

S: Genuine emotion, but we show it differently to different teachers.

N: You exaggerate.

School is very much a terrain that students navigate. For most students, some of the time, learning is accomplished as they traverse the school landscape. However, a significant part of students' lives is spent occupied with activities that don't result in learning. These activities, a subset of the corpus of behaviours that comprise *studenting*, are the focus of this paper and will be discussed in more detail below.

STUDENTING

The term *studenting* was first used by Gary Fenstermacher (1986). He describes this concept as a parallel to that of teaching.

> Without students, we would not have the concept of teacher; without teachers, we would not have the concept of student. Here is a balanced ontologically dependent pair, coherently parallel to looking and finding, racing and winning…there is much more to studenting than learning how to learn. In the school setting, studenting includes getting along with one's teachers, coping with one's peers, dealing with one's parents about begin a student, and handling the non-academic aspects of school life. (p. 39)

Fenstermacher essentially describes studenting as what students do to help themselves learn. He explains that these student activities include recitation, practice, seeking help, reviewing, checking, locating sources and accessing material, among others but his definition goes beyond the activities and tasks that the student performs in order to learn. In 1994 he expanded this definition to include behaviours that students exhibit in learning situations that do not help them learn.

> The student becomes proficient in doing the kinds of things that students do, such as 'psyching out' teachers, figuring out how to get certain grades, 'beating the system', dealing with boredom so that it is not obvious to teachers, negotiating the best deals on reading and writing assignments, threading the right line between curricular and extra-curricular activities, and determining what is likely to be on the test and what is not. (p.1)

There is a noticeable shift from the primary goal of student learning to the non-academic aspects of studenting. In this later definition much of the work of studenting consists of 'beating the system'. These non-academic features, as well as the academic aspects of student behaviour can be found in several contemporary theories.

The notion of studenting has strong links to the didactic contract, classroom norms, behavioural economics, and game theory, all of which have been used to explain student behaviours in the classroom. Brousseau (1997) explains student behaviour in relation to an implicit didactic contract, negotiated between teacher and student, but confined to behaviours relating to the student's learning of mathematics and neglects those behaviours not related to learning. The concepts of classroom norms and sociomathematical norms (Cobb, Wood, & Yackel, 1991; Yackel & Cobb, 1996) were introduced as constructs for understanding the socially constructed aspects of student behaviour within the classroom. These focus on collective behaviour (rather than individual) and mathematical aspects relating to classroom discourse. Concepts in behavioural economics such as minimisation of effort or economy of action, bounded rationality (Simon, 1955), hyperbolic discounting and present bias, loss aversion, risk aversion, and context and persistent error can be applied to explain what may seem to be irrational student behaviours. Research related to game theory involves the role of performance goals when students 'game the system' (Baker, Roll, Corbett, & Koedinger, 2005), the behaviours and related consequences when students engage in 'playing the game' or 'playing the system' (Dryden, 1995), and students' behaviour in response to inventive grading systems (Newfields, 2007).

Although none of the above research uses the term studenting, it does appear infrequently in the literature, though the research tends to be limited to only some aspects of studenting, and then only within particular situations. For example, the term appears in the dissertations of two doctoral students, Wendy Aaron and Simona Goldin, at the University of Michigan. Goldin (2011) explores the work of students from a historical and sociological perspective but limits her study to the teachers' perspectives of studenting, with respect to the nature of student work, the politics of studenting, and what the student brings to the work. Aaron looks the rationality of student behaviour in her investigation of the work of studenting in high school geometry instruction (2010). She looks at this from the perspective of the student, but only in the context of geometry instruction, and only those behaviours relating to the work students do in instruction and the tacit knowledge they bring to it. This neglects the aspects that Fenstermacher discusses in 1994; namely the work that students do to 'beat the system'.

For the purposes of this article, I take a broad view of the notion of studenting. Drawing from Fenstermacher as well as Fried and Sizer I put forth a definition of studenting as: the work that students do in the context of learning, the non-academic aspects of student activities such as getting along with peers and teachers, as well as the behaviours that students exhibit as they attempt to negotiate the system, including those related to subverting the system, whether consciously or nonconsciously. I propose to look at the broader interpretation of studenting from the perspective of the student, and rather than exploring this in the context of particular situation or topic I look more holistically at the gamut of student behaviour within the mathematics classroom.

It is exactly these aspects of studenting that I am interested in. More specifically, I am interested in the studenting behaviours that are not in alignment with the teacher's goals and expected actions, yet are missed by the teacher during the activities of teaching. I refer to this class of studenting behaviours as *gaming* behaviour, as in the students are *gaming* the system.

GAMING – A SUBSET OF STUDENTING BEHAVIOURS

Several works based on research within schools and classrooms discuss the nature of the student role, primarily focussing on the non-academic aspect that I refer to as gaming.

Robert Fried critiques the state of schooling and the development of student practices (2005). Throughout their early school experiences, Fried claims that students learn to disassociate, to tune out, to disengage from the arena of learning. He describes how, in early elementary school, students lose enthusiasm and motivation for learning and instead focus on pleasing the teacher and being obedient. Unfortunately these aspects tend to undermine and shift the focus away from the goal of learning, resulting in students having little motivation to understand concepts beyond performing well on assessments and achieving other goals. Other largely undesirable behaviours that develop as a consequence of schooling include: conformity, obedience, cheating, copying homework, teacher shopping, skipping tests and faking illness in order to write the test at a later date, etc.

Theodore Sizer echoes and extends Fried's conclusions as he describes the deplorable state of affairs in schooling (1984). The researchers lament the "…weakness of incentives for serious learning that the culture as a whole lends – the signals that hard work and high standards are expected of everyone. Kids need strong incentives to engage at school..." (p. x). Sizer focuses on what he calls the 'triangle' of students, teachers, and subject. His use of the triangle highlights the unique effect each aspect has on the learning process. Each child will engage in studenting differently depending on the teacher and the subject area. For example, the same student will engage in different studenting activities in mathematics than in philosophy, even if the teacher is the same.

Tim Newfields claims, "From a game theory perspective, school classes can be seen as interactional matrices in which teachers and students try to adopt optimal behaviours in order to minimize losses and maximize returns" (2007, p.33). Further, "most classrooms should be regarded as Bayesian game systems (Nurmi, 2005) in which information about the interactants involved is incomplete and outcomes are uncertain" (p. 35). This results in "a tendency to conform to acceptable performance levels, conserve energy expenditures, and avoid risk" (p.35) rather than put the maximum effort into learning.

Gaming in the Literature – Applications in Analysing Student Behaviour

Historically, the application of game theory to mathematics (and school in general) has been limited to mathematical games and mathematical modelling. This reflects a more formal approach to gaming. More recent research investigates the links between game theory and student actions. The notion of 'gaming the system' has been used in the context of off-task behaviour, performance goals, grading, and group work, among others (Baker et al., 2004; Baker et al., 2005; Newfields, 2007).

Off-task behaviour can be detrimental to learning. Baker et al. (2004) assert that of all off-task behaviours they identified through studying student interactions with an intelligent tutoring system, 'gaming the system' is the one most strongly associated with reduced learning. 'Gaming' included such activities as quickly and repeatedly asking the tutor for help until the student is given the correct answer, and inputting answers quickly and systematically until the correct answer is identified. Results showed that low pre-test scores were positively correlated a high incidence of gaming. In addition to this, the authors hypothesize that students engaged in selective gaming on questions where they have the most difficulty, and thus "exactly where it would most hurt their learning" (p. 388). They conclude that they don't know why students game the system, but their evidence shows lack of interest in the material is not a good explanation.

Gaming does not have to have negative consequences for learning. Baker (2007) gives an example of gaming behaviour intended to increase efficiency and decrease effort, but does not inhibit learning. If a student engages in gaming in order to skip "time-consuming but easy steps, in order to focus more time on more challenging material" (p. 1064) that is a positive instance of gaming. The results of the study also indicate that engaging in negative gaming is correlated with disliking mathematics, and not being educationally self-driven.

For a particular student, the presence or absence of gaming behaviour depends on that student's motives, or goals. Therefore to explore these behaviours in more depth it is necessary to discuss student goals.

GOALS

Research has taken different approaches to the study of goals. Some theories focus on students' goal orientations, which are generally accepted to be either mastery or performance (Dweck, 2000; Lemos, 1999), although it is also common to categorize performance goals as either performance-approach or performance-avoidance (Cury, Elliot, Da Fonseca, & Moller, 2006; Pantziara & Philippou, 2009). Hannula developed a framework for motivation as a structure of needs and goals that identifies three categories of socio-emotional orientations: learning or task orientation, performance or socially dependent orientation, and ego-defensive orientation (2004). Hannula's orientations can be seen as analogous to the mastery, performance-approach, and performance-avoidance categories respectively. A mastery goal indicates the student has an intrinsic need to understand and "know" the material. A student with a performance-approach goal is more concerned with the extrinsic factors involved such as getting a good grade or pleasing the teacher or parents, whereas a performance-avoidance goal arises from a desire to avoid looking bad and to avoid negative consequences.

Goals are influenced by a student's needs, beliefs, and emotions. While goals are directed towards specific objects a need is more global. A student's different needs will also serve to determine different goals. Following this, beliefs about self-efficacy will also have an effect. For example, a student's beliefs about intelligence (fixed or malleable) will play a determining role in whether that student has a mastery or performance goal orientation. Lastly, an individual's emotions will influence goal choice (Hannula, 2002, p.3).

These are broad (global) categorizations; there are also specific (local) goals within contexts. Context is used in the global sense and includes the subject area, teacher, classroom environment, and social elements, among others. Goals also vary depending on the particular situation, which is a more specific and immediate factor than context. A student may have a mastery goal in the context of mathematics class, but in a specific situation, such as an unanticipated quiz or homework check, the student may exhibit performance-approach or performance-avoidance goal by memorizing a procedure or copying from the back of the book.

Goals have a significant contribution to make in the further exploration and explanation of student gaming behaviour. I believe that links between gaming behaviours and particular goal orientations will surface upon further study and analysis of these behaviours and that in the future these results could be used to help teachers and students achieve intended results.

CONCLUSION

To teachers and parents, student behaviour often appears irrational. From the perspective of the student, however, there is a certain rationality to their actions that I am trying to understand using theories from behavioural economics and game theory.

Based on the evidence in the literature, personal experience, and the experiences of colleagues, it appears that a significant subset of the activities of studenting seems to involve attempts to game the system at some level. The prevalence of these behaviours, and the possibilities for influencing student behaviour warrants the further investigation of studenting as gaming.

A future study involves developing a taxonomy of student behaviours that fit the description of gaming. My goal is to take the threads I've pulled from contracts, norms, conventional and behavioural economics together with game theory and weave them into a coherent theory to explain the subset of student behaviour I call gaming.

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PERSPECTIVES ON THE SAUDI GENERAL APTITUDE TEST

Lyla Alsalim
Simon Fraser University

The General Aptitude Test (GAT) is a standardized ***test designed*** *to be used as an* ***admissions*** *tool by colleges and universities in Saudi Arabia. Since its establishment, the debate about the GAT has been growing. Different viewpoints and* ***perspectives*** *on the issue have resulted in conflicting opinions, some supporting the test and believing in its value and others opposing it. The purpose of this paper is to explore the perspectives of different stakeholders about issues related to the GAT in order to highlight any disparities and similarities in perceptions and how these differences and/or similarities affect the way in which the GAT is being applied. The paper highlights the key themes gathered from interviews.*

INTRODUCTION

Saudi Arabia has been experiencing an increased demand for higher education in recent years. In 2011, the number of Saudi high school graduates exceeded **340 000**, and it is expected to continue to increase for several more years (NCAHE, 2012). More high school students than ever before have the desire to continue their education by attending four-year colleges and universities. The increased number of high school students who are ***pursuing higher education*** makes it more competitive to get into top universities and graduate schools. For decades Saudi schools focused only on a student's overall high school average as the criterion for admission to post secondary education institutions. For most Saudi universities and colleges, especially the elite ones, a high school percentage was not enough for making admission decisions. Therefore, most Saudi universities established their own admission test to be used to evaluate applicants along with the high school percentage. This movement had an impact on the application process, which became complicated for those applicants who applied to more than one university and for those who did not live close to the university to which they applied.

As a result of the debate over the validity and reliability of high school percentage for university admission, the Ministry of Education in Saudi Arabia felt the need to offer fair, valid instruments that accurately predict the level of students' performance in post secondary institutions. Therefore, the National Center for Assessment in Higher Education (NCAHE) was established in 2001, a movement in educational reform in Saudi Arabia. Its mission is to "assure fairness and equal opportunity in higher education and contribute in the efficiency of higher education institutes based on solid scientific grounds" (NCAHE, 2012). It administers and develops different standardized tests, including the GAT. The General Aptitude Test (GAT) was first administered by NCAHE in 2004.

According to the NCAHE, the GAT is required for admission to all institutions of higher learning in Saudi Arabia.

Standardized testing has been used to evaluate the accumulated knowledge of an applicant pool since early times. The earliest evidence of standardized testing comes from Imperial China. The system of civil service examinations is considered one of the most noteworthy contributions from Chinese history to the world. The history of Chinese civil service examinations is attached to the history of civil service itself. Imperial dynasty leaders recognized the fundamental usefulness of these exams and relied on them to ensure required services were being offered by qualified individuals to the large population in China. The ***civil service*** examination system originated ***during*** the ***Han*** dynasty (206 B.C. – 221 A.D.). The idea of examination came from the notion that government officials should be extremely educated in order to maintain a ***stable government***. This idea was heavily influenced by Confucian ideals, which focused on the principle that moral guidance, courtesy, and filial piety could maintain a thriving government and social system (Menzel, 1963). The original purpose for the Chinese ***civil service*** examinations was to guarantee appointees to civil service positions were not selected based on inherited privilege, but on an individual's own abilities, talent, and education (Menzel, 1963; Miyazaki, 1976).

Deciding who the most qualified person is for a particular job can be a complicated task and may require applicants to take tests which measure their skills and knowledge in a particular area. Colleges and universities face the same perplexing task when accepting new students. The ***ancient Chinese notion of choosing the most qualified people based on the result of examinations has found its way into many education systems around the world.*** In the United States, the SAT is the ***most commonly taken*** test by high school students as a reliable, effective measure of their readiness for post secondary education. According to the College Board, the SAT measures student ability in critical reading, mathematics reasoning and writing skills developed in and out of school. It also tests their ability to apply that knowledge to participate positively in higher education institutions (2011b). The SAT has been a valuable tool for students and admission officers in the competitive admission process for almost a century, helping students prove they possess the skills required for college and university life. In 2011, approximately 1.65 million students participated in taking SAT (College Board, 2011a).

FRAMEWORK

According to the descriptions of the GAT and the SAT in terms of their purpose, it is notable that these two tests share the same purpose. Also, the early versions of the SAT are similar in terms of its content and sections to the GAT. These similarities allow me to draw on the history, development and critiques of the SAT in my research and analysis of the GAT.

The story of the conception of the SAT goes back to 1923 when Carl Brigham became the chairman of the College Board organization. He was interested in adopting a standardized test, which was used at that time for the selection of the United States military members for use in college admission. In 1926, the SAT was born under its original name the Scholastic Aptitude Test. In 1933, when James Bryant Conant became the president of Harvard University, he introduced a new scholarship program to attract academically gifted students. Conant knew that Brigham had already designed a test to evaluate the academic potential of high school students. Conant was convinced that the SAT was a reliable intelligence test, which could predict academic success; as such, the test was administered to students who wished to be awarded scholarships to Harvard. The fact that these students were succeeding encouraged Conant in 1938 to influence the presidents of the other Ivy League schools to adopt the SAT and the scholarship program. A few years later, the SAT was no longer administered exclusively for the function of selecting scholarship students, but several colleges depended on the test results to assess students for admission into their schools (Lemann, 2004).

The SAT examination has never been static. The test has evolved to meet the educational standards of the best colleges and universities and to reflect the subject matter highlighted in the mainstream of American high schools (Epstein, 2009). Lots of the changes made are the result of ensuring the test is better able to measure the abilities of the participants, especially taking into account the changing goals of education in the United States. A number of the changes make certain the test is as comprehensible and effective as possible, whereas other changes ensure that students coming from certain backgrounds do not experience an unfair disadvantage (Stringer, 2008).

Changes to the SAT come in ***response*** to ***criticism*** that the test has received. One of the common criticisms of the SAT is that it is not truly reliable in predicting students' academic performance (Geiser, 2009; Stringer, 2008; Atkinson & Geiser, 2009). Stringer (2008) notes that the test is failing to measure students' actual knowledge, which influences the ability of the test to predict future success in college. One of the main reasons behind not considering the test as a valid predictor of college success is that the test does not take into account some other elements that impact students' achievement in higher education such as motivation and study skills. Geiser (2009) explains that the test does not measure factors such as personal discipline and perseverance, which are considered "key to achieving and maintaining a strong academic record over the four years of high school" (p. 21).

Another main critique of the SAT is that students differ in their mental abilities, which are often associated with their background. Some educators, Lemann, Atkinson, and Gilroy to name a few, argue that the way the SAT is designed and presented provides some students more of an advantage over the content of the test than others. It confines education equity and hinders access to higher education for otherwise qualified students

(Gilroy, 2007). “The perception of the SAT, as a test of basic intellectual ability, had a perverse effect on many students from low-performing schools, tending to diminish academic aspiration and self-esteem” (Atkinson & Geiser, 2009, p. 667).

The SAT ***has*** also been ***criticized for being*** vulnerable to the impact of elements related to students’ socioeconomic status, such as schooling excellence and SAT coaching (Atkinson, 2001; Atkinson, 2004). While some researchers, such as Powers and Rock (1999), argue that coaching has minimal effects on SAT performance and it only ***marginally*** improves test ***scores***, others indicate that SAT tests can be highly coachable (FairTest, 2007). The level of effectiveness of coaching programs on SAT performance does not prevent some educators from arguing that SAT coaching is not reasonably available for all test takers which provides an unfair advantage to some test takers over others (Epstein, 2009; Stringer, 2008). Higher-income test takers who can afford to spend $800 or more on test preparation classes are more ***advantaged*** compared to low-income test takers students (FairTest, 2007).

The SAT is also accused of being culturally ***and*** ethnically ***biased*** since white students tend to score higher than minority students as a group (FairTest, 2007; Geiser, 2009; Atkinson & Geiser, 2009). “African American, Latino, new Asian immigrant and many other minority test takers score significantly lower than white students” (FairTest, 2007, p. 4). According to a study done at the University of California, “the SAT had a more adverse impact on poor and minority applicants than traditional measures of academic achievement [such as high-school GPAs]” (Geiser, 2009, p. 19). Therefore, strict utilization of the SAT for college admissions has a great effect on reducing the number of minority students in higher education, especially those who have a good potential to succeed academically since they apply with strong academic records but comparatively low SAT scores (FairTest, 2007; Geiser, 2009).

The General Aptitude Test (GAT)

The General Aptitude Test (GAT) is an aptitude test used to assess the level of general ability in verbal and quantitative areas mastered over time. It is designed to specifically measure general comprehension, as well as analytic and quantitative abilities in language and mathematics. NCAHE claims that the GAT is not a subject oriented test; it is not developed based on specific standards related to particular subject materials (NCAHE, 2012). According to the NCAHE, the GAT is not an achievement test; “GAT is based on skills related to logical thinking, analysis and relationship. These skills have been acquired by test-takers throughout their education and through exposure to different experiences in life” (NCAHE, 2011, p. 3).

The GAT measures students’ abilities in “reading comprehension, logical relations, problem-solving behaviour, inferential abilities, inductional abilities” (NCAHE, 2011, p. 2). The test consists of 120 multiple-choice questions. It has two main components, the verbal which is presented in 68 questions and the quantitative which has 52 questions. The verbal section contains questions in three areas: sentence completion, analogy, and reading comprehension. The questions in the quantitative section consist of 40% arithmetic, 23 % algebra, 24% geometry, and 13% interpretation of graphs and tables (NCAHE, 2012).

METHODOLOGY

Eight semi-structured interviews were conducted with eight different stakeholders. Stakeholders were selected based on their involvement, experience, and knowledge of the GAT. The interviews took place with a recent high school graduate, a high school mathematics teacher, a parent of recent high school graduate, a teacher who is working in a GAT coaching institute, a faculty member in a higher education institution, a university registrar employee, a person who works at the Ministry of Education, and a lay member of the public. The aim of the interviews was to obtain points of view, reflections and observations about different issues related to the GAT. The semi-structured interviews began with an **interview guide** of a list of previously prepared questions to help provide the needed focus. The **interview guide** opened with questions geared at getting a general sense about the participants' background as it related to the GAT. Interviews lasted between 40 and 60 minutes.

Interviews were audio recorded and transcribed. A qualitative analysis approach was undertaken using content analysis as the favored method. Content analysis was useful for grouping the ***transcribed text*** into meaningful key themes. The main key themes gathered from the interviews are the purpose of the GAT, the quality or adequacy of the GAT, ***the fairness of the*** GAT, mathematics and the GAT, preparation for the GAT, GAT as a predictor of academic success, and the future of the test. However, for the purpose of brevity, this paper will only present three themes: the purpose of the GAT, the quality, or adequacy, of the GAT, and mathematics and the GAT.

DISCUSSION

The purpose of the GAT:

The analysis revealed that participants have different views of the issues related to the purpose of the GAT. The member who works for the Ministry of Education is strongly influenced in his opinion by his association with the National Center for Assessment in Higher Education (NCAHE), believing that the tests provides fairness and equal opportunity in higher education. He added, "I think that most Saudi educators agree that student's high school percentage is not enough for making admission decisions." The university registrar also explained that post secondary institutions cannot be certain that high school grades accurately represent the abilities of their applicants and entering first-year students. "It is hard to make admissions decisions with a satisfactory level of confidence based on high school GPA alone." She believes that the GAT provides some valuable information about applicants.

The lay member of the public and the associate professor agreed partly with this view. They both indicated that although students' high school grades can generally predict how a student will do in college, there is a place for standardized tests. However, the lay member of the public indicated the she is not sure if this test can actually reveal enough information about students' ability to succeed in university. The associate professor raised the issue of grade inflation. She stated, "Grade inflation is a real problem. Standardized tests can minimize its negative effects." The high school mathematics teacher admitted that student evaluations, especially in senior high school, are often deceptive. "Teachers are put under too much ***pressure*** to help students achieve good results which increases the ***probability*** of ***grade inflation.***"

The teacher who works in GAT preparation programs indicated that students generally don't believe in the purpose of the test. They mostly believe that the test is imposed on them. This belief is clear when considering the high school student's opinion. "I think the GAT was established so the government could find a reason not to offer everyone the opportunity to continue their education; so it would be the student's fault, not the government, if the student could not find a place in a public university." The student also added that introducing the test is a kind of confession from the government that the 12 years students spend in school is not enough to prepare them for university life, which indicates the deficiency and ***failure*** of the ***education system*** in Saudi schools. The mother of the student also ***attacked*** the Ministry of Education for introducing the test. She claimed that this test puts students and their families under more pressure during this critical time in students' lives. She added, "I can't understand the purpose of introducing this test."

The quality, or adequacy, of the GAT:

The Ministry of Education member firmly indicated that the GAT is carefully designed to measure basic concepts and skills that student should have in order to succeed in university. He explained that most of the language questions are to test comprehension. For example, students read a small paragraph then answer questions related to that paragraph. These skills reveal if a student will be able to understand and process the material that s/he will encounter in university courses. The quantitative questions also test some basic and fundamental mathematics skills, such as basic algebra and geometry concepts. "The test only measures the basic skills. A student who has not mastered these basic skills, I am afraid, will not be able to succeed in the first year."

The associated professor noted that the test filters out students who will not be able to perform successfully in college. "The test is strictly and tightly timed. It is necessary that students encounter this type of test before they encounter university tests." She believes that students usually suffer in the first year of university because they cannot perform quickly during tests, and this test provides students an opportunity to learn about their test taking abilities. The high school mathematics teacher also commented on the issue related to test timing. In her view, most of the mathematics questions provided in the test are not difficult but they are not straightforward. She noted, "Usually, students are not used to dealing with these type of questions, and giving them only 45 seconds to answer a every question is not adequate**."**

Although the university registrar indicated that the GAT can provide some information about applicants, she believes that the system used in the past, where every department designed their own admission test, was more adequate than the GAT. "Every department would design a test to measure the skills that students really need to be accepted to that department. I don't think it is adequate to have all students who are interested in applying to different schools take the same test." The teacher who works in GAT preparation programs is also not convinced about the test's adequacy. She explained that the GAT, more than anything else, shows how well a student can take a standardized test. She added that it is very hard to finish every section. The test was designed in a way to provide a nice range of scores for college comparison. "I think the test does not measure student's raw math or verbal ability; the students know that if they can't perform quickly, they will screw up."

The mother of the student considers test taking as a skill in its own right which can be ***acquired*** with time and practice. She could not see the GAT, in any aspect, as an indicator of the likelihood of success in college. "It's an instrument for 'inattentive' admissions administrators to defend their choices." The lay member of the public also believes that the test is great at predicting the students' ability to take standardized tests. "I think it is a very faulty test if we expect it to expose a good deal about students' content skills, personal discipline, and perseverance. These are also accomplished and maintained through an excellent academic record."

The high school student also questioned the type of skills the test measures. He believes that the skills the GAT tests are not necessarily the skills needed to succeed in college. "When we ask ourselves about the skills students need to have before they start college, we find that GAT or any other standardized test, does not measure most of them." He also indicated that the GAT puts students in a pressurized environment, and students who can handle the pressure in testing situations will do extremely well. In his opinion, some brilliant students get a very average score just because they cannot handle the pressure of the idea that this test, where in less than three hours, their future is determined.

Mathematics and the GAT:

Questions about mathematics as a subject and its relation to the GAT were not asked to all participants, but only to those who are in a position which enables them to reflect on this issue. Those include the mathematics high school teacher, the teacher who works in the GAT preparation programs, the associate professor, and the high school student. The negative attitude that is usually associated with mathematics appeared when the high school student described his experience with the test. "I find the mathematics section in the test much more difficult because it consists of numerical calculations that involve memorization of rules and application of formulas. If I forget the formulas, there is nothing I can do to deal with the question. And even if you can get the correct formula, if there is a small arithmetic mistake you get the question wrong." He also explained that to prevent the use of calculators in the test puts students under more pressure. He indicated that students need to have ***some*** knowledge of some ***mathematics*** calculation ***tricks*** for faster ***mathematics*** calculation, knowledge which mathematics school teachers don't provide for their students. According to him, the experience of taking the GAT makes students realize their actual level in mathematics. Some students usually get high grades in mathematics in school but do poorly in the quantitative questions on the GAT. He believes that mathematics teachers in general are not exempt from blame for not preparing students with strong mathematics skills.

The associate professor also commented on the poor teaching practices used by most mathematics teachers in schools and its negative effect on students' mathematics skills. In her opinion, the difficulty students face when dealing with the quantitative questions in the GAT is an indicator of a serious problem in mathematics teaching and learning in schools. The quality of ***mathematics teaching*** and ***learning*** in schools is not good enough to provide students with proper mathematics understanding. "Mathematics is seen as a collection of rules and procedures. Most mathematics concepts are taught without meaning." She hopes that this test encourages the establishment of a new movement toward more effective mathematics teaching and learning.

There is another essential issue raised by the teacher who works in the GAT preparation programs.   She explained that a considerable number of students are terrified or anxious about taking the test and student anxiety appears more when students are working on quantitative questions than when working on the verbal or language questions. She attributed this anxiety to the negative experiences students' encounter with mathematics in schools. She also noted that students usually have poor mathematics knowledge and weak arithmetic skills compared to their verbal or language skills, which seem to be much stronger. She believes that students' anxiety and bad skills in mathematics affect students' ability to do the quantitative section effectively in the GAT test.

The high school mathematics teacher admitted that mathematics teachers in high school do not make enough of an effort to support their students going through this tough experience. She explained that some mathematics teachers in high school believe that they are not required to provide any kind of support to their students related to taking the GAT since this support requires working extra time and it is not indicated in the curriculum. According to her, on the GAT, students never encounter questions with really advanced geometry; the mathematics questions are about basic mathematics concepts which students are ***supposed*** to be familiar with. "I believe that mathematics teachers in every level should always go back to the basics. Things like fractions, decimals, ratios, and areas of basic figures. Students always need to be reminded of this ***basic*** mathematics knowledge."

CONCLUSION

The diversity of participants' position and level of involvement with the test enriched this research by providing different points of view, reflections and observations about some basic issues related to the GAT. Overall, participants welcomed the questions and a high level of engagement in the interviews was noticeable. The high school student and the mother of the high school student had generally negative views about almost all aspects about the GAT as discussed in this paper. This negativity about the GAT may come from the high pressure they both felt during their experience engaging with the test. On the other hand, the ministry member had a generally positive view about the test. His view may come from his position as a ministry member, which encourages him to side with the test, defending it most of the time.

Also, one of the surprising findings is that all the participants agree that in the Saudi education system depending on student's overall high school average is not enough criterion for admission to post secondary education institutions. Most participants believe that some of the common practices of high school teachers, such as providing students with summaries or notes before tests and giving tests that are more about memorization than understanding, result in student grade inflation, meaning an increase in students' grades with no accompanying enhancement in their academic achievement. As a response to this perceived problem, some participants suggested that instead of introducing new tests to evaluate high schools graduates, a serious reform to school education should be done in order to produce more prepared students for post secondary education.

This research tells us a lot not just about the test itself, but about the whole education system in Saudi Arabia. Students take the test with the assumption that they have already acquired the basic knowledge requirements in school subjects to prepare them to take such tests. Unfortunately, this is not the case; the poor performance of school results in producing students who are not intellectually and psychologically prepared to take such test. In actuality, we have to focus on improving the education system and the teaching performance then improved testing would come next.

It has been nearly a decade since the GAT was established; however, there are not enough published studies about it. Perspectives of different stakeholders about issues related to the GAT were explored in this paper. It is expected that the test will continue to be a controversial topic among people who are interested in educational issues in Saudi Arabia. ***There is a pressing need for more*** extensive ***research in*** all aspects related to the GAT discussed in this paper**.** The test clearly has a great impact on the life of high school students. ***This research is a first step in understanding the impact of this test on*** high school students, especially as it relates to their mathematics learning experience.

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USE OF PHENOMENOLOGY THEORY PERSPECTIVE IN THE ANALYSIS OF PROFESSIONAL DEVELOPMENT SESSIONS: TEACHERS' ENGAGEMENT AND USE OF RESOURCES

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This research paper considers how Phenomenology Theory can be used to explain the factors that affect teachers' engagement throughout a professional development experience and what are the factors that are more likely to contribute to sustained engagement in acquiring new knowledge and pedagogical skills.

INTRODUCTION

The purpose of this article is to describe a model that can help us explain the factors that affect teachers' engagement throughout a professional development experience, and what are the factors that are more likely to contribute to sustained engagement in acquiring new knowledge and pedagogical skills which will contribute to teachers' effective re-sourcing of pedagogical materials.

As the professional developer working with groups of teachers, I systematically reflected on the live experience of professional development sessions: teachers' behavior, level of activity, interaction with resources, and so on. The goal was to find ways to promote in teachers what Rogers (1986) has labeled 'emancipatory learning', to help them find productive learning frameworks where they reflect, ask questions and try to figure out on their own how to make sense of the problem at hand. I looked for ways to analyze the challenges faced in each session with a particular group of teachers, in order to develop this kind of learning in them while at the same time supporting the kind of engagement that would be pedagogically productive.

I analyzed what occurred at particular professional development sessions: who was engaged and how? And who was not and why? How teachers' knowledge, belief and values were exposed by behaviours, and how this helped or deterred the purpose of the each session. What are also the challenges from the professional developer perspective: barriers, self-doubt, etc.?

FRAMEWORK

Phenomenology uncovers the meaning of a phenomenon by revealing the many layers that socially and culturally influence a person's experiences in their lifeworld; where lifeworld is defined by Van Manen (1997) as "the world of immediate experience", the world as "already there" (p.182). As such phenomenology studies events as they arise in our lifeworld and how we interpret and interact with those events according to our own individuality; for this reason phenomenology must first "describe what is given to us in immediate experience without being obstructed by pre-conceptions and theoretical notions" (Van Manen 1997:184). While doing phenomenological research and analysis one must get "rich descriptions of phenomena and their settings" (Kensit 2000:104) in order to get closer to true essence of the phenomenon. As the professional developer, I would describe how teachers engage with, understand and use the curriculum that is being implemented; being able to engage the teachers is the initial key point of any professional development session.

Sierpinska (1994) pointed out that to be able to describe an act of understanding, the person who describes it must pay attention to three features of the act: what is being understood, the context in which the act occurs, and the mental processes galvanized in the act of understanding.

In order to analyze this process I developed a modified version of Remillard's (2012) model to analyze the various ways a professional developer tries to position teachers in an experience that engages them. She developed a model to analyze various ways curriculum developers use to make their manuals and textbooks materials attractive to teachers. In my case, I realized early on that the use of the manuals and the textbook was not enough to engage all the teachers in the use of the curriculum to be implemented, so I tried to find other ways.

Remillard's model is inspired by Ellsworth's book Teaching Position based on analysis of film studies where assumptions about the audience background influences the structure of the film's narrative in order to maximize their interaction or attention to the film; we find that the main goal is to position the audience in a way where interaction is possible and furthermore this is something that the audience wants to do on their own volition. This model consists of four main parts: Mode of address, Forms of address, Modes of engagement, Forms of engagement.

Modes of address are ways of positioning an audience that are needed to initiate an interaction. To position the teacher in a place where she can enter in a relationship with the resources the professional developer would like her to work with, and not only that, to do it in a way so that the teacher wants to be in that position. However, as Remillard points out, this positioning is "problematic in its shaping of the relationship around power and authority in the interaction… modes of address do not merely speak to an intended audience, but actually seek to assert control over that audience or to enlist a particular kind of participation" (Remillard 2012, p107). A main point in Remillard's framework is that "teachers are positioned by and through their encounters with curriculum materials as particular kinds of users of them" (Remillard 2012, p106). Researchers like Lloyd (1999), Adler (2000), Remillard (2005), and Guedet & Trouche (2009), have studied how curriculum resources are used by teachers and they concur that their implementation is not a straightforward process, but depends very much on the interaction of the teacher with the resource. These forms are what teachers interact with as they engage in professional development activities. Remillard calls them forms of address, and according to her there are four main characteristics to consider as we analyze them or describe them (Remillard 2012, p110):

Structure: here we look at the content, the organization and the activities included in the resources.

Look: mainly refers to the physical appearance of the resource.

Voice: how the designers of the resource communicate with their users about their intentions in the use of the materials.

Medium: how the resource is delivered: print, video, website, artifacts, etc.

Genre: Remillard usually uses this term to refer to textbook use, using Otte's (1986) idea "that texts have both objectively given structures (what can be seen) and subjective schemes (ways of being understood or expectations upheld about them)." (Remillard 2012, p113). Teachers' expectations about specific texts or curricula influence the way they approach them. In this research, genre will not only refer to text but to ideas or concepts (fractions, decimals, multiplication, etc.) to be discussed during the professional development sessions and the expectations regarding those particular genres, which will influence the way teachers will engage particular Forms of Address: "Genre precisely presupposes much of what can be expected in the kind of communication in question" (Ongstad 2006, p. 262).

According to Remillard "Forms of address are powerful mediators of teachers' engagement with a particular curriculum resource", and she bases her analysis on Vygotsky (1978, p.40) theoretical ideas founded on a model of artifact-mediated and object-oriented action. Remillard points out that as such we can see curriculum resources as artifacts to carry out a "goal-directed activity" and therefore forms of address matter. Form of address can strengthen or weaken modes of address and the way teachers will interact or engage with particular resources, what Remillard calls modes of engagement, how teachers inject meaning to these forms.

Remillard only analyzes engagement through text forms; I expanded this model to include other forms like the use of videos and workshop activities. I use Peter Liljedahl's (2012) taxonomy based on teachers' wants which affect teachers' modes of engagement and can provide a way to analyze teachers' behavior during the professional development sessions.

Resistance: there are teachers who do not what to participate in professional development sessions, their contributions are minimal if any, and in some cases they are defensive or challenging. This is not necessarily a permanent condition, there are ways the professional developer can change resistance.

Do Not Disturb: Here teachers want to improve their practice by learning some new things or teaching strategies, and adding some additional activities to their repertoire; but they do not want to change their practice much.

Willing to Reorganize: Here teachers are willing to look at new curricula and resources and to reorganize their teaching around these new resources but the changes are more "clerical" in nature, not deep pedagogical change.

Willing to Rethink: teachers are open to “a complete rethinking of significant portions of a teaching practice” (Liljedahl 2012 p6). Teachers are open to changes in pedagogical styles and to be critical of their own practice.

Out With the Old: These teachers have a sense that what they have being doing is not working. They are looking for new pedagogical ways of approaching learning and they are more than happy to work with completely new materials and ideas.

Inquiry: Here teachers are more interested in learning and questioning new ideas about teaching, to have a better understanding and knowledge about a variety of teaching practices and their possibilities.

Forms of engagement: just as modes of address are connected to particular forms or resources (forms of address), modes of engagement also connect to forms of engagement, which reflect on how teachers act/react and their ‘take-up’ of the process. I will state that forms of engagement are the ways the teachers re-source the resource. Here I use Adler’s “conceptualization of ‘resource’ as both a noun and a verb; as a verb “re-source” will connote to source again or differently, and “source” will denote origin (p 6 from text). By looking it this way a resource can be an artifact but in addition we are also able to look at how teachers select, interact and work with resources by adapting them, revising them and re-organizing them; how “design and enacting are intertwined.” (Adler, 2012)

This model is a circular model, the professional developer plans and uses particular modes and forms of address during the professional development session and sees what modes and forms of engagement result from her plans. The next session will be planned depending on the engagement and forms produced in the previous session.

METHODOLOGY

Participants and setting

A new school-wide math program was being implemented at a school in the Lower Mainland in British Columbia, Canada. Thirty of its teachers teaching grades K to 6 participated in the research. The teachers were divided in seven groups corresponding to the grade they were teaching. Each professional development session tended to a particular grade. There was never a session where two or more grades worked together, except for the introductory session where the professional developer gave an introductory overview of the program and all the teachers at the school attended this session.

Process

Each group of teachers met with me between 10 and 15 hours distributed between 3 to 7 sessions.

My main concerns were:

1. - To be able to address the needs and concerns of the teachers in order to engage them in a pursuit to further develop their practice, and math content knowledge.

2. - Opportunities for teachers to take risks in sharing their beliefs among their peers.

3. - Being able to have sufficient time for the teachers to learn and maintain their learning.

Throughout the professional development sessions I supported the teachers with readings, activities, resources and explanations of particular math concepts and ideas that came up during the session. I planned each session according to implementation expectations and what occurred on previous sessions.

Data and Analytical Tools

Data regarding the actions of teachers in a professional development setting was gathered in order to find if they were engaged in the discussion and how. This was done by recording most of the professional development sessions and by notes made by the researcher during and after each session. After every session I made a few notes reflecting on the effectiveness of the session, which concerns were addressed and the level of engagement. Most observations of teachers' behavior were made during the professional development sessions; however I had the opportunity to visit classrooms to observe the teaching practice of a few of the teachers.

For the data analysis I used the following protocol, which is a simplified version of Hycner's (1999) explicitation process used by Groenewald (2004) together with some steps delineated by Van Manen's (1997) methodical structure for hermeneutic (interpretative) phenomenological inquiry:

1. - Investigating experience as we live it rather than as we conceptualize it;

2. - Phenomenological reduction: reflecting on the essential themes, which characterize the phenomenon;

3 - Delineating units of meaning by extracting those narratives, which throw light on the researched phenomenon (Creswell 1998; Hycner 1999).

2. - Clustering of units of meaning to form themes: units of significance are created by grouping units of meaning together. (Creswell, 1998; Moustakas, 1994; Sadala & Adorno, 2001)

PRELIMINARY RESULTS

The first session was very telling, and it is useful in showing how this model can be used to analyze engagement and learning.

This was the plan for the first session: teachers were first asked how they would teach a particular mathematical concept; after the ensuing discussion teachers were exposed to the books and workbooks of the program without the benefit of the manual. Teachers were to compare between their initial take on the concept and how the materials in the program dealt with it. Afterwards the manual was introduced with the goal of furthering discussions regarding the teaching of the concept, teachers were asked to read a particular session of the manual referring to the discussed concept. Teachers were asked about what additional ideas were included in the manual, and how useful it was to better teach the ideas students found in the book and workbooks. For some groups the same procedure was done for several sessions, and for others the process changed according to what I as the professional developer thought would be a better way to elicit engagement on the part of the teachers.

As you can see most of the modes of engagement were active in that I actively sought teachers' input in working with the materials, however not all the teachers were actively participating as I expected. As teachers were asked to look at the manuals and to discuss the mathematical and pedagogical content I found the following.

Those teachers who had been using the program for several years and who didn't use the manual had a very hard time incorporating this new resource into their practice. They believed that because they had been using the materials for a while, in some cases years, they knew how to use those materials. They resisted a thorough reading of the manual even when some important ideas were missing which were important for the flow of the program.

Even for some teachers who had never used the program, if the mathematical content looked 'too familiar' they did not take more than a few seconds looking at the manuals. However if the ideas were new to them or if I pointed out some interesting ideas or key points that were new to them, they were more willing to take a second look at the materials than the group discussed above.

As a result of what happened in these initial session, as I planned some future sessions I tried to engage teachers by making them aware of new ideas within the mathematics that they thought they knew, teachers became engaged in deeper mathematical discussions and were more enthusiastic about conveying these new ideas to their students and giving the manuals a second look.

Here is an example: the researcher used some interesting insights from Ron Aharoni's book Arithmetic for Parents and she gave the teachers some excerpts from his book to read. Here is Aharoni's (2007:69) discussion on the meaning of addition:

"The expression 3+2 applies to the joining of two groups … Joseph has 3 flowers, Reena has 2 flowers. How many flowers they have altogether? … However, before we go any further, we must discern a subtlety of meaning. There are actually two different forms of addition: dynamic and static. In dynamic addition, to join means to change the situation: 3 birds were sitting on a tree, 2 joined them. How many birds are there now? In static, joining signifies grouping of types: A vase contains 3 red flowers and 2 yellow flowers.

How many flowers are there altogether? ... I do emphasize difference, especially because of the link to subtraction…children find static subtraction difficult".

This passage created quite a discussion among teachers in grades K to 2. Many of them could give examples of students who could add and subtract but had difficulty with problem solving and discussed the possibility of what would change in their students' understanding and problem solving abilities if they emphasized the difference. They wondered how they never thought about this difference in meaning before: teachers were engaged and learning.

There are many other examples like this in the data, with concepts that teachers thought they knew well, where subtleties like the one above make the concept "new" in a way, and more engaging.

CONCLUSION

The process of phenomenological inquiry provides a way to analytically reflect on the process that will help professional developers to support teacher learning.

As an example, it is interesting to point out that in this study I was able to show that even with support, teachers will not necessarily read the manuals that will support the implementation of a new curriculum/math program. This is an interesting finding given that manuals are supposed to be the main if not the only resource used by developers and schools to implement new programs.

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WHAT ARE WE SURE ABOUT? WHAT DO THEY TELL ABOUT OUR PROBABILISTIC THINKING?

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In this study the prospective teachers' understanding of extreme probabilities is studied via their examples. Watson and Mason's Learner Generated Examples (LGE) theory is employed to justify the type of data used in this study and to emphasize the importance of examples in learning about different levels of the learners' probabilistic thinking.

Keywords: Probability, Learner Generated Examples, Teacher Knowledge

ABOUT THE LGE FRAMEWORK

Watson & Mason (2005) considered Learner generated Examples (LGEs) -an approach in which learners are asked to provide examples of mathematical objects under given constrains – as a powerful pedagogical tool, through which learners enhance their understanding of the concepts involved. Watson and Mason also introduced the construct of example space as collections of examples that fulfill a specific function, and distinguished among several kinds of example spaces. When invited to construct their own examples, learners both extend and enrich their personal example spaces, but also reveal something of the sophistication of their awareness of the concept or technique (Bills, Dreyfus, Mason, Tsamir, Watson, & Zaslavsky, 2006). In accord with this observation, Zazkis and Leikin (2008) suggested that LGEs provide a valuable research tool as they expose learner's ideas related to the objects under construction and examples generated by students mirror their understanding of particular mathematical concepts. Of my interest in this study are personal example spaces, triggered by a task as well as by recent or past experience, and collective example spaces, local to a classroom or other group at a particular time.

Mason in an analysis of the phenomenology of example construction (Mason, 2011) describes what takes place through the process of mathematical example construction as: A strong tendency to combine the simplest possible with maximum generality, constructing lots of examples and tinkering with examples to modify them so that they meet some particular constraint, experiencing dimension of possible variation and range of permissible change associated with the examples constructed and explore deeper aspects of the notion, and drawing attention to the playful aspects of example construction and the ways of tinkering with a basic construction that might be of benefit the future use. Vinner (2011) finds the role of examples in everyday and mathematical thinking to be very crucial. Unlike in mathematics in which the concept formation is aided by definitions, examples and proofs, in everyday thinking, examples are the only tool by which we can form and verify concepts and conjectures. Even in mathematics there are important notions such as "proof" that have no (undergraduate level) definitions and the students are supposed to acquire the concept of proof by the many examples they are exposed to.

Zazkis & Leikin (2008) suggest that The task of constructing examples of mathematical concepts can be quite a complex task for students and teachers, but several researchers find it a well worth effort since the example generating task provides rich educational potentialities: providing a window into learner's mind through which significant aspects of conceptualization could be observed, raising the students' awareness of features of examples that can change and of the range where they can vary (Mason, 2011), and the richness and complexity of processes involved in constructing examples (Antonini, 2011).

Examples also may be used to identify, mirror, and confront learners' incorrect mathematical inferences. Building on ideas of cognitive conflict and conceptual change, Zazkis and Chernoff (2008) extend the considerations of dimension of possible variation and range of permissible change to counterexamples and discuss the role of counterexamples with respect to those theoretical constructs while helping students face their misconceptions.

Many different ways to look at the examples are introduced into the research. From the generating point of view, there are two types of examples: those generated by learners upon invitation (learner generated examples) and the examples used by teachers in a classroom setting (instructional examples). With regard to the availability of examples to the generator one can distinguish between situated, personal, personal potential, accessible, and conventional example spaces; discussed in Watson and Mason 2005. With regard to the specific functioning of examples one can put them into examples-of, examples-for (Michener, cited in Watson 2011), pivotal examples, bridging examples (Zazkis & Chernoff 2008), non-examples, counterexamples, ...

THIS STUDY

In this study I consider student generated examples of an event with 100% probability and address the following question:

To what extent do examples generated by participants reveal their understanding of the mathematical concept of probability and more specifically of the certain events?

About 100% probable events:

Extreme probabilities have mathematical significance. Also known as tail probabilities, the extreme probabilities create additional complexity to the probability estimation methods and techniques. Every computer simulation method has limitations and problems when the probability sought after is around the extremes. For example the central limit theorem allows for a binomial distribution to enjoy the normal approximation when np and n(1-p) are both greater than 5, even if the sample size is small. For very extreme probabilities, though, a sample size of 30 or more may still be inadequate and the approximation works at its worst when the sample proportion is exactly zero or exactly one.

From an educational perspective distinguishing between the binary opposites of certain-uncertain and possible-impossible is often located at the very introductory phases of a typical probability education. For example Van de Walle (2011), suggests that young children come to class with all sorts of bewildering ideas of probability, “to change these early misconceptions, a good place to begin is with a focus on possible and not possible and later impossible, possible, certain” (p. 474). Thus extreme probabilities are the type of events that the learners are familiar with since the very early grades.

Participants of the study

The participants of the study are 29 undergraduate students taking a mathematics education course in Simon Fraser University, Vancouver, with a diverse mathematical background including arts, social studies, biology, and computer science. However, all of them have taken an equivalent of an introductory probability and statistics course at some point before. They are asked to give examples (in writing) of events with 100% probability of happening. They produce 45 examples in total. The task is presented to them in written form and the time for answering has been unlimited.

Method of data analysis:

The Framework used to analyze the data is a tool for analyzing personal or collective example spaces based on (a) correctness, (b) richness, and (c) objective-subjective duality. The first two elements of this framework are adopted form Zazkis & Leikin (2008) the last part is borrowed form Gillies (2000) and Chernoff (2008).

Correctness: In the *correctness* category I consider whether the examples satisfy the condition of the task, which is fulfilling a 100% probability of happening from a reasonably acceptable mathematical point of view. There is an important decision to be made before we go through a discussion of correctness of data. If a student expresses a belief that there is a 100% chance that the next roll of a die will be six and to prove himself correct he rolls a die and a "six" shows up, is his assessment of the probability correct or not? It is while we do not have any knowledge of the die, it could be a fair die or it could be loaded to show six all the time. The same issue comes up in several examples from the participants: "there is a 100% chance that I will take the bus back home today" is this a correct example of a certain event or not? The student that has presented this example possesses certain knowledge of her transportation options and habits and perhaps she sees this as the only event in the sample space and perhaps she is right to assign a 100% probability to it.

The criteria for assessing the validity of the probabilities assigned to events is whether common sense (to be more specific: accounting for all of the possible scenarios/outcomes) and knowledge that is reasonably accessible to everyone is used and wherever applicable the background information necessary to make the judgment appeal to other people is presented or not. For instance in the bus example it is reasonable to take into account that a bus is a vehicle prone to accidents or general failure and it simply might break, so there is a chance however small that she might have to call a friend to give her a ride back home today. That marks this particular example incorrect and I have coded them as *lacking key information or common sense (Lack).*

Another group of examples that have been identified as incorrect are what I call examples of "non-random events". In this group of examples the participant holds a vision of 100% probability as a fact that no one can challenge or refute, and finds such facts from situations which are not subject to randomness at all simply because they either refer to events in the past or they deal with definitions. Among such examples are: "There is a 100% probability of me having the same color eyes as someone in this class because I see some people with the same eye color", "There is a 100% chance that tomorrow is Tuesday given that today is Monday", and "There is a 100% chance that I went to bed before 10 last night, because I did so". This group of examples is coded as "Non-random situations" (NR)

Table 1: correctness of the participants' examples (n=45) for events with 100% probability

Example of a certain event	Condition
Correct (n=21)	N/A
Incorrect (n=24)	Lack of key information or common sense (n=19) Non-random situation (n=5)

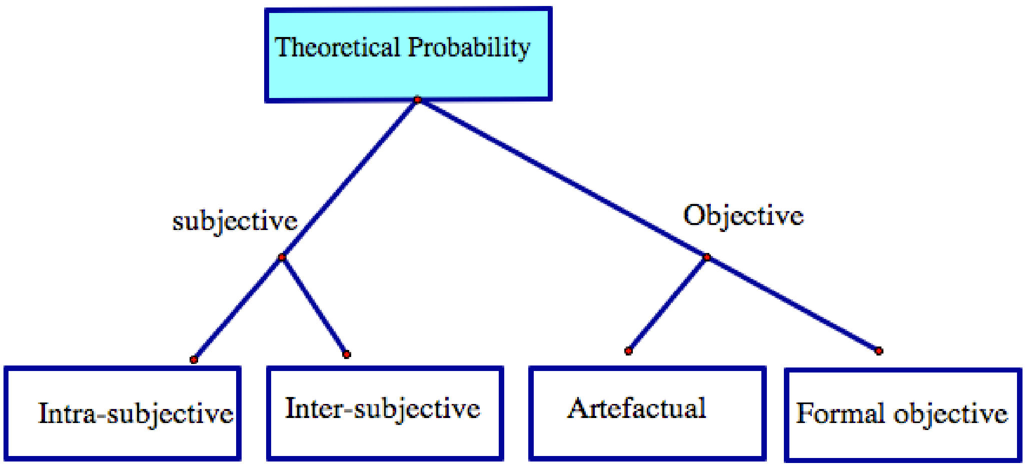
Richness: In richness category I consider the context from which the example is generated. Everyday experience and mathematical experience are the two main contexts that have been looked for. However it is not an exhaustive partitioning of the possible contexts for the examples and also the two are not mutually exclusive since mathematics both comes from (not all of it though) and is applied to the real life. In order to decide on whether the context of an example is mathematical or not, I have looked for evidence of combinatorial reasoning, meaningful use of numbers or standard randomizers such as coin, dice, spinners, urn of balls, etc. 16 examples are marked as including mathematical context and the other 29 examples are describing less mathematical, and more real life situations.

Table 2: richness and objective-subjective duality analysis of the examples (n=45)

Mathematical (n=16)	12 correct 4 incorrect	14 Artefactual Objective 2 Formal Objective 0 Inter-Subjective 0 Intra-subjective
Everyday life (n=29)	10 correct 19 incorrect	3 Artefactual Objective 0 Formal Objective

		16 Inter-Subjective 10 Intra-subjective

Subjective-Objective: In the next step, the examples are put into two main categories of objective and subjective inside which four refined categories of "formal objective", "artefactual", "inter-subjective", and "intra-subjective" are recognized.



These expressions are adopted from Gillies (2000) and used as informative and distinctive probability terminologies by Chernoff (2008); here is a very brief description of each:
Artefactuale: "probabilities can be considered as existing in the material world and so as being objective, but they are the result of interaction between humans and nature. Probabilities in coin tossing and other games of chance, as well as the probabilities associated with repeatable experiments in science, are artefactual", Gillies (p.171).
From the 45 examples provided by the participants, 17 examples were identified as referring to artefactual type of probability. Evidence of combinatorial reasoning (e.g. "10 red apples and 1 green apple in a basket, you are guaranteed with picking 1 red apples with 2 chances without replacement"), references to the games of chance (e.g. "Probability of getting heads or tails when tossing a coin is 100%"), and addressing statistical or scientific findings (e.g. "An earthquake here in BC in the next hundred years occurs with 100% chance. The experts have been predicting it for decades but no one knows when it will happen") are used as the main criteria for this group of examples.

Formal objective: Those events that are independent of humans—to the greatest extent that they can be, for example events related to the hypothetical problem of dropping a needle on number line and finding the probability of certain set of numbers being hit (rational numbers for instance) is divorced enough from the human context to be categorized with "formal objective".

Within the collected data there were only two incidents of formal objective type of probability:

1) “There is a 100% chance that in flip of a coin it is 50% probability that it flip heads”.

2) “The probability of rolling a 1 on a 6-sided die being 1/6”.

It could be argued that these examples are more of an artefactual type since they refer to the well-known facts. It is apt to distinguish between the mathematical facts and statistical facts and the mode of inquiry of those. I contend that if the participants consider “Probability of heads or tails are each 50%” as a result of experiment or as what statistics suggests, then these examples are merely artifacts and hence pertain to artefactual probability. But if we look at these facts as results of mathematical theorems (e.g. ($\lim_{n\to\infty}(p(\left|\frac{x}{n}-\frac{1}{2}\right|<\varepsilon))=1$ for the coin tossing experiment), then the two examples above are assigning an objective probability to an event which is far away enough from the human context to sit with Formal objective probability. This itself is a fascinating example of how the perspective of the person who is examining these examples (that would be me in this case) can affect on the probability stance of a single probability assessment.

Inter-subjective: probabilities that represent the degree of belief of a social group that has reached a consensus. In other words it includes probabilities that are assigned on a subjective basis but in the light of some evidence that are clear to a group of people. For example the probability of Sara taking an umbrella on a cloudy November day of Vancouver could very well be assigned on an inter-subjective way. The followings are two examples from the 16 examples identified as bearing indications of Inter-subjective probability. In these examples the probability proposed is perceived (by me) as containing no formal calculation, but close to what might be akin to the belief and knowledge of a group of people.

“There is a 100% chance of having three girls in a lecture room that contains 100 students”. This example is marked as incorrect because of the lack of accounting for a possible scenario, which is “an all boy class”, but nevertheless it tacitly referrers to the experience of students from their large classes in a typical university/college.
“I am 100% sure that most of the class is right-handed.” Once again this example is not referring to any statistical finding or a ratio of right-handed to the total, but reasonably it is believed that most of people are right handed and a class is an appropriate representative of the whole population with regard to this feature.

Intra-subjective: it is more of a personal belief-type of probability. The probability that I’ll take the bus tomorrow (and no further evidences or information provided) is put into this category. Examples include: “There is a 100% probability that I will take the bus home”, “Going to bed tonight”, and “It is 100% probable that at least 2 people in this class will be born in the same birth month”.

It is necessary to note that the four above-mentioned types of probabilities (formal objective, artefactual, inter-subjective, and intra-subjective) mostly describe the extremes and indicate the upper and lower bounds of the probability continuum. In total, 10 examples fit this category all of which are marked as "incorrect" in the first run of examining the examples. This brings us back to the issue of introducing subjective probabilities into the k-12 mathematics curriculum. When it comes to marking tests or using other forms of evaluation, we need to decide whether it is possible to develop a consistent criteria to mark the students' intra-subjective arguments or not. Or it could be the case that any intra-subjective probability assignment by definition carries a connotation of "wrong or insufficient explanation.

CONCLUDING REMARKS

The absence of 'expert' example space (as described in Zazkis & Leikin, 2008) that displays rich variety of expert knowledge is apparent. All of the mathematical examples obtained from the participants are textbook examples of sure events typically used by the teachers at the very first phases of the introduction of the notion. Examples related to basic coin tossing or die rolling events as well as statements such as "the sun will rise tomorrow" (which is yet another textbook example of certain events, p.474 Van de Walle). The later performance of the participants of this study on the probability test (not reflected in this paper) shows that they have a reasonably good grasp of laws of probability and that they have an above average performance with the probability related tasks and problems. Yet their examples of certain events don't reflect the same level of development and expertise. An overwhelming 26 out of 45 example referred to in this study proved to be subjective probability statements, pointing out the fact that the frequency and classical (objective) approaches to probability are less widely applicable than the belief interpretation. A person can hold beliefs about any event, but the frequency interpretation applies only when a well-defined experiment can be repeated and the ratio always converges to the same number. Many events for which we would like to have probabilities clearly do not have probabilities in the frequency and classic sense. For example consider the most frequently mentioned sure event: several participants presented the "I will die" example as an event with 100% probability of happening. Let's try to assign a classical probability to this event: we first need to define a sample space consisting of *equiprobable* events, count the number of events in which "I will die" and divide it by the total number of the events in the sample space. The inherent difficulty in doing so may lie in the idea that the sample space is either S={I will die, I will not die} or S={I will die}. The former is suffering from the absence of equiprobability and the latter is acceptable only if we have made up our mind (in an a-priori fashion) that nothing else is possible and thus the "I will die" event is the 100% sure event. This conceptual difficulty is not specific to the extreme probabilities, subjective aspects of making decisions about assigning or calculating mathematical probability remains the same all over the probability continuum, but they are more noticeable in the case of impossible and certain events.

I propose that students of probability at all levels need to experiment with probability tasks in which they are not only asked to calculate/assign the probability of an event but also they are encouraged to uncover and discuss the underlying assumptions that are made about the event in question and the knowledge of different individuals about the event. The dynamic process of taking in new information and adjusting the previously formed beliefs and judgments creates not only a bridge between frequency based and subjective probability measurements but also creates valuable opportunity for students to develop a new perspective on uncertainty.

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A POST-HUMANIST PERSPECTIVE ON A GEOMETRIC LEARNING SITUATION

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This research report presents a post-humanist approach to analysing a geometrical activity involving grade 9 students. In looking at students' practices in using mathematical tools in different contexts, this study considers the range of components involved in a learning situation, rather than focusing only on the learner, taking into consideration the student, the tool (the Geometer's Sketchpad) and mathematics, all of which can be considered to have influence or agency in such a learning environment. I use the construct of intra-acting agency to examine the relation between the components of the situation.

INTRODUCTION

Traditional perspectives on human practice are being challenged by researchers within a post-humanist paradigm (Barad, 2007; Sorensen, 2009; Malafouris, 2008). Post-humanists view the individual as important but not as the only "participant" or "agent." In contrast, many learning theories, like constructivism, focus on the individual as the main source of action and agency. Socio-cultural theories acknowledge the role of others in shaping an individual's actions, but are still principally about the human. Technology-based theories like instrumental genesis aim to understand the way in which tools affect human action, but still subordinate the tool to the epistemic subject. These anthropocentric perspectives position the subject as an external author; a post-humanist perspective adopts the idea that non-human elements can "participate" in various forms of practice.

In this study, the mathematical practice of a classroom of students will be considered. The focus will not be solely on the students, but on the intra-actions between subject, their tools and the mathematics. Agency will be granted to the non-human elements of this environment to help identify forms of activity. This is not a study of individual parts collected together but one of a mutual co-constitution of emerging agencies. The ultimate goal of this study is to show how this intra-action might look in a mathematics setting.

THEORETICAL FOUNDATION AND FRAMEWORK

A post-humanist perspective does not view learning as an individual achievement (Sorensen, p. 5). This challenges an anthropocentric perspective, which can be limiting in that it dismisses the physical world around us and how it shapes us. In her studies on quantum physics, philosopher and physicist, Karen Barad (2007), describes her observations of Bohr's work on particle physics indicating that actants become defined in the emergence of activity: "Objects are not already there; they emerge through specific practices" (p. 157). She uses the term intra-action, as opposed to interaction, so that the focus is on things emerging and not on capacities or attributes of things before they come together. I contend that there is a tendency to think that individuals are fully formed and stable but this perspective can lead to a focus on individual capabilities. But instead of focusing on what an individual is bringing to an interaction, I suggest that the question should be what kind of distributed activity occurs across the human and the non-human actants. The focus is not on pre-action nor on post-action, but on action itself.

In addition, Barad (2007) challenges the idea of analysing individuals or things outside of context. The very notion of identifying individuals or things distinctly involves creating divisions or boundaries. According to Barad, these cuts are arbitrary, subjective and continually shifting. For example, traditionally, to speak of an individual would typically include a person bounded by their skin. But if a blind person is using a walking cane to help navigate an environment, their "self" is clearly extended. The tip of the cane might be considered the extent of their "touch".

In her analysis of Bohr, she describes how concepts are dependent upon apparatus, or modes of observation: "Concepts, in Bohr's account, are not mere ideations but specific physical arrangements" (p. 54). Although Barad is using Bohr's model of observation in a context of quantum mechanics, I contend that the context is analogous to a learning environment for a mathematics student. In any educational context there are different arrangements of mathematical tools. I propose each has its own emerging outcomes and corresponding concepts, such that where an apparatus begins or ends is a matter of subjectivity. Certain arrangements bring forth different features, ways of looking at, or constraints of observation or action. The thinker or rational being needs to be redefined, not as an individual but as a subject immersed in activity intra-acting with other things: "Knowing is a matter of intra acting" (Barad, p. 149). Therefore, mathematical activity is considered to be an assemblage of human and non-human agencies.

Using Barad, agency is operationalized as a construct to identify methodologically what emerges from the intra-action of a student with a mathematical tool or concept. Agency has traditionally been conceptualized as a human capacity but many researchers now see it as emerging from intra-action, thereby granting non-humans the ability to act (Malafouris, 2008). Barad states that agency is not an attribute but the ongoing reconfigurings of the world (p. 141). Agency can be thought of as an action or a doing. Intention is not synonymous with agency, for otherwise it becomes a human-centered construct.

Although Pickering's model is based within a humanist paradigm, his perfomative idiom is helpful in identifying assemblages of agency. In his study of practitioners in science studies, Pickering highlights the cycle of resistance and accommodation, which occurs in scientific work with machines. Within an education setting, this model may be analogous to a student using, say, a dynamic geometry software (DGS). In any task, the DGS may provide a resistance or a challenge, and the student will then need to accommodate their action to overcome this challenge. Although Pickering focuses on the individual, we can name the resistance as a material agency. The material, non-human element imposes a restriction upon the user. Further, a DGS may extend possibilities or distribute activity of the person, as the walking stick had done for the blind individual.

As Sorenson posits, "…to decenter we can still emphasize the individual" (Sorensen, p. 57). Given the setting of this study, I have chosen to emphasise the individual by introducing the notion of self-agency. The human is an exceptional figure and how she acts can be acknowledged so as to keep analysis clear. Self-agency is the degree of agency a person has, when using an "I" voice such as "I am driving this car" they are enacting a self-agency. According to Knox (2011), a developmental psychologist, self-agency is necessary to development; I contend this development of self-agency parallels Barad's idea of becoming.

Providing students opportunities to act, they come to see themselves as participants, which may lead them to experience self-agency. Opportunities for self-agency do not necessarily evoke self-agency, nor is self-agency guaranteed or even linear. What is important here is that what results from exercising self-agency is a "sense" of agency. An individual may or may not have a sense of agency in a particular context. This is an important feature of this study because one must have a sense of agency in order to participate in a performative idiom (Pickering, 1995).

The question of this study is based on a change of physical arrangement. A geometry activity is observed in two different contexts. The first involves a traditional classroom; the second includes a newly introduced digital tool. In observing the two contexts, I identify significant changes in the students (their self- and sense of agency), their practices (actions) and the resulting mathematics. These actants are in the process of becoming. The mathematics adopted in this study is a discipline of negotiation, conjectures and exploration, not one of infallibility. In this study, I have chosen to focus more on the co-constitution of student and the tool, leaving their co-constitution with mathematics for another study.

METHODOLOGY

The theoretical framing of this study demands close attention to the back and forth and integrated intra-action of the student using tools in a mathematical activity. Attention to discourse, written or verbal, provides the means by which I identify activity. I use James' (1983) distinction of the "I" voice as expressions of self-agency and his distinction of the "me" voice as the objective self, as that which is being acted upon. I will use these distinctions of voice to identify resistance and extensions. These will be examples of material agency. Student discourse will be a major source of identifying intra-action between themselves and the software.

RESEARCH ACTIVITY AND PARTICIPANTS

The data for this study was collected in a Vancouver high school in a grade 9 (14 years old) classroom during a geometry unit. Mathematics 9 in British Columbia has an extensive geometry component that involves rotations, symmetries, circle properties as well as coordinate geometry. Students had last worked explicitly with triangles and squares in grade 6.

The teacher introduced a two-phase activity based on what he had done in previous years. I requested a third phase. The teacher's two phases of instruction corresponded to my interest in looking at different practices and using different tools in different environments. In phase one, the teacher drew (freehand) what looked like a triangle and a square on the whiteboard for all of the class to see. He requested that students try to identify how they might determine whether these geometrical figures were, in fact, as claimed, a triangle and a square. Students worked in pairs to encourage discussion and wrote their responses. For phase two, the teacher took all the students to a computer lab, sat them in pairs and requested the students use *The Geometer's Sketchpad* (GSP) (Jackiw, 1988) to construct both a triangle and a square. During phase two, the teacher allowed students to explore the software's environment, as this was the first time the students had used the program. He also went around and gave guidance and support by approaching pairs of students who seemed to be having difficulty or who were asking questions. In addition, he challenged student "constructions" to see if dragging would break them. Although the triangle was constructed by almost all students, the square provided more of a challenge. Students most commonly "fit" four segments together, but when the teacher dragged one of the vertices of the "almost-square" (Figure 1), the "square" would morph into another shape. Students were given more time to try to construct the square over the course of the 80-minute class in the computer lab. For phase three, the teacher brought all the students back to the classroom and requested that they again write, in pairs, how they would determine whether a given figure is a square. The researcher was present during all three phases; he also interacted with students, lent support and "challenged" their constructions.

All written work for phases one and three were collected and analyzed. Data from the computer lab was collected by using SMRecorder, which records all the digital activity on the screen as well as verbal utterances of students.

ANALYSIS

The analysis of the data is based on identifying examples of changes in students' conception of themselves or of the mathematics. The majority of data in this study is based on data from the computer lab because this is where the agency emerged and made itself known. I identify examples of both self-agency and material agency in working with GSP. I also contrast the transition from phase one to phase three identifying significant changes in students' conceptions of geometrical shapes. I then present four examples rich in intra-action and agency.

In phase one, students written work in the classroom, almost exclusively, listed properties of the geometrical shapes. Their conception of these geometrical shapes was based on properties. Although the figures drawn at the front of the room did not have these properties (they were drawn freehand), the students discussed, recalled, using the diagrams to guide their memories of grade 6 geometry. In all of the written work there was no reference to the "I" voice, nor were there references to the shapes as imposing themselves in any way. It is relatively clear, in this activity, where the boundaries were drawn. The mathematics was represented on the whiteboard, and the students were the subjects expected to absorb or recall the knowledge.

In the computer lab, the proposed activity supported a process of exploration which in turn actualized enactments of agency. For example, in constructing an "almost-square" (Figure 1), multiple pairs of student could not get the lengths of the sides to equal. One way to deal with this was to draw one segment and copy and paste three more. This was a good idea (although this still did not "construct" a square), for the segments were all the same length, but the lengths did not remain constant under dragging, as Laura found out.

Laura: ohhhhh, how come it changes length?

This back and forth attempt to make the square is an example of Pickering's model of resistance and accommodation.

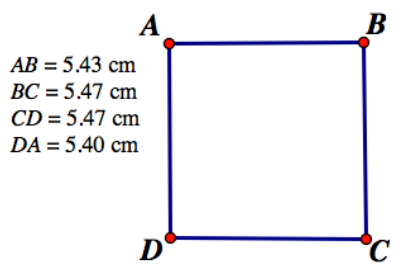

Figure 1: An "almost-square"

There were multiple examples of self agency in the computer lab that were evident as the students worked on the task:

Ricardo: I want to see what moving this will do.

Alice: I want to know what happens when I try this…

There were also examples of resistance, where the software did not do what the student expected:

Mitchel: It won't let me drag the point.

Heather: How come this part is not moving?

The transition between phase one and three is significant. Data from the classroom after the intra-action with GSP, in contrast with the phase one activity, was distinct in that the conceptions of squares and triangles were different. In general, their descriptions of the square from phase three included new vocabulary, new metaphors and new forms of engagement. In their written activity new words were used such as: pull, put all, flip, adjusted, drag, copy and paste, angle and locked.

The following four examples were chosen because they were rich in intra-action and agency. The first three are occurrences from the computer lab. The fourth example was an occurrence in the classroom during phase three.

Justin and David constructed a triangle and then translated it partly off the screen and the question "Is this a triangle?" was posed.

Justin: Is this a triangle?

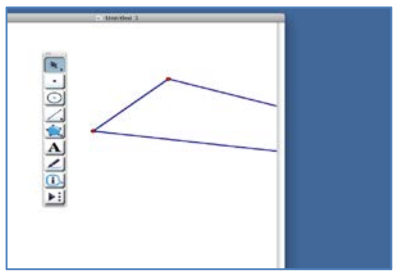

Figure 2: Justin's triangle

In this particular activity we see an example of students generating new questions; there were new opportunities for negotiation. Unlike a drawing of a triangle on a whiteboard or a sheet of paper, this triangle was initially fully visible and then translated off the screen. The limitation of the screen became negotiable due to the intra-action of the student and screen agencies. The agency of the screen limits visibility but also the tools allows for easy access to translate the triangle back. The boundaries of the triangle are challenged. The students seem to be the ones asking the question, but the screen and the triangle occasion this situation. Justin's half triangle is an example of the relationship between humans and negotiation, a challenge not available without the tool. Justin challenges the perspective of the student and introduces the question of where the mathematics lives. Does it exist off the screen?

Also in the computer lab, another pair of students, Luna and Michel, described to the teacher how they constructed the square using the grid option in GSP. They thought they had constructed a perfectly good square (Figure 3). Most other students were getting their square pulled apart by the teacher, but Luna and Mishel were confident that their square would hold up since it lined up with the coordinate grid. The teacher, however, changed the scale on the grid and the square became a rectangle (Figure 4). They did not try to figure out another way to construct the square; instead they based their construction on the limitation of not being able to change the scale.

Luna: This created a 1x1 square and no matter how you move the point, it stays a square – unless you change the grid.

As long as someone did not change the scale, the square that they had made was a square. Luna and Michel's definition of their virtual square illustrates an assemblage of human and non-human forms for they based their definition on a particular situation in GSP which included software, agency and mathematics. The definition held all components together.

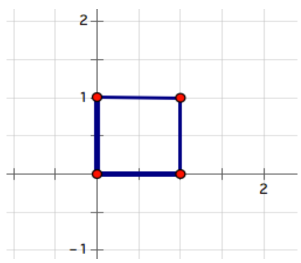

Figure 3: Luna's square

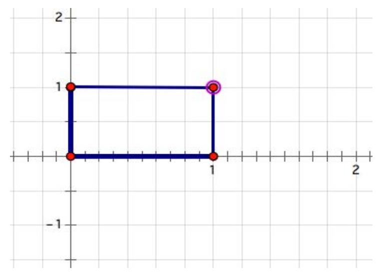

Figure 4: Luna's rectangle

According to James the diagram in Figure 5 is not a triangle. James discussed, with the teacher, how GSP expected endpoints to be connected properly otherwise segments could be dragged away from each other and the shape did not retain invariant features. James challenged the idea of endpoints and intersections. A new way of categorizing intersections was introduced; intersections did not become "points" without self-agency and the tool.

James: The four sides must be touching but not intersecting.

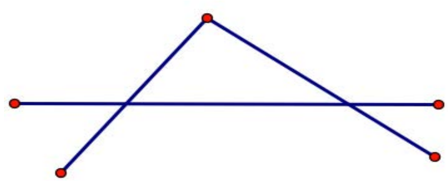

Figure 5: James' non-triangle

The last example draws from phase three. One male student said the following while explaining to the teacher what a square was.

Leo: It has four corners, 90 degree angle, four equal sides, has 360 degrees. You can move it around it is still 360 degrees. The four points are attaching perfectly so you can move it around.

When the teacher asked him what it meant to move it, he moved his hands around in the air as if he was turning a steering wheel. The mathematics was changing because the object had changed definition – it had become accessible and he had developed a sense of agency with it in that he knew he could move a square and it would hold its invariance. With the tool, the square became available for empirical challenge, thus radically affecting student's acceptance to what a square was.

DISCUSSION AND CONCLUSION

In the computer lab, the students used *Sketchpad* to test whether a shape is a square. The shape became a figure to move around, push; an object with hinges. But a student needs a sense of agency to begin the enactment and a self-agency to endorse the square. Without the ability to flip, move, drag, the determination of whether the figure is a square is not possible. Only in the combination of invariance and movement could a square be actualized. The boundaries in such an intra-action are difficult to identify. In the classroom, boundaries were easy to identify but with Sketchpad, possibilities were enhanced, for the students were doing things with squares and triangles that they had not conceived. Dragging the triangle off the screen, challenging its existence outside of perception was something not possible in the classroom. Moving his hands in the air, Leo's sense of agency is actively trying to access the square. The possibilities of engagement were extended for the square did not exist without intra-action. Otherwise there would be no way to determine the difference between an "almost-square" and a proper square. The square depends on the student to act and the student depends on the tool to act and the boundaries of agency continually shift.

If we are to accept Bohr's statement that concepts are physical arrangements we should consider that Sketchpad is such an arrangement. Thus, the concept of a triangle is different than its representation on the whiteboard. The concept of a triangle is not based on properties of a transcendental platonic geometrical figure but an actualized digital form that necessitates student engagement. In phase three when students were describing the triangle in terms of gestures, new words, and new metaphors, the tool did not just draw attention to different aspects of the triangle but reconceptualised the triangle. As de Freitas and Sinclair (2012) write: "A concept of this kind, with logical and ontological functions … resists reification while carving out new mathematical entities and forming new material assemblages with learners" (p. 12).

This study troubles existing, humanist assumptions about the role of tools. If the tool can alter the way we look at simple geometrical figures as well as the way we look at our own involvement in mathematical activities, both the way digital tools are designed as well as the way they are presented can have very important effects on our mathematical experiences.

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QUALITATIVE LEARNER PROFILING USING BASIC CONCEPTS OF ELEMENTARY NUMBER THEORY

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The objective of this study is to look in depth into personal factors affecting metacognitive monitoring and control in self-regulated study and restudy of basic concepts of elementary number theory. By incorporating a wide spectrum of observational methods such as behavioural and physiological, and self-reporting techniques and demographics of the participants, I aim gaining deeper insights into personal factors implicated in learners studying a mathematical text. My ultimate objective is to provide "learner profiles" with the help of these qualitative tools that can be used to better inform assessment and tailor instructional design in mathematics education.

OBJECTIVES AND PROPOSES

My focus here on study and restudy of basic concepts of elementary number theory that include the division theorem, divisibility, divisibility rules, factors, divisors, multiples, and prime decomposition (Campbell & Zazkis, 2002; Zazkis & Campbell, 2006; Campbell, Cimen, & Handscomb, 2009). My aim is to gain deeper insights into personal factors affecting study and restudy of this material, interjected with self-reports of judgments of learning (Nelson, Dunlosky, Graf, & Narens, 1994), and eventually to generate learner profiles.

THEORETICAL FRAMEWORK

Learner profiling is a method used by researchers especially in the field of education for grouping learners (who are the subjects of a study) based on their specific characteristics or behaviours, such as their cognitive thinking, motivation orientations, abilities, strategies, or levels of stress. Learner profiling in quantitative studies are done based on determining and extracting these types of components out of the data using statistical methods such as cluster analysis (Alexander & Murphy, 1999; Csizer & Dornyei, 2005). In qualitative studies, clustering learners are more depended on researchers' observations with lenses of a framework, such as 2x2 achievement-goal theory (Elliot and McGregor, 2001). In such studies, learner profiles are generally more detailed and personal, not only defined by a set of characteristics of the cluster they belong. This qualitative case study focuses on participants' motivation orientations based on Elliot & McGregor's (2001) framework with the help of eye-tracking methods to have insight on participants' cognitive abilities (calculation, understanding and reasoning), in combination with their levels of stress based on heart rate and respiration data those are used in some contemporary educational research (Campbell, S. R. with the ENL Group, 2007).

It is important to state that this study neither intent to make general claims, such as describing all characteristics pertaining to a specific learner profile, nor tries to identify all possible profiles exist when a learner confronts with a math text. Rather this study aims providing a better insight into possible learner profiles and possible characteristics of such learners while studying and restudying math text, using a theoretical framework (2x2 Achievement-Goal Framework) and methods such as eye-tracking, heart rate and respiration analysis.

METHODOLOGY

I have chosen to focus on some basic concepts of elementary number theory for a number of reasons. First, I hold the view that concepts of elementary number theory, especially with regard to division and divisibility, have a natural role to play in helping elementary and middle school students make the transition from arithmetic to algebra (Campbell, 2001).

The instrument for investigating metacognitive monitoring and control of study-restudy of basic concepts from elementary number theory, comprising six pages of subject matter content delivered using gStudy (Perry & Winne, 2006). This subject matter content for study-restudy was specifically designed to involve three levels of learning: the first involving computation (C), the second involving understanding (U), and the third involving reasoning (R). The participant was allowed to study this material at her leisure. The study material was then presented to my participant in a manner that highlighted different parts thereof (Figure 1), enabling her to provide a judgment regarding her learning (JOL), i.e., whether she understood that content very well, well, or not well. Once this was done, she was given an opportunity to restudy the material in preparation for a test.

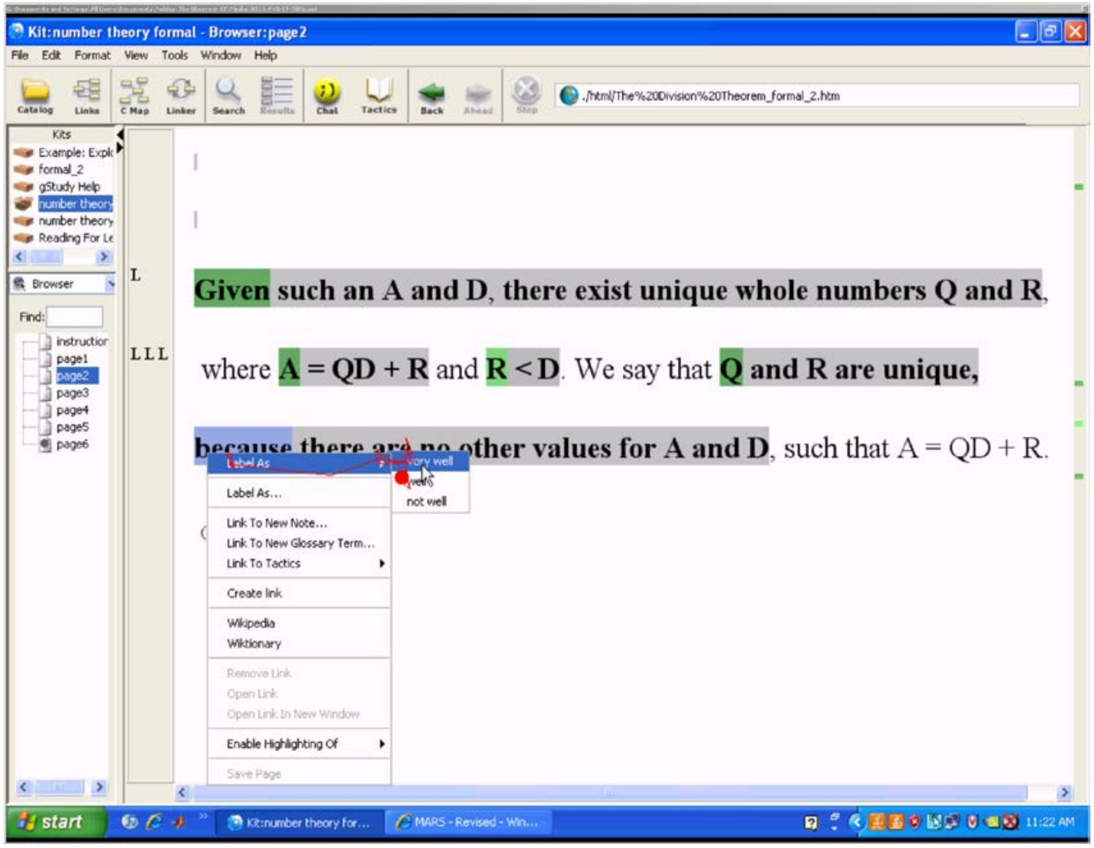

Figure 1: Screen capture of Page 2 of study material with participant indicating JOL

The model for interpreting the data on metacognitive monitoring and control is an adaptation of Elliot (1999) and Elliot and McGregor's (2001) motivational distinctions between mastery-performance and approach-avoidance, fused with Nelson, et al's (1994) notion of self-reported judgments of learning (JOLs) resulting from metacognitive monitoring (Figure 2).

	Mastery / intrinsic motivation	**Performance / extrinsic motivation**
Approach / taking time	JOL: not well understood	JOL: very well understood
Avoidance / not taking time	JOL: very well understood	JOL: not well understood

Table 1: Metacognitive monitoring and control model for interpreting motivation in restudy

According to my model, I interpret mastery-approach represents taking time in restudy to learn something for its own sake judged to be not well understood, whereas mastery-avoidance represents not taking time for restudy of content judged to be well understood. Performance-approach serves to better consolidate content judged as well understood, whereas performance-avoidance represents not taking additional time restudying content considered poorly understood. Accordingly, mastery and performance represent intrinsic and extrinsic motivation respectively, and approach and avoidance represent time allocated to study-restudy.

DATA SOURCES AND EVIDENCE

Behavioral data

The participant was wired up to monitor fluctuations in heart and respiration rates. She was presented the gStudy stimulus using a Tobii 1750 eye-tracking monitor, which detects reflections of infrared light pulses on a participants' retina to precisely trace what is being looked at from moment to moment. An ultra sensitive microphone allowed for highly sensitive recordings of think-aloud narratives. Infrared video cameras record important aspects of the participant behavior, such as facial expressions and body movements, from three vantage points. Several steps were taken to maximize the accuracy of eye tracking data of the study-restudy material such as increasing font size and spacing of the study material. Data streams were integrated; time synchronized and analyzed using Noldus's Observer XT (Figure 3). The data was cross calibrated and synchronized with the help of audiovisual, eye-tracking to ensure that the behavioral data for analysis was coded at the appropriate times (Campbell & the ENL Group, 2007).

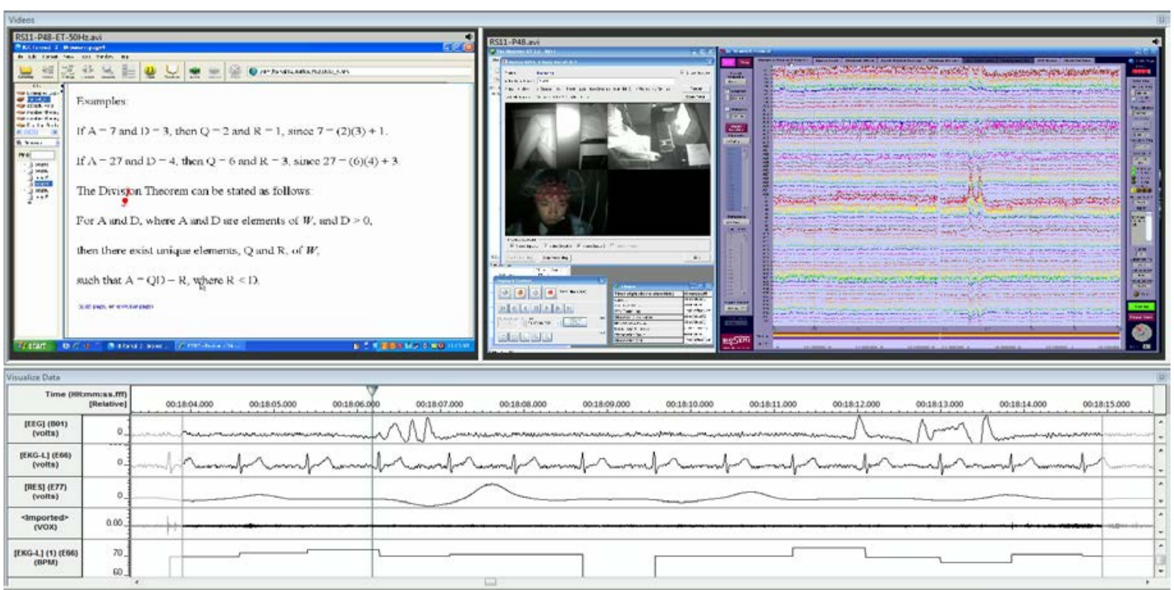

Figure 5: The integrated and synchronized data set using Noldus's Observer XT

Self-report data

The participant was given informed consent. She filled out a demographic questionnaire. Pre- and post-questionnaires were used prior to and after engaging the participant in the study-restudy activity. Pre-questionnaires, I do not go into detail here, included the Motivated Strategies for Learning Questionnaire (MSLQ) (Duncan & McKeachie, 2005), the Epistemic Belief Inventory (EBI) (Schraw, Bendixen & Dunkle, 2002), the Metacognitive Awareness Inventory (MAI) (Schraw & Dennison, 1994), the Math Anxiety Rating Scales (MARS) (Hopko, 2003).

A pre-questionnaire designed to gain insight into how comfortable the participant was with their abilities regarding calculation, reading, recall, comprehension, and reasoning. After completing the pre-questionnaires, the participant engaged upon the study component of the experiment, following completion of this initial study period, the participant labeled their judgments of learning (JOLs) pertaining to how well she learned computational, conceptual, and inferential aspects of the study material. After labeling the JOLs, the participant was given a 10-question true/false test on the study material and asked to rate her confidence in her answers on a scale of 0-10. Following a short rest, the participant engaged in restudy of the material, and then rewrote the same test. Finally, the participant filled out a metacognitive post-experiment questionnaire pertaining to her experiences in the experiment.

Participant

The participant was a 22 years old Female undergraduate student (in Molecular Biology) with Vietnamese background. Her overall health was self-reported as good (No anxiety disorders of symptoms, no physical problems). After the observation she reported being "a little worried that it was going to be hardcore math theory that was being tested on the exam part" before the observation (The study also involves data from other three participants which are currently being analyzed).

RESULTS

The participant's average heart rate for the study period was ~75.1 beats per minute (bpm), reduced to ~69.0 bpm for the self-report period, and reduced further to ~67.0 bpm for the restudy period of the same subject content material. Her respiration rates were ~20.3, ~18.0 and ~17.8 breaths per minute for the study, self-report and restudy periods, respectively, while her respective eye blink rate over those three time periods were 37.5, 16.0, and 34.3 blinks per minute. These values are summarized in Table 1.

	Time Spent (seconds)	Heart Rate (beats per minute)	Respiration rate (breaths per minute)	Eye Blink Rate (blinks per minute)
Study	608	75.1	20.3	37.5
Self-Report	278	69.0	18.0	16.0
Restudy	98	67.0	17.8	34.3

Table 2: Time and physiological data summary for study, self-report, and restudy periods

Considering that heart rate is a strong indicator for the level of stress and anxiety (Kelly, 1980; Dew, Galassi, & Galassi, 1984); the results clearly indicate that the participant was less anxious, i.e., more relaxed, for the restudy period, in comparison with the study period.

During the self-report period, the participant was re-shown the six pages of study material with items highlighted and she was asked to report her judgment of learning (JOL) regarding them (35 in total). She was asked to choose among three options per case for her self-reporting: not well, well and very well (Figure 1). I substituted scores of -1 for not well, 0 for well, and +1 for very well. I then tallied this scoring to give us a total JOL confidence indicator of +11.

All the JOLs labeled by the participant as "not well" learned involved calculations, and my data indicates she did not spend much time on these tasks. Hence, in accord with Table 1, the participant can be classified as having a performance-avoidance orientation in this regard. The participant reported she learned most of the understanding tasks very well, while reporting most reasoning tasks she had learned well or very well.

Question	Question Type	Test 1 Results	Test 1 Confidence	Test 2 Results	Test 2 Confidence	NTPreQ
1	Calculation	Incorrect	7	Correct	8	3
2	Calculation	Correct	10	Correct	10	3
3	Understanding	Correct	10	Incorrect	10	4
4	Understanding	Correct	10	Correct	10	4
5	Reasoning	Incorrect	9	Correct	10	3
6	Reasoning	Incorrect	10	Incorrect	10	4

Table 3: Number theory test and NTPreQ results

Table 2 summarizes the results from the test that was administered after the study period with the results from the same test, which was administered once again after the restudy period. These results align well with results of her self-assessment from the Number Theory pre-questionnaire. She reports her level of comfort on a scale of 1 (not comfortable at all) to 5 (completely comfortable), with calculation tasks as 3, while reporting her level of comfort with understanding involving recall and comprehension as 4, and with aspects of reasoning as 3.5. Test results substantiate these reports, reiterating she is less confident with her answers with calculation tasks compared to understanding and reasoning tasks. Although she reports a higher confidence for reasoning tasks, she is less successful on this type of task compared to understanding, which she self-reported prior to the study/restudy periods as being most comfortable with. Again, my JOL indicates she spent less time on the pages that involved calculation.

Another interesting result was the answers provided for Number Theory Post-Questionnaire, which was directed to her after restudying the material. She stated that learning the task was not interesting for her (ranked 0 out of 7) and it was not challenging for her (ranked 2 out of 7). She indicated that, she restudied the items she found most difficult to understand. These answers indicate in this regard that she is a mastery-oriented learner when it comes to subject content involving understanding and reasoning.

DISCUSSION AND CONCLUSIONS

Based on the comparison of heart rate for the self-report and restudy periods, the results of this specific study indicate that reporting JOLs, might help reduce the level of anxiety. The self-report data substantiates itself and the behavioral data substantiates itself. These preliminary results, which will be soon expanded to data from four participants, evidences how qualitative learner profiling using contemporary methods can help researchers in mathematics education to gain much better understanding of learners' cognition and attitudes towards mathematics. Results from similar future studies can also help other players in education system, such as teachers and policy makers, in instruction, lesson planning, designing curriculum materials, and making policies in education system with deeper understanding of the learners themselves.

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EXAMINING CONSTRUCTS OF STATISTICAL VARIABILITY THROUGH A SEMIOTIC MEDIATION LENS.

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This paper reports on the way students work with notions of statistical variability in a dynamic computer-based learning environment. Taking a semiotic mediation perspective, I explored introductory statistics students' understanding of variability as expressed through spoken word, gestures, and inscriptions. Participants interacted with dynamic graphs designed with the aim of making more explicit the notion of variability. Based on the analysis of the changes in their multimodal communication, I argue that the use of dynamic mathematics environments can help promote a more physical and temporal understanding of statistical variability.

INTRODUCTION

Statistical variability or variation is a key concept in introductory statistics and has attracted a lot of attention from statistics educators and mathematicians since the1990s. The concept of variability is considered as a foundation of statistical thinking (Wild & Pfannkuch, 1999; 2004). Variability is also emphasized in statistics curriculum documents (e.g. GAISE, 2005) as a key concept in developing students' statistical reasoning, thinking and literacy (Cobb, 1992; Garfield &Ben-Zvi, 2008). Despite its importance, variability has attracted fewer research studies at the tertiary level than at the school level.

Studies carried out at the undergraduate introductory statistics courses reveal that variability is not well understood by students (e.g. delMas & Liu, 2005; Reading & Reid, 2005, 2006). Reading & Reid (2005) designed a sequence of teaching strategies with assessment activities (short quizzes, assignment, and test) that enabled them to develop a structure of students' initial understanding of variation. They analysed students' responses to the assessment activities and developed four hierarchies of consideration of variation that include no consideration, weak, developing, and strong consideration of variation (Reid & Reading, 2008). According to them, the four levels can be used to assess learning activities in the statistics curriculum. Although Reid et al.'s (2008) consideration of variation hierarchies were developed in a specific learning environment with a relatively small sample of students, they provide helpful ways of assessing consideration of variability in introductory statistics.

In another study, delMas et al. (2005) investigated tertiary students' ability to coordinate how the mean and standard deviation varied in different distributions represented by pairs of histograms. They report that most students used a rule-based approach to compare variability across distributions instead of reasoning from a conceptual representation of the standard deviation. Moreover, students' explanations during the testing phase were often focused on finding a single feature between distributions rather than recognizing how data were distributed around the mean. By rule-based approach, delMas et al. (2005) meant that students relied on some few patterns that they observed in the distributions and used them to generalize for other distributions which were different. The authors recommended changing the design of the learning tasks so that students can explore the relationships between the mean and standard deviation rather than focusing on single values of the mean. The current study builds on that work by re-designing the graphs and using them to report on the way tertiary students consider notions of variability in a dynamic computer-based learning environment. The study is different than delMas et al.'s (ibid) in that I design the tasks in a dynamic geometry environment (DGE). Secondly, I take semiotic mediation perspective to inform the study.

THEORETICAL PERSPECTIVE

Given that dynamic computer- based tools are used by students to explore notions of variability, I take a Vygotskian socio-cultural framework, which considers artefacts, (including modern computer technologies) as products of human activity that can play a role in cognitive development (Vygotsky, 1978). Vygotsky (ibid) assumes a relation between practical tools and symbolic tools as instruments of semiotic mediation:

> […] the use of signs as means of solving a given psychological problem (to remember, compare something, report, choose, and so on) is analogous to the invention and use of tools in one psychological respect. The signs act as an instrument of psychological activity in a manner analogous to the role of a tool in labour (p.52).

Human activity with an artefact produces different signs (e.g., speech, gestures, drawings, inscriptions) that for instance, the student can draw on to develop mathematical meanings (Bartolini Bussi & Mariotti, 2008; Falcade & Laborde, 2007; Healy and Sinclair, 2007). For example in a dynamic geometry environment, a student may use a dragging tool to move data points from position A to B. The movement produces a system of signs that he/she can use to develop meaning of variability. According to Falcade at al. (2007) a specific tool can be considered as an *instrument of semiotic mediation* (original emphasis) if the user gains an awareness and develops new understanding from the physical action on the tool. The new meanings can evolve for instance in a classroom discussion facilitated by the teacher (Bartolini Bussi & Mariotti, 2008) into stable mathematical meanings. These perspectives informed the design of the learning tasks that are summarized in the methodology.

METHODOLOGY

Design of graphs

With semiotic mediation framework in mind, I designed two dynamic sketches using The Geometer's Sketchpad (Jackiw, 1989). I chose Geometer's Sketchpad (GSP) software because of its flexibility and its Dragging tool that can support learning concepts through student participation. GSP's graphic and arithmetic representations also support visualization of abstract concepts such as distribution, mean and standard deviation.

The first sketch (below Figure 1a— which I call the dynamic mean and standard deviation —*dyMS Sketch*) illustrates the variability of the mean and standard deviation with the variability of a data set on the number line. Six draggable data points B, A, D, E, F, and C are shown on the horizontal line. As the points are dragged on the horizontal line, the squares vary in size, and the arithmetical values of the mean and standard deviation also change. The design of the dyMS Sketch uses statistical principle of data variation from the centre. Only six data points were used to simplify the tasks but the design allows for adding in more data points. Moreover, the focus of the design was on demonstrating how the mean and standard deviation dynamically vary with varying data distributions.

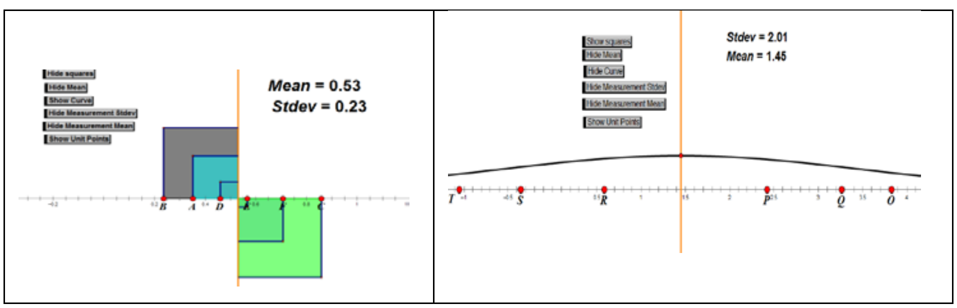

Figure1a. dyMS Sketch   b. gC Sketch

The second sketch (Figure 1b that I called the Gaussian curve – *gC Sketch*) depicts the variability of the mean, standard deviation and the Gaussian curve[1]. As shown in Figure 1b, the curve near the horizontal line is designed such that its peak rises or falls as the data points are dragged on the horizontal line closer to, or away from the mean line. Apart from introducing the Gaussian curve, all features of Figure 1a are available in Figure 1b, but hidden. The gC Sketch extends studies on variability of the mean and standard deviation in descriptive statistics into some elements of variability in inferential statistics.

1. Gaussian distribution or Normal distribution is a continuous probability distribution with a density function (Gaussian function) given by:

$$\xi(x,\mu,\sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

The parameter μ is the *mean* (the peak of the Gaussian curve) and σ^2 is the variance (the square of the average distance from the mean). σ is the standard deviation (the average distance from the mean). The Gaussian distribution with $\mu = 0$ and $\sigma^2 = 1$ is the standard normal distribution.

The interview tasks

I adapted Falcade & Laborde's (2007) method of analysing variation tasks in a dynamic environment which enabled me to consider variability i) as relation between different signs (numerical signs, squares, curve etc.) all of them linked to the other in the dynamic environment; ii) I considered movement—the change in space according to change in time—to constitute the general notion of variability, and iii) The dynamic geometry environment (DGE) provided space and time within which participants could experience the notion of variability.

Using Figure 1a, (with the arithmetical values of the mean and standard deviation first hidden), participants were asked to *predict* how the mean, standard deviation, and the squares would *behave* as they dragged the data points on the horizontal line away from or toward the mean line. After making their predictions, participants were asked to check using the Dragging tool. In the second task that was based on the gC sketch (Figure 1b), participants were asked to predict how the mean, standard deviation and the curve peak would change as data points moved on the horizontal line closer to the mean or away from the mean line. Then they were asked to use the dragging tool and check their predictions. All actions, speech and bodily movements by participants were videotaped and transcribed for analysis.

Participants

Eight undergraduate students (4 male and 4 female) who were enrolled in introductory statistics in a North-western Canadian University volunteered and were interviewed. I conducted one -on-one tasked based clinical interviews at the end of the semester when participants had covered all the topics in their statistics courses. Out of the eight, three participants whose data were considered too brief or unrelated to the topic of variability were not included in the analysis. Of the five participants (2 male, 3 female) that were included in the study report, two (1 male, 1 female) were in their second year of Actuarial science course and the other three were in their third year in the health sciences program. The two participants from Actuarial science had stronger background in statistics than others. In this paper, I present partial results for two participants: Anita from health sciences and Boris from Actuarial science (the names are not real ones).

RESULTS

The tasks presented below relate to above Figure 1a. Anita and Boris were asked to predict and later check prediction how the mean, standard deviation and the squares would vary if the data points were dragged on the horizontal line.

1 I: "… can you predict what the squares, the mean and standard deviation will be like as you move any of the data points along the horizontal axis?"

2 Anita: "… If I move it [point B] away from the centre … the line of the square will move this way toward this other line, so it would get a bit bigger…"

When Anita said, "If I move [point B] away from the centre ..." she pointed to the left side of the mean line with her right index finger (Figure 2a) to indicate the direction of movement of the square and she predicted that the square "would get bigger."

3 Boris: If you move [data points] away from the centre, the square is getting bigger and bigger because the square is the distance from the centre.

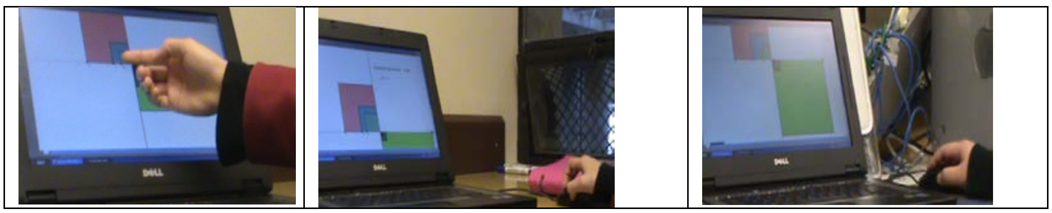

Figure 2 **a**. Move this way **b**. Away from the centre **c.** The boxes are just scales

Boris' prediction was similar to Anita's, except that he assigned the moving of the points to someone else by saying, "*if you move away […] the square is getting bigger and bigger […]*" [line 3]. Boris also depicted the continuity of the square "getting bigger and bigger" as data points were moved away from the centre. He was more specific about the size of the squares "getting bigger and bigger" [line 3] than Anita who predicted that the square "would get a bit bigger" [line 2]. I conjectured Anita was less sure of her prediction than Boris although none of them mentioned changes in the mean and the standard deviation. I asked Anita to check her prediction using the dragging tool.

4 Anita: "The mean gets smaller; standard deviation gets larger."

5 I: "… Move it again and see how the mean and standard deviation are changing."

6 Anita: "Oh, so both of them got larger yeah ok, so I thought the mean would get smaller and the standard deviation would get larger but actually both of them are increasing" (Figure 2b).

When Anita said, "*The mean gets smaller, standard deviation gets larger*" [4], she had used the dragging briefly and then stopped. I asked her to do it again (line 5), and when she dragged one data point farther to the right side (Figure. 2b) of the mean line, she said, "*oh, so both of them got larger …*" [line 6]. Anita's surprise reaction "*oh, so both of them* *got large*" suggests that she saw something different than she had predicted. Boris checked his prediction but his reaction was different than Anita's:

7 Boris: "Yeah, yeah, they are getting bigger, the boxes are just scales, and they are actually just scales to see how the relative differences are. That was interesting" (Figure 2c).

Boris had predicted that the squares would get larger because "the square is the distance from the distance from the mean." His response "yeah, yeah, they are getting bigger" after checking seemed to confirm his prediction. He called the squares the "boxes are just scales …to see how the relative differences are" [line 9]. What Boris probably meant was that the size of the squares indicated how far away the corresponding data points were from the mean. A smaller square showed that the data corresponding data point was nearer to the centre than a bigger square.

DISCUSSION

Anita's statement [see line 6] *"Oh, so both of them got larger … I thought the mean would get smaller, but actually both of them are increasing"* suggests that she gained some new understanding of the variability of data after using the dragging tool to check her prediction. She said, "*I thought the mean would get smaller, but actually both of them are increasing."* The dragging was in that sense was transformed into an instrument of semiotic mediation that brought about a new realization for Anita (Bartolini Bussi & Mariotti, 2008; Falcade & Laborde, 2007). Although a full understanding of the mathematical meanings involved in an activity takes time to develop (Bartolini Bussi & Mariotti, 2008), Anita's realization of the variability of the mean in relation to the variability of standard deviation suggests some evidence of the internalization of the dragging tool (Vygotsky, 1978).

Similarly Boris' statement, "Yeah, yeah, they are getting bigger, the boxes are just scales, and they are actually just scales to see […] the relative differences …" [7] also provided evidence of the dragging tool being transformed into a psychological tool. Boris described the squares as "the boxes" that were "actually just scales to see […].the relative differences" [7]. By that he meant the size of the square represented how far away a data point was from the mean. Although Boris' language was not strictly mathematical, for instance by naming the "squares" the "boxes", he coordinated the size of the size of squares with the deviation of data points from the mean. Indeed the squares were just scales that represented the variance (the square of the average distance from the mean) of the data points. Anita and Boris used informal language to describe their ideas of mathematical meanings of variability. There are pedagogical questions in literature whether or not informal, personal meanings students express about concepts should be encouraged by teachers. However, many researchers and educators (e.g., Maker & Confrey, 2005; Healy & Sinclair, 2007) have encouraged teachers to pay closer attention to the ideas students come with about specific concepts to inform teachers' design of the learning tasks.

Semiotic mediation considers particular tools and objects that participants interact with in a dynamic geometry environment as signs. This study explored how the dragging tool can be used as an instrument of semiotic mediation for learning statistics.

The different signs produced during the interaction with the dragging tool had some relationship. For instance, the sizes of the squares related to the distances of the individual data points from the centre. Moreover, the arithmetical values of the mean and standard deviation also increased (or decreased) as the data points were dragged farther away from (or closer to) the centre, and participants used the signs to check their predictions. From these relationships, it can be argued that the DGE provided space and time within which participants experienced the notion of variability.

The study provides some evidence of the potential of dynamic mathematics environments in helping student make meaning of variability. The design of the tasks responds partially to delMas et al.'s (2005) work by drawing students away from using formula and calculation, to focussing more on exploring the relationships between the mean and standard deviation in the learning environment. The study contributes toward building students' strong consideration of variability (Reid & Reading, 2008) which they need as a foundation for statistical reasoning, thinking and literacy

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INTERPLAY BETWEEN CONCEPT IMAGE & CONCEPT DEFINITION: DEFINITION OF CONTINUITY

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This study looks at the interplay between the concept image and concept definition when students are given a task that requires direct application of the definition of continuity of a function at a point. Data was collected from 37 first year university students. It was found that different students apply the definition to different levels, which varied from formal deductions (based on the application of the definition) to intuitive responses (based on rather loose and incomplete notions in their concept image).

Keywords : Continuity, Concept image, Concept definition, Cognitive conflict

Among others, functions, limit, derivative and continuity have been widely recognized as some of the advanced mathematical concepts that not only students but also teachers find somewhat hard to grapple with. In addition to research carried out on the understanding of these concepts individually (Bezuidenhout, 2001; Vinner, 1987; Cornu, 1991), there has also been research done on understanding of the relationships between some of these concepts (Aspinwall et al., 1997; Duru et al., 2010). Further, the presentation of these concepts in a particular text book is discussed by Tall & Vinner (1981). This paper aims to look at how students work with the concept of continuity. Concept image and concept definition by Vinner (1991) will serve as a theoretical framework for the analysis of the data. '*Concept image*' is the name given to the total cognitive structure of a concept which includes all the mental pictures, properties and processes related to it. The *definition* of a concept on the other hand, is a form of words used to specify that concept and is termed as '*concept definition*'(Tall & Vinner, 1981).

This study is driven by the following questions: To what extent do students recall and apply the definition of continuity when handling tasks involving continuity? What notions of continuity are present in their concept images?

RESEARCH METHOD

Thirty seven student responses to the following question were collected and analyzed for this study.

Let
$$f(x) \begin{cases} \dfrac{x^2+x-2}{x-1} & ; x \neq 1 \\ a & ; x = 1 \end{cases}$$

Which value must you assign to a so that $f(x)$ is continuous at $x = 1$?

The students were in their first year of undergraduate studies specializing in the biological and medical sciences and were taking a Calculus course. They had covered the topics functions, limits, limit laws and continuity at the time of data collection. Based on the definition that was taught in this course "A function $f(x)$ is said to be continuous at $x = c$ if $\lim_{x \to c} f(x) = f(c)$", a complete answer to the above question may include three distinct points.

- Identifying the condition that must be satisfied for $f(x)$ to be continuous at $x = 1$.

  For $f(x)$ to be continuous at $x = 1$, $\lim_{x \to 1} f(x)$ must be equal to $f(1)$ which is a.

- Finding the limit of $f(x)$ when x approaches 1.

  $\lim_{x \to 1} f(x) = \frac{x^2 + x - 2}{x - 1} = \lim_{x \to 1} \frac{(x-1)(x+2)}{(x-1)} = \lim_{x \to 1} x + 2 = 1+2 = 3 \quad ;x \neq 1$

- Concluding that a must be 3.

  $f(1) = 3$ and $f(1) = a$ hence $a = 3$.

These steps need not be in this same exact order but there must be some logical sequence in the way the students organize their answer. The consultation of the definition in the first step requires them to proceed to the second step where they need to find the limit of $f(x)$ when x approaches 1. A student may do this step first 'knowing' it needs to be done in their head and may state the condition afterwards. Because without calling on the definition, there will not be a necessity to find the limit. The second step is a matter of finding the limit of a function where the function is a rational which produces an indeterminate form with direct substitution. This step hence, may not call on the definition of the limit but only on the procedures of finding the limit. Last step is the conclusion of the answer.

RESULTS

Five different types of answers could be identified. Only the first four categories are presented in this paper since the page limitation does not allow the fifth category to be discussed which was special cases that showed the need of individual analysis. The first four categories which show the application of the definition to different degrees are listed in a certain order which is from a poor answer to a good answer from a marker's perspective.

Type 1 - *The correct answer for a is obtained but taking the limit of $f(x)$ when x approaches 1 is not explicitly shown.*

Four (out of 37) students in the group gave the answer in this category as shown in figure 1. It is hard to say whether these students are *thinking* of taking the limit but not showing it or they are merely doing an algebraic manipulation of the expression. The line $f(1) = \big((1) + 2\big)$ can be interpreted at least in two ways.

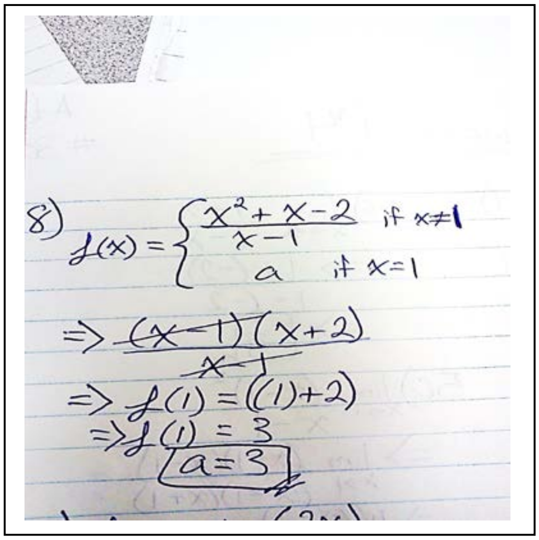

Figure 1: Type 1

Case 1 : ‘plugging a value into the function’

$f(x) = (x + 2)$ and hence $f(1) = ((1) + 2)$ or

Case 2 : applying the condition for continuity and hence stating an identity

$\lim_{x\to 1} f(x) = \lim_{x\to 1}(x + 2) = \big((1) + 2\big)$ & this must equal to $f(1)$, $f(1) = ((1) + 2)$

The way they have presented their answer it appears as though the students meant the first case rather than the latter. This is because if they meant the second case, the way the argument is ordered, it should be written as $\big((1) + 2\big) = f(1)$, not as $f(1) = ((1) + 2)$. The concept definition of ‘continuity of a function at a point’ *contains* the concept of ‘limit of a function’. If students have trouble understanding the concept of limit and hence possess a blurred concept image of limit, then, this has a significant impact on the concept image of continuity. The portion of their concept image, which is evoked by this problem, does not seem to contain or have any overlap with the concept of limit. Their working can be best described as an effort to merge the two pieces of the function. This can be pointing to the notion that students were found to have by Tall and Vinner (1981) too, of the need for a function to be in *one piece* to be continuous. It appears that they simplify the case when $x \neq 1$ which is $\frac{x^2+x-2}{x-1}$ to $(x+2)$ and then assign it to the case when $x = 1$.

The features of the responses of this type also suggest that the task has not made these students to consult the concept definition but that they have worked on certain notions in their concept image of continuity. This intuitive response is modeled by figure 2 as illustrated by Vinner (1991, pg. 73).

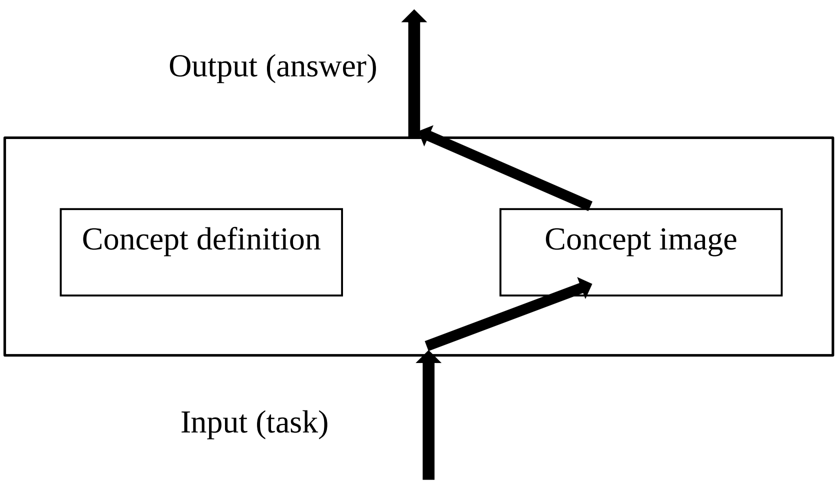

Figure 2: Intuitive Response

Type 2 – *The limit is taken and the value for a is given without noting that the limit must equal to* $f(1)$.

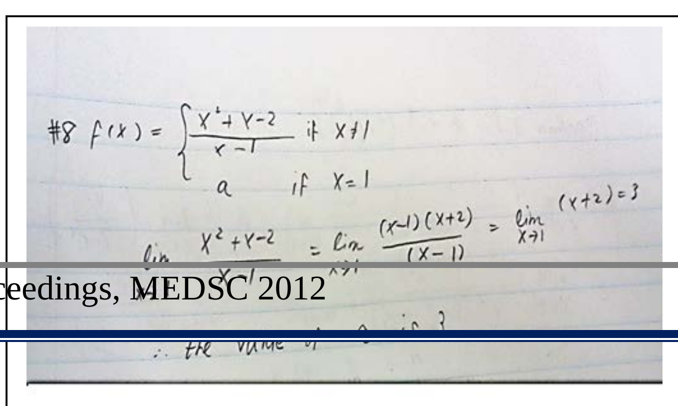

Figure 3: Type 2

In this category (12 out of 37) the students have taken the limit and have just concluded that it is equal to a (see figure 3). This kind of an answer can come from a correct reference to the definition. What is lacking in terms of writing is, not explicitly showing or stating that the calculated limit must equal the function value at $x = 1$. And it is not acknowledged that $f(1) = a$. However, this may have been thought through to obtain the answer as $a = 3$.

Another possible process that may be on work here is a rote memorization of a procedure rather than any attention given to the definition. Since this is a familiar and 'routine' kind of question, students may have developed an algorithm for it, as part of the concept image. It may be a rule like 'find the limit of the function given and assign it to the letter'. Only this procedure, in that case, may be evoked when presented with this style of a question.

Type 3 - *The limit is taken and notes that it should be equal to* $f(1)$ *and hence to* a.

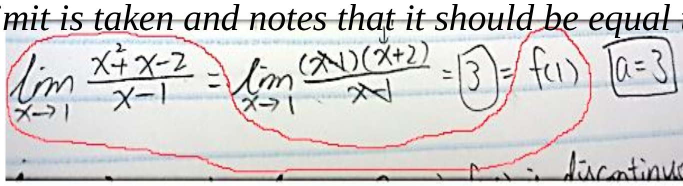

Figure 4: Type 3

These students (4 out of 37) have explicitly stated that $f(1)$ is equal to the answer they obtained for the limit and hence have exhibited an important part of the definition before concluding the final answer for a (see figure 4). And as shown in figure 4, the definition is embedded in their answer. It can be concluded that, in their concept image they have a complete concept definition image, which they have been able to appropriately apply in this task. Based on the presented written work, students in this type are a step ahead of the students under type 2. Even if one argues that these students too can be applying a mere memorized algorithm, it is evident that their 'algorithm' is more closely grounded to the definition.

Type 4 – *A complete logical answer with all reasons is given.*

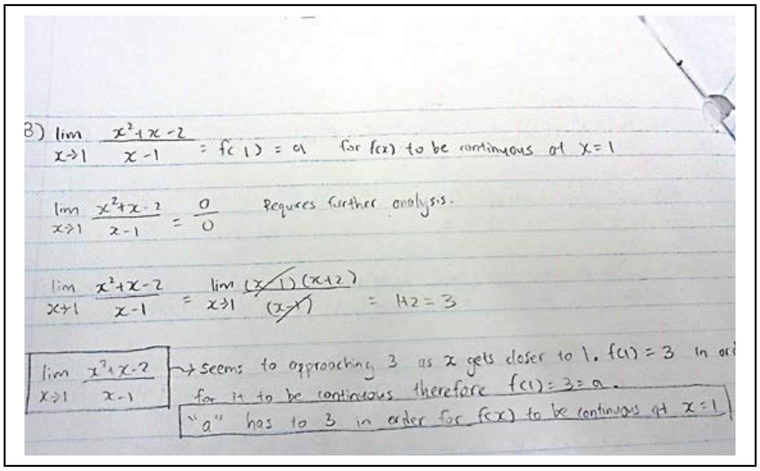

Figure 5: Type 4

The answers were with a good logical sequence of reasoning without missing any points as shown in figure 5. Thirteen of the students had given answers in this category. It is clearly demonstrated how they formulate their answers by consulting the concept definition. And no sign of side tracking or being disturbed or intervened by unnecessary notions that *may* be present in the concept image is visible. Hence, this can be modeled by figure 6 as illustrated by Vinner (1991, p. 72) of a purely formal deduction.

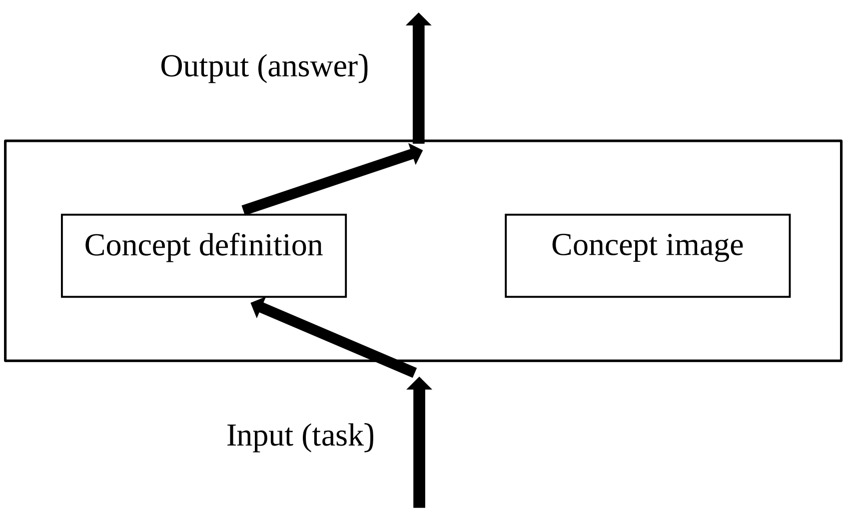

Figure 6: Formal Deduction

DISCUSSION & CONCLUSIONS

Response types 2, 3 and 4 show clear attention given to the definition in different degrees. Vinner (1991) claims that the majority of students do not use definitions when working on cognitive tasks in technical contexts and that college courses do not develop in the science students, not majoring in mathematics, the thought habits needed for technical contexts. However, as far as using definitions goes, this preliminary study suggests that, majority of students who are not majoring in Mathematics do refer the definition but in different levels. They seem to have a concept definition image developed to different levels as part of their concept images. Or, if the assumption- that their writings reflect their cognitive processes- is removed, this can be pointing to a different category of levels in transforming their cognitive processes into writing.

What seems to emerge from type 1 is the tendency of some students to tackle problems in ways that they have built for themselves with little rigor which works and produces the correct answer. Vinner says that 'as long as referring to the concept image will result in a correct solution, the student will keep referring to the concept image since this strategy is simple and natural' (Vinner, 1991, pg. 80). This brings us to the following questions yet to be answered. Can this be overlooked as they produce the correct answer and be satisfied about their performance, as these students are not majoring in Mathematics? Or should these be resolved by creating cognitive conflicts that make students confront these erroneous methods?

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EFFECT OF DYNAMIC GEOMETRY ON CHILDREN'S PERFORMANCE IN ANGLE COMPARISON TASKS

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This paper examines the effect of the use of dynamic geometry environments on children's thinking about angle. Using a driving angle model in Sketchpad, kindergarten children were able to develop an understanding of angle as "turn," that is, of angle as describing an amount of turn. After the classroom lessons with dynamic sketches, students were interviewed on various angle comparison tasks. It emerged out that gestures and motion played an important role in their developing conceptions of angles as well as in their decision making on angle comparison tasks.

Keywords: Dynamic Geometry Environment, Angle, Gestures, K-2.

INTRODUCTION

Being multifaceted concept of angle can pose challenges to learners, even into secondary school (Close 1982, Mitchelmore & White 1995). Despite these difficulties, children show sensitivity to the concept of angle from very early years (Spelke, Gilmore and McCarthy, 2011). Angles are normally introduced to children quite late in formal school settings. For example, in British Columbia, they are introduced in grade 6 (12 years old), even though students are expected to describe, compare, and construct 2-D shapes, including triangles, squares, rectangles and circles in grade 2. The strong capacity of young children to attend to and identify angles in various physical contexts motivated us to see whether a more dynamic conception of angle —namely, angle-as-turn—might support their developing understanding at an earlier age.

We have been investigating other geometry-related concepts at this age too, using DGEs, including shape identification, symmetry and parallel lines (Sinclair, Moss & Jones, 2010; Sinclair & Kaur, 2011). Previous research reports on the effectiveness of Turtle Geometry (Logo) for teaching the concept of angle (Clements, Battista, Sarama & Swaminathan 1996; Simmons & Cope, 1990). However, we believe that dynamic geometry environment (DGE) might be helpful in thinking of angles as turns and rotation more effectively. In this paper, we report on a one-on-one interview with a split class of kindergarten/grade 1 children (ages 5-6) who were taught concept of angle using *The Geometer's Sketchpad*. We focus on the emergence of the concept of angle-as-turn and its use in angle comparison tasks as well as discuss the specific mediating role of the use of the software on this thinking.

CHILDREN'S UNDERSTANDING OF ANGLE

In the research literature, the concept of angle is shown to have different perspectives, namely: angle as a geometric shape, union of two rays with a common end point (static); angle as movement; angle as rotation (dynamic); angle as measure; and, amount of turning (Close, 1982; Henderson & Taimina, 2005). Due to different prevalent definitions of the term angle, teachers often face difficulty in knowing what definition of angle to use (Close, 1982). Mitchelmore (and colleagues) and Clements (and colleagues) have done abundant research in the area of angle concept over the past twenty years. Much research has been conducted on the development of the concept of angles, focusing at the grades 3, 4 and higher levels. Mitchelmore & White (1995) suggests that angles occur in a wide variety of physical situations that are not easily correlated. Despite the excellent knowledge of all situations, specific features of each situation strongly hinder recognition of the common features required for defining the angle concept (Mitchelmore, 1998).

Later works of Mitchelmore involved teaching experiments (White & Mitchelmore, 2003) in which they divided angle situations into three clusters—2 line angles (corners of room, intersecting roads, pairs of scissors), 1-line angles (doors, windshield wipers), and 0-line angles (the turning of a doorknob or a wheel). The situation is more problematic for students where the two arms (of angle) are not clearly visible. Research using *Logo* shows that students tend to visualize the turn of turtle as turn of their body but making these turns involves writing numerical commands (Clements, Battista, Sarama & Swaminathan 1996). The DGE does not involve the writing of the commands and can thus be used at an earlier age to develop more qualitative understanding of angle.

Research has reported about the young children’s difficulties in understanding the turn as an angle (relating turning to angles in general) as well as connecting static angles to turns (Mitchelmore, 1998; Clements, Battista, Sarama & Swaminathan, 1996). Thus, young children do not spontaneously conceptualize turning in terms of angle and they don’t naturally connect static angles to turns. Other popular misconception about angles is related to the relative size of angles. Students think that the length of the arms is related to the size of the angle (Wilson and Adams 1992; Stavy and Tirosh 2000; Clausen-May 2005; Munier, Devichi & Merle, 2008). The emphasis on quantity aspect of angles leads students to think ‘the longer the rays, the greater the measure of the angle’ (Keiser, 2004, Stavy and Tirosh 2000, Clausen-May 2005). Stavy and Tirosh (2000) reported this misconception could develop among children as a result of intuitive rule ‘More A - More B’. Another reasons for such misconception are the introduction of angle as a shape rather than a measure as well as the limited experience of angles as shown in textbook (Clausen-May, 2005). This misconception seems to be very hard to overcome. Lehrer, Jenkins, and Osana (1998) conducted a longitudinal study involving children in grades 1–3 who were followed through grades 3, 4, and 5. Their results show that “the length of the line segments had a substantial influence on children’s judgments of similarity … the effects of length on children's judgments about angles did not diminish during the three years of the study (1998, p. 149)". We believed that the DGE approach would be helpful in developing the dynamic as well as static concept of angle. Kaur and Sinclair (2012) reported about children's readiness to learn about angle using the Sketchpad.

THEORETICAL PERSPECTIVE

In previous research, we have found Sfard's (2008) 'commognition' approach is suitable for analysing the geometric learning of students interacting with DGEs (see Sinclair, Moss & Jones, 2010; Sinclair & Kaur, 2011). For Sfard, thinking is a type of discursive activity. Sfard's approach is based on a participationist vision of learning, in which learning mathematics involves initiation into the well-defined discourse of the mathematical community. The mathematical discourse has four characteristic features: word use (vocabulary), visual mediators (the visual means with which the communication is mediated), routines (the *meta-discursive rules* that navigate the flow of communication) and narratives (any text that can be accepted as true such as axioms, definitions and theorems in mathematics). Learning geometry can thus be defined as the process through which a learner changes her ways of communicating through these four characteristic features. We have previously presented a developmental trajectory related to identifying shapes in terms of different levels of discourse and now we are trying to do the same thing with angles, but we will look first at how the different components of the discourse change as the students work within the DGE. We are particularly interested in investigating how the students might move between different word uses and to examine the informal language they use to talk about angles.

Similarly, given the importance of gestures in communication of abstract ideas (Cook & Goldin-Meadow, 2006), and their potential to communicate temporal conceptions of mathematics (Núñez, 2003; Sinclair & Gol Tabaghi, 2010), we chose to extend Sfard's approach to incorporate gestural forms of visual mediators. Kita (2000) focuses on the cognitive functions of gestures, which play an important role in communication. He points out that the production of a gesture helps speakers organize rich spatio-motoric information, where spatio-motoric thinking organizes information differently than analytic thinking (which is used for speech). We thus expect that children will use gestures to convey spatio-motoric information, even though they might not be able to convey the analytic thinking used in speech. Moreover, children's gesture might be non-redundant with their speech. Our goal in looking at the gestures will be to see how they communicate different ideas about angles; particularly the mobile ones associate with angle-as-turn.

EXPLORING THE UNDERSTANDING OF CONCEPT OF ANGLE

Participants and tasks

We worked with kindergarten/grade1 children (aged 5-6) from a school in a rural low SES town in the northern part of British Columbia. There are 20 children with diverse ethnic backgrounds and with a wide range of academic abilities. We designed lessons related to angle along with the classroom teacher, who has a Masters degree in mathematics education and has been developing her practice of using DGEs for a couple of years. The teacher and students worked with angles in different ways using Sketchpad for six lessons in a whole class setting with an IWB (Interactive Whiteboard). Each lesson lasted approximately 30 minutes and was conducted in a group with the children seated on a carpet in front of a screen. After five months, seven students were interviewed on angle comparison tasks. The students were presented with the triads of angle sketches (Table1) and asked to determine "which two are most alike and give a verbal justification for your choice?" Total seven triads were printed on 7 different sheets with one triad on each. Each triad was shown to students one by one. Four angle triads (1,2,4,5) are adapted from Lehrer (1998) longitudinal study. Three (3,6,7) additional triads were included in this study. Interviews were videotaped and some of them were transcribed. Present paper analyses the responses of one student on the angle comparison tasks.

Dynamic angle sketches

We used two different sketches to explore the concept of angle with the children. We began with a simple angle diagram (Fig.1). In the sketch, dragging the vertex of an arm of the angle changes the angle. The research suggests that children have difficulty seeing a static angle as a turn. The second sketch used is a 'driving angle model,' which shows both a static as well as dynamic sense of angle (Fig.2). It includes a car that can move forward as well as turn around a point. The turning is controlled by a little dial (which has two arms and a centre). There are four action buttons (Turn, Drive Forward, Erase Traces and Reset) that control the movement of the car.

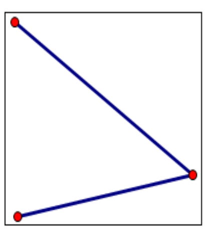
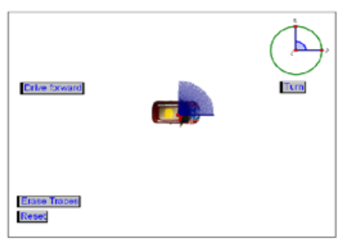


Figure 1: Angle as a Shape



Figures 2: Driving Angle Model

The traces offer a visible, geometric record of the amount of turn. (For details See Kaur & Sinclair, 2012).

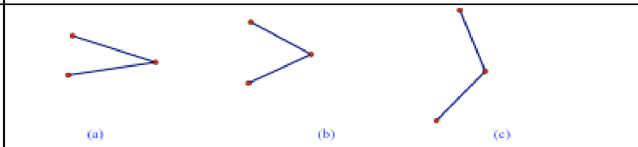
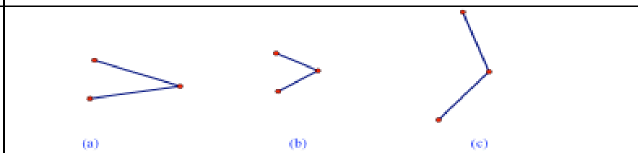
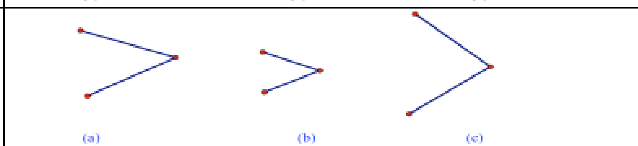
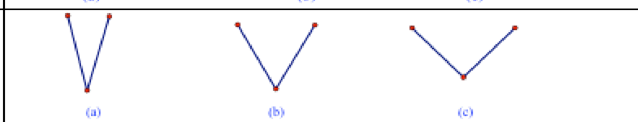
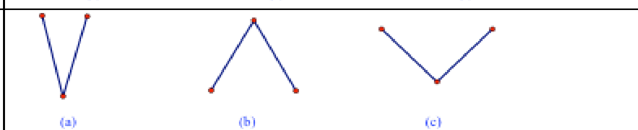
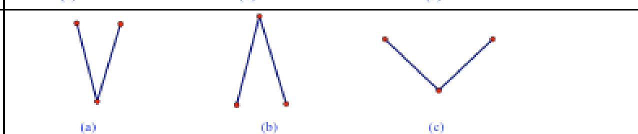
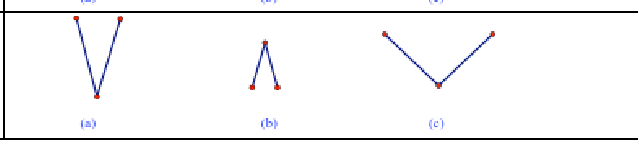
No.	Triads	Description
1		Angles 30, 60 and obtuse angle (120)
2		Angles 30, 60 and obtuse angle(120). The length of arms is reduced in 2(b)
3		Angles 50, 50 and 120. Angles in 3(a) and 3(b) are same, but arm lengths are different.
4		Angles 30, 60 and 90.
5		Angles 30, 60 and 90. The orientation of 5 (b) is changed.
6		Angles 30, 30 and 90. The orientation of 6(b) is different
7		Angles 30, 30 and 90. The orientation as well as length of arms is different in 7(b).

Table 1: Angle comparison triads

Working with angle comparison tasks

Present paper will present the interaction with a girl named Chloe on angle comparison tasks. Teacher presented the first triad to Chloe and asked:

Teacher: Have you seen things like that before? (*Showing angle triad*)

Chloe: Yeah on the smart board we do that?

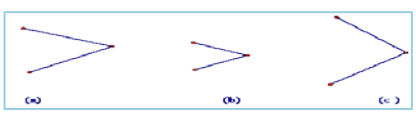


Figure 3 (50^0, 50^0, 120^0 triad)

Teacher: Oh yeah…which one out of these three you think is most different?

Chloe: I think that one (*Pointing at 3b*)

Teacher: That one is most different? (*Pointing at 3b*)

Chloe: Yeah, that one is the smallest

Teacher: Okay (*Taking out the next sheet of triad*)

Chloe: And those two are pretty big

After looking at the triad with angles 50,50 and 120 degrees, Chloe instantly said that

1(b) is different. The use of the words ‘smallest’ and ‘pretty big’ indicates that initially she was paying attention to the length of the arms of the angles. She started the comparison of triads based on the visual appearance of the length of arms of the angles. But as soon as she is presented with the next sheet with other triad, she changed her mind and gave a different answer with entirely new explanation.

Teacher: How about these three? (*Showing next triad sheet*)

Chloe: Well in a different way all of those can be different. (*Flipping the next sheet halfway, looking at and then talking about the previous task*)

Teacher: What do you mean in a different way all of these can be different? (*Taking away the second sheet*)

Chloe: Well, that one is wider. (*Pointing at c and drawing the angle over the c with index finger of right hand extended…tracing over the 3(c) starting from top point to the lowest point and then repeating the drawing with index figure from bottom to top backwards*))

Chloe: And that one is just like that one kind of (*Drawing b and a respectively with index finger of left hand tracing over the 3(b) and then 3(a) starting from top point to the lowest point*), except that one is just smaller (*talking about 3(b)*). That one is tiny bit bigger (*Using her left hand thumb and four fingers together as arms and bringing them close together and then taking them further from each other*). That one is wider (*Again drawing 3c with right hand index finger*)

Chloe’s use of words ‘in a different way all of these can be different’ shows that she analyzed the triad from more than one perspective. Later she started to think in terms of angles being different in terms of wideness. She used the words like ‘wider’, ‘that one is just like that one kind of’, ‘just smaller’, ‘and tiny bit bigger’ for the comparison of same triad second time. Chloe’s use of the words ‘that one is just like that one kind of’ while gesturing for drawing a and b with left hand’s index finger, shows that she is talking about angles a and b being same due to same amount of wideness, except the fact that length of arms are different. It is worth noting that in case of all triads, Chloe drew a and b with left hand’s index finger, while the c figures were drawn with right hand index fingers. The drawing of angles with index fingers acted as a visual mediator for Chloe to decide about the wideness of angles.

Teacher: Okay, that makes sense. How about these three? (*Showing the next sheet with another triad*)

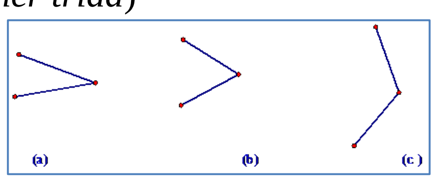


Figure 4 (30^0,60^0,120^0 triad)

Chloe: That one is most different (*Drawing 4c with the right hand index figure from top to bottom point*) because if you could stretch that one (*Pointing at 4a*) a tiny bit further, it will be just like that (*Pointing at 4b*).

While comparing a 30, 60 and 120 degrees triad, Chloe's use of words 'stretch that one a tiny bit further' again reveals that she is paying attention to the relative position of arms of the angle. This kind of dynamic thinking might be invoked due to DGE based instruction where the car turns from one position to another with a particular angle showing the traces. Also, It is also interesting to note that Chloe could recognize the triads 1 and 2 (in Table 1) being same. This shows that the length of arms does not effect her conception of angles and she has understood the angle as a turn. Chloe used the drawing of angles with index fingers as the visual mediator to decide the wideness of angles.

Teacher: Okay…how about here? (*Showing the new sheet*)

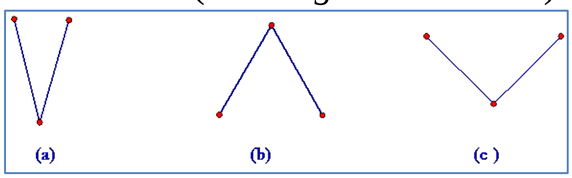


Figure 5 (30^0,60^0,90^0 triad)

Chloe: Umm…(*thinking*)...I think that one (*Pointing at 5(a) and drawing with the left hand index finger*)

Chloe: Because that one *(pointing at 5c and drawing with right hand index finger*) all you have to do to put a tiny bit more like that way (*opening wide hands and then turning them inwards to make the angle smaller*) and turn it over (*gesturing with hands to flip the angle*) And in that you have to put a lot farther and turn it over.

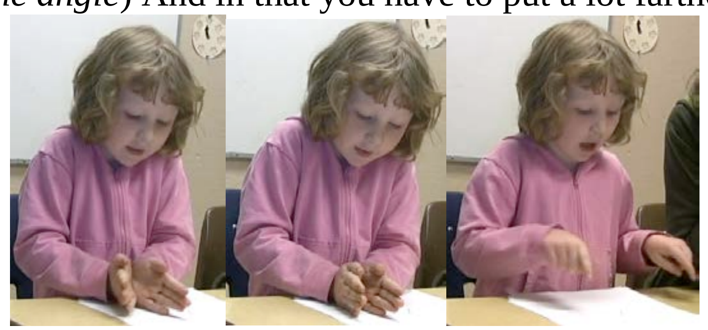


Figure 6a 6b 6c

Chloe didn't compare the angles being different on the basis of the orientation of the angles. This time for comparison of angles, Chloe looked at angles in terms of the turn involved. She opened her hands wide enough to show angle 5c (figure 6a) and then suggested 'all you have to do to put a tiny bit more like that way' (figure 6b) and turn it over to get angle 5b. The use of words 'put a lot farther' along with gestures (Figure 6c) shows that she is again thinking in terms of angle being dynamic. The Chloe's use of motion in her explanation of static angle figures shows that instruction of angles within dynamic geometry environment can enable children to link the static angle concept with the dynamic turns. She considered her hands being the arms of the angles and stretched them inwards to show a smaller angle. Clearly, her comparison involves the motion and transformation. Chloe used the embodied visual mediators for deciding which angle is different. Her routines for comparing the angles involved use of motion and transformations focusing more on the relative position of the arms as well the amount of turn involved from one arm to another.

Teacher: How about this? (*Showing another triad*)

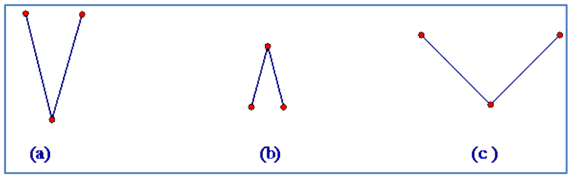


Figure 7 (30,30,90)

Chloe: Hmm…is this upside down?

Teacher: Haha…(*laughing turning the page other way*)….you clever girl

Chloe: Umm..(thinking)… ….that one (c)…because you have to put that lot more together, about that much (*opening hands wide open and bringing them inwards to make smaller angle*) (Figure 8 a and 8 b)

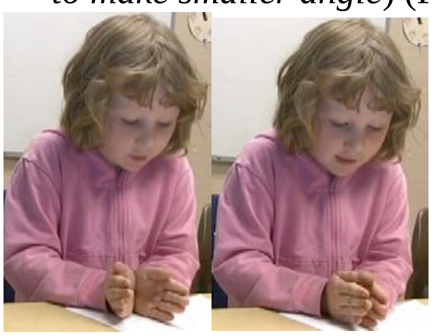


Figure 8a and 8b: Gesture for reducing the wideness of angle

Figure 9a and 9b: Gesture for flipping the angle

And then you have to phhh.. (*gesturing to flip the hands together*)(Figure 9a and 9b).. turn it over. And in these two, well you have to just make that one (*pointing at 7b*) a little bit bigger and turn this way or turn that one over.

For the comparison of the angles, Chloe developed a routine of assuming her hands as the arms of the angle and then turning them inwards or outwards to compare the amount of turn of one or more angles. It is interesting to note that Chloe used drawing with index fingers as visual mediators in case of triads 1 to 3 and she used gestures of hands as arms of angles in case of triads 4 to 7 of Table 1. This implies that she preferred to use index fingers for sideways oriented angles and hands for the upward or downward facing angles.

DISCUSSION AND CONCLUSION

Chloe's frequent use of gestures for drawings angles with index fingers shows that she is making frequent use of embodied visual mediators for comparing the angles. Chloe's use of words ‘‘wider', ‘stretch that one a tiny bit further', ‘have to put that lot more together', ‘turn it over' shows her propensity to reason in terms of motion and transformation, which can be attributed to the dynamic geometry environment. Clearly, her routines of assuming hands as arms of angles and then stretching it in and out (figures 6a, 6b, 8a, 8b) to match the angles shows that she is very comfortable in connecting static angles to turns as well as understanding the turn as an angle. This gives an initial evidence that DGE can help in overcoming young children's difficulties in relating turning to angles as reported by Mitchelmore, 1998; Clements, Battista, Sarama & Swaminathan, 1996. Also, Sfard's framework can be extended to incorporate the embodied routines as Chloe used the routine of assuming hands as arms of angles and then moving it inwards and outwards to compare the angles again and again. Chloe's thinking of angles in terms of turning and wideness of arms provides the evidence that DGE might be helpful in overcoming the misconceptions related to effects of length of arms on children's judgments about angles.

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NUMERACY EXPERIENCES AND MODELLING BEHAVIOURS

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Literature has shown that models and model-eliciting activities are important aspects in the learning of mathematics and a powerful tool in promoting students' higher order thinking. On the other hand, numeracy and numeracy tasks are vaguely defined and are not fully recognized as useful tools to promote students' higher order thinking. In this article, I examine the similarities and differences between model-eliciting activities and numeracy tasks, and examine the possibility to integrate numeracy tasks into the curriculum.

INTRODUCTION

Numeracy plays an important role in developing students' ability to interpret mathematical information, solve problems, and make mathematically informed decisions (Steen, 2001). However, while there exists much discussion about numeracy and the closely related mathematical literacy, these terms lack agreed upon definitions. As such, there is little to help us delineate numeracy, numeracy tasks, and numerate behaviour. On the other hand, modelling, model-eliciting activities, and modelling behaviour are well represented in the research literature. In this article, I look at the possible similarities between numeracy tasks and model-eliciting activities, and try to map the characteristics of model-eliciting activities on to numeracy tasks.

In what follows I briefly discuss numeracy and mathematical literacy, and examine the characteristics of numeracy tasks, models, and model-eliciting activities. I focus on models used for the purpose of learning mathematics and model-eliciting activities found in secondary mathematics classrooms.

NUMERACY, MATHEMATICAL LITERACY, AND NUMERACY TASKS

In general, numeracy involves the making sense of numbers and carries with it quantitative and numerical aspects. It can be loosely defined as "a concrete skill embedded in the context of real-world figuring." (Cohen, 2001, p. 25) A numerate person is able to use, understand, apply, and interpret mathematical information, and effectively communicate this information with others. Such a person also has an awareness, respect, appreciation, and understanding that mathematics is relevant and important, and possesses an ability to learn and do mathematics when needed (Steen, 2001).

While both numeracy and mathematical literacy involve using mathematics in the real world setting in a practical and functional manner, these terms emphasize slightly different focuses. Numeracy focuses on the numerical and quantitative aspect. Meanwhile, mathematical literacy emphasizes a strong link to being literate, and highlights the important goals and components of numeracy, including "mathematical thinking, constructing understandings, monitoring and regulating mathematical thinking and technology in mathematics." (Yore et al., 2007, p. 574) Mathematical literacy also relates a mathematically literate person to those who are knowledgeable and educated, proposes that those who are mathematically literate are able to identify and use the resources of mathematics in their everyday lives, and implies "an integrated ability to function seamlessly within a given community of practice." (Ewell, 2001, p. 37) Furthermore, literacy carries with it a social and communicational aspect, and suggests the ability to understand and communicate context-based contents effectively. While these terms emphasize slightly different focuses, they are very similar in the sense that they both promote students' mathematical thinking and understanding, communication skills, and allow students to develop into functional and numerate individuals. Although slightly different, in this article I use the terms numeracy and mathematical literacy interchangeably.

Numeracy tasks can be used to promote students' numeracy skills. Numeracy tasks can be loosely defined as novel problem solving questions situated in real life situations. These tasks are generally ambiguous, require students to make sense of the question and data provided to generate suitable solutions, present these solutions in a mathematical way, and sometimes extend or generalize these solutions to similar problems. As such, these tasks rely on the sophisticated use of mathematics, and require students to extend their thinking beyond computation. Also, numeracy tasks require students to work in teams and promote their communication skills.

There are various ways to classify numeracy tasks. McAskill et al. (2004) suggest the organization of numeracy tasks based on topics we find in the mathematics curriculum. Although this is a convenient method to organize numeracy tasks, I feel that this organization method may further contribute to the belief that mathematics can be broken down into isolated units, where these units are not connected or related to each other. Another way to classify numeracy tasks is by task goals (Liljedahl, 2010). It shifts the focus of numeracy tasks away from specific mathematical topics found in the curriculum to solving the problem by applying various mathematical knowledge and skills. It also gives students the freedom to solve the problem using whichever method they believe to be the most appropriate and encourage them to make connections between various mathematical skills they learn in class and outside of class. Based on this classification scheme, there are four types of numeracy tasks: fair share tasks, planning tasks, estimations across a large number of variable tasks, and modelling tasks (Liljedahl, 2010). Fair share tasks require students to divide something fairly but not necessarily equally. Since fairness is not well defined in the task, the strategy for sharing is often challenging. Planning tasks require students to make sense of a situation and do some planning based on, for example, spatial orientation and budget. The third type of numeracy tasks, estimation across a large number of variables tasks, involve making estimations across a wide range of contexts in order to arrive at possible solutions. One common characteristic between these three types of numeracy tasks is that they do not require students to make generalization or to apply their solutions to similar problems. The fourth type of numeracy task is modelling tasks. Modelling tasks require students to construct a suitable model based on their interpretation and manipulation of the data given in the question, refine and make adjustments to their models to make sure they are applicable to the data given and to other sets of data in similar contexts, and produce a final report which presents the strategy used to solve the problem.

The following numeracy task, 2004 Summer Olympics Results, is an example of modelling tasks. The first part of the 2004 Summer Olympics Results task asks students to rank 20 countries based on the number of gold, silver, and bronze medals they won during the 2004 Summer Olympics games. In order to approach this task, students need to invent a model to weigh the number of gold, silver, and bronze medals, and to quantify this data to give each country a final score. In part II of the task, students are asked to rank another set of data using the model they develop in part I. In order to construct a meaningful way to rank the countries, students need to account for the various scenarios that are not presented in part I. They also need to adjust and refine their models to break ties and to avoid situations where ties occur, and to generalize their models developed in part I to be applicable to part II of the task. Eventually, students come up with a model that allows them to rank countries efficiently and effectively. Numeracy tasks are a great way to help students understand the role of mathematics in diverse contexts and situations. They allow students to reflect on the mathematics they learn, make well-founded judgments, and to apply this mathematics to solve problems in meaningful ways.

MODELS AND MODEL-ELICIATING ACTIVITIES

Mathematical models are conceptual systems that focus on structural characteristics of a system, including the relationships and interactions between various objects within the system and objects outside of the system. As such, models allow us to explain, manipulate, and predict the behaviours of the system they describe. Models are not restricted to written language or graphs and equations. They come in various forms, including spoken language, written symbols, drawings, experience-based metaphors, mathematical equations, computer simulations, etc., and they range from simple to sophisticated, concrete to abstract, and single to multiple representations. Since various representations emphasize (and de-emphasize) different aspects of a system, multiple representations are often needed for learners to understand, describe, and explain the system as a whole.

Lesh and Doerr argue that models reside in both the minds of the learners and are embodied in the various representations. While models as conceptual models are comparable to schemas for interpreting experiences, which rest in the minds of the learners, the representations of these models take various forms, such as "spoken language, written symbols, concrete materials, diagrams or pictures, computer programs, experience-based metaphors, or other representational media" (Lesh and Doerr, 2003, p. 11), and are used as ways to communicate with others. They allow learners to project their conceptual systems, which reside internally within them, into the external world. The term "model-eliciting activities" originates from Lesh's work on modelling activities (Lesh and Doerr, 2003; Lesh et al., 2000; Lesh et al, 2003). They are thought revealing activities that evoke the construction of models. They are open-ended problem solving activities that are contextualized in a real world setting. These activities allow students with various mathematical knowledge and abilities to interpret and approach the problem in meaningful ways. There are six principles to effective model-eliciting activities:

1. The Model Construction Principle
2. The Reality Principle
3. The Self-Assessment Principle
4. The Construct Documentation Principle
5. The Construct Shareability and Reusability Principle
6. The Effective Prototype Principle

The model construction principle states that the goal of model-eliciting activities is the construction of models that are able to describe, explain, and predict situations. The reality principle ensures the activity is situated in a real life setting and allows students to make sense of the problem based on their experiences and knowledge. The self-assessment principle allows students to make judgments regarding their approaches to solve the problem. Proper assessments allow students to modify, refine, and improve on their solutions. The construct documentation principle requires students to record their thoughts and responses to the problem and foster self-reflection. These documents reveal the progress and possible changes in students' ways of thinking and their understanding. They also shift the attention from the end product to the process which gives rise to the solutions. The construct shareability and reusability principle ensures the models students create are transportable, modifiable, and reusable. A non-transportable model that is created specifically for a specific situation is definitely not as useful as one that can be generalized to other situations. Finally, the effective prototype

principle provides students with “rich and memorable contexts for learning and for discussing important mathematical ideas.” (Lesh et al., 2000) Problems that follow the effective prototype principle allow students to go back and reflect on the way they solve the problem and build important mathematical ideas from the problem.
Model-eliciting activities are very different from traditional problem solving activities. Instead of making sense of classroom mathematics and expanding this mathematics into the real world, model-eliciting activities make sense of the world through mathematical models.

Another important aspect about model-eliciting activities lies not in their characteristics but in the instructions given to the problem solvers. Based on Lesh’s work, model-eliciting activities always involve an audience and a proposal as one of the end products, which requires problem solvers to explain and demonstrate their model to an audience and convince the audience their model is effective and suitable in solving the problem. Model-eliciting activities serve as great learning opportunities for mathematics students, while models function as tools for students to understand, approach, and solve problems. Model-eliciting activities promote the use of models while revealing students’ thinking as they develop and refine their solutions. Furthermore, model-eliciting activities foster students’ communication skills. Due to the nature of these activities, students often work in teams. This requires students to interact and communicate with their team members, other teams, and the problem poser. Students need to clearly articulate their ideas, the problems they encounter, possible solutions, etc. to their audiences in order to be successful (Diefes-Dux et al., 2004; Lesh and Doerr, 2003).

Lesh and Doerr (2003) provide examples of model-eliciting activities in their book chapter, *Foundations of a Models and Modeling Perspective on Mathematics Teaching, Learning, and Problem Solving*. In the next paragraphs, I briefly describe the volleyball problem, which is one of the model-eliciting activities in the chapter, to give the readers a sense of these activities.
The volleyball problem begins with a description of a volleyball camp in a small town, where there is no systematic procedure to divide players into equal teams. Students are given a set of tryout scores of a small group of volleyball players (n = 18) and are asked to create a systematic way to divide the players into equal teams based on these tryout scores. The six tryout scores include: height of the player, vertical leap (height), running ability (time), serve results (accuracy), spike results (accuracy), and coaches comments. Students are also asked to write a letter to the organizers explaining the procedures so the organizers can apply these procedures in the upcoming volleyball camp where they are expecting a large group of volleyball players (n > 200).
In order to solve this task, students need to analyze the six tryout scores and create a volleyball playing potential index by weighing and combining both quantitative and

qualitative measurements. This is similar to creating consumer guides for products in the market, where a number of qualitative scores are quantified and combined with other quantitative scores. In this task, for example, students need to come up with a meaningful way to deal with opposite scoring systems, such as vertical leap (high scores (height) for good results), and running ability (low scores (time) for good results). Depending on their importance, the scores may be weighed differently, and students need to justify their decisions. Moreover, students need to create a model that allows the camp organizers to divide volleyball players into equal teams accurately and efficiently. In other words, not only does the model need to work for the small sample given, it also needs to work for large samples.

This example closely follows the criteria listed for model-eliciting activities. The task is situated in the real world (volleyball camp in a small town) and requires the construction of a model (volleyball playing potential index) where data (the tryout scores) are interpreted and manipulated. The quantification of data allows the organizers to determine the potentials of volleyball players accurately and thereby divide large groups of volleyball players into equal teams efficiently. Students also need to reflect on the effectiveness of their models and make necessary adjustments and refinements. Finally, students are asked to write a letter (or a proposal) to the organizers to describe their model. This allows students to explain and justify their choices made, and demonstrate and communicate their thinking in the process of solving the task.

As I compare the volleyball tasks to the 2004 Summer Olympics Results task, I can see many similarities between them. For example, both tasks are situated in the real world. Also, students are required to interpret and manipulate data, and develop a way to combine and quantify data. They need to modify and refine their model and to account for possible ties. In addition, the model constructed need to be transferable to solve similar problems. Finally, students need to present their model and convince others this is a suitable way to solve similar problems.

There are many similarities between model-eliciting activities and modelling type numeracy tasks. For example, these tasks are both situated in the real world. Also, students are required to interpret and manipulate data, and develop ways to combine and quantify data. They need to modify and refine their models in order to arrive at a suitable solution to the problem. In addition, the models constructed need to be transferable to solve similar problems. Finally, students need to present their models and convince others these models are suitable ways to solve similar problems.
Based on these similarities, I argue that modelling tasks, which are a specific type of numeracy task, can be described or classified as model-eliciting activities. If this is the case, there should be similarities between modelling behaviours and numeracy task

behaviours. This also allows the possibility to map what is already known about models and model-eliciting activities on to numeracy, and provide numeracy and numeracy tasks with more structures, and develop concise definitions for these terms.
Lesh et al. (2000) describe three stages of observable modelling behaviour. During early stages of model-eliciting activities, students often interpret the problem in different ways, and attempt to simplify and reduce the amount of information. At this stage, students tend to focus on specific relationships, patterns, and trends, but ignore other information. They may also express concerns that additional information is required to solve the task. Therefore, at this early stage, students may aim at different goals, and envision different strategies and approaches to achieve these goals. Furthermore, students often switch unconsciously between different ways of thinking.

As students progress in solving the task, they develop more sophisticated ways of thinking. Instead of looking at isolated pieces of data, students look at the data as a whole, and begin to identify various relationships, patterns, or trends in the data. They organize data and information in meaningful ways. Students may also realize the flaws in their initial ways of thinking and the need to modify their strategies to the problem. Students then create models to make meaningful and useful mathematical descriptions and explanations of the situations given in the problem, make predictions to the system's behaviour, and test and refine their models. During the process, students also reinterpret the data and go beyond their initial ways of thinking. They make meaningful judgments of the various approaches to the problem, thereby sort out the weaknesses and combine the strengths of particular approaches. Lesh and Doerr (2003) refer to this as multiple modelling cycles.

Eventually, as students arrive at their conclusions, they take into account the various possibilities that may affect the way data can be processed. Their concluding solution may include supplementary procedures to gather additional information, or a series of telescoping procedures to organize data into groups and apply a different set of procedures to each group. While suggesting these three stages as the process of solving model-eliciting activities, Lesh et al. (2000) also recognize that students' behaviours may not closely follow these stages. Nonetheless, these three stages of behaviours provide researchers with a possible framework for observable behaviours.

CONCLUSIONS

In this study I looked at the similarities between modelling tasks and model-eliciting activities. Based on Lesh's framework around model-eliciting activities and Liljedahl's classification of numeracy tasks, I find that there are interesting similarities between modelling tasks and model-eliciting activities. Further research directions include empirical studies on modelling behaviours during model-eliciting activities and numeracy tasks. If modelling tasks are a type of model-eliciting activities, then we should be able to see modelling behaviours during these numeracy tasks. Another research direction is to look at the possibility to turn other categories of numeracy tasks into modelling tasks or model-eliciting tasks by modifying the instructions to the task, and examine the possibility of integrating numeracy tasks into the curriculum.

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GESTURES AND TEMPORALITY: CHILDREN'S USE OF GESTURES ON SPATIAL TRANSFORMATION TASKS

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This paper discusses findings from a task-based interview with 5 elementary school children working on a spatial transformation task. The paper focuses on children's gestural and verbal communication when engaging in the task. Findings suggest that children use gestures as multi-modal resources to communicate temporal relationships about spatial transformations. Although research has shed light on the use of gestures to represent functions deictically, iconically and metaphorically, this work has not addressed aspects of temporality and the dynamic nature of gestures. This paper raises questions for further research in the area of gestures and communication to address the temporal aspects of mathematics.

Keywords: Geometry and Geometrical and Spatial Thinking, Elementary School Education, Cognition, Reasoning and Proof

GESTURES AND MATHEMATICAL THINKING

The goal of my research has been to extend Presmeg's (1986) work on "dynamic (moving) imagery" for describing the dynamic roots of certain geometrical concepts that are visualized and understood by high school mathematics students. My own work on the effects of teaching with dynamic geometry technology also points to learners' conceptions of mathematics in a dynamic, temporal sense. Lehrer (1998) shows that children are capable to think dynamically before they receive formal schooling, further leading me to attend to the dynamism of young children's geometry. What does dynamic thinking look like, and what mathematics might be involved?

The embodiment of mathematics has led to growing interest and development in the field of gestures in mathematics education research. Recent gesture studies in a mathematical context have focused on different functions of gestures: iconic, indexical and symbolic (Radford, 2003), how gestures accompany and synchronize with speech when communicating mathematical meanings (McNeil, 1992), and in the way students mimic teachers' gestures in mathematical communication (Singer & Goldin-Meadow, 2005). While these studies have provided insights into the multidimensional nature of mathematical thinking, there is a lack of common framework theorizing the relationship between gestures and mathematical thinking. Moreover, much of this work focus on the relationship between speech and gestures and study gestures as independent from cognitive processes. With this dualist approach, gestures are external acts that represent the mathematical thought from within and embodied acts that make cognitive processes explicit. In contrast, Sfard's (2008) commognitive approach provides a lens to combine gestures and thought as one process. In this view, thinking and communicating, in the form of utterances or gestures, are two parts of one entity. Moreover, Sfard (2009) suggests that gestures and utterances complement each other by serving different functions in communication. In this paper, I utilize Sfard's commognitive framework to analyze children's gestures in their communication.

Although gestures have been widely examined in mathematical discourse in recent years, studies have yet to address the temporal functions of gestures in mathematical discourse. McNeil (1992)'s categorization of gestures into deictic, iconic, metaphoric, and beat, only broadly characterizes the type of functions served by gestures. For example, deictic gestures serve as pointing devices, while metaphoric gestures serve to represent the mathematical objects themselves (McNeil, 1992). These categories have not capture the dynamic nature of gestures, in particular, when gestures are used to convey temporal relationship. Given a paucity of research in the area of gestures and temporality, my study aims to address the use of gestures to communicate temporal relationships in mathematical thinking.

THEORETICAL PERSPECTIVES

Gestures, utterances, and thinking

Sfard's (2008) commognitive framework is helpful for examining the relationship between gestures and mathematical thinking. Her non-dualist approach disobjectifies *thinking* as part and parcel of the process of *communicating*. She defines *language* as a system that includes all kinds of symbols in communicational acts, and *gestures* as bodily movements fulfilling communicational function (Sfard, 2009). "Language is a tool for communication, whereas gesture… is an actual communicational action." (p.194) This communicational act can be interpersonal when it is directed to another person, or intrapersonal when directed towards the actors themselves (communicating to oneself). With this view, the actors may be conscious or unconscious of their gestures. Using Sfard's communicational approach will enable studying children's mathematical thinking: "talking and gesturing stop being but 'expressions' of thinking and become the process of thinking in itself." (p.195)

Furthermore, Sfard's (2009) commognitive approach suggests that gestures and utterances take on different roles in mathematical thinking. "Utterances and gestures are the building blocks of commognitive process… Each of these modalities contributes to commognitive processes at large and to mathematical commognition in particular" (p.195). Recursivity is a linguistic feature in mathematical discourse offered by utterances. The unlimited possibility to expand linguistically allows human to work in meta-discourse, or thinking about thinking. On the other hand, gestures enable effective communication to ensure all interlocutors “speak about the same mathematical object” (p.197). Gestures are essential for effective mathematical communication.

> Using gestures to make interlocutors’ realizing procedures public is an effective way to help all the participants to interpret mathematical signifiers in the same way and thus to play with the same objects. (p.198)

Furthermore, gestures can be realized *actually* when the signifier is present, or *virtually* when the signifier is imagined. Sfard illustrates how a student uses ‘cutting’, ‘splitting’, and ‘slicing’ gestures to realize the signifier *fraction.* Since these gestures were performed in the air where the signifier *fraction* is imagined, it was an instance of virtual realization.

Gestures and temporality

The classifications of gestures proposed in gesture studies are problematic in mathematics education for its inability to capture the dynamic nature of gestures. In particular, leading gesture specialist David McNeil’s (1992) classification of gestures into deictic, iconic, metaphoric, and beat, are useful to identify the general functions of gestures, yet they do not distinguish between the static and dynamic nature of gestures. These categories of gestures are too broad and do not consider *how* the message is communicated by the gestures. For example, when a person makes a metaphoric gesture to realize the signifier, *a linear function*, it could be a static one, with the arm or hand representing the function, or a dynamic one, with the hand or finger tracing the motion of the path. In the latter case, the dynamic gestures communicate temporal relationship of the linear function as opposed to the shape of the linear function statically.

Núñez (2006) in his book chapter “Do Real Numbers Really Move?” studied how mathematicians use hand gestures as a way to express dynamic thinking of functions, continuity, and other abstract mathematical ideas. Furthermore, these mathematicians say “approaching,” “tending to,” “going farther and farther,” to express a sense of motion, while producing metaphoric gestures tracing the trajectory of the point or particle with their fingers. Sinclair and Gol Tabaghi (2010) also examine motion in gestures, in particular, mathematician’s hand gestures depicting movement of vectors, providing evidence of time and motion-based conceptualization of vectors. These are two of few studies that examine the use gestures to communicate temporal relationship in mathematics:

> The gestures (and the linguistic expressions used), however, tell us a very different conceptual story. In both cases, these mathematicians are referring to fundamental dynamic aspects of the mathematical ideas they are talking about. (Núñez, 2006, p.177)

To summarize, I use Sfard’s commognitive framework to study children’s gestures as part of their mathematical discourse (thinking) while they engage in mathematical tasks. In addition, I focus on children’s use of dynamic gestures in spatial transformation tasks. I aim to observe the type of gestures that they communicate and instances that their gestures realize aspects of temporality in mathematics.

METHODOLOGY OF RESEARCH

Participants and tasks

I adapted the spatial transformation task used by Goldin-Meadow et al. (2006) and Levine et al. (1999) with minor alteration for my task-based interview. The task contains two halves of a shape that could be spatially transformed to form a vertically symmetric figure. From Goldin-Meadow et al. (2006) and Levine et al. (1999), I selected two questions from each of the four problem types and ordered them randomly to form a set of eight questions. The four types of spatial transformation are: direct translation, diagonal translation, direct rotation, and rotational translation. A 2x2 choice array accompanies each question; it contains four whole shapes of which one of them is the correct answer. Figure 1 shows the problem types and a sample choice array.

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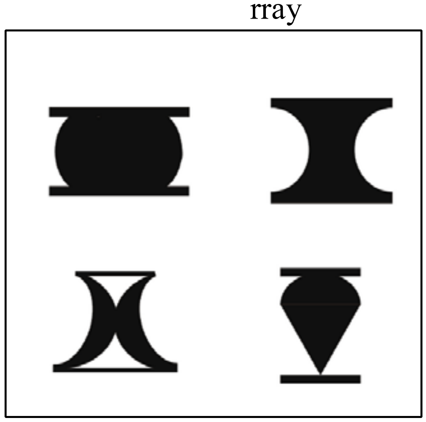


Figure 1: Example of a spatial transformation task and problem types. The choice array is given to the participant along with one of a) to d). The participant is asked to choose which figure in the choice array can be transformed from the two halves.

During the interview, subjects were asked, “if you were to put these pieces together, which one of those will they make?” After the children provided their answers, they were asked to justify their choices or show their reasoning.

My research associate conducted a semi-structured interview with five children in the classroom of their elementary school. The subjects are kindergarten (age five to six) students who live in a low socioeconomic, suburban neighborhood in Northern British Columbia, Canada. Each interview was roughly ten minutes long and was videotaped with the camera facing the interviewer and the subjects. All voice and bodily movement during the interview was recorded. The interviews were transcribed for data analysis.

I adapted the spatial transformation task used by Goldin-Meadow et al. (2006) and Levine et al. (1999) because the task allows children to express static features of the figures as well as dynamic movement in their reasoning. The task is age and level appropriate; data from their studies shows that children of 5 to 6 years of age used a variety of strategies when completing the task. These strategies include:

- Movement: Any indication of movement of the pieces
- Feature: Any indication that the child is focusing on a particular feature of either the piece(s) or the whole
- Whole: Any indication that the child is seeing the pieces as a whole
- Alignment: Placing the piece(s) on top of the corresponding portion of the whole
- Other: Any strategy other than one listed here

Goldin-Meadow et al. (2006) found that children used the "movement" strategy most frequently, on average 5 times out of 8 questions. The children frequently used strategies expressed in gestures without accompanying speech. Their study focused on observing sex difference in using the "movement" strategy as well as comparing overall girls' and boys' performance on the tasks. Their methodology was to record the frequency of each type of strategies used; they did not report that children used more than one strategy at a time when completing a given question.

ANALYSIS OF DATA

Children's strategies for solving spatial transformation tasks

I observed a significant amount and wide variety of gestures used across all five children engaged in the task. They gestured whether or not they answered the questions correctly, and some gestures were expressed with accompanying speech while some were not. In general, the children gestured both deictically to point to the *feature* or the *shape*, and metaphorically to explain *movement*, findings that resonate with Goldin-Meadow et al. (2006). Table 1 illustrates some examples of children's gestures and utterances in their reasoning.

Another interesting observation was that, overall, the children used more gestures in the beginning of the task than when towards the end. All children, except for one (who did eventually begin to make that gesture in the last four questions), used a "movement" gesture in the beginning of the task. This movement gesture involves moving the fingers or hands to mimic moving the pieces together. Some used this gesture extensively, and in general, all children used less of this gesture as they progressed in the task.

Table 1: Examples of Strategies

Categories	Utterance examples	Gesture examples
Movement	"Because them shapes, when they stick together, they can	Place one index finger on each piece, and then move the fingers so

	make that thing." (Child 3)	that the tips of the fingers touch. (Child 5)
Features	"Because of the pointy things." (Child 1)	Point to the specific feature (corner) of a shape with the index finger. (Child 3)
Whole	"…like a fortune cookie." (Child 4)	Make a circle with her arms above her head (Child 1)
Alignment	"If you put that… you can see through the paper, and you can measure the sides, and you will know what shape it will make." (Child 2)	Place the choice array sheet on top of one of the original piece. (Child 2)
Other	"Because I know this." (Child 5)	Gestures that are not categorized in the above list.

Dynamic gestures and temporality

As mentioned above, I observed that all children made a "movement" gesture in the beginning of their interviews. Some used one hand, connecting their fingers to gesture the moving together, while some used two hands. This sliding gesture is a metaphoric one: it expresses the sliding together of the two pieces to form one whole, where the fingers enact the pieces and the transformation. This gesture provides evidence that the children were dynamically thinking about the movement of the pieces. It expresses temporality by tracing the motion of the pieces and their location in time from the beginning to the end of the movement.

All children in the present study used different strategies when completing the task: by pointing to the specific *features* of the shapes ("because of these corners"), perceiving the shape as a *whole* ("it's a trophy"), and suggesting *movement* of the pieces ("because if you put them together, it makes this shape"). They also made the "movement" gesture most frequently, findings that resonate with Goldin-Meadows et al. (2006). However, when I compare children's use of "movement", "features", and "whole" gestures in my study with Goldin-Meadows et al. (2006), some aspects of temporality emerge. The children in my study were making "movement" gestures first and then followed by another strategy. Some children even made this "movement" gesture twice in the same question, first when they attempted to answer the question, and second when they were asked to justify their answers. This suggests that they had been *dynamically* thinking about the movement of the pieces before they provided their reasoning. The key here is that they were not just reasoning in one category at a time. The fact that they used the "movement" gesture before justifying with "features" and "whole" suggests that they were *both* thinking dynamically and attending to details of the shape. This was not discussed in Goldin-Meadows et al. (2006).

Example: child 1's gestures

In the following excerpt, I illustrate Child 1's sequence of gestures and utterances in response to her third question in the task (see Table 2). The child was given the same question as seen in Figure 1, with problem type "b)".

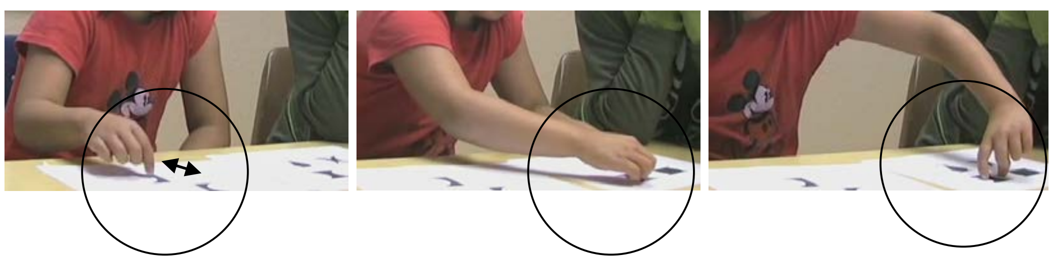
Table 2: Excerpt of Child 1's Utterances and Gestures

		Utterance	Gesture
1 2 3	Interviewer:	How about this one here, these two, which one will it make?	
4 5 6 7	Child 1:	<2 second pause>	Uses right index finger to touch the two half pieces on the question sheet back and forth three times, moving from one piece to another alternatively.
8 9		It will make this one.	Makes pointing gesture with right index finger to touch the top right figure on the answer sheet.
10	Interviewer:	That one?	
11	Child 1:	Yea.	
12	Interviewer:	How do you know?	
13 14	Child 1:	'Cause, see these pointy things?	Makes pointing gesture with right index finger to touch the sharp features of the figure.
15 16 17		Then that goes down,	Makes pointing gestures with right index finger to touch the top left side of the figure and then move downwards towards the bottom left.
18 19 20		and this goes down,	Continue with the same gesture but now move from the top right of the figure towards the bottom right.
21 22 23		and that's how it makes a… <3 seconds pause> window!	Place left index finger and thumb at the middle, narrow part of the figure.
24	Interviewer:	Oh they are	

		windows.	

Child 1 first made a “movement” gesture (line 4) by moving and touching the pieces back and forth with her finger. Then, when asked how she solved the question, she explained that, “’cause see these pointy things? And that goes down, and this goes down, and that’s how it makes a window!” Her utterances, in “pointy things” (line 13) and “windows” (line 24) were accompanied by gestures and exemplified using “features” and “whole” strategies respectively. Therefore, Child 1 has used all three strategies, “movement”, “features”, and “whole” in solving this question. This observation is consistent with the other four children, who occasionally used a combination of strategies both verbally and in their gestures after initially gesturing the movement of the pieces. Figure 2 shows Child 1’s gesture sequence in the above excerpt.

a) “Movement” gesture (line 4) b) “Feature” gesture (line 13) c) “Whole” gesture (line 21)



Utterance: none Utterance: see these pointy things? Utterance: it makes a window!

Figure 2: Child 1’s Gesture Sequence.

DISCUSSION

My data provides strong evidence that children rely on gestures to communicate mathematically: to communicate both with the interviewer and with themselves to internalize their thinking. This claim can be supported by the amount and variety of gestures that the children produced, as well as in the way that the children frequently gestured without accompanying speech. Data also shows that the children used a combination of “movement”, “feature”, and “whole” strategies in the same question, as illustrated in Child 1’s excerpt. Child 1 first produced a “movement” gesture, followed by utterances that characterized “feature” and “whole” strategies while using her fingers to refer to the parts of the figure that she was speaking about (see Figure 2). Sfard’s theoretical perspective provides a lens for explaining this phenomenon. Using Sfard’s definition of gestures as a communicational act that serve to “speak about the same mathematical object” (Sfard, 2009, p.197), the children effectively used a combination of utterances and gestures to communicate their mathematical thinking. Their "movement" gestures enable them to effectively communicate their dynamic thinking by enacting the transformation of the pieces metaphorically. On the other hand, the children also made use of deictic gestures to point to the features and trace the outline of the figure as they spoke about them in speech.

In Sfard's term, the "movement" gestures are *actual* realization since they signified the pieces on the paper actually moving towards each other. These dynamic gestures were used most frequently by the children and exemplified aspects of temporality in children's communication about mathematics. By analyzing the sequence of the children's gesture, I found that the children were both thinking about the dynamic movement of the pieces and attending to the static properties of the shape. This constitutes a very important finding in my study; one that has yet been discussed in relevant studies in the field. My study suggests that young children are not only capable of thinking about spatial transformations dynamically, but they can also communicate the dynamic nature over static properties of geometry. This discussion aligns with Núñez (2006) who argues, "motion… is a genuine and constitutive manifestation of the nature of mathematical ideas" (p.168). Given that temporality is not captured by formalisms and axiomatic systems in the mathematical discourse, my findings discuss the possibility for more opportunities for young children to interact with temporal mathematical relationships and dynamic geometry environment.

Finally, more research is needed to explain the finding that children used fewer "movement" gestures during the course of the interview. I speculate that this could be due to the children's compromise to move towards an adult discourse: the mathematical discourse. In Sfard's (2008) term, there exists a commognitive conflict between the children's and the adult's discourse; the children may well have compromised the way they communicate mathematically as a result of negotiating with the leading discourse of the adult, one that dismisses temporality and dynamism. Further research that examine the change of children's gestures in expressing temporality over time will be needed to warrant this claim about the commognitive conflict between children's and adult's discourse.

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MATHEMATICS AS DESIRE: THE LIFE OF ANDRÉ WEIL

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When mathematicians write about their involvement with mathematics, what lies beneath? What do such accounts tell us about the nature of the discipline and the attendant demands, costs, and rewards? Working from the autobiography of the French mathematician, André Weil (1906-1998), and using the Lacanian notion of desire, I examine the forces that shape and influence engagement with mathematics. I contend that, at an elemental level of human development, these forces turn on the notion of subjectivity and the forms of desire.

Keywords: Affect, Desire, Emotions, Subjectivity

In a simple yet powerful sentence, Tony Brown (2008, p. 91) encapsulates wherein mathematics lies: "Mathematics is accessed through the accounts of others." My interest in general is in seeing what we can learn about the discipline of mathematics from the testimonies of others who have engaged with it. In this paper, I want to investigate what we can learn about the discipline of mathematics from the autobiography of one mathematician, André Weil (1906-1998). My purpose is to consider the following questions: When mathematicians write about themselves and their work, what lies beneath? What do they offer as insights about the discipline or about those who engage in it? While mathematics is often considered as cold and conscious, rational and logical, I contend that engagement with the discipline is a function of the Psyche and the psychosocial Self, and that in our wide range of emotional responses to mathematics (Hersh and John-Steiner, 2011) lie the keys to understanding our relationships with the discipline and hence in eventually shaping how we present and engage with the discipline.

Relationships with and responses to mathematics were first conceptualized in mathematics education research as affect (McLeod, 1992) where the affective domain was defined as the area of emotions, attitudes, beliefs, moods, and values relating to mathematics. Blanchard-Laville (1992) gives an early application in this vein in the in-service training of mathematics teachers. Then in 1993, a landmark issue of the journal, For the learning of mathematics, edited by the late Dick Tahta, brought to the fore the psychodynamics of mathematics education, teasing apart aspects of the self, conscious and unconscious, that we bring to bear on our responses to mathematics including defence mechanisms (Nimier, 1993) and transformations (Nicodemus, 1993). Further the notion of the unconscious was explored in various papers including Skelton (1993) on the work of the Chilean psychoanalyst, Ignacio Matte Blanco, The unconscious as infinite sets

(Skelton was unbelieving when he came across this text as he thought that his two interests of mathematics and the unconscious were so disparate), and Sutherland (1993) on the "emotional side" of mathematics and her inability to find "theoretical explanations for what [she] observe[d] about students when [she] work[ed] in the classroom." (p. 43)

The research has since moved laterally from the theories of Piaget (psychological) and Vygotsky (sociocultural) of the individual as a cognizing subject immersed in cognitive relationships with other "objects", namely, students, teachers, classroom, and curricula to psychoanalytic perspectives and notions of the subjectivity, focusing on the subject who confronts and responds to the discipline. Notable among these researchers are Baldino and Cabral (1998, 1999, 2005, 2006, 2008), Bibby (2009), Brown (2008, 2011), and Walshaw (2004) This paper follows in this qualitative interpretive tradition of examining the self not as a coherent, stable identity but as a Subject immersed in a web of forces and influences, shaping a subjectivity that impels our various responses to the discipline of mathematics. Walshaw (2004, p. 127) writes:

> Psychoanalysis presents complex and well-developed theories of subjectivity. Arguably, psychoanalysis has many shortcomings, yet it does provide us with the most promising theories of how the subject is at once fictional and real.

My points of reference are the Lacanian notions of subjectivity and desire. I present a brief sketch of the relevant notions of Lacanian theory and then show how these figure in Weil's life as rendered in his autobiography.

THEORETICAL PERSPECTIVE

I employ as a critical method some of Lacan's psychoanalytic theory as he sought to 'reread'/rewrite/reformulate Freud. As Žižek (2006, p. 4) writes, "Lacan enlisted a motley tribe of theories from the linguistics of Ferdinand de Saussure, through Claude Levi-Strauss's structural anthropology, up to mathematical set theory and philosophies of Plato, Kant, Hegel and Heidegger." In his 'return' to Freud, Lacan presents a theory of human development that excavates and elaborates the notions of subjectivity and desire.

The Three Registers

Lacan posits three psychic registers or orders of experience: the Imaginary, the Symbolic, and the Real. These registers are not to be understood as developmental stages as they obtain at every sphere of human activity. They function interdependently as they work to shape the constitution of the subject. There are three caveats: 1) Lacan's registers do not line up neatly with Freud's categorization of the id, the ego and the superego, 2) there are various renditions and reformulations of these three as the theory evolves over time by Lacan and his followers, and 3) the order in which the three registers are listed varies among those who take up Lacan's work. I will consider them as they appeared historically over the years of his work: The Imaginary (1936-1952), the Symbolic (1953-1962) and the Real (1963-1981).

The Imaginary is the realm of “images, conscious or unconscious, perceived or imagined” (Lacan, 1973/1981, p. 279) of the people and objects in the world present to us. These idealized images are formed in childhood and persist even into adulthood. One’s sense of ‘self’ starts from the mirror-stage, i.e., from the child seeing its specular image in a mirror or in beholding another child. This marks the beginning of a méconnaissance or misrecognition of ‘self’ as the child imagines the image to be whole and coherent while perceiving itself as fragmented. The Symbolic, derived from the “laws” of the wider world in its structure and organization, disturbs the shaping or the interpellating (Latin: inter/between, within, pellere/push) of the subject. The Symbolic is enabled by language as it is language that gives us the structures for the signifiers for the “I” and the Other, for loss, lack, and absence, for the misidentification of the self with the Other, and for the formation and experience of desire. The Real is the unmarked backdrop against which the Imaginary (image-based) and the Symbolic (word-based) come into play, the screen on which images and words unfold and move.

Towards the end of his work, Lacan saw these three interlinked as a Borromean knot (three circles linked so that if one is cut the other two fall apart). These knots occupied him to such an extent that his later seminars always included careful drawings of knots (Turkle, 1976).

The ego and the Subject

Lacan distinguishes between the ego and the Subject, S, in his L-schema (1966/2006, p. 40). For Lacan, the ego is an Imaginary function, both a defensive and inauthentic agency. Starting from the mirror stage, there is always a lack, a sense of loss and alienation, a splitting in the subject on encountering the ‘other’. Lacan distinguishes between the small ‘other’ and the big ‘Other’. The small other is the Imaginary others, the ones we see as whole and coherent, the ones whom we see as a reflection of ourselves, the ones whose desires we think are completely fulfilled by us. The big Other is the Symbolic order and language into which we are born and into which we must insert ourselves if we are to become subjects. Separation from the Symbolic order and language produces a lack and hence, Lacan describes the subject as the barred subject, $, divided and decentred. Alienation and separation are two constitutive forces for the subject. Homer (2005) writes: “[A]lienation, for Lacan, is unavoidable and untranscendable” (p. 71). Separation, distinct from alienation, marks the beginning of the differentiation of the subject from others and the big Other, and, indeed, marks the beginning of desire.

The objet petit a, the object cause of desire

For Lacan, desire is to be distinguished from need and demand. Examples of need are hunger and thirst in that they can be satisfied. Greater than need is the subject’s demand in its dawning recognition and search of self in relation to others and the Symbolic order. From the lack of the subject and the lack of the Other, desire emerges. In Lacanian arithmetic when need is subtracted from demand, what remains is desire. “[I]t is this irreducible ‘beyond’ of the demand that constitutes desire”. (Homer, 2005, p. 77)

Desire, created and expressed through language, is borne out of the desires of others (imaginary and symbolic) and out of the two lacks in the subject and the Other. The subject seeks to find its place in the Other’s desire and to differentiate itself and its desire from the Other’s desire. While the subject cannot realize the Other’s desire (there is something always unattainable or exceeding in the Other’s desire), there is something in the subject that remains, the remainder, the objet petit a, the object-cause of desire that makes the subject a desiring subject. Lacan provides an elaborate graph of desire built up in successive stages, culminating in the ‘che vuoi’ (literally, what do You (meaning the Other) want, what is it that the Other is asking of the subject?). Hence one of Lacan’s dicta: Desire is desire of the Other.

In the analysis of desire in Weil’s account that follows, I adapt the forms of desire articulated by the Lacanian theorist, Bracher (1993, pp. 20-21) based on the distinction between the desire to be and the desire to have, and consider the following three: 1) The desire to have (possess) the Other as a means of enjoyment; 2) The desire to become the Other (this desire takes the form of identification or love/devotion); and 3) The desire to be the object of the Other’s love (admiration, idealization or recognition).

DESIRE IN WEIL’S LIFE

To summarize, André Weil (1906-1998): French mathematician of Jewish parents (who provided no observance of or instruction in Judaism); educated at the Ecole Normale Supérieure (ENS) with notable teachers including Hadamard and peers including Cartan, Delsarte, and Dieudonné; founded a major new field in mathematics (the algebraization of geometry); formulated conjectures that were related to key conjectures that contributed to Wiles’ proof of Fermat’s last Theorem; served time in prison as a conscientious objector during the War; occupied positions at two prestigious US universities, the University of Chicago and the Institute of Advanced Study at Princeton; one of the founding members of the Bourbaki group; inveterate traveler; writer of one the few autobiographies of mathematicians.

The primary force in Weil’s life in becoming and being a mathematician and spending a life in mathematics can be seen as Desire. Jameson (1977, p. 340) writes of the “logic of wish-fulfillment, le désir, as the organizing principle of all human thought and action.” By considering the three desires above, I show how Weil’s account may be read as a study in Desire.

Desire to have (possess) the Other as a means of enjoyment

Weil's passion for mathematics begins at an early age and continues through his education. The following extracts from his autobiography show how the Imaginary aspects of his engagement with mathematics (the content, the means of acquiring the content such as textbooks and teachers, and his own attempts at doing mathematics) contribute to his enjoyment of mathematics. His first observation about mathematics comes when he is eight years old: "Once when I took a painful fall, my sister Simone could think of nothing for it but to run and fetch my algebra book, to comfort me." (Weil, 1992, p. 23) This is an astounding sentence as small children in times of pain generally turn to a parent or a favourite teddy bear. That an algebra book is a transitional object (to use Winnicott's term) for Weil is indeed singular. Textbooks greatly influence his developing interest in mathematics:

> … I still have an algebra text written by Bourlet, for third, second, and first form instruction, which was given to me in Menton in the spring of 1915. Leafing through it now, I see it is not without its defects; but it must be said that this where I derived my taste for mathematics. (pp. 21-22)

This will resonate with anyone who has preserved a similar textbook over the years as a keepsake of times of great delight and persuasive power.

Weil is further swayed by a teacher, Monsieur Collin who impressed on him the demands and fascination of mathematics:

> I do not think that any teacher could have been better than Monsieur Collin in developing both rigorous thinking and creative imagination in students… definitions had to be memorized and Mr. Collin was merciless towards any gap in solutions or proofs. With him, mathematics was truly a discipline in the fullest sense of this beautiful word. (p. 26).

From Mr. Collin, Weil experiences the triumph of logic and rationality in capturing and rendering the discipline.

All of this is nothing compared to the intense and powerful enjoyment of creative mathematical activity.

> Every mathematician worthy of the name has experienced, if only rarely, the state of lucid exaltation in which one thought succeeds another as if miraculously, and in which the unconscious (however one interprets this word) seems to play a role. In a famous passage Poincaré describes how he discovered Fuschsian functions in such a moment. About such states, Gauss is said to have remarked as follows: 'Procreare jucundum (to conceive is a pleasure'; he added, however, 'sed partuire molestum (but to give birth is painful)'. Unlike sexual pleasure, this feeling may last for hours at a time, even for days. Once you have experienced it, you are eager to repeat it but unable to do so at

> will, unless perhaps by dogged work which it seems to reward with its appearance. It is true that the pleasure experienced is not necessarily in proportion with the value of the discoveries with which it is associated. (p. 91)

Weil writes of two such moments, one relating to working on Mordell's conjecture relating to his thesis and another relating to a discovery of a result resolving a problem on polynomial series. By now, Weil is versed in the highs and lows of mathematical discovery and activity and their attendant emotions.

Desire to become the Other as a form of identification or love/devotion

The subject forms itself in the images of others and of what others expect and desire of him/her. Lacan writes of the subject as a signifier for another signifier (1966/2006, p. 713). Both "mathematics" and "mathematician" are master signifiers for Weil; they loom large as the big Other for Weil. From early on, Weil receives clear signs of what it means to be in or to do mathematics, and to be a mathematician. These are instrumental in shaping him as a mathematician and a mathematical subject. To what does Weil aspire in his conceptions of a mathematician and in his desire to identify as a mathematician?

From Monsieur Collin, Weil learns to appreciate the rigour and precision in mathematics: "What I remember most about Monsieur Collin's lessons prior to entering the first form is that he showed me once and for all that mathematics operates by means of rigorously defined concepts." (pp. 26-27) When it came to definitions: "I do not recall in what terms Monsieur Collin taught me the definition of the word "function"… once the definition was given, he would not tolerate anyone's using the word "function" for anything not corresponding to the definition." (p. 27) Hence, in the Imaginary order, the conception of mathematics that Weil received and took to heart was a very particular one of rigour, firm foundations, and precise rendering. This was to be borne out in the way that the future Bourbaki project was conceived and executed, its express aim being to place mathematics on a careful axiomatic basis and to set the standard for rigorous exposition with pedagogical intent.

With regard to identifying with the discipline and aspiring to be a mathematician, Weil's admiration and recognition of Monsieur Collin's efforts in "making a mathematician of me" is great: "I think there is no one, with the sole exception of Hadamard, from whom I learned more about mathematics from Monsieur Collin. Before I became his pupil, I was basically self-taught: he made a mathematician of me, and he did so above all by means of his unrelenting criticism." (p. 27) Later on Weil writes that "[t]he bibli[othèque] and Hadamard's seminar are what made a mathematician out of me." (p. 40) Weil further credits Monsieur Collin for teaching him "how to write-up mathematics" where he learns to limit himself to "two pages into which everything had to fit" and not to take shortcuts such as saying "it is obvious that…" (p. 27). This is reminiscent of a remark by Alain Connes [also educated at the ENS] about communicating a particular mathematical discovery ("and because I had been taught by Chevalley, I wrote this up in half a page") to a colleague who found it somewhat terse.

How is Weil formed as a mathematical subject? For Weil, a big part is his exposure to analysis, both grammatical and propositional, in a "non-trivial symbolism".

> [Monsieur Monbeig at the lycée] was an exceptional teacher, full of unconventional ideas. For the purposes of grammatical analysis, he had invented a personal system of algebraic notation, perhaps simply to spare both himself and his students time and effort; but it seems to me, looking back, that this early practice with a non-trivial symbolism must have been of great educational value, particularly for a future mathematician… At one time it has been thought that young children should be primed for the study of mathematics by being forced to speak of sets, bijections, cardinal numbers, and the empty set. Perhaps I was no less well prepared by my study of grammatical analysis – both verbal and, as it was called at the time, "logical" (that is, propositional) analysis – at the hands of Monsieur Monbeig. I must say in any case that nothing I later came across in the writings of Chomsky and his disciples seemed unfamiliar to me. (p. 20)

As preparation for being a mathematician, Weil also writes about as the geometry of the triangle and the focal theory of conics as sharpening the "geometric imagination" and of the method of "complete enumerations" as something that is disparaged today but leaves him with favourable memories. He notes, "a facility with algebraic manipulation as something a serious mathematician is hard put to do without." (p. 35) Weil also mentions a wide range of other subjects such as poetry, history, French literature, Greek, and Latin as part of his education. Most of all, he learns that it is "best to learn the rules before breaking them." Here Weil has a clear sense of the demands that mathematics imposes.

Weil fashioned himself in the example of his teacher, Hadamard: "I had formed the ambition of becoming, like Hadamard, a 'universal' mathematician: the way I expressed it was that I wished to know more than non-specialists and less than specialists of every mathematical topic. Naturally, I did not achieve either goal." (p. 55) Hadamard has been referred to as the last 'universal' mathematician – "the last that is, to encompass the whole of the subject, before it became so large that this was impossible". (Derbyshire, p. 159) This romantic notion of a 'universal' mathematician, of one who understands every topic in mathematics is appealing to Weil in his wish to conquer the field. To this end, he combined his passion for touring with "a specific mathematical variety", that of visiting and meeting with mathematicians "in their natural habitat". Indeed the list of mathematicians whom he met and visited in various cities is staggering [the list includes Berlin (Brouwer, Hopf, Schmidt), Finland (Ahlfors, Nevanlinna), Frankfurt (Dehn, Epstein, Hellinger, Siegel, Szász), Gottingen (Courant, Noether), Hamburg (Artin), Moscow (Pontrjagin), Rome (Lefschetz, Mandelbrojt, Volterra, Zariski), and Stockholm

(Cramér, Mittag-Leffler)]. While this may be interpreted as a mere gratification of a scopic drive to see (and hence to possess in some way) people and landscapes, Weil describes it as a way of determining whether they [mathematicians] are worth reading ("Despite all the errors to which this method exposes one, it actually saves considerable time."). This also hints at a possible fear of competition and a wish to see if others were gaining on him, as it were.

Desire to be the object of the Other's love (admiration, idealization or recognition).

Why does Weil write an autobiography? Very few mathematicians have done so. I argue that Weil's account of his life in mathematics can be seen as a quest for recognition from the Other as devotion and service to a Cause. There is great pride in that 'glorious day' when he sees his name in print for the first time in recognition of him and his work. From then on, Weil's various mathematical results and especially his work as a major force in Bourbaki can be seen as offerings to the big Other of mathematics. Weil has a clear sense of the Symbolic order from early on:

> One day my father, taking a walk with me along the boulevard told me that my first name came from the Greek word meaning "man", and that this was one reason it had been given. Did he go on to say that I must prove worthy of this name? I do not recall; but certainly that was the intention of his words, and it is thus that the meaning sticks with me. (p. 13)

To be worthy of his name is then the root of Weil's desire to be the object of the love of the big Other of mathematics. Hence Weil has an early sense of duty in his life and his autobiography is his address to that call. Weil feels a duty to himself as a mathematician of a country that has sustained serious losses of its mathematicians: "Already while at the Ecole Normale, I had been deeply struck by the damage wreaked upon mathematics in France by World War I" (p. 126). A second duty for Weil is his duty to the discipline of mathematics and the community of mathematicians; his account may also been seen as a letter addressed to mathematicians. Weil seeks to reproduce and to enhance the knowledge system in which their sense of themselves as mathematicians is inscribed. He seeks to take his place in that community and to ensure that his legacy is remembered on his own terms. Žižek (interpreting Lacan) writes that a letter always gets to its final destination, even when there is no addressee. Weil's autobiography is his attempt to not choose death and obscurity but to assure and secure his place in the history of the discipline on his own terms.

CONCLUSION

> Insofar as a cultural phenomenon succeeds in interpellating subjects – that is, in summoning them to assume a certain subjective (dis)position – it does so by evoking some form of desire or by promising satisfaction of some desire. (Bracher, 1993, p. 19)

A cultural phenomenon such as mathematics survives and flourishes by inducting its practitioners in its ways, by offering rewards (praise and ignominy), and by stirring up desire, both attracting and repelling. From the above analysis, Weil's interpellation as a mathematical subject is then a product of forms of desire. For Weil, "mathematics" is a powerful complex of notions that functions as a master signifier in the Imaginary, the big Other in the Symbolic, and the object-cause of desire in the Real. Indeed, Weil's account is part of his answer to the che vuoi of mathematics as the big Other.

What are Weil's desires? That he becomes a mathematician and engages in a life in mathematics is primary for Weil. That his life was to be in mathematics or that he valued mathematics was not evident in the beginning: "It was not yet obvious, either to my family or to my teachers, or even to me that I was destined for a career in mathematics." (p. 28) His father had planted the seed of his name meaning 'man' in Greek. When Weil writes of Hadamard making a mathematician of him, Hadamard becomes his mathematical father and for Weil, the expression 'making a mathematician of him' is very nearly synonymous with 'making a man of him'. Weil chose for the title of his account in French, Souvenirs d'apprentissage, which has more of a flavour of a training or apprenticeship. It is only in the English rendering of the title that we get some hint of the work having to do with being in mathematics. My reading of this is that mathematics is so elemental a signifier for Weil that it is not necessary for him to include such a sign. Lacan refers to this as "disappear[ing] as a subject beneath the signifier [that] he becomes." (1966/2006, p. 708)

What does it take to lead a life in mathematics? Weil's life in general has mostly an even tenor despite the hardships caused by grave forces such as war and prejudice. For Weil, mathematics requires degrees of both isolation (he completed some of his best work while in prison) and collaboration (he thrived on getting to know and keeping up with the developments of mathematics around him). It also requires episodes of creativity that cannot be summoned at will but perhaps are given as rewards for sustained effort, and unswerving one-pointed dedication to a goal whose sights keep coming in and out of focus. Žižek (1991) writes: "The real source of enjoyment is the repetitive movement of this closed circuit [the path to and from the goal]." (p. 5)

Reading Weil's autobiography, one gets the sense of Mathematics as Being; that he was a consummate mathematician and could be no other. There is the sense of existential destiny and inevitability, a sense of little else that absorbed him. Do we find mathematics or does it find us?

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NUMBERS ON FINGERS

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This paper describes TouchCount application (designed for iPad) and its two Counting and Adding worlds. We explore how a five-year old child (Kindergarten level) builds meaning through communicative-touch based activity involving talk, gesture and body engagement. The main goal of this paper is to show the impact of touch-based interactions on the development of children's perception and motor aspects of ordinality and cardinality of numbers. In this case study, we found a strong value of mathematics embodiment in emergent expertise in producing and transforming numbers, which can be supported with the Perceptuomotor integration approach theory.

Keywords: Counting, Adding, perceptuomotor, Digital technology

Modern-day learners expect information to be at their fingertips. They interact, play, source information and socialize through instantaneous mediums such as mobile phones, tablets, iPads, etc. Today's children use technology not only to entertain themselves and their peers but also to learn. There is a number of Mathematics educational software that is developed for computers and laptops; learners interact with them via mouse, keyboard or/and electronic pens. Interacting with computers via those devices, needs hand-eye-coordination which is a hard task for young children. Moreover, they" indirectly" manipulate objects through the computer via keyboard and mouse. Alternatively, touch-sensitive interfaces feature of iPad enables children to "directly" interact and manipulate objects via both their hands and all their ten fingers. Additionally, auditory, visual, tactile senses and kinesthetic touch through gestures (flicking finger, sliding finger, taping, nudging, pinching, spreading, etc.) let to learner to be engage bodily in learning strongly.

Finger-number interaction

Fingers play a vital role in developing number sense for children. They use their fingers for counting while simultaneously thinking and saying the numbers. In fact, the use of ***fingers*** to represent ***a*** number is ubiquitous across ages and cultures. Children use finger counting, even if they are discouraged to do so, sometimes even before they are able to utter the number word sequence. Butterworth (1999) argues that fingers play a functional role in the development of a mature counting system. Researchers have found at least five distinct brain areas, including somatosensory cortex, are involved in representing the fingers. In addition, Finger movement and finger positions are associated with numerical meaning (Butterworth, 2000). Researches on neurosciences have revealed that there is a strong relation between fingers and numbers. In fact, There is a relation between number and neuro-functional fingers (Andres, Seron, & Olivier, 2007). Gracia-Bafalluy and Noël (2008) found that improving finger gnosis in young children (grade one) provide provides more useful support to number sense than grade two and three students (Gracia-Bafalluy & Noël, 2008).

Vision also plays a very important role in counting and showing quantities when young children develop conceptual understanding of counting(Crollen, Mahe, Collignon, & Seron, 2011). Therefore, using fingers to create numbers when it is supported by audiology and visual provision will support and augment cardinality and ordinality of numbers for counting and adding.

There are many number-related apps on iPad, though most of them are game-based. In most of these games, the learner is not able to create and manipulate numbers with all his/ her fingers and both hands. Thus, in order to develop one-to one correspondence between numbers and fingers and enhance number sense and mathematical relation TouchCount has been designed.

TouchCount:

Currently, there are two sub-applications, one for Counting (1, 2, 3,…) and the other for Adding (1+2+3+…). The app runs in three different languages: French, English, and Italian (Sinclair, 2013).

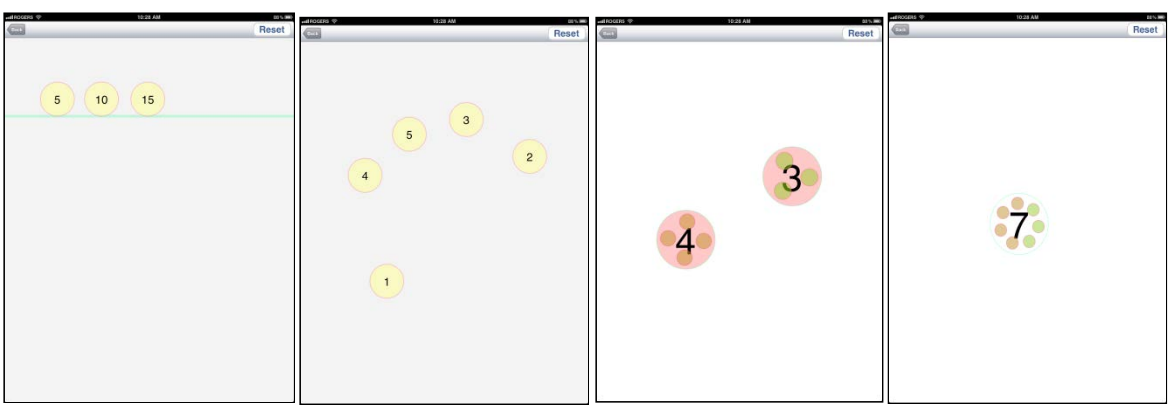

Figure 1: (a) Default Counting world with numbers on the horizontal bar; (b)No gravity Counting world; (c) Two groups in the Adding world; (d) resulting sum with colour-based record of the addends.

Counting and Adding World (1, 2, 3…)

The initial goal of counting world is to enable development of one-to-one correspondence between ordinal numbers and objects. Small dots appear on the screen when user taps their finger on it. The numbers (dots) will be represented in both written form and spoken word. In the default mode, numbers fall down off the screen, unless they are placed on the horizontal bar. User can also swipe number (dots) off the screen or reset the application by its designed key (Figure 1-(a) and (b)).

As the Counting world emphasis on ordinality of the numbers, Adding world is designed to target cardinality of numbers. In this world, a group of numbers will be created each time that user taps on the screen that is labelled by its cardinality (See Figure 1 (c)). Pinching these groups makes the fundamental metaphor of addition, which is gathering together. The Adding world is intended to provide an embodied practice for the addition operation (Sinclair, 2013).

As it is stated in Sinclair (2013), Nunes and Bryant (2010) claimed that children "need to understand cardinality; they need to understand ordinal numbers, and they need to understand the relation between cardinality and addition and subtraction (p.8)." as three type of connections between quantities and number words. Therefore, "while the Counting world focuses on ordinality and the objectification (Sfard, 2008) of number, the Adding world specifically targets the idea of cardinality" (Sinclair, 2013).

THEORETICAL FRAMEWORK

In general, we are fascinated in emergent of learning and the role of tools and artifacts in a technology-based environment. Moreover, having the nature of the interface, we are interested in using a theory, which emphasizes on the role of embodied practice on learning mathematics when a learner is engaged in an activity with digital technology. Therefore, we adapt Nemirovsky's Perceptuomotor integration approach. In this theoretical framework, "mathematical expertise involves the systematic interpenetration of perceptual and motor aspects of playing mathematical instruments." The perceptuomotor integration approach lays on a non-dualism view of human cognition and therefore, a strong version of mathematical embodiment. It assumes mathematical learning occurs through a transformation of a bodily engagement of a learner in a specific mathematical activity. Therefore, we are interested in developing fluency and associated changes in the way that learners move, pause, gesture, talk and, etc. "As the perceptuomotor fluency emerges, greater interpenetration of perceptual and motor aspects is revealed by retentions and pretentions that each includes both perceptual and motor aspects" (Nemirovsky, 2011, p. 21).

METHODOLOGY

Several Kindergarten children were interviewed, all aged between 5 and 6 near to the end of the academic year. Each interview took about 20-30 minutes. Having the perceptuomotor integration lens as theoretical framework besides being interested in emergent instrumental expertise as an embodied phenomenon, we chose some episode that could appreciate our goals with that represented a range evolving perceptuomotor expertise for one of the children named Sarah. Sarah is selected for discussion in this paper because she showed higher lever of tool fluency and engagement in the interview.

RESULTS

The session starts in Counting world with the interviewer asking a question: “Can you put some numbers there?". Without any instruction, Sarah starts by placing her right index finger on the screen.

10 iP One

11 S [Smiles, puts all right hand fingers and palm on screen]

12 iP Six

13 S [Pauses, looks at interviewer and smiles]

14 S [looks back to the screen, pinches all fingers and puts them on the screen while her palms touches screen as well]

15 iP Thirteen

16 S [looks at interviewer, smiles]

17 S [looks back to the screen, taps repeatedly with all her fingers]
17 iP Twenty eight
18 S [Starts taping continually with all fingers like playing a piano]
19 I WOW, it’s raining number, isn’t it?
20 S [laughs]
21 I So, If you put your fingers up here…[Pointing to horizontal bar line]

It seems she does not listen to the interviewer instruction instead she listens and repeats the number that iPad is saying.

22 iPad Two hundred and twenty seven
23 Sarah [laughing] two hundred and twenty seven!

The interviewer then gives Sarah an instruction to pull out a number and put on the horizontal bar. Then she asks Sarah

30 I what is your favorite number?
31 S A hundred
32 I [Rests the App] How would you get just 100?

Although, it seems Sarah is able to find 100 but she fails to put 100 on the bar and misses 100. She tries again but she passes and misses 100. In the third try, Sarah’s tapping become purposeful, regular and rhythmic which suggests she can anticipate when 100 will come.

In another episode Sarah explore the Counting world to find “infinity”. She taps on screen rapidly with her two hands:

41 S : Let’s see when I stop what big answer is” [smiles, tilts her head down and taps quickly on screen.]
42 I Okay
43 S If I get to [smiles tilts her head up and looks aside]

44 S Infinity! [smiles looks at interviewer]
45 I Oh!
46 S She continues tapping when she looks down on iPad and sometimes aside. When she stops tapping, all raining numbers has fallen and no number is seeing on screen. Sarah waits to iPad says the number! She is looking for the cardinality of Infinity.
50 I [laughs] It's too big. Let's see, if you tap up here will get to see what the number. Tap up here. You got to one thousand one hundred and eighty five.
51 S [Astonishes, looks around and smiles]
52 I WOW!

The interviewer then switches to the Adding world.
60 I this is one you may make the groups and put them together.
61 I [Using pinch gesture on the air]

Immediately; without direct instruction, Sarah uses pinch gesture to "gather" numbers together. Interviewer asks Sarah to make a Seven. In the first attempt, Sarah make a 7 by gathering 5 and 2. Then, the interviewer asks for a group of seven in another way than adding 5 and 2, as she perceived at that time Sarah made 7 by the chance. Sarah's try on making 7 by adding 2 and 4 fails and she realizes that 2 and 4 makes 6 and not 7. Thus, she continues:

70 S 3 and 4?
71 I OK. Let's try it
72 S [Taps with index, middle and ring fingers.] (Figure 2- A).

73 iP Three
74 S [Taps again with same three right-hand's fingers.] (Figure 2-B).
75 iP Three
76 S [Looks at her three fingers to check] Ops!

Sarah forget to make a 4; thus, she taps on the screen and make a 1 (Figure 2-C).

77 iP One
78 S [At the same time Sarah whispers] One

Then she pinches 1 and 3 by index and thumb fingers and makes four following by gathering 4 with 3.

79 S (Whispering) Put together (Figure 2-D).
80 S (Smiles tilts her head back). I made it!

Figure 2: Making 7 by adding three different groups 3, 3, and 1

81 I Good job, you made it with three different groups, a group of 3, a group of 3 and a group of 1.

82 S One, three, three.

83 S (Moves on chair, bends on screen)

Then Sarah stops speaking with a long pause, touches upper edge of iPad, looks around, puts her hand under her chin.

85 I what are you thinking about?

86 S [looks around, puts hand under her chain]. I am thinking (pause) making (pause) adding, 10 and 10 to see what that makes.

88 I Wow, that's a good idea. Can I show you a trick? As long as you have your fingers down [Interview put her five left-hand fingers on screen]; see how it says 5 there, I can keep adding to that group [interviewer keeps adding to 5 by tapping with her right-hand index finger on screen]. And let it go. That might be an easier way for you to do it (Figure 3-A).

Consequently, Sarah resets the app and makes 10s by the same method rapidly (Figure 3-B). Then she pinches 10 and 10 and makes a twenty.

89 iP Twenty

90 S That's why they say, 5, 10, 15, and 20.

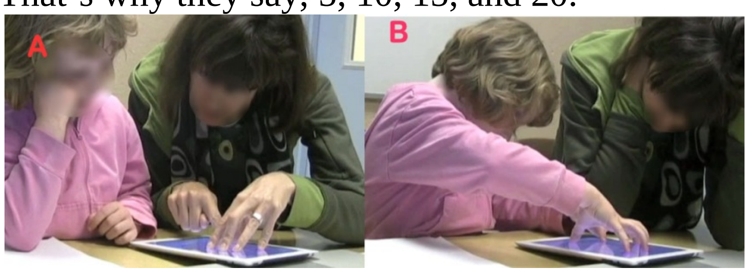

Figure 3: making big numbers with both hands

It seems Sarah has found the importance and role of "Fiveness" in producing numbers and its relation with number of fingers in each hand.

After about 5:04 minutes spending time in "adding World" Sarah became an expert in using both hands and all fingers to make numbers and gathering them up, which is in the line of perceptuomotor theory. Her justification and gestures in the upcoming episode reveal her competence in achieving both mathematical concepts of addition and motor actives.

91	S	[puts her five fingers on iPad]
92	iP	Five
93	S	[taps with all fingers]
94	iP	Ten
95	I	Oh, cool you made 10 really nicely there.
96	S	[Pinches 5 and 10]
97	S	Fifteen
98	I	Cool!

99 S I did a 5 [putts her right hand fingers into left hand] (Figure 4-1), 10 [puts her left hand fingers into right hand] (Figure 4-2) and then I just made a 15(grabs numbers on the air and putting them together, Figure 4-3), added them together.

100 I Yea, you added them together, good job!

Figure 4: Sarah's gesture for making 15.

Previously, gestures were made on the screen; now they are in the air, maybe because Sarah comprehended the metaphor of adding.

DISCUSSION

In this research Sarah was actively engaged in Counting and Adding worlds by responding to interviewer questions, exploring the worlds and reasoning. Nemirovsky's Perceptuomotor integration approach suggests that learning is understood in terms of alternative shifts in bodily experience so we traced such movements in our study. In the starting point of the interview when Sarah was exploring in counting world, she realized that she can make numbers with all her fingers, and each finger is the correspondent (10 to 20) to a number. There are several occasions that she expresses her pleasure and proud of making and creating many (e.g. rain of numbers) or specific number (100 as her favorite number). Sarah showed evidence of fluency in a sense of ordinality of numbers once her tapping came purposefully and rhythmic. The results presented that she could anticipate what numbers come before and after one hundred. Sarah's success in the pull out a number putting on the horizontal bar reveals a kind of objectification. In addition, her attempts to reach and objectify infinity can be an evidence of her strong bodily engagement in the activity.

Moreover, we found several dimensions of tool fluency and associated changes in the way that Sarah interacted with the device that was demonstrating in her pauses, gestures, talks and acts. She showed high level of shifts in bodily experiences from using one hand to both hands and eventually gesturing on the air (Figure 4) (Robutti, 2006), which is a result of her learning (Kim, Roth, & Thom, 2011). Sarah's expertise developed further as she came to integrate relatively all fingers, and both hands use with small errors.

The displayed episode in Adding world, demonstrated noticeable shifts in Sarah's bodily engagement. Sarah showed a development of tool fluency in this world. For example, when she tried to produce a group of 7, she successively added three groups of numbers with each other (82). Sarah's gesture in last episode indicates her integrated understanding on adding numbers in a sense of making or creating numbers and then "gathering" them together in the air which its initial idea arose from working on Adding world (figure 4). Moreover, making big numbers with 5s and 10s suggests Sarah's awareness on Fiveness in creating number.

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BRINGING 'REALITY' INTO CALCULUS CLASSROOMS: MATHEMATIZING A PROBLEM SIMULATED IN VIRTUAL ENVIRONMENT

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The study explores how students, who had completed the AP calculus course, mathematized the optimal navigation real-life problem simulated in the Virtual Environment (VE). The particular research interest was to investigate the factors determining the ways of students' horizontal and vertical mathematizing, including the role of their empirical activity in VE and the role of intuition. It was found that empirical knowledge prevails over intuitions and that horizontal mathematizing is fully grounded on empirical activity.

INTRODUCTION

A troubling problem with current education is the practical application of knowledge. Graduates do not know how to apply knowledge to many problems that arise outside the walls of school (Ilyenkov, 2009). A serious mismatch exists and is growing between the skills obtained at schools and the kind of understanding and abilities that are needed for success beyond school (Lesh, & Zawojewski, 2007). The problem of 'the practical application of knowledge' is especially significant for calculus, which has numerous applications.

The idea of including the out-of-school world in mathematics education, implying that focus be put on real-life applications, is not new and was emphasized in education policy in many countries (Palm, 2006). Regarding teaching and learning calculus, in the late 1980s the Calculus Reform movement began in the USA. One of the desired characteristics of calculus course was that students and instructors would find the applications real and compelling. Palm and Burman (2004) reported that, in Finland and Sweden, in many of the tasks encountered by students in school mathematics the situation described in the task, is a situation from real life.

A traditional way of description of the contextualized tasks containing out-of-school real life situations is a so-called '*word problems*'. Word problems are firmly entrenched as a classroom tradition, particularly in North American schools (Gerofsky, 1996). And yet, there has been long lasting debates about the reasons for the lack of word problems' effectiveness as a link between abstract mathematics and real-life phenomena.

The contemporary computer technologies can provide much better simulations of real world situations in mathematical classrooms for connecting the mathematical abstract with out-of-school situations, which is a point of this research. The purpose of this study is to utilize Virtual Environment (VE) as a method of simulating real-life situations so that to bring the reality to the calculus classrooms. The task for the students was to find the optimal path in VE empirically, and then to transfer the simulated in VE real-life situation into a mathematical formal task. The particular research interest was to investigate whether/how students' empirical activity in VE influences their mathematical activity.

THEORETICAL BACKGROUND

More than forty years ago Freudenthal (1968) posed the problem of lack of connection between mathematical knowledge and its real-life object. The Freudenthal Institute has developed a theoretical framework of Realistic Mathematics Education (RME) (Freudenthal, 1968, 1991), which is based on Freudenthal's idea that mathematics must be connected to reality. The use of realistic contexts became one of the determining characteristics of RME approach to mathematics education. The most general characteristic of RME is mathematizing, wherein the realistic contexts must be used as a source for mathematizing.

Treffers (1986) formulated the idea of 'progressive mathematizing' as a sequence of two types of mathematical activity – horizontal mathematizing and vertical mathematizing. He suggested that horizontal mathematizing consists of non-mathematical real world situations, transforming the situations into mathematical problems. Vertical mathematizing is grounded on horizontal mathematizing and includes reasoning about abstracts within the mathematical system itself.

Another important aspect of RME is a special role of models. According to Streefland, cited in (Van den Heuvel-Panhuizen, 2003), models can fulfil the bridging function between the informal and the formal level: by shifting from a *'model-of'* to a *'model-for'*. At first, the model is a model-of a situation that is familiar to the students. By a process of generalizing and formalizing, the model eventually becomes an entity on its own. It becomes possible to use it as a model-for mathematical reasoning.

The RME theory has been accepted and adopted by some educational institutions of England, Germany, Denmark, Spain, Portugal, South Africa, Brazil, Japan, and Malaysia. In spite wide acceptance and adaptation of RME, the recent research shows that there is still a wide gap between the world of knowledge obtained at school and the world of conceptions found in real-life experiences. The claim of this paper is that the reason of why students do not connect the mathematical world with reality is because they continue mathematizing only 'word problems' but not real-life situations which include the students' activities directed at the objects to be mathematized.

MATERIALS, METHODS, AND PARTICIPANTS

Second Life VE was used for programming an interactive setting for the empirical real-life optimal path finding task. The simulated setting includes a pond with shallow water and two platforms: one platform is located on land near the water's edge; another is located in the water (Fig.1).

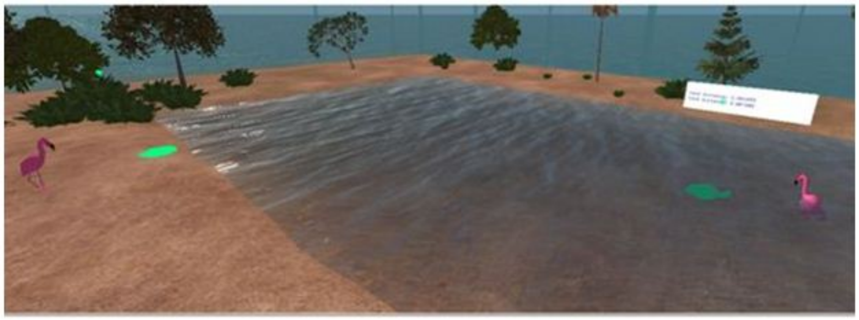

Figure 1: Simulated in the Second Life VE interactive setting.

The setting was programmed so that walking/running speed on land is twice as fast as walking/running speed in water.

The task for the student in this VE was to travel between the two green platforms trying to find the path, which would give the shortest time of travel. The environment was programmed to record the time spent for each trip with a corresponding distance travelled by land and to display this information on the banners (Fig.1). After each trip the student had to transfer the data into a specially designed guiding–reflecting journal, which was an integral part of the instructional/experimental design.

The aim of the guiding–reflecting journal is to connect the student's optimal navigation practice in the VE with the calculus optimal path finding task, which in turn is available in almost every calculus textbook (Pennings, 2003). According to RME instructional design theory, the teacher provides guidance, playing a 'proactive role' within the classroom setting. In this study every student decides whether and to what extent he/she needs guidance. In other words, the students had a free choice: either to construct and develop their own models-of the situational problem or to accept and develop the journal's model. The journal contains blank space for independent reasoning and provides help/hints/directions for those students who need guidance and/or additional information.

The model offered in the journal corresponds to the calculus optimal path-finding task described in (Pennings, 2003). Namely, the task is to reach an object B, located in water, from the position A, located on land close to water edge, and to find such a path that would minimizes the time of travel from A to B (Fig. 2).

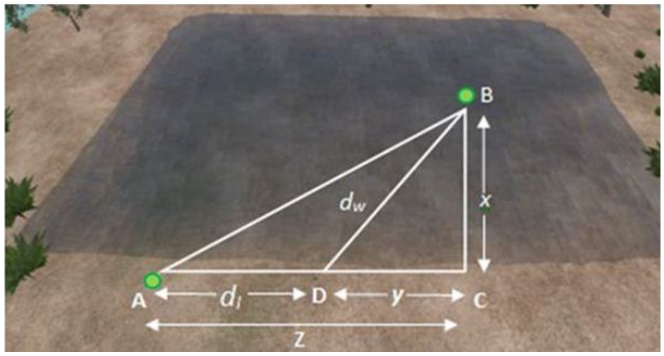

Figure 2: Possible paths: from location A on Land to location B in water.

Path AB is the most direct and shortest, but also is of the longest water distance. Since the speed in water is slower than speed on land, the choice could be to use the shortest water distance which means sprinting down the beach to the point on shore closest to the 'water' platform, which is C, and then turning a right angle and moving to B. Finally, there is the option of using a portion of the land path, up to D, and then entering into the water at D and moving diagonally to the water platform.

Let z denote the distance between A and C. Let $y = z - d_l$, and x represents distance between B and C. Speed on land is s_l; speed in water is s_w. Then time spent for the trip is

$$T = \frac{z-y}{s_l} + \frac{\sqrt{x^2+y^2}}{s_w} \qquad (1)$$

The condition of minimal time:

$$T'(y)=0, \text{ or } \quad \left(\frac{z-y}{s_l} + \frac{\sqrt{x^2+y^2}}{s_w}\right)' = 0 \qquad (2)$$

$$y = \frac{x}{\sqrt{\frac{s_l}{s_w}+1}\sqrt{\frac{s_l}{s_w}-1}} \qquad (3)$$

The designed study contained three main stages. The exploration trial was the first stage of experimental design that allowed students to explore the pond with its shallow water and to feel the speed difference on land and in water. The goal of exploration trial was to let students get feeling of 'being' in the environment before starting the next, second, stage of the designed study which is an optimal navigation in the VE. The third stage of the designed study is mathematizing the VE activity, which implies the journal work only.

The students ranging in age from 17 to 18 years, who had completed the AP calculus course at a high school of participated in the research study. The participants' exploration of the computer environment was screen recorded by SMR software. Their work with the journals was video-recorded.

RESULTS

The first participant, named Kenneth, performed fully independent, non-guided mathematizing. His VE activity was characterized by deliberate planning, realizing the trip strategies, and collecting empirical data for ‘transforming a problem field into a mathematical problem’ (Treffers, 1997). I call such activity ‘empirical mathematizing’.

Kenneth spent 3.45 min exploring the environment and had an opportunity to feel the speed difference in two different mediums. Moreover, he was informed by journal instructions that the speed on land is two times faster than the speed in water. Nevertheless Kenneth’s first two trips were straight lines between the platforms, which maximize lower speed water part of his path (Fig. 3).

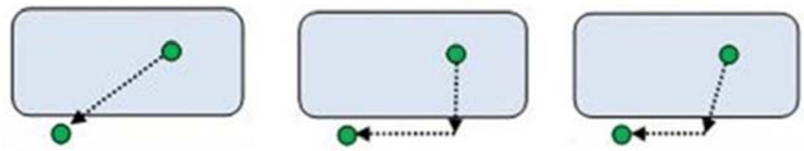

Figure 3: Kenneth’s three strategies. The first is of maximum water distance (left diagram); the second is of minimum water distance (middle diagram); the third strategy is Kenneth’s best time trip (right diagram).

Obviously, Kenneth chose the straight line as a shortest distance having in mind an intuitive model that the shortest distance would give him the shortest time. Kenneth’s tacit intuitive model prevailed over his knowledge about speed difference due to its robustness (Fischbein, 1989). Remarkably, that already after two trips in VE Kenneth asked, “Actually, can I do math?” which means that his level of confidence in calculus application was very high and not typical for the school student (Ernest, 2002). Not relying on this intuitive approach, Kenneth decided to continue empirical mathematizing. Kenneth used the strategy of minimizing distance in water in third and fourth trips in VE. His best time strategy was the path between minimal and maximal water distances (Fig. 3 above). After testing all three strategies Kenneth constructed his graphical model-of the situational problem which was fully grounded on his empirical activity (Fig.4).

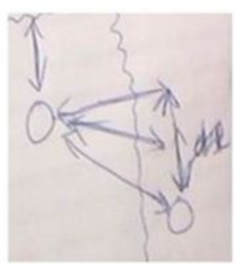

Figure 4: Kenneth’s graphical model-of the problem.

Interesting moment happened when Kenneth’s vertical mathematizing resulted in “plus or minus” land distance, which obviously was a mathematical abstraction. This turned Kenneth back to situational horizontal layer for verification the plausibility of the obtained formal results.

The second student, named Jason, spent less than 30 sec for it being in water during a very short time. He did not want to explore the environment longer, so, obviously he didn't feel the difference between speeds in water and on land. Nevertheless, he chose first trip strategy under the influence of information that speed on land was faster than speed in water (Fig.5).

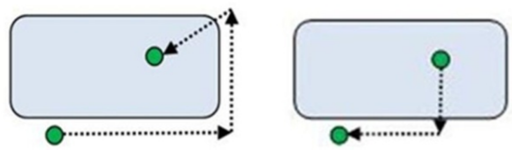

Figure 5: Jason's two first trips in VE.

Jason received this information before his first trip from the journal, and from the researcher. He commented this strategy as "Land is faster" and completed the next trip according to the same strategy of trying to minimize the distance in water (Fig.5). Overall, Jason completed 10 trips the strategies of which, according to his comments, were either maximal or minimal water distances. The result of Jason's empirical mathematizing did not allow him to construct such model-of the situational problem, which would allow him to develop a model-for mathematical reasoning (Fig. 6).

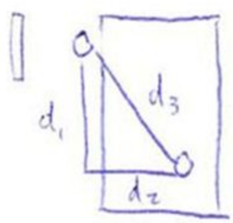

Figure 6: Jason's graphical model-of the problem.

As a result, Jason decided to develop the journal model. He demonstrated excellent independent vertical mathematizing.

The third student, named Nick, spent 3.5 minutes for the exploration trial and had enough time to feel the speed difference in two mediums. Nevertheless, Nick's first trip was of the same strategy as the Kenneth's first trip-the strategy of a straight line between the platforms giving the shortest distance between the platforms but longest water distance (see Fig.3 above). Kenneth's and Nick's tacit intuitive model that shortest distance should give the shortest time prevailed over knowing that the speeds were different in different mediums. The remarkable change in Nick's empirical mathematizing approach happened after 6 trips of different strategies, when he wrote in his comments, "I noticed the angle in which I enter the land from water is key in reducing the time". He planned all the other trips according to his new 'angle' approach. Nick demonstrated how empirical mathematizing can result in the construction of his own original model-of the situational problem which he developed into model-for mathematical reasoning (Fig. 7).

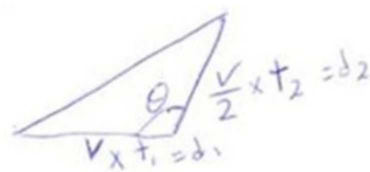

Figure 7: Nick's graphical model-of the problem.

Nick was persistent in mathematical verification of his empirical finding. He spent more than 10 minutes working on his original model independently, and another 6 minutes working on the journal model development. He performed all stages of mathematizing: from empirical to horizontal, grounded on empirical; from horizontal to vertical grounded on horizontal.

All three students showed the crucial role of empirical mathematizing in construction of the models-of the situational problem. During empirical activity the participants demonstrated the interference of intuitive cognition. Particularly, Kenneth's and Nick's first trip strategies were determined by tacit intuitive model, which prevailed over newly received knowledge. Nevertheless, all three participants demonstrated that empirical knowledge obtained from their empirical mathematizing prevailed over intuitive cognition and fully determined the models-of the situational problem. Therefore, if students are provided with opportunity for empirical mathematizing, their new empirical knowledge prevails over intuitions; their horizontal mathematizing is fully grounded on empirical mathematizing.

The students' way of mathematizing depends on their stage of knowledge, according to the epistemological empowerment model, described by Ernest (2002). Particularly, Kenneth demonstrated the Constructed Knowledge Stage, characterized by confidence for integrating Connected Knowing (the intuitive knowing) and Separated Knowing (the impersonal rational reasoning). Jason's epistemological empowerment relates to the earlier, Separate Knowing stage. This earlier stage of knowing is a rational mode in which the subject realizes that there are objective logical rules, impersonal rational reasoning and uses them. Nick's confidence in developing his own model allows relating his empowerment to the Constructed Knowledge stage. Since at some point Nick lost his confidence and decided to develop the journal model, his epistemological empowerment may correspond to the stage between the Connected Knowing and the Constructed Knowledge.

CONCLUSIONS

The study showed that instead of real-life situations described by 'word problems' with ready-made images to be mathematized, the real-life activity can be simulated in a VE.

A background assumption which was made at the beginning of the research was that the VE technology provides simulation on the computer screens close to the reality; and the real-life problems simulated in VE can be considered as the problems of the real physical world. The fact that all three students developed their models-of the situational problem on the basis of their empirical activity in VE suggests that VE indeed provides simulation close to reality; close enough to meet the purpose of this study. This, in turn, contributes to fundamental principle of RME by making formal mathematics as a natural extension of students' experiential reality. Another contribution to RME is connected with identifying the role of empirical mathematizing and empirical knowledge in construction of the models-of the situational problems and as such, in horizontal and vertical mathematizing.

Interesting finding concerns the role the intuitive cognition plays on different stages of mathematizing. Particularly, on the basis of the research results it was suggested that new empirical knowledge obtained from empirical mathematizing prevails over intuitions formed from a previous experience.

It was also shown that the way of mathematizing depends on the stage of epistemological empowerment. Therefore, the instructional design based on utilization of VE simulations should develop students' epistemological empowerment through the development of their applicable skills.

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SECONDARY MATHEMATICS TEACHERS' TINKERING: HOW TO TEACH SOLVING RADICAL EQUATIONS

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This report presents findings from a collaborative teaching experience on the topic of solving radical equations in a Grade 11 mathematics classroom. An in-service professional development process was employed in a K-12 suburban school over an extended period of time in which teachers created, implemented, and reflected upon their mathematics lessons in the traditions of "community of inquiry" and "lesson study". Teachers' discourse during the phases of planning for instruction and reflecting upon the teaching experience were analyzed with respect to what teachers notice about students' mathematical thinking. Through the process the teachers became attuned to critically examine their practice and how it affected what students are doing, thinking, and understanding.

Keywords: Teacher Education-Inservice/Professional Development, Mathematical Knowledge for Teaching, Instructional activities and practices

INTRODUCTION

This report comes from an ongoing study on the development of mathematics teaching practices at an independent, K-12 coeducational, nondenominational, university preparatory school, the West Coast Academy (WCA) in British Columbia. Departments at the school work closely together to improve students' learning. All the 16 teachers that teach mathematics at any grade level at the school participated in the study voluntarily through a type of situated professional development practice (Chazan, Ben-Chaim, & Gormas, 1998) known as "lesson study". Lesson study is a well-defined process for ongoing professional development of teachers that originated in Japan over 50 years ago, but it first gained attention in North America with the publication of The Teaching Gap (Stigler & Hiebert, 1999). It has been documented by several authors who have researched it and led a number of its implementations across North America (Fernandez & Yoshida, 2004, Fernandez, 2005, Lewis, 2006). In the traditions of "lesson study" and "community of inquiry" (Jaworski, 1998), the teachers at WCA worked together as a professional learning community (Wenger), engaging in considerable shared planning, observation, and discussion of lessons. Typically, in each lesson study cycle, a team of teachers who teach the same or neighbouring grade levels collaboratively designed a mathematics lesson to challenge a difficult topic or to learn about an aspect of students' ways of thinking. Next, the lesson was taught by one of the teachers in the team while the other team members observed and documented students' learning; then, the teachers collectively reflected upon and revised the lesson, and finally implemented it again in another classroom with a different teacher teaching it. In some of the lesson study cycles the researcher acted as a facilitator (in teams of teachers teaching K-5 grade levels), in some also as a participant (in the team of teachers the 6-8 grade levels), and in some solely as an observer and a researcher (in the team of teachers teaching 9-12 grade students).

It is not the lesson study itself that is the research focus. Instead, the research study seeks to find out what is needed for teachers to be able to initiate, recognize, stimulate, and sustain the mathematical thinking of their students. As such, this study aims to foster sensitivity among teachers to recognize and create a culture of mathematical thinking in their classrooms, as well as to raise the level of mathematics instruction and learning in general. Lesson study acts as a window into the "full act of teaching", in which teachers can learn about how students think and learn. This report focuses on a single lesson in a Grade 11 mathematics classroom, developed by a team of 3 experienced secondary mathematics teachers. It discusses the ways in which teachers speak and think about students' mathematical thinking, and how they see their role in building a culture of mathematical thinking in the classroom through their instructional practice. The main source of the data is from the collaboration and discussions (audio taped and transcribed partially), the lesson implementations (videotaped), and the artifacts that were created during the process (lesson plans and instructional materials).

DATA COLLECTION AND METHODOLOGY

The three teachers, Sam, Florence, and Fred met twice to develop the lesson on teaching Grade 11 students about solving radical equations. Florence, who taught the lesson in her class for the first implementation, wrote the lesson plan, which included the input from the other two teachers. This artifact included details, such the goal of the lesson, the learning tasks that will be offered to students and the questions that the teacher would ask, as well as some of the anticipated student reactions and evaluation points that the teacher would make at the critical points of the lesson. At the end of the school day, the teachers discussed how the lesson worked, paying special attention to how students learn and what makes the topic difficult for the students, as well as how they might engage students' interest in what seemed like a dry and rather technical topic. The slightly revised lesson was then implemented by Fred with his class of Grade 11 students on the following day. The teachers discussed the revised lesson and evaluated the improvements that they had made in how the lesson affected student learning. Both lessons and the two post-lesson discussion were videotaped, and the lesson planning sessions were audiotaped.

Lessons can be planned, observed and reflected upon from a variety of perspectives, such as social, the discourse that goes on, lesson structure, resources being used, modes of presentation, and so on. Since the overarching goal of the lesson study implementation at WCA was to foster the development of students' mathematical thinking, the author is interested in the mathematical analysis of the lesson, as this seems to be the key for finding out about what is available for students to learn in terms of the nature of the object of learning.

FRAMEWORK

Mathematical thinking has been a major aim of mathematics education, and also a major research topic of lesson study. In the literature there are two traditional references for describing mathematical thinking, one from the perspective of mathematical processes (Polya, 1945, Mason, 1982, Schoenfeld, 1992), and the other with the focus on conceptual development of mathematics (Freudenthal, 1973). Besides these two major trends there is also attention to disposition with regards to mathematical ways of thinking and attitude (Kilpatrick et al., 2001). These three perspectives are embodied in Katagiri's theory of mathematical thinking which has been used in the context of classroom practice and lesson study for developing mathematical thinking (Isoda & Katagiri, 2012). In his theory there is a distinction between "mathematical attitude", "ways of thinking", and "ideas", which could all be used in different ways as a driving force for mathematical thinking. They are the classical distinctions of affect, processes, and content which all play an important part in instruction. We take this framework as a lens through which to study what teachers do in their practice, and how they think about cultivation of a classroom culture where mathematical thinking is the norm, and not just something that happens accidently to some students.

On the other hand, Watson devised a way of mapping the development of mathematical ideas in lessons, which is based on the 7 key features, not hierarchically ordered (Watson, 2007). The author shares her perspective that "all task-types are available for deep analysis of mathematical affordances, and that such analysis can help teachers develop sensitivity to variations of presentation, layout, symbol use, language, diagram and hence to variations in perception, recognition, interpretation on the part of learners." (Watson, 2008)

Table 1: Key Features of Lessons

Focus of episode	Shifts of mathematical activity
Teacher makes or elicits declarative / nominal / factual / technical statements	*remembering*

Learners copy, imitate, follow instructions	*developing fluency, reporting/recording actions*
Teacher directs learner perception/attention	*public orientation towards concepts, methods, properties, relationships*
Teacher asks for learner response	*personal orientation towards concepts, methods, properties, relationships*
Discuss implications	*analysis, focus on outcomes and relationships*
Integrate and connect mathematical ideas	*synthesis, connection*
Affirm/ act as if we know …	*rigour, objectification, use*

The lesson on radical equations is analyzed in relation to these 7 features and discussed it from the perspective of Katagiri's theory of mathematical thinking. What was available for the students to experience in this lesson, mathematically? What improvements were the teachers in this team able to enact from one lesson to the other, in their efforts to evoke their students to think mathematically?

FINDINGS

Here we present the development of the lesson, in a form of a blending between what actually happened and what the teachers intended in the lesson plan. The middle column represents actual and/or anticipated student reactions, and also what is made available to the learners, mathematically (including what appears on the board). The third column captures the ways in which teachers spoke and thought about student thinking. It contains teachers' comments from pre and/or post lesson discussions selected by the researcher as representative of teachers' talk regarding students' mathematical thinking. As this is a study of teachers, and not of students, we do not make the distinction between anticipated and actual student responses, as long as they are consistent with each other. The whole lesson unfolds as a conversation between the teacher and the students in an interactive whole class instructional style.

In the first part of the lesson, the teacher announces the goal of the lesson, writes up the title "Radical Equations", and begins with reminding the students of the different kinds of equations the students are already familiar with. The teacher is setting the stage for the lesson by drawing students' attention to the mathematical ideas of domain, restricted values, and when an equation might not have a solution. This part belongs to the first and third category of the lesson features, presented in Table 1. A small excerpt from the last sequence of this part is presented in Table 2.

Table 2: Setting the Stage

What the teacher does and says	**Students' responses**	**Comments**
Rational equations: Would you be able to write an example of a rational equation? What is the set of numbers for which this equation is defined? Or, does this type of equation have any limitations in its domain? Or, do you see any 'red flag' in this equation? What will happen if x = 0?	$\frac{x}{x+1} - 2x = \frac{2}{x}$ We know that division by zero is not possible, so we need to exclude that possibility. It means that the denominators in this equation have to be different than zero: $x+1 \neq 0 \Rightarrow x \neq -1$ and $x \neq 0$ It is not possible to divide by zero, so for x = 0 the equation is not defined!	Domain is always some kind of a problem for students. Limitations and the reasons to look for them in any equation, especially in a rational function is important, and it is expected that students will have some difficulties with this.

In the next stage of the lesson, the teacher focuses the students' attention to the notion of radical, and addresses some well-known students' difficulties surrounding the interpretation of the root symbol (key features 3 and 4). A teacher who works mathematically can identify the difficult bits of mathematics and address those heads on; otherwise, the students will likely generate their own conceptions as a result of thinking mathematically: pattern-seeking, generalizing, interpreting, applying. The teachers in this team are all experienced mathematics teaches, who deeply care about their students, and this was one of the key points they identified as requiring more attention in the reimplementation of the lesson. The leverage they got from this adaptation was enormous in the sense of accomplishing the main goals of the lesson.

Table 3: Dealing with a potential difficulty proactively

Before we move forward, the question is – what does the term *radical* in mathematics mean? Give me an example! Here we will explore roots of perfect "numbers"; this means, square root or any other root that turns out to be rational.	That means the result of any root. Examples: $\sqrt{15}$, $\sqrt[3]{29}$, $\sqrt[8]{12}$, $\sqrt[11]{6}$, etc. Students had a few seconds to think about the possible result, and the first student who	Students had learned about radicals before, and about irrational numbers, but we don't expect everybody to know them. It is known that students have difficulties when

But first, we have to know the sign of the square root of a positive number. What is the result of $\sqrt{9}$?	volunteered to respond said: The result of $\sqrt{9} = \pm 3$	square root is in question. The most common mistake students make is to say that the root of a number is $\pm$ (positive or negative).

The teacher proceeded to deal with this error, explaining that, "value under the root has to be nonnegative in order for square root to exist in the set of real numbers, so the result has to be nonnegative too." We see this as a case of epistemological obstacle; that is, an inherent phenomenon arising from the need to learn notation and interpretation. Is the square root an operation, or is it a number? One of the desirable aims of formal education is to be able to read the meaning of an expression. In literature, conceptual development in relation to the reading of a meaning through the symbols is described using the notion of "procept" (Gray & Tall, 1994).

Table 4: Relating square roots to solutions of quadratic equations

Now take a look at this quadratic equation: $x^2 - 9 = 0$ What would be the solution of this quadratic equation? Compare this result to $\sqrt{9}$. Do you notice the difference? And why is it so?	We will start with: $x^2 = 9$ Consider first these two equalities: $(+3)^2 = 9$ and $(-3)^2 = 9$; looking at them, we can conclude the solutions x = 3 and x = - 3 or $x = \pm\sqrt{9} \Rightarrow x = \pm 3$ As we said, $\sqrt{9}$ is always equal 3, (not $\pm$3).	The given quadratic equation has two solutions, one positive and one negative, while $\sqrt{9}$ is just a positive number. We need to pay attention to this problem and explain it better, so that students don't make that mistake in the future.

Next, teacher poses the question about the value of $\sqrt{(-3)^2}$, justified by, "This is also students' usual mistake – to cancel out square and square root, so the result is -3". The discussion of permissible values for radical expressions was considered an essential component for establishing the "ways of thinking" for the part of the lesson that followed next.

Table 5: Establishing the permissible values

What could be the value of the root: $\sqrt{x+4}$? This is called a radical expression. Is this root positive, zero, or negative? When does this number exists? I hope that we now know what a radical is.	Again, the value of a square root is always nonnegative, so the value of $\sqrt{x+4}$, if it exists, has to be nonnegative. That number exists when the value under the root is nonnegative, (there is no real value of the root if the value under the square root is negative number). That means: $x + 4 \geq 0 \Rightarrow x \geq -4$	At the beginning, students usually forget to make the statement about the value under the root – to be sure that it is nonnegative. To reinforce that, we need to question them again and again about the possible values for the value under the root.

Discussion about the permissible values for radical expressions was seen as a necessary component for establishing "ways of thinking" about the topic. Mathematical activity in the next segment of the lesson relates to considering the constraints for possible solutions. This is one of the psychological functions essential for successful mathematical reasoning. Some authors who studied how concepts are formed in a mathematics classroom concluded "students' mathematical failure is often triggered not by the lack of specific mathematical knowledge but by the absence of cognitive functions of analysis, planning, and reflection" (Kinard & Kozulin, 2008, p7). This segment of the lesson falls into the 5th category of Watson's key features from Table 1, which pertains to the mathematical activity of analysis. It is worth mentioning that in her study of practices of teachers, Watson concluded that the 5th and the 6th categories are underrepresented in the lessons of non-specialist teachers, and yet they are essential for the integration of the learners' mathematical activity for the development of their mathematical repertoire and ideas (Watson, 2008, p5).

Table 6: Why does a solution "crash"?

Consider this radical equation. What is the set of numbers for which this equation is defined? What are the limitations? What we have said for the values of a square root? Then, under this condition we can start solving the given equation.	$\sqrt{x-3}+x=5$ We have to look at the expression under the root first. We know that it has to be positive or zero in order for the root to exist: $x - 3 \geq 0$ or $x \geq 3$ $\sqrt{x-3} = 5 - x$	We predict that students will try to square both sides of this equation first, without making any limitations for the right side. To show them how serious this mistake is, we let them square it first, to

In order to solve the equation we will keep the root on the left side of the equation, and move x to the right side. Next, to solve this equation we have to eliminate the root. How to do that? May we do that? What do you think? Before we answer that question, let’s eliminate the root without any limitation. Are we satisfied? Are x_1 = 4 and $x_2 = 7$ really the solutions of the given radical equation? Check it! How to know if the zeros x_1 = 4 and $x_2 = 7$ really are the solutions of the given equation? Check x_1 = 4 first. Now, check for $x_2 = 7$. What that means? Could we know it beforehand? OK, we could, but seems that we did not care. How come? Where is the mistake? Yes, we could know that if we did our work properly.	If we square both sides of the equation, the root will be eliminated. $\sqrt{x-3} = 5 - x \quad /^2$ (square both sides of the equation) $x - 3 = (5 - x)^2$ …students solve the quadratic equation using the quadratic formula… Seems that the solutions of this radical equation are: x_1 = 4 and $x_2 = 7$ If we substitute these solutions into the given equation, we will know whether these are the solutions. Substitute the solutions into the given equation: $\sqrt{x-3} = 5 - x$ for x_1 = 4 $\Rightarrow$ $\sqrt{4-3} = 5-4 \quad \Rightarrow \quad \sqrt{1} = 1$, and the equation is satisfied, (because 1 = 1). On the other hand, for $x_2 = 7 \quad \Rightarrow \sqrt{7-3} = \sqrt{4} = 2$ $\Rightarrow$ 2 = -2, which is not correct. That means that $x_2 = 7$ is not the solution of the given equation. Is it possible to predict that a solution will “crash” ahead of time?	see what the consequences are if we don’t pay enough attention. Students really didn’t realize that they made a mistake by not making statements about the right side of the equation. Knowing that students will have difficulties in deciding what to do with equations like this, we offered one very simple example, end asked questions first, before explaining how to approach the solution of this equation. We want to intrigue students to get a sense that there is predictive power in mathematics, which they can access if they think carefully.

In terms of the lesson features we qualify this segment as being in the 5th and 7th category of key features described in Table 1, while in terms of Katagiri's framework, this is where the teacher instigated a shift in mathematical attitude. Students wanted to know what is happening, how come that solution did not work. They wanted to know this from a structural perspective, what was missed in the process, and they were not satisfied to be left with "having to check" and then "getting a crash", but wanted to know instead how they could reach this conclusion beforehand. The teacher used this opportunity to tell a story about the failed Iron Workers Memorial Bridge, where the engineers at the time overlooked a constraint and failed to include it in their analysis. The students were very ready to hear the rest of the story.

Table 7: Solving radical equations under the analysis of constraints

Now, the limitation under which we may square both sides of the equation is x $\leq$ 5, and the solution $x_2 = 7$ does not satisfy that limitation. That means that we could know much earlier that $x_2 = 7$ is not the solution of the given equation. This way, putting limitations through the process of solving the equation, we know in advance what could and what could not be the solution, and we will not fall into the trap of taking the result what is not the solution. Even more, sometimes, like in this example, $\sqrt{x-1} = -2$, we see the solution immediately, namely, the solution is the empty set or, the equation does not have the solution. That means that we solved this radical equation even before we started the process of solving it. That way, putting the limitations first, we shorten the solving process.	We made the mistake when squaring both sides of the given equation without checking the right side, because: Left side of the equation is a square root, and we know that it has to be nonnegative. If so, then the right side of the equation has to be nonnegative too, because it is an equation, and left side have to be equal to the right side, or: $5 - x \geq 0$ If the right side of the equation was negative, by squaring we made it positive. We cannot just change the sign of one side in the equation without changing the other side too. So, we had to make limitation that the right side is nonnegative too. That	Now we underline the problem, making sure that students really see the need for making statements (limitations) before squaring both sides of the radical equation. We think that students now have elementary sense about the procedure and ideas regarding solving radical equations, but that is just a beginning. In order to truly understand the process, more examples have to be done, and these examples should vary in range. This was just one of them, a very simple kind.

	means: $5 - x \geq 0 \Rightarrow 5 \geq x$ or $x \leq 5$	

The teacher then proceeded with 6th and 7th type of lesson episode focus, and expanded the example space with two additional tasks, always working on the analysis of constraints first.

Fred: This was a discussion of the solution of one of the radical equations. The same principle we have to apply whenever we need to solve a radical equation. It means that we have to take all the precautions whenever we do the next step in the process of solving such equations. It applies not only to the roots and the signs of the expressions in the equation, but also to the denominators of the fractions, if they exist, in the equation. We always challenge the students by questioning them about the possible solutions, making them aware of possible shortcuts in the process of solving radical equations. This was such example, and we can again remind the students about the possibilities we need to explore. Students will eventually find the examples in the textbooks, for example Mathpower 11, where they give the only way of solving radical equations by checking the solutions et the end by substituting the solution into the given equation. But this is not a way of developing mathematical thinking in students' math education.

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TEACHER JUDGEMENTS IN THE CLASSROOM: WHAT IS IT WE ATTEND TO?

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When meeting a group of students for the first time teachers can often make judgements, wittingly or not, about the students' ability. In this paper I will examine some possible clues teachers attend to which may be enabling them to make this judgement. In this instance I am considering the feedback a teacher receives from observing a group of students problem solving. I show that certain features of the dialogue, along with the body language of the students, can offer clues as to the level of understanding the students have regarding the material. Using tools of Conversation Analysis and an analysis of gesture, I show that certain features are recognizable amongst students that are successful in their problem solving, and suggest that the experienced teacher may develop a subconscious recognition of such traits.

Keywords: Understanding, Conversation, Gesture, Teacher attention

INTRODUCTION

Reform based teaching encourages the development of a mathematics discourse within the classroom. Correspondingly, discourse based research has often been used as a way to probe student understanding (e.g. Williams & Baxter, 1996). In this research I use conversation analysis combined with gesture analysis to gain insight to the differences between groups of students as they work on mathematics problems. I examine the structure of the discourse with the intent of looking for similarities and differences between the students, which may serve as signals to the classroom teacher. In particular, I examine the question as to how a teacher may claim to quickly determine a student's understanding without having to carefully analyse the discourse.

THEORETICAL FRAMEWORK

Wittgenstein (1967) notes that the possibility of a student *understanding* something will depend on whether the student can go on to write independently (143), while *getting* someone to understand requires changing their way of looking at things. Importantly, understanding is considered to be a *source* of correct usage (146) and Wittgenstein suggests not thinking of understanding as a 'mental process' at all, but as the set of circumstances in which a student is able to 'go on'. If we relate Wittgenstein's ideas to a conversation, then understanding could manifest itself in terms of the ability of a student to 'go on' with the particular exchange. This can be tied in with the idea, from Conversation Analysis (CA) (Sacks, 1984), of the adjacent pair, in which a second utterance of an exchange is functionally dependent of the first. This connection is clearer if we adopt the distinction between a conversation, as an exchange of ideas of which the interlocutors are willing to change, and a discussion, as statements of ideas which the interlocutors make but do not alter (Davis, 2001). The onset of simultaneous speech, overlapping, and intonation are also important indicators of information in such a conversation (Wooffitt, 2005).

Rorty (1979) views conversation as the ultimate context within which knowledge is to be understood, while Ernest (1998) suggests that conversation is central to learning. In a reform-based classroom, where group work and collaborative problem solving is a focus, conversation between students may be seen as central to the process. While these conversations have been studied by researchers (as sumarised by Walshaw & Anthony, 2008), the lens is often on the detailed utterances of the students. However, as Morgan (1998) points out with regard to examples of student writing (p. 154), such content is subject to interpretation, suggesting that such analysis can only be done by someone practiced in this area. Morgan also notes that in assessing written work 'teacher judgements are largely formed in an intuitive way' (p. 127).

Halliday (2009) has observed that if communication is to take place then students must make intelligent guesses, based on their interpretation, about the meanings to be exchanged. This view is supported by Pask (1975, p. 49), who considered that if a student is able to explain a topic then this is evidence for a concept being present. If the explanation is agreed upon by the interlocutor then that is evidence for a concept equivalent to a concept entertained by the other person. If the person can further explain how the concept is constructed, and if the explanation is agreed upon by the interlocutor, then that is evidence for shared understanding. While we might take leave to further examine the notion of what is meant by a concept, I will take a dictionary definition of a concept as 'something conceived in the mind', 'an abstract or generic idea generalized from particular instances' (Merriam Webster). This view of a conversation, then, is one wherein two or more students are exchanging explanations of what they are doing. In order to do this effectively they must have some degree of understanding with regard to the topic in hand. This would also suggest that the more students are able to 'go on' with the conversation, then the deeper their understanding. Where students are in a discussion the dialogue is characterised by isolated statements. This does not necessarily mean the speaker has no understanding, but that we cannot easily recognise it.

I further suggest that CA can be enhanced by including an analysis of body language and, in particular, gesture analysis (e.g. McNeil, 1992, Gerofsky, 2010). Roth (2001), building on work by Kendon (1997) and Levinson (1997), suggests gesture to be a central feature in cognition. Although the writers in these fields are often viewing through different lenses, the analysis of gesturing, what McNeil describes as 'a window to our thoughts', is about trying to make sense of the subconscious thinking that is occurring beneath the level of the spoken word. The implication from the work of

McNeil and Goldin-Meadow (2003) seems to be that we can use gesturing to help, not only to help interpret meaning, but to be aware of things that even the gesturer may not be. Goldin-Meadow, for example, has researched extensively on the speech-gesture mismatch as an indicator of what she frames as a 'readiness to learn'. While CA is often used to give a very detailed analysis of conversations, the original research of Sacks was important in asking what it is that the structure of a conversation tells us about the speaker. Sacks was able to determine if a caller was a determined suicide case from the way they participated in a conversation. Factors in the way callers responded to questions and in the structure of their reply rather than in the responses themselves, told Sacks more information than the utterances. I see this as being similar to the way we can use gesturing in that it can provide us with something more than the words in use. It seems that there is a natural link to be found in this area. Such a combination of these two areas I refer to as *Conversation-Gesture Analysis* (CGA). This process has the potential to analyse a dialogue in great detail, but in this study I am more concerned with particular aspects of the introductory group speech rather than the discourse as a whole.

METHODOLOGY

This report is part of a larger study of two grade 5 classes over the period of a school year. The analysis selected is for first encounters with groups of students during the first few weeks of the year. Each group was selected by the classroom teacher as a normal part of their classroom procedure, and worked on mathematics relevant to the curriculum content at that time. A camera was set up on a tripod at a distance from the group in order to capture body language, while a second video recorder was placed face-upon the desk to pick up language more clearly. The classroom teacher's practice of stationing some groups outside of the classroom helped with the quality of recording by reducing background noise. The students in this study attended a midsize independent school in metro Vancouver, Canada. As is typical in the Vancouver area, the school contains a broad mix of nationalities with several students being ESL. Several hours of video was analysed in the course of this study, with each of the students being captured at sometime early in the year. I focus on two particular groups in order to illustrate more general results.

I began analysing the transcripts by looking for trends which might suggest something about the students' understanding. I was initially interested in how/if conversations developed and their relations to success in the problems. The videos were first transcribed for text using software which allowed the video to be slowed down considerably to help determine utterances. By comparing the two recordings of each session the text could be determined to a high level of accuracy. The video portion of the recordings was then viewed again in order to add the gesturing and body language to the transcript. The transcript was then examined using the tools of conversation analysis to look for particular points of interest as suggested by ten Have (2007). The matter of this report arises from observations of conversational structure and gesturing.

EXAMPLES AND DISCUSSION

Case 1. The class has been given the problem of calculating whether or not they have been alive for a) a million seconds b) a million minutes c) a million hours. A group of four boys have been recorded (N, M, S, and E). One boy, N, immediately speaks up.

Table 1: Transcription notation used

[ ] indicates an overlap	(.) indicates a short pause (n) indicates a pause n seconds long
capitals indicate louder tone	º indicates the start/stop of a quietly spoken section
. indicates a falling tone	: indicates stretched sound or elongated sounds in repetition
? indicates a rising end tone	↑ indicates a rising shift in intonation immediately before the rise
Underline is an emphasis	= indicate there is no gap between utterances
(h) indicates breath intake	((*used to add comments added*))
~ indicates the preparation stage of a gesture	***** indicates the stroke/hold stage of a gesture **.-** indicates the retraction stage of a gesture

Table 2: Case 1 transcript

	time		Markup	Images
1	12.1	N:	umm (.) well what I think we should do is go like 24 x 60 so we can figure out the minutes in a da?y (hh)	

			[~/*****/-] ((*S nods 3 times*)) and my guess is that its close to a tho.usand ((*He looks down at the question sheet and then up to S*))	
2	21.2	M:	=24 times 60? (0.3) ((He looks at N in a puzzled way; N returns his look. Both boys have their hands below the desk top))	
3	22.7	N:	Yeah (.) or 60 x 24 [~/***hold] ((*turning to face M*))	
4	24.2	E:	°same answer [no matter what]° ((*He looks through his binder but pauses and looks at N as he speaks; his hands still. Immediately after speaking he continues to sort through his binder*))	 
5	25.8	M:	[why 60] (.)	
6	27.1	N:	cuz there's 60 minutes in an h↑our (0.5)	
7	29.2	M:	**O**↑:::h ye::ah. ((*rolling `head gesture*)) [~~/***/--] ((*N turns to S, smiles, then raises his eyebrows*))	 

This opening exchange was selected because it illustrates a common start to a discourse. N takes on a leading role by expressing his opinion of how the problem should be attempted. In this case, however, he also includes an estimate and this helps to give the impression of a student who is thinking about what he is saying. Even the way he speaks, beginning with ‘well’ and ending with a falling intonation suggests this to be a confident statement. In most cases like this the group would then continue to follow N’s lead and continue with the problem. Here, however, M immediately presses N for an explanation, resulting in a conversation indicated by adjacent connected utterances. The two boys continue their conversation over the comment of E (4) who illustrates a commonly seen behaviour of occupying himself with a related but diversionary task, only adding safe comments to maintain part of the discourse. S, in this clip adds nothing verbally to the initial exchange but nods in agreement to N’s suggestion. He seems content to follow N’s lead. The body language of both N and M suggest confidence in what they are doing, as they look relaxed and speak without hesitations or the use of fillers. E’s contribution, in contrast, is added sotto voice and his body language is more reserved, sitting slightly back and to the side. S sits on the adjacent corner of N’s desk but turns his shoulders to face N and M. (fig. 1) He is clearly involved in the exchange although he does not add to the opening conversation.

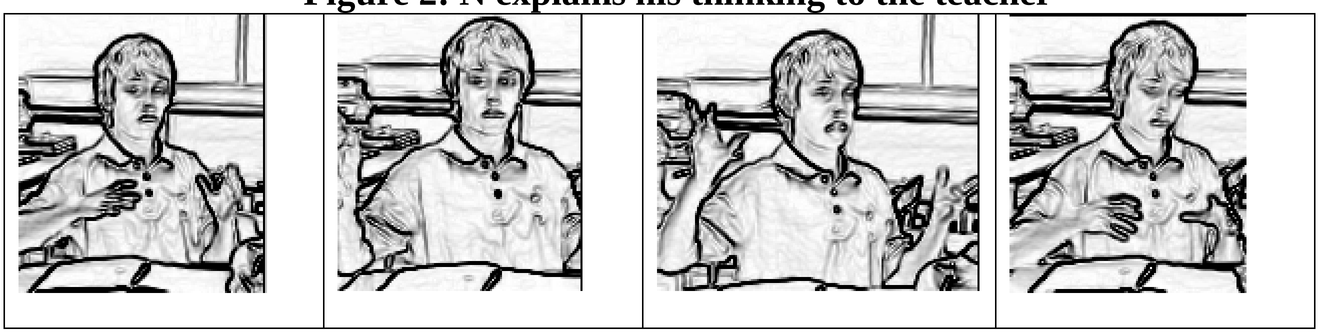
Figure 1: Group Posture

Pausing at this opening conversation, I was aware, without realizing why, of having formed impressions about these four students’ abilities, even though they had only been working for 30 seconds. This reaction piqued my interest in paying closer attention to the start of any session when a student appears for the first time. I had the impression of N as a student who has an understanding of how to solve the problem. M’s questioning seemed indicative of a student who is seeking to understand rather than simply follow N’s lead, and he continued to press N until he had an understanding of what N was suggesting. N, for his part, was able to ‘go on’ with the problem when questioned. In other recorded sessions, it was noticeable if a student was willing to press for an explanation and, when a student was pressed, if s/he was able to ‘go on’ with their initial thought. I suggest that this ability to engage in a conversation in this manner was a trigger to think favourably of such students. In an analysis of all recordings, in the majority of cases, students who began in this manner were able to successfully complete the problem or make substantial progress towards completing it.

In the continued session M takes the lead in attempting the calculation, a role N is happy to observe. S will later attempt to duplicate (unsuccessfully) M's calculations. At no point in the discourse did S or E offer anything new to the solution. On closer observation, when M was calculating, E poised his pencil over his notebook but did not write (field observation). As M performed his calculations, N would lean over to his work and ask for clarifications or offer advice. He did not perform the calculations himself. S exhibited behaviours which suggested that he was confident in his ability but weak in his execution. He also did not seem to be open to change in the same way M demonstrated. His later contributions were more in the nature of a discussion of his calculation rather than a conversation; even when N explained to him that his calculation could not possibly be correct, he found an excuse for this rather than accepting his mistake.

Interestingly, there is no hand gesturing in this opening exchange. The only noticeable gestures are facial. This may be because the students are positioning themselves relationally to their understanding at this early stage. As the exchange continued, only small beat or deictic gestures were evident. For N, this changed when he was asked by the teacher to explain his ideas; he then used full arm gestures (see fig. 2) as he pointed out how you could move from minutes to years and know if he had lived for a million minutes. This was a common feature of students who seemed to have an understanding of the material, in that their understanding seemed to be correlated to broad gestures.

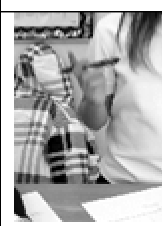
Figure 2: N explains his thinking to the teacher



Case 2. In this example, two students, D and R, have been asked to discuss an estimation problem they have previously been working on individually. D immediately begins with confessed inability, spoken softy with a posture low to the desk. There is sense that she is embarrassed by this; in contrast, in many examples from the same class other students state their inability with no obvious sense of misgiving. Fortunately, R, whose posture is tall, responds in a positive and open manner, offering to help. D

response, however, is not conversational and I suggest that this is due to her continued lack of understanding; she cannot yet follow R’s explanation so that her response to R is to offer an excuse (3). Note that, although R is explaining her process to D (2), R’s posture is separate and turned away. Her hand to her mouth indicates that she is thinking through her own process. She is describing what she did rather than conversing. D ignores R’s explanation and instead describes what she tried (5). R now closes the space between the two girls and as a result D’s posture gradually straightens. The two girls begin to converse, as seen by the adjacent pairs which follow on. R is engaged in D’s process and makes comments to support her. There is also overlap (6) as R connects with D’s comments. D begins to gesture as she outlines her work, at first using deictic gestures (5) combined with a beat which may indicate that D is relaxing. She then uses a flat hand gesture which is raised (7) in a suggestion of recognition, indicating that she is now responding to R. In addition, her posture remains upright. This gesture is of a class which McNeil (1992) classifies as *metaphoric mental*, indicating a state of mind. This suggests engagement with the material.

A first impression of this exchange suggests D to be a student who lacks confidence in herself in this area, but who expects to be doing better. R comes across as a student who has confidence and feels that she understands the work – she is able to explain her thinking and ‘go on’ with this explanation when pressed by D. When compared to M’s questioning from case 1, however, there is a different sense of purpose. M processes N’s response in a conversational way while D, although supported by R in a conversation, is primarily concerned with what she did. In both students, however, there is a sense of being open to further learning.

1	00.0	D	°I didn’t really g::et it?° ((*She looks around the room before looking up to M, her posture is closed and low*))	
2	08.0	M	oh. (.)well (..) I will expla:in what I did (..) so I::(4.1) [~/*/--.] [~~~/***/-.-.-.] ((*A spread hand*)) ((*She raises her hand then sweeps it to her mouth before resting her elbows on the table*))	
3-4: *((D tells M that she has just come back from*				

being off school. M responds but her reply is inaudible.))			it was (.) says estim↑ating so (.) I (..) went down to (..) th?irty (.) divided by six..and I got four remainder o::ne (..) so I just I just like rounded to the nearest..umm.. te?n. = ((*Her forearms flatten to the desk*)) same with this on?e..then I got five rem.ainder 8 (4.9) [~~~~~~~~~~~/*****/-.-.-.] ((*Rolls her hand to point with her pen*))	
5	40.9	D	umm.. like (2.1) I have d::ifferent .. (1) di.fferent (2.3) and what I tried to do? was I tried to do umm like (1) umm (2.4) ((*She picks up her paper and is looking down at it*)) you know it's (.) eight divide by fi-sixty?five (..) [~~~~~~~~~~~/******/hold] ((*She moves her hand to point at her paper*)) I said that I could do two divided by sixtyfive but do that. f↑our ti?mes (.) and then just add up [all of]	
6	1:04	M	[you] can't do two divided by sixtyfive ri.ght It's sixtyfive divided by t::wo. ((*rests her chin on her hands and looks at D*))	
7	1:08	D	(0.9) sixtyfive divide by two (.) sorry (1) and [~~~/*** hold /-.-.-.]	

			((*moves her right hand to her ear and raises a flat left hand; her eyes widen*)) and then umm (1.5) and then I could have d.one like [~~ hold umm (3)and then I could have done like umm added all /***/-.-.-.] the an::sw.ers so. ((*raises her left hand to her ear and holds it there then rolls her hand out on the stroke*))	

Table 3: Case 2 transcript

Following this initial exchange, D asks R if she can '*try rounding the way you did*'. This results in D confronting the question of whether to round 65 to 60 or 70 for the purpose of the estimation. In this query she uses iconic gestures which indicate that she is now connected to the mathematical process (fig. 2). Her hand first indicates herself, followed by a representation of 5 and then a chopping gesture for first sixty and then seventy.

Fig 2: D uses iconic gesturing as her confidence grows

As the conversation continues, both girls begin to gesture more, and the gestures become more expansive. Throughout the conversation D seeks to understand what she is doing and R tries to help her. This case differs from the example in case 1 where both N and M helped each other and tried to see the other's point of view.

DISCUSSION

Having made field notes on my initial sense of the students' competence in the subject, and analysed the recorded data, I consulted the classroom teacher on his early sense of how the students were doing in the subject. I sought this opinion without speaking of my own findings to this point. It was interesting to note that my initial reaction matched that of the classroom teacher, and this was generally true for other recordings. Those students who did not matchup tended to be the quieter ones. It may be that quiet students can give mixed signals to a teacher at an early stage. Following this I then examined a diagnostic test each student in the grade wrote at the start of the year, based upon the BC Government curriculum for grade 4 (developed by Vancouver IslandNet). I again found that the majority of students matched my initial impressions. At the end of the academic year I followed up by asking the classroom teacher about the progress of the students, including which student he thought had improved the most. His response was to suggest that D would be his choice. My intent here is not to claim accurate knowledge of the students from such an early setting, merely to suggest that an experienced teacher may develop an awareness of what to look for in successful students, even if they are unable to express what it is they attend to. Such awareness may be used to assist students learning, but also has the danger of labelling. It is worth noting that the way the teacher interprets these markers to suggest understanding may not directly relate to understanding.

In developing an awareness of students' understanding of a topic I suggest that there are several factors, which might be attended to. In keeping with the ideas of Wittgenstein and others, as discussed above, the ability of a student to 'go on' in a conversation about a topic, in which a meaningful exchange of ideas occurs and where both parties are willing to adjust their position, is a key factor. Students, who are involved in a discussion, where ideas or opinions are stated but where there is no willingness to accept changes, demonstrate a lower level of understanding. They are unable to 'go on' with an idea and often, when pressed, withdraw the idea or make an excuse for any inconsistencies pointed out. (I would point out that sometimes a student gets flustered by a question and is later able to respond more adequately.) A second marker seems to be the student's willingness to press an interlocutor to explain a statement. A third marker may be suggested by the body language of the student. In particular the gesturing a student does when explaining work seems to be more positive and on a larger scale when the student is more confident about what they are saying. A careful analysis of these three markers can give the classroom teacher important information about the level of understanding the student has. Perhaps more significantly, however, is that these markers are able to be observed on a more casual basis and may be used by the teacher to intervene in group discussions in order for them to be more effective.

Determining the potential of a student from a brief encounter is clearly fraught with danger, yet it seems to be something many teachers admit they do. An obvious issue is that such an early conclusion funnels the teacher into treating students to match their expectations, perhaps being harder on some students and more forgiving to others as a result. It would be prudent for any teacher to be aware of what is potentially feeding these conclusions and to be willing to work with and around them. By recognizing our own thought processes I suggest we are better equipped to treat each student in a fair and consistent manner.

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PARADOXES OF INFINITY – THE CASE OF KEN

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Previous studies have shown that the normative solutions of the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation are difficult to understand even for learners with advanced mathematical background such as doctoral students in mathematics. This study examines whether this difficulty is due to the way they are set in everyday life experiences. Some variations of the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation and their abstract versions set in the set theoretic language without any reference to everyday life experiences were given to a doctoral student in mathematics. Data collected suggest that the abstract versions can help learners see beyond the metaphorical language of the paradoxes. The main contribution of this study is to reveal the possible negative effect of the metaphorical language of the paradoxes of infinity on the understanding of the learner.

Keywords – Infinity, Paradoxes, Cognitive conflict

PARADOXES OF INFINITY – THE CASE OF KEN

INTRODUCTION

Historically it took a long time for the concept of infinity to have a proper place in mathematics. Perhaps more than any other concept in mathematics it is the concept of infinity that troubled the mathematicians most. It was Cantor in the late nineteenth century who came up with a mathematical theory to explain certain aspects of infinity.

Paradoxes involving infinity have been used as a lens in mathematics education research for identifying students' difficulties in understanding infinity. One study was conducted by Mamolo and Zazkis (2008) who used the paradoxes Hilbert's Grand Hotel and the Ping-Pong Ball Conundrum. This study is a part of PhD thesis research of Mamolo (2009) which also included the Ping-Pong Ball Variation. In Mamolo (2009) the author reports that even students with advanced mathematical background including some doctoral students in mathematics had trouble understanding the normative solutions of the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation. In Mamolo and Zazkis (2008) the authors say

> What we believe is desirable is an instructional approach that will help students separate their 'realistic' and intuitive considerations from conventional mathematical ones. This is in accord with recommendations made by Dubinsky and Yiparaki (2000) in their study of quantification. They observed that using 'real life' intuitive contexts to teach evaluation of mathematical statements is more harmful than helpful. Having noted that ''the conventional wisdom to teach by making analogies to the real

world can fail dramatically'', they advised the reader ''to remain in the mathematical realm''. (p. 283)

In this study I examine the interplay between some variations of the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation set in everyday language and their abstract versions. I examine whether the abstract versions set using only mathematical entities can help learners see beyond the metaphorical language of the paradoxes. The main contribution of this study is to reveal the possible negative effect of this metaphorical language of the paradoxes on the understanding of the learner.

I first give a brief introduction to the concept of infinity in mathematics. Then the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation, and their normative solutions are given. Then I give the questions in the questionnaire: a variation of the Ping-Pong Ball Conundrum and a variation of the Ping-Pong Ball Variation and their abstract versions, and their normative solutions. A brief summary of research regarding students' intuitive understanding of infinity based on the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation in Mamolo (2009) follows next. Then I describe the theoretical perspectives underlying our investigation which lay the foundation for the subsequent presentation of the research design and the findings in this study. I conclude with some pedagogical and research considerations.

Infinity in Mathematics

Cantor's mathematical theory of cardinality of infinite sets is based on the very simple idea of one to one correspondence. A one to one correspondence is a one to one and onto function. Two sets *A* and *B* have the same cardinality or more informally the same size or the same number of elements if there is a one to one correspondence between them. But this idea leads to very counter intuitive results. Two infinite sets can have the same cardinality, or more informally the same number of elements, even though one is a proper subset of the other.

For example, the set of positive integers and the set of positive even integers have the same cardinality as $f(n) = 2n$ is a one to one correspondence between the sets. This is very counter intuitive as the set of positive even integers is a proper subset of the set of positive integers. This is startling when one realizes that for any n, $\{1, 2, \ldots , n\}$ contains only $\lfloor \frac{n}{2} \rfloor$ even numbers. This counter intuitiveness troubled some of the best mathematicians in the history. For example, Galileo established a one to one correspondence between perfect squares and natural numbers but he reasoned that the attributes 'equal,' 'greater,' and 'less,' are not applicable to infinite, but only to finite quantities. Even Bolzano reasoned that set [0, 12] has more elements than the set [0, 5] even though he saw that each element in [0, 5] corresponds with exactly one element in [0, 12] and vice versa. So the knowledge of finite sets constitutes an obstacle to the understanding of cardinality of an infinite set – in other words the knowledge of finite sets is an epistemological obstacle that a learner has to overcome in understanding the cardinality of an infinite set.

According to Hilbert, the idea of infinite is not in our lived experience. In Hilbert (1926), he says "Our principal result is that the infinite is nowhere to be found in reality. It neither exists in nature nor provides a legitimate basis for rational thought — a remarkable harmony between being and thought." (p. 140).

Aristotle distinguished between two types of infinity: potential infinity and actual infinity. One can think of potential infinity as a process which at every instant of time within a certain time interval is finite. Actual infinity describes a completed entity that encompasses what was potential.

The Ping-Pong Ball Conundrum An infinite set of numbered Ping-Pong balls and a very large barrel are instruments in the following experiment, which lasts one minute. In the first half of the minute, the task is to place the first 10 balls into the barrel and remove the ball number 1. In half the remaining time, the next 10 balls are placed in the barrel and ball number 2 is removed. Again, in half the remaining time (and working more and more quickly), balls numbered 21 to 30 are placed in the barrel, and ball number 3 is removed, and so on. After the experiment is over, at the end of the minute, how many Ping-Pong balls remain in the barrel?

In this thought experiment there is an infinite sequence of time intervals of length $\frac{1}{2},\frac{1}{4},\frac{1}{8},...$ and all these time intervals are contained in the time interval of $[0,1]$. Since in the time interval of length $\frac{1}{2^n}$ the ball numbered *n* is taken out, at the end of one minute the barrel is empty.

The Ping-Pong Ball Variation An infinite set of numbered Ping-Pong balls and a very large barrel are instruments in the following experiment, which lasts one minute. In the first half of the minute, the task is to place the first 10 balls into the barrel and remove the ball number 1. In half the remaining time, the next 10 balls are placed in the barrel and ball number 11 is removed. Again, in half the remaining time, balls numbered 21 to 30 are placed in the barrel, and ball number 21 is removed, and so on. After the experiment is over, at the end of the minute, how many Ping-Pong balls remain in the barrel?

In this variation the same infinite sequence of time intervals of length $\frac{1}{2},\frac{1}{4},\frac{1}{8},...$ is there and all these time intervals are contained in the time interval of $[0,1]$. But in the time interval of length $\frac{1}{2^n}$ the ball numbered $10(n-1)+1$ is taken out. So at the end of one minute the barrel will have the balls numbered 2, 3, … , 9, 10, 12, 13, … , 20, 22, … - this corresponds to the set $N-\{10(n-1)+1/n\in N\}$. Even though in each time interval of length $\frac{1}{2^n}$ one ball is removed, just like in the Ping-Pong Ball Conundrum, the outcome at the end of the minute is very different from the Ping-Pong Ball Conundrum.

Mamolo (2009) contains a research done with 8 participants with advanced mathematical background on the understanding of the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation. Two of the participants were doctoral students in mathematics at the time. All of them were familiar with comparing infinite sets via one to one correspondences, and also with Cantor's diagonal argument establishing the set of real numbers as having larger cardinality than the set of natural numbers. But she found that despite the sophisticated mathematical knowledge of participants only 3 participants provided a resolution to the Ping-Pong Ball Conundrum that was consistent with the normative solution. Two of these 3 participants and another one was presented with the Ping-Pong Ball Variation. One of them had taught Cantorian set theory to prospective teachers in the past. But he reasoned that the barrel should be empty in the Ping-Pong Ball Variation. One of the other two came to an understanding that the barrel would have infinitely many balls but that that would be a bigger infinity of balls than the balls taken out. Only one was able to come to a resolution that was consistent with the normative solution.

Mamolo (2009) says that the cognitive leaps facing an individual who attempts to develop an understanding of actual infinity include a leap from the intuitive to the formal. She further says that infinity is a concept for which no 'real world' analogy can do justice and that her research suggests that the ability to clarify a separation between intuitive and formal knowledge is an important leap toward accommodating the idea of actual infinity.

So it is worth investigating if abstract versions of the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation can help students make the leap from the intuitive to the formal. As it is difficult make abstract versions of the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation I came up with the following questions:

1. A large barrel has Ping-Pong balls numbered 1, 2, 3 … The following task is done in one minute. In the first half of the minute the ball number 1 is removed. In half the remaining time the ball number 2 is removed. Again, in half the remaining time the ball number 3 is removed, and so on. At the end of the minute, how many Ping-Pong balls remain in the barrel?

2. A large barrel has Ping-Pong balls numbered 1, 2, 3 … The following task is done in one minute. In the first half of the minute the ball number 1 is removed. In half the remaining time the ball number 11 is removed. Again, in half the remaining time the ball number 21 is removed, and so on. At the end of the minute, how many Ping-Pong balls remain in the barrel?

3. Let $A_n = A_{n-1} - \{n\}$ for n = 1, 2, 3, … where A_0 is the set of positive integers. Describe $\bigcap_{n=1}^{\infty} A_n$.

4. Let $A_n = A_{n-1} - \{10(n-1)+1\}$ for n = 1, 2, 3, … where A_0 is the set of positive integers. Describe $\bigcap_{n=1}^{\infty} A_n$.

Questions 3 and 4 are the abstract versions of Questions 1 and 2 respectively. In Question 3, A_n corresponds to taking the ball numbered n from the barrel in the time interval of length $\frac{1}{2^n}$ in Question 1. But in Question 3 there is no apparent sense of time. What is in the barrel at the end of one minute in Question 1 corresponds to $\bigcap_{n=1}^{\infty} A_n$ in Question 3. Now, $\bigcap_{n=1}^{\infty} A_n = \{x / \forall n \in N : x \in A_n\}$. As $\forall n \in N : n \notin A_n$ and $\forall n \in N : A_n \supseteq \bigcap_{n=1}^{\infty} A_n$, it follows that $\bigcap_{n=1}^{\infty} A_n = \varnothing$. There is a similar analogy between Questions 2 and 4. Question 1 and the Ping-Pong Ball Conundrum have the same normative solutions and likewise Question 2 and the Ping-Pong Ball Variation have the same normative solutions.

THEORETICAL PERSPECTIVES

What kind of thinking is involved in understanding paradoxes like the Ping-Pong Ball Conundrum? Barbara, Dubinsky and McDonald (2005) suggest that it is advanced mathematical thinking. They define Advanced Mathematical Thinking as thinking that requires deductive and rigorous reasoning about mathematical notions that are not entirely accessible to us through our five senses. They say that comparing |N| with |2N| may require Advanced Mathematical Thinking and the ability to understand that there is a one-to-one relationship between N and 2N is probably not available through experience in the physical world.

I consider two other theoretical frameworks to analyze the data. One is APOS analysis of conceptions of infinity by Dubinsky, Weller, McDonald, and Brown (2005). They suggested that interiorizing infinity to a process corresponds to an understanding of potential infinity - infinity is imagined as performing an endless action. The ability to conceive of the process as a totality occurs as a consequence of encapsulation of the process to an object, and corresponds to a conception of actual infinity. I chose APOS as it describes well how one might think of a concept like infinity.

Can Questions 1 and 2, the variations of Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation, help in some way to understand the abstract versions? So the other theoretical framework is reducing abstraction by Hazzan (1999). According to this perspective abstractness of mathematical concepts can be reduced by connecting them to real-life situations and establishing a right relationship (in the sense of Wilensky) between the learner and the mathematical concept.

SETTING AND METHODOLOGY

Participants Ken is a PhD student in mathematics at a big ten university in the Midwest in the USA. He has already passed qualifying exams and done the advanced topics exams. He is currently doing research in C* algebras. The author taught him Measure Theory in his final year of undergraduate studies in Sri Lanka. He learned Cantorian set theory in his third year of undergraduate studies.

My questions require Advanced Mathematical Thinking defined by Dubinsky et al. (2005): their normative solutions do not relate to experience in the physical world and require deductive and rigorous reasoning about mathematical notions such as one to one correspondence, transfinite arithmetic and intersection of infinitely many sets. It is reasonable to say that Ken is capable of advanced mathematical thinking and so he is a good selection to investigate the interplay between the concrete versions and the abstract versions in the questions.

First Ken was emailed a questionnaire that contained the research questions to get his written responses. After that he was interviewed by the author over skype and it was audio recorded. Ken answered in both Sinhalese and English. Later the interview was transcribed by the author.

RESULTS AND ANALYSIS

He answered all the questions correctly. So my selection of him for this investigation is justified. He wrote the following at the end of the questionnaire:

The way I thought: Problem 4 formalises the process described in problem 2. I thought of A_n as the set of balls remaining in the barrel in the nth step of the process described in 2. So the intersection corresponds to the set of balls that remain in the barrel after completing the whole process. For problem three I thought the same way as I gave the solutions. Actually, I realised the similarity of problem 3 to one when I came to problem 4 and noticed its similarity to problem 2. The picture in 2 helped specially in part (iv) of problem 4.

He clearly has seen the connection between the concrete versions and abstract versions. When he says "The picture in 2 helped specially in part (iv) of problem 4." I see that he reduced the abstraction in Question 4 by going back to Question 2. More evidence of this can be seen in the following excerpts from the interview:

Interviewer: and then number two last part helped you in number 4 last part

Ken: yeah in the sense that in number 4 I knew it is going to be an infinite set so but when I thought about what is it going to be the way I got the answer was thinking like in number 2 problem 4

Even though arguably Ken is capable of Advanced Mathematical Thinking, he has trouble understanding the process in Questions 1 and 2:

Ken: yeah, yeah [in Sinhalese] mata eke penne eke digatama karan yanna puluwan vadakda kiyala api hithanna onda kiyala prashnayak thibuna eke problem one eke last part eka kiyavanakota e process digatama karanna yanna puluwanda kiyala mata hariyatama sure vune ne [what I see in that is I had a problem that whether we should think whether it can be done continuously when I read the last part of problem 1, I was not sure whether this process can be continued.]

He describes in Sinhalese as he finds it easy to explain in Sinhalese his thoughts about a difficult point. The abstract versions helped him to see that the process can be continued:

Interviewer: yeah then when you look at number 3 that means?
Ken: Rather than Question 3 giving me an answer to problem 1 last part it helped me to see that we can assume that the process can be continued.
Interviewer: yeah yeah
Ken: that means to answer that question we have to assume that the process can be continued. This I understood when I read the third question

What is effect of these abstract versions on him?

Interviewer: what if you did not get number 3 and 4 you got 1 and 2
Ken: [in Sinhalese] ow [yeah] then … I would still probably I need to take more time I will probably end up assuming that I have to think that this process can be done and I would still give the same answer but after I mean it take bit more time to kind of assume that to take that

So without questions 3 and 4 he thinks he would have answered question 1 and 2 the same way but it would have taken him more time.

Ken never questioned the plausibility of the questions 3 and 4. As an advanced graduate student in pure mathematics he knows the mathematical language well. He can work in the mathematical realm. So he did not have any trouble with questions 3 and 4.

Though he interiorised the action of removing the ball numbered n in question 1 as a process he could not encapsulate this process to an object.

Ken: I first looked at I mean I started from the first question but I didn't write down answers because er at some point I was little bit confused about problem 1 because since it was kind of a practical procedure although it was clear what was going on I mean specially answering the last part

Interviewer: aha
Ken: what to be or what is left after one minute

APOS analysis can be applied to questions 3 and 4 as well. Apparently Ken did not have any trouble with encapsulating the intersection of infinitely many sets to an object – he got little help from question 1 and 2 in describing this object.

CONCLUSION

Paradoxes involving infinity can provide a window to infinity. The cognitive conflict elicited by a paradox is difficult for a learner to resolve. Resolving this cognitive conflict requires the learner to make a cognitive leap from the intuitive to the formal or from the real world to the mathematical realm.

But some of the paradoxes make this cognitive leap difficult as they are too far away from the reality but yet set in the everyday life experiences. If we compare Zeno's paradox of Achilles and Tortoise and the Ping-Pong Ball Conundrum we can see that Achilles and Tortoise is about a real life situation and the Ping-Pong Ball Conundrum is not a real life situation though it involves real life objects. Even in the mathematical realm the concept of infinity is a difficult concept to grasp. Bolzano and Galileo could not grasp infinity though they considered abstract mathematical entities like intervals and sets of numbers. So when the concept of infinity is presented through everyday life experiences but far away from reality situation it adds to the difficulty of grasping infinity. We can see it from Ken. My findings agree with Mamolo (2009) who found that even students with advanced mathematical background including some doctoral students in mathematics had trouble understanding the normative solutions of the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation. There is further evidence in Mamolo and Zazkis (2008):

Based on the results of our research, and specifically acknowledging the similarity in responses of students with different mathematical sophistication, we suggest that a formal mathematical view of infinity implied in conventional resolutions of the paradoxes may not be reconcilable with intuition and 'real life' experience. (p. 180)

The Ping-Pong Ball Conundrum and the Pin-Pong Ball Variation have the appeal to draw learners to the concept of infinity. This is an important aspect of an instructional tool. The use of paradoxes as effective instructional tools in mathematics is well documented. But for them to be a window to infinity, I believe, they should be presented with the mathematical underpinnings.

The variations of the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation I included in the questions have the same effect of the Ping-Pong Ball Conundrum and the Ping-Pong Ball Variation with much less complication. They have the same normative solutions. We believe that they provide a bigger window to infinity. With their abstract versions they could be excellent instructional tools.

The concept of infinity in mathematics is very mathematical and counter intuitive. This research reveals that the metaphorical language of the paradoxes could have a negative effect on the understanding of the learner. It also indicates that my variations of the Ping-Pong Ball Conundrum and the Pin-Pong Ball Variation and their abstract versions could be excellent research tools in investigating the student understanding of the concept of infinity through paradoxes.

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