

Faculty of Education
Simon Fraser University

MEDSC 2011

Proceedings

**PROCEEDINGS OF THE 6TH ANNUAL
MATHEMATICS EDUCATION DOCTORAL STUDENTS
CONFERENCE
November 19, 2011**

8:30 – 8:55	WELCOME AND COFFEE	
	Blusson Hall 10051	Blusson Hall 10061
9:00 – 9:35	Armando Paulino Preciado Babb Conceptualizing role and position in interactions among teachers engaged in Collaborative Design of mathematics learning artifacts	Maria Luisa Melania Alvarez Use of the Phenomenology Theory perspective in the analysis of professional development activities for in-service teachers
9:40 – 10:15	Veda Abu-Bakare Presenting Tom: A life in mathematics	Shiva Gol Tabaghi The interplay between diagrams and gestures
10:20 – 10:55	Olga Shipulina Calculus beyond the classroom: Application to a real-life problem simulated in a virtual environment	Simin Sadat Chavoshi Jolfaee Learners' vague notions of the Law of Large Numbers
10:55 – 11:05	Break	
11:05 – 11:40	Sean Chorney Interacting agencies and their influence on emotional experiences	Darien Allan Gestures in a Grade 12 classroom
11:45 – 1:00	Plenary Speaker: David Robitaille International Studies in Mathematics Education: TIMSS and PISA	
1:00 – 2:10	Lunch	
2:15 – 2:50	George Ekol Undergraduate stat students' conceptions of variability in a dynamic computer environment	O. Arda Cimen A case study: Studying, self-reporting, and restudying basic concepts of elementary number theory
2:55 – 3:30	Krishna Subedi Mathematics, Abstraction and Teaching: Revisiting TIMSS 1999 video lessons	Kevin Wells Using a conversational approach: Can this inform a teacher about students' understanding?
3:30 – 3:40	Break	
3:45 – 4:20	Ike Udevi-Aruevoru Word problems in business mathematics education: Impact on students' mathematical achievement	Harpreet Kaur Effect of dynamic geometry environments on children's understanding of the concept of angle
4:25 – 5:00	Susan Oesterle Tensions in teaching math for teachers: Managing affective and cognitive goals	
5:00 – 5:10	Final Remarks	

Plenary Session:

**INTERNATIONAL STUDIES IN MATHEMATICS EDUCATION:
TIMSS AND PISA**

David Robitaille

For the past 50 years or more, more and more countries have become involved in large-scale studies focusing on the teaching and learning of a number of subjects, but in mathematics rather than any other area, who runs these studies? Why do so many countries participate in them? Can the result of such studies be used for anything more than seeing who came first?

Research Reports:

PRESENTING TOM: A LIFE IN MATHEMATICS

Veda Abu-Bakare

Relationships with mathematics cover the continuum from enchantment and engagement to accommodation and disillusion in myriad shifts and turns. What can we learn about mathematics from studying relationships with mathematics and what can we use as evidence? Partly biography and partly philosophy of mathematics, this research explores how we come to terms with mathematics and what mathematics demands of us. In this paper, I consider the phenomenon in general and report on one life in mathematics.

GESTURES IN A GRADE 12 CLASSROOM

Darien Allen

The goal of this research is to investigate the impact of teachers' gestures on students' mathematical understanding. This paper reports preliminary results of a larger study of the extent to which secondary students notice mathematically-relevant gestures that teacher make in the classroom while presenting concepts and, if so, whether they use these gestures themselves in their own mathematical communication. Prior studies have shown the important role spontaneous gestures play in student learning but this study explores how students mimic gestures used

by the teacher. Preliminary results show that students will spontaneously mimic gestures when explaining a particular concept and those students who do so tend to have a better understanding of the concept than students who do not mimic the target gesture.

USE OF THE PHENOMENOLOGY THEORY PERSPECTIVE IN THE ANALYSIS OF PROFESSIONAL ACTIVITIES FOR IN-SERVICE TEACHERS

Melania Alvarez

This research paper considers how Phenomenology Theory can be used in the analysis of professional development workshops for in service teachers in order to gain a better understanding of the challenges faced in this endeavour by the professional developer (PD) in order to elicit change in teachers' practice. The goal of the professional development was to introduce teachers to a new math program, which is mathematically and pedagogically different from what most of the teachers involved in these sessions have been exposed to in the past: this is a problem-solving centered program which demands a deeper knowledge of mathematics from teachers and a constant awareness of students' understanding throughout the lessons.

LEARNERS' VAGUE NOTIONS OF THE LAW OF LARGE NUMBERS

Simin Sadat Chavoshi Johlfaee

The law of large numbers is at the heart and soul of probability at any level. In this work I have tried to look into the participants' understanding of the law of large numbers in one of its most basic and ubiquitously addressed forms: the probability assigned to the outcome of a fair coin. Five individual interviews reveal the inadequate understanding of this key concept manifested in the contradictory arguments and poor judgment of probability of certain events made by the participants. The interview is then followed by a task that is aimed to help the participants develop a feel and an insight for the law of large numbers, which come to life through the results of many simulation trials. Computer simulation of coin flipping experiment is used to aid the interviewees who couldn't comment on the probability questions otherwise (due to not remembering formulas) connect with

the problem and to invite them to reflect on their beliefs by comparing their expectations with the simulation results.

INTERACTING AGENCIES AND THEIR INFLUENCE ON EMOTIONAL EXPERIENCES

Sean Chorney

This research report adopts a theoretical perspective of interacting agencies when a student engages with a mathematical digital tool. Adopting the notion that external activity is analogous to inner activity originally posited by Vygotsky, students' emotional experiences outlined by Leont'ev are analysed. Using the tools of discourse analysis as well as the implementation of written reflections outlined by Sanino is used to access the inner workings of the student while they engage with an exploration activity based within a dynamic geometry environment.

A CASE STUDY: STUDYING, SELF-REPORTING, AND RESTUDYING BASIC CONCEPTS OF ELEMENTARY NUMBER THEORY

O. Arda Cimen & Stephen R. Campbell

The objective of this case study is to look in depth into personal factors affecting metacognitive monitoring and control in self-regulated study and restudy of basic concepts of elementary number theory. We incorporate a wide spectrum of observational methods enabling us to record overt behaviour, psychometric questionnaires, and covert behaviour related to various psychophysiological responses. All this is applied toward an attempt to gain deeper insights into personal factors implicated in motivation, metacognition, and beliefs, pertaining to self-regulated learning and mathematics anxiety. Ultimately, our aim is to provide "learner profiles" that can be used to better inform assessment and tailor instructional design, and mathematics education research.

UNDERGRADUATE STAT STUDENTS' CONCEPTIONS OF VARIABILITY IN A DYNAMIC COMPUTER ENVIRONMENT

George Ekol

Students were provided with five key words related to variability and asked to describe them in their own words. Then they applied the same concepts to measure variability using dynamic computer models designed in Geometry Sketchpad. Students' descriptions and discursive expressions were videotaped and analysed using Sfard's (2007) commognition framework, while Rabardel's (2001) instrument-mediated activity theory provided a framework for the dynamic computer models. Partial results show that students have more developed and sophisticated metaphors when describing concepts not directly related to statistics. For concepts directly related to statistics, they are likely to struggle between using their own words, and using formula and procedure in the books. I argue that having fewer conceptual models and metaphors is one main source of difficulty in learning variability. The study's on-going objective is to address this gap by designing and applying dynamic computer models.

THE INTERPLAY BETWEEN DIAGRAMS AND GESTURES

Shiva Gol Tabaghi

This paper reports on the two students' interactions with the dynamic diagram that designed to illustrate vectors and their image under different linear transformations. The interactions with dynamic diagram caused the participants to use gestures and to sketch diagrams in describing their imageries of the behaviour of vectors and their image. We refer to Châtelet's theory (2000) on diagram/gesture relationships to emphasize the capacity of dynamic diagrams that enabled the participants to capture and to arouse gestures and diagrams.

EFFECT OF DYNAMIC GEOMETRY ENVIRONMENTS ON CHILDREN'S UNDERSTANDING OF THE CONCEPT OF ANGLE

Harpreet Kaur

This paper reviews literature on children's understanding of the concept of angle and proposes some ideas for a future study which aims at investigating the effect of

computer-based Dynamic Geometry Environments (DGEs) on young learners' understanding of angle.

TENSIONS IN TEACHING MATH FOR TEACHERS: MANAGING AFFECTIVE AND COGNITIVE GOALS

Susan Oesterle

This paper presents partial results of a study which investigated the experience of teaching mathematics content courses to preservice elementary teachers. Interviews with ten mathematics instructors who teach these courses revealed several major tensions, including one that arises as instructors strive to set priorities and balance their affective and cognitive goals for their students. An analysis of two of the instructors' expressions of this particular tension will provide insight into the factors that contribute to it and how it is managed. Implications for practice are considered.

CONCEPTUALIZING ROLE AND POSITION IN INTERACTIONS AMONG TEACHERS ENGAGED IN COLLABORATIVE DESIGN OF MATHEMATICS LEARNING ARTIFACTS

Armando Paulino Preciado Babb

The collaborative design of mathematics teaching and learning artifacts by teachers and other educators has proved to be effective as both developing curricular material and teacher professional development. Teachers' collaborative design, in this paper, refers to the design of these artifacts that includes: (1) the collaborative design of the artifact based on negotiated goals or purposes, (2) its implementation in the classroom, and (3) the debriefing of the results. The purpose of the paper is to conceptualize role and position of participants in teachers' collaborative design from a social perspective framed in embodied cognition. Such conceptualization would help to understand the dynamics and interactions—co-determinations—of teachers, and other educators, engaged in this mode of collaborative work.

CALCULUS BEYOND THE CLASSROOM: APPLICATION TO A REAL-LIFE PROBLEM SIMULATED IN A VIRTUAL ENVIRONMENT

Olga V. Shipulina

This study concerns the correlation of mathematical knowledge with a corresponding real life object within the theoretical framework of Realistic Mathematics Education. By simulating the interactive milieu in the Second Life Virtual Environment (VE), this study explores how students find a ‘real-life’ optimal path ‘practically’, and how they then re-invent the corresponding calculus task. The instructional design, based on simulation in VE allowed students to explore mathematical solutions relative to their intuitive findings in VE. By mathematizing their own ‘real-life’ activities, students connected them with corresponding mathematics at an intuitive level.

MATHEMATICS, ABSTRACTION AND TEACHING: REVISITING TIMMS 1999 VIDEO LESSONS

Krishna Subedi

Mathematics is an abstract subject. When teachers plan, one of their most important challenges is to figure out ways of translating abstract concepts into understandable ideas. This paper explores the notion of mathematical abstraction from teaching view point and proposes a theoretical framework of Reducing Abstraction in Teaching (RAT). By analysing mathematics classroom practices from the public release video lessons of TIMMS 1999, this paper illustrates various tendencies of teachers dealing with mathematical abstraction. It also exemplifies some instances where ‘reducing abstraction’ seems to be an effective teaching strategy while in other cases it may go unsupportive for the development of student’s mathematical understanding.

WORD PROBLEMS IN BUSINESS MATHEMATICS EDUCATION: IMPACT ON STUDENTS' MATHEMATICAL ACHIEVEMENT

Ike Udevi-Aruevoru

Business mathematics is usually an introductory mathematics course in post-secondary institutions for students specializing in business administration, accounting, and other finance programs. Its primary objective is to equip the students with the mathematical knowledge and skill set they need to solve mathematical problems common in business operations. Usually, these are text descriptions of business transactions requiring mathematical solutions, and are as such, word problems.

USING A CONVERSATIONAL APPROACH: CAN THIS INFORM A TEACHER ABOUT STUDENTS' UNDERSTANDING?

Kevin Wells

In the classroom, students may engage in several casual conversations with their teacher or peers regarding mathematics. Typically, these conversations are informal and may not be consciously used by the teacher as a means of formative assessment. This paper investigates the possibility of analysing the structure of casual classroom conversations to question if a student's ability to hold a conversation reflects on their conceptual understanding of the topic in hand.

PRESENTING TOM: A LIFE IN MATHEMATICS

Veda Abu-Bakare

Relationships with mathematics cover the continuum from enchantment and engagement to accommodation and disillusionment in myriad shifts and turns. What can we learn about mathematics from studying relationships with mathematics and what can we use as evidence? Partly biography and partly philosophy of mathematics, this research explores how we come to terms with mathematics and what mathematics demands of us. In this paper, I consider the phenomenon in general and report on one life in mathematics.

BACKGROUND

Of all the disciplines in which we (as human beings) engage and of which we demand that our youth and citizens have some degree of knowledge, mathematics is one of the few that engenders extremes of emotions. In a recent book, *Loving + Hating Mathematics: Challenging myths of mathematical life*, Hersh and Jon-Steiner (2011), a mathematician and a psychologist respectively, explore the breadth and scale of responses to and relationships with mathematics and the resulting effects on ourselves, those with whom we interact, and the wider society. Feelings for and about mathematics set in at an early age, and the relationships taken on and forged as we wrestle with the subject are often so strong that they lead to expressions of anxiety and pain (Black et al., 2009). The famous psychoanalyst, Jung, expressed his own anxiety and dismay at being defeated by mathematics: “The teacher pretended that algebra was a perfectly natural affair, to be taken for granted, whereas I didn’t even know what numbers were. [...] No one could tell what numbers were, and I was unable to even formulate the question. [...] All my life it remained a puzzle to me why it was that I never managed to get my bearings in mathematics when there was no doubt whatever that I could calculate properly. Least of all did I understand my own *moral* doubts concerning mathematics” (cited in Pimm, 1994, p. 115, original emphasis).

My own interest in this little-explored area of mathematics education approach arose from reading Burton (2004) in which she related the results of a survey she conducted with 76 UK research mathematicians to discern from their trajectories and experiences what is involved in becoming and being mathematicians. From a review of the sociological, psychological and pedagogical literature relating to mathematics (Burton, 2001; 1999a; 1999b; 1995), Burton had proposed an epistemological model of mathematics: person- and social/cultural-relatedness, aesthetics, nurturing of intuition and insight, recognitions of different approaches, and connectivities. In a pilot-study, I tried out her questionnaires with two mathematician-colleagues and found that only the first component figured in the

interviews - their trajectories and experiences were intensely personal and related more to internal factors and to the surrounding social and cultural milieus. In this paper, I tease out the trope of engagement with mathematics and report on one life in mathematics.

THEORETICAL CONSIDERATIONS

The phenomenon of interest is the scope and diversity of relationships with mathematics, how we come to terms with it, and what it demands of us. Mathematics itself almost defies definition and description. At one end, Devlin (1996) defines mathematics as the science of patterns and the other end, Hersh (1997) asks eponymously: What is mathematics, really? One approach that we may adopt is that of sociology of science studies since the 1980s which has put aside the question of asking what science is, and instead asks, what can we learn about science by studying what scientists do? And what can we learn about science by following scientists around (Latour, 1999; Pickering, 1992). Hence my questions of interest are: What does studying engagement with mathematics tell us about mathematics? What can we learn about the subject and engagement with it from following mathematicians around? What are the forces (personal, psychological, psychoanalytical, political, and social) that shape the engagement? What does studying engagement with mathematics tell us about what mathematics demands of us as individuals?

This research may be placed within three theoretical discourses: psychoanalytic, sociocultural, and discursive according to the factors that are emphasized and examined. I will focus on the psychoanalytic approach which focuses on identity, the interplay between conscious and unconscious processes, and the role of emotions and relationships (Boaler and Greeno, 2000; Britzman, 1998).

In this paper, I present and discuss an interview carried out with a male mathematician, Tom, of long standing at a major university. The interview was taped using video and audio, and lasted just over two hours. It was semi-structured and organic; I had a few prepared questions but for the most part, my purpose was to hear his reflections about the subject and his journey and experience of it.

PRESENTING TOM, ONE LIFE IN MATHEMATICS

I begin by presenting the agents (people and objects) at play in Tom's journey and then his feeling for mathematics as it developed. Then I consider more of the psychoanalytic dimensions of the experience.

Agents (People and Objects)

Tom, at the outset, declares the primary influence in his trajectory to be "the other people". In the careful naming and description of his teachers from elementary

school to high school in his former country, then to high school in Canada and to his years at university, Tom indicates the persisting power of the influence of his teachers. Because of his family's relocation in his childhood, school is a place of refuge for him. He lingers on the memory of his teachers in elementary and high school. In high school in Canada, his mathematics teacher "was another kindly lady in her fifties, she looked at this [his mathematics book that he had brought with him] and it started with cubic equations, Cardano's formula and stuff like this and she said, oh, we don't do this here ... so she just said to me, just go ahead, dear, and finish your book (laughs at the memory)".

Tom describes the other two students who started with him in his Math and Physics program at university and the teachers who influenced him. In second year, they had "two new teachers who had brought the New Math with them. All of a sudden we were faced with sets and the famous symbol for being a member of a set."

Besides the other people in Tom's trajectory, many objects 'conspire' greatly in his becoming a mathematician. For the interview, Tom had brought a slide rule, an abacus, and a few pages of an interview with M__ (one of his two classmates who had gone on to be a great mathematician). The four objects I consider here are books, the slide rule, the abacus, and millimetre paper.

Despite what Tom says in the following sequence about not having much experience with books and not being "very bookish", books (always "a little book", perhaps to minimize the influence?) figured prominently in his recounting [I had asked about the prompts, the desire, so to speak, in coming to know mathematics]: "... so how does it start? I mean, you hear something or you are looking at a book, M__ always said to me, don't try to read a math book from cover to cover,... so I wasn't very bookish, I didn't have much experience with books, I think Halmos' book, *Finite Dimensional Vector Spaces*, was the only book I read cover to cover without jumping around".

Besides the math book that he had brought with him from his old country, Tom mentions among others, *You and Nature*, which he listed as "another one of these influences, haphazard influences", Bertrand Russell's autobiography, "a little book" given to him by a theoretical physicist who had come to see him after he had done well in the Provincial exams and was written up in the papers (he had told the reporter he wanted to be a theoretical physicist), "a little book by Landau which you may know, *Foundations of Analysis*", a book by Loomis that had just come out on factor harmonic analysis, and "a little book" by Schrödinger.

Tom describes the slide rule as "a big help, BIG HELP"; with it he makes his second self-discovery (his first was carrying in elementary school) of how

logarithms worked and how to find the square root of 2 (“it gave me this feeling that in mathematics you can figure out things by yourself”). He notes that the abacus “was something that was missing in my upbringing because the discovery about the carrying and all that and the place-value system (moves the beads on the abacus) would have been trivial had I had this thing in hand (smiles, puts down abacus, sits back and crosses arms)”. A final object Tom describes is millimetre paper, “that kind of thing you know, it gives more room to the imagination, when you have a piece of paper which just has a coordinate grid on it just for guidance, you are much freer than any kind of blocks or rods”.

Tom’s enthusiasm for these objects attests to their agency in his journey (he insists that today we are using the wrong ones). They gave him the realization that mathematics is an individual activity; that it could be done by one’s self without anyone’s help. Further, while Tom acknowledges his teachers, he uses the metaphor of learning tumbling to indicate the gradual development of skill, “at first you need teachers to help you and then you don’t”.

Feeling for Mathematics

First, Tom expresses his feelings vividly for the Math he was learning as people came and went. In high school in Canada, he “disdained the kind of math they were doing. They were rationalizing surds (in a drawn-out bemused manner), basically doing a little bit of number theory and quadratic fields without saying so, and I already understood it, of course, and I kind of looked at it disdainfully and continued my thing”. In his first year at university, Math is a “hodge-podge and unpleasant”.

Tom describes passion in Math: “..but there is something about mathematics that eventually becomes passionate, a kind of cognitive passion that is aroused or something and it doesn’t leave you”. From this remark, it was a surprise to hear him say that he was “lukewarm” about mathematics even at the Master’s level and remained so for a long time.

Second, Tom’s reluctance to accept the label of a mathematician even by second year is another surprise. “Oh, hell, no (emphatically)... it happens very late (emphatically)...it takes a long, long time, if you read M__’s essay you see that even for him it took until second year at university until he saw himself as a mathematician”. Here is a description that I had not foreseen: “I became just a sort of a run-of-the-mill mathematician ... but I think that if Sputnik hadn’t happened, maybe I wouldn’t have been a mathematician”. Tom is also torn by his father’s wishes that he follow in his footsteps and become an engineer.

Third, with respect to what mathematics is and how mathematics happens, Tom ventures that “it is an experience” and that if we were to speak for another two

hours, “we would still draw a blank in the end”. At one point, he threw his hands up and said that the question of what is mathematics cannot be answered and is unknowable, that it was like asking what is life or what is the unconscious. As for how mathematics happens: “...somewhat, it’s a start, I mean, you hear something or you are looking at a book ... something that you find curious and you start thinking about it, and then you go backwards and forwards until you fill it all in, and so (slowly) this is how it starts, you see something that’s a little odd and you say to yourself, well, that’s strange, how might that come about, and then if you have an idea, if you are lucky you have an idea, oh yeah, it’s probably because of this, and then you start fiddling around on paper or on the blackboard, try this calculation out, no, this is not quite it, oh, but it may be that and so, you carry on like this and if you get more deeply into it with some partial success and so on, then you stick with it another day and so on, so it draws you in and finally you are obsessed with it”.

Fourth, Tom presents himself as a curious onlooker, on the “outside” looking in. He describes being on the train to and from high-school in his old country and helping boys who were a year ahead of him with Geometry, and being in the room with mathematicians at university and supplying a lemma. He calls himself “a camp follower: but not so much out of cowardice, I think, but by curiosity, those kids over there are playing with a different kind of object, I want to go there and see what they’re playing with, why are they having so much fun, there was so much like that coming out in mathematics”.

The insights in to what mathematics is and how it is experienced provided here point to the process of engagement with mathematics as being slow, gradual, and uncertain with no flashes. There are small victories but mostly a sense of desultoriness. In the reference to passion, it is not the “uncomfortable raciness” in Sinclair (2010) but a “cognitive” passion that holds you tentatively at first, then in more gradually sustained manner and finally “it doesn’t leave you”.

While these threads deal with Tom’s relationship to mathematics, they cannot be considered as summative because it seems to me that the keys to understanding the relationship lie elsewhere. As I listened to Tom and reflected on the interview later, it was clear that there were phantoms that could not be ignored. I cast these as: Standing in the Shadow, the Notion of Dislocation, and the Notion of the Unsayable.

Standing in the Shadow

Throughout the interview there was a palpable presence of a third person in the room, namely Tom’s classmate at university, M__. I had not heard the name before and since I believed that I had indicated to Tom that I was interested in *his* journey,

I was startled by how often and with what regularity Tom brought M__ into the conversation, especially when I had no thought that the response would include M__. [M__ is an eminent Canadian mathematician, well-known for his research program and winner of many prizes in mathematics].

The first reference to M__ came in the first two minutes of the interview in response to my thanking him for his time. Tom says, “and I actually appreciate it too because, for example, I read this essay by M__ which is 22 pages long”. In the next two minutes after remarking on the influence of “other people”, Tom says, “if I compare (crosses arms), because I looked at M__’s essay again (gestures at the document), it was very well thought out obviously and obviously he did some research ... it’s an interesting one for me to hold up as a kind of mirror (holds up his hand to simulate). He’s become a great mathematician, truly great (voice rising), I mean as I told you, in a category almost by himself, I mean the people that would be mentioned as the greatest mathematicians of the last half of the 20th century would be apart from him would be people like...um...I just said Andrew Wiles would perhaps not quite make it, but Serre, Jean-Pierre Serre and Grothendieck, Alexander Grothendieck and uh, a few people like that, you know at most ten of that stature and so I was very lucky to have somebody, someone like that as a fellow student and I had no inkling (emphasis in voice) that he would become so famous”.

I cite at length here to show the elevated position in which Tom holds M__. Much later in the interview, Tom repeats the same phrasing about M__ with a telling gesture: “I just told you how the math department here was wrong about M__ and me (holding up his hand with his index and middle fingers in a vee). M__ became a great mathematician (with a downward twist of his hand, the positions of the fingers were reversed), truly great (voice rising)... but the faculty had it the other way around”.

Notion of Dislocation

I had come away from the interview mystified as to Tom’s having spent a life in mathematics and coming across as indifferent to it. I didn’t understand the interview until I read the word, *dislocation*, in Fulford (1999). Of Nabokov, Fulford writes: “He passed along some his most poetic reactions to dislocation in *Speak, Memory*, one the great autobiographies of the century. [...] In many other books he led his readers through the special uncertainties of immigrants, adrift in new worlds, threatened by failure, loneliness, and poverty, threatened even by madness if they cannot accept calmly the radical change that is their fate.” (p. 119)

The sense and power of place, the movement from place to place, and the affective events surrounding the transitions in Tom’s narrative provide compelling keys to

understanding much of Tom's trajectory. Fulford (1999) citing Paul Auster, the American novelist, writes: "We construct a narrative for ourselves, and that's the thread we follow from one day the next". As Auster sees it, each more or less healthy man or woman has a story that helps create and sustain the necessary integrity of the personality" (p. 13). Tom's narrative as whole, in its careful and delicate rendering, preserves that need for wholeness of Self.

The Notion of the Unsayable

In searching for an understanding of this interview, I happened upon a chapter in a handbook of narrative inquiry, Rogers (2007): *The Unsayable, Lacanian Psychoanalysis, and the art of narrative interviewing*. Similar to the discovery of 'dislocation', the discovery of the word, the Unsayable, in the literature, unlocked much of what I had been pondering in Tom's interview. Rogers writes: "It's a curious truth that even as we speak, we circle around what it's not possible to say, reading one another about what to elaborate, what to revise (and even try to erase), coming, almost inevitably, to what eludes any possibility of being heard. This is what I call the unsayable" (2007, p. 99).

Dissatisfied with interpreting interviews according to various research stances, Rogers formulated her own method of Interpretive Poetics as a means of listening to the Unconscious with five layers: story threads, the divided "I", the address, languages of the unsayable, and signifiers of the unconscious (2007, p. 109). I will consider only two here: story threads and the languages of the unsayable here (though the others are worthy of study, especially the divided "I", opposing voices of the ideal self, "I", undermined by the faltering "i"). The story thread is like the melody of a song that one listens for instead of the message. Listening for it in an interview is to note where it disappears and re-emerges, where it leaves its trace. The two story threads I can see in Tom's narrative are that of dislocation/movement and the seemingly contradictory dispassion for mathematics in a life of mathematics. The languages of the unsayable are writ in "negations, revisions, smokescreens (diverting attention to a safer place), and silences. Tom's "you see I didn't have an idyllic childhood like M__'s", his references to M__ with unerring regularity, and the unknown consequences of the events of his childhood all point to the Unsayable. As Pimm (1993) writes in a moving essay, "We all have mastered some aspects of mathematics. What do we truthfully know of the processes and the psychic costs involved?" (p. 38)

CONCLUSIONS AND IMPLICATIONS

In drawing these threads together, I consider Pimm's (1994) argument for another psychology of mathematics education that leaves "the social and the rational" behind and points inwards (p. 112), and for the role of 'unconscious activity' in

mathematics. Pimm notes that meaning (one of the two major themes of mathematics education, the other being existence) is more than referential but about associations of all kinds, and partly about “unaware associations, about subterranean roots that are no longer visible even to oneself, but are nonetheless active and functioning” (p. 112). Tom’s particular narrative of a life engaged in mathematical activity reveals that overshadowing the mathematics is the person, the experiences and the influences, and the relation to others, and ultimately, the ways the unconscious lays bare the self and identity.

With respect to the Unconscious and mathematics, Pimm (1994) wonders about Freud’s “windows into the unconscious” and asks if there any particularly mathematical ones. While his quotation of Turkle on Lacan may be taken as related to the mathematics of creation, I think some of it is still helpful here: “For Lacan, mathematics ... is constantly in touch with its roots in the unconscious” (p. 114). With respect to Tom’s presentation of self and engagement with mathematics, it was interesting to me that he did not list his accomplishments but talked about them in minimizing, off-hand ways, and that he spoke clinically about something in which he had engaged for nearly all his life. From his website later, I saw a list of 32 publications in mathematical journals. For all his modesty and diffidence, there had been ample opportunity and time for what he had achieved, or perhaps he may have thought it was beyond me.

This interview confirmed to me that engagement in mathematics is intensely personal and is inextricably bound to the sense of self and the psyche. Indeed I register here my gratitude to Tom for his generosity of spirit in speaking with me. I note the psychiatrist, Coles (1989) quoting one of his teachers, William Carlos Williams: “We have to pay closest attention to what we say... Their story, yours, mine – it’s what we all carry with us on this trip we take, and we owe it to each other to respect our stories and learn from them” (p. 30).

The implications for mathematics education from this research are twofold. One is that we continue to expose and uncover the myths of mathematics and of engaging in mathematics. The second is that we continue to explore what engagement with mathematics demands from us and does to us as individuals. Tom, speaking of his second year of university and the excitement of new teachers and of encountering the famous symbol of inclusion for sets shows that coming to terms with the language and symbols of mathematics is new and challenging, requiring much effort on the part of those who undertake mathematics. Do we find mathematics or does it find us? How do we as teachers enable or guide our students to a lasting and rewarding engagement with mathematics?

References

- Black, L., Mendick, H. & Solomon, Y. (Eds.). (2009). *Mathematical relationships in education: Identities and participation*. New York, NY: Routledge.
- Boaler, J. & Greeno, J. (2000). Identity, agency, and knowing in mathematics worlds. In J. Boaler (Ed.). *Multiple perspectives on mathematics, teaching and learning*, pp. 171-200. Westport, CT: Ablex Publishing.
- Burton, L. (2004). *Mathematicians as enquirers: learning about learning mathematics* (Chap. 6 in collaboration with Nathalie Sinclair). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Burton, L. (2001). Research mathematicians as learners: And what mathematics education can learn from them. *British Educational Research Journal* 27(5), 589-599.
- Burton, L. (1999a). The practices of mathematicians: what do they tell us about coming to know mathematics? *Educational Studies in Mathematics*, 37(2), 121-143.
- Burton, L. (1999b). Why is intuition so important to mathematicians but missing from mathematics education? *For the learning of mathematics* 19(3), 27-32.
- Burton, L. (1995). Moving toward a feminist epistemology of mathematics, *Educational Studies in Mathematics*, 28(3), 275-291.
- Britzman, D. (2003). *Lost subjects, contested objects: Toward a psychoanalytic theory of learning*. Albany, NY: State University of New York Press.
- Coles, R. (1989). *The call of stories: Teaching and the moral imagination*. Boston: Houghton Mifflin.
- Devlin, K. (1996). *Mathematics: The Science of Patterns*. New York, NY: Holt Paperbacks.
- Fulford, R. (1999). *The triumph of narrative: Storytelling in the age of mass culture*. Toronto: House of Anansi Press.
- Hersh, R. (1997). *What is mathematics really?* Oxford: Oxford University Press.
- Hersh, R. & Jon-Steiner, V. (2011). *Loving + Hating Mathematics: Challenging the myths of mathematical life*. Princeton, NJ: Princeton University Press.
- Latour, B. (1999). On recalling ANT. In K. Knorr-Cetina & M. Mulkay (Eds.). *Science observed: perspectives on the social study of science*, pp. 15-25. London: Sage.
- Pickering, A. (Ed.). (1992). *Science as practice and culture*. Chicago: University of Chicago Press.
- Pimm, D. (1994). Another psychology of mathematics education. In P. Ernest (Ed.), *Constructing mathematical knowledge: Epistemology and mathematics education*, pp. 111- 124. London: Falmer Press.
- Pimm, D. (1993). The silence of the body. *For the learning of mathematics* 13(1), 35-38.

- Rogers, A. (2007). The unsayable, Lacanian psychoanalysis, and the art of narrative interviewing. In J. Clandinin (Ed.), *The handbook of narrative inquiry: Mapping out a methodology*, pp. 99-119. Thousand Oaks, CA: Sage.
- Sinclair, N. (2010). Knowing more than we can tell. In B. Sriraman & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers*, pp. 595-612. Heidelberg: Springer-Verlag.

GESTURES IN A GRADE 12 CLASSROOM

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The goal of this research is to investigate the impact of teachers' gestures on students' mathematical understanding. This paper reports preliminary results of a larger study of the extent to which secondary students notice mathematically-relevant gestures that teachers make in the classroom while presenting concepts and, if so, whether they use these gestures themselves in their own mathematical communication. Prior studies have shown the important role spontaneous gestures play in student learning, but this study explores how students mimic gestures used by the teacher. Preliminary results show that students will spontaneously mimic gestures when explaining a particular concept and those students who do so tend to have a better understanding of the concept than students who do not mimic the target gesture.

INTRODUCTION

Hand gestures are present everywhere in our lives. They can be defined as the hand movements that occur with speech. Gestures can add emphasis, additional and complementary information, or even contradictory information to our speech. In the classroom, gestures can allow students to express ideas that they cannot yet express in words, and thus are an important communicatory tool.

The gestures used by students and by teachers can have significant impact on students' learning. Teachers' use of gestures in conjunction with speech has been shown to influence and improve student understanding. "Teachers produce gestures that can have an impact on what their students take from a lesson" (Goldin-Meadow, 2004, p. 320). Singer and Goldin-Meadow (2005) summarize other researchers' findings in stating that "spoken instruction presented with gesture promotes learning better than the same spoken instruction presented without gesture" (p. 85). Gestures that teachers use can enhance student learning by providing additional information to students and also by giving students an example to imitate and thus improve their own understanding.

More recently, significant research has been done on the role of gestures in mathematical thinking and learning (e.g., Núñez, 2003; Radford, 2009; Sinclair & Gol Tabaghi, 2010). The production of gestures allows students to organize information and over time learn the language to describe them (Robutti, 2006). For example, when students first learn about slopes they may not know or remember the terminology for vertical and horizontal lines, but can use their hands and arms to demonstrate what a line with a slope of zero or an undefined slope would look

like. A student describing the motion of a soccer ball while tracing a parabolic arc in the air is giving the observer a visual image of the path the ball takes, but is also concretizing the concept for herself. Not only do gestures help students convey meaning, the use of gesture actually improves student understanding.

The role of gesture is especially important in mathematics in part due to the often significant difficulties students have with formal mathematical language. Formal mathematical writing tends to be dehumanized and information-dense. As Núñez (2003) writes, “Formal language in mathematics [...] is not as rich as everyday language and cannot capture the full complexity of the inferential organization of mathematical ideas. It is the job of embodied cognitive science to characterize the full richness of mathematical ideas” (p. 70). Using gesture to communicate abstract ideas allows for richer communication than what language can provide. A number of studies have shown that students who use gesture are more successful on mathematical tasks than those who do not (Cook & Goldin-Meadow, 2006; Singer & Goldin-Meadow, 2005). Students who might struggle to find the appropriate language to describe a concept can alleviate their difficulties by showing what they mean. Indeed, quantitative studies in brain and cognition research have shown that “gestures supply a concrete context for what are often viewed as abstract mathematical concepts” (Lim et al., 2009, p. 312).

Teachers’ use of gestures in conjunction with speech has been shown to improve student understanding. Singer and Goldin-Meadow (2005) show that learning improves when instructions are presented verbally and with gesture, rather than verbally alone. Cook and Goldin-Meadow (2006) explicitly taught children gestures to copy and had the students practice using them. They found that children who used gesture to express a correct problem-solving strategy were more likely to solve a problem correctly on a later assessment than those who did not use gesture or did not express a strategy. This shows that students can benefit from using explicitly taught gestures.

THEORETICAL FRAMEWORK

This research and analysis comes from a multi-perspective approach. First, I draw from constructivism in the sense that students cognitively construct knowledge individually and that social interaction and social context play a significant role in students’ formation of knowledge and understanding. Interaction with both peers and the teacher is vital. Second, I believe that students understand and express their learning in many ways. That is, I take an embodied and a multimodal approach (Arzarello, Paola, Robutti, & Sabena, 2009) to teaching and learning. A multimodal approach will serve more students and promote understanding at a deeper level than any single method of instruction. Specifically, I assume that

gesture has a role in students' understandings and explanations and that this role is linked to speech and language (Goldin-Meadow, 2004).

The idea that gesture has a valid role in learning seems to be generally accepted, however, the specific role that gesture plays is still under investigation and debate. This particular study is designed to extend the work of Cook and Goldin-Meadow (2006) to more advanced learning situations and to focus on a more naturalistic context in which students are not directly taught specific gestures. The goal of the research is to explore and investigate the impact of teachers' gestures on students' mathematical understanding. This paper is a report of preliminary results of a larger study of the extent to which secondary students notice mathematically-relevant gestures that teachers make in the classroom while presenting concepts and, if so, whether they use these gestures themselves in their own mathematical communication. Thus the role that these gestures play in students' learning in relation to particular mathematical concepts will also be examined.

THE STUDY

Participants

The participants were 31 students in a grade 12 International Baccalaureate Mathematics Standard Level class. The students are generally above average in terms of ability and have positive attitude towards mathematics and learning in general. The teacher is one of the researchers and has been teaching this course for five years and the majority of these particular students for the past year.

Method

Three specific gestures were devised to use while teaching a particular lesson on horizontal, vertical, and slant asymptotes. The lesson was taught twice, first to one half of the class, without using the target gestures, then to the second group with the target gestures.

The three target gestures were used for three particular concepts. These concepts were chosen for different reasons, but one common connection is that the concepts are all abstract concepts that, according to the teacher's experience, students have trouble with.

The first was to indicate the difference between a secant and a tangent line; specifically, to indicate how a secant line joins two points on a curve, and when those points become arbitrarily close the secant line becomes the tangent line (see Figure 1). This gesture was chosen because the teacher wanted to illustrate the connection between the secant line and the tangent line and to reinforce the concept of the tangent line.

The third still-shot in Figure 1 shows a curve approaching a horizontal asymptote, which was not one of the target gestures, but was a gesture that was used in both lessons. This gesture was used in both lessons because it had been used in previous lessons for both groups of students.



Figure 1: Tangents, secants and a horizontal asymptote

The second target gesture depicted a curve crossing a horizontal asymptote (see Figure 2). The left arm shows the horizontal asymptote and the right index finger draws the curve starting above the horizontal asymptote, crossing it, and then curving back up to approach the asymptote. As the curve approaches the asymptote, the teacher switches from using the index finger to using her palm to show the behaviour of the function approaching, but not crossing, the horizontal asymptote. This concept was chosen to address a common misconception that students have based on the types of functions they experience in the traditional curriculum. Students believe that a curve cannot cross any asymptote ever, whether it is horizontal or vertical. The teacher wanted to give an example showing that a curve can cross a horizontal asymptote.



Figure 2: Crossing a horizontal asymptote

The third, and final, target gesture was chosen to introduce a new type of asymptote to students: a slant, or oblique, asymptote (see Figure 3). The left arm of the teacher is held on a diagonal to indicate the oblique asymptote and the right index finger draws the curve. As the curve approaches the asymptote the teacher switches from the index finger to the palm.

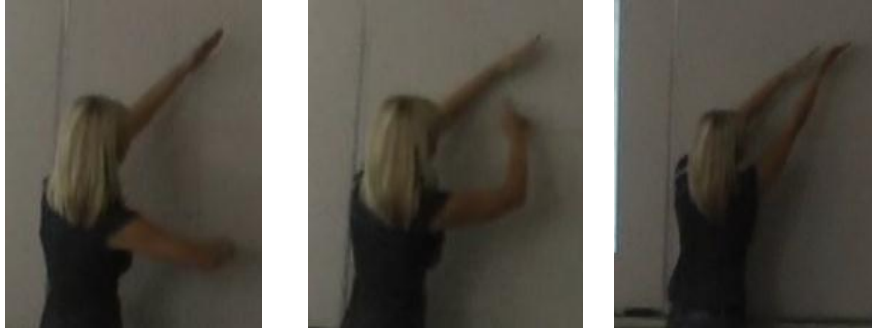


Figure 3: Showing the slant asymptote

Immediately following the lessons six students from each group were interviewed individually. The interview questions were:

- What was the topic of today's class? (How would you describe what you did today?)
- How are a secant and a tangent line related? (How is a tangent line different from a secant line?)
- How did Ms. Allan describe the idea of a horizontal limit? Can you cross a horizontal asymptote? (Why not?) What would that look like?
- What do you mean by a slant asymptote? (What would that look like?)
- How would you explain what you learned today to someone who was in the other half of the class? (Well, can you explain it again?)
- Was there anything difficult or hard about the idea of a horizontal limit or slant asymptote?

Questions in brackets were asked if the student needed more prompting.

A few days following the lesson, a short video clip (without sound) of the teacher using a particular gesture was shown to each of the twelve students (individually) who were interviewed. Students were asked to identify what concept the teacher was demonstrating in the clip.

The videos of the lesson were analysed to ensure that the target gestures were not used in the first lesson, and to determine the number of occurrences of the gestures in the second lesson. The videos of the interviews were analysed to determine what gestures (if any) students used in their responses to the interview questions and during the lesson.

RESULTS

None of the three target gestures were used in the lesson without gestures. The only gestures that appeared in the lesson without the target gestures were the normal everyday hand movements of the teacher as well as gestures indicating a curve approaching a horizontal asymptote and pointing to the left and right to

indicate a value of x approaching negative infinity and positive infinity respectively. Vertical, horizontal, and slant asymptotes were shown with various arm positions and hand movements; these occurred rarely in the non-gestures lesson and more frequently in the gestures lesson.

The three target gestures were each used multiple times during the with-gesture lesson (see Table 1). The lesson with target gestures lasted twenty-five minutes with the bulk of the gestures in the first ten to fifteen minutes and then a few at the end of the lesson. The gestures were used primarily to introduce and illustrate a concept before addressing the algebraic portion of the lesson.

Student usage of gestures during the lessons (both with and without target gestures) was limited. Students spent the majority of the lesson taking notes and asking an occasional question. One student used a gesture showing a horizontal asymptote by moving his right hand back and forth in a horizontal motion (palm down) and then later showed a curve approaching a horizontal asymptote by drawing an arc with his right hand, fingers extended, rising to the horizontal. The same student also showed a vertical asymptote using a vertical “chopping” motion with his right hand. This student performed these gestures while seated at a desk. He was in the class taught with gestures.

The students were interviewed within one hour of the lesson (both the non-gesture and gesture), with the exception of one student who was interviewed a few days later. During the interview, students were seated in a chair facing the interviewer without a table so that they would not be inhibited in their gesture use. At times the interviewer had to try to prompt students to explain by asking what something would look like or asking the students to show her. Sometimes this elicited gestures and sometimes it did not.

Four of the students replicated the gesture used in the class where two points are shown to be approaching each other, indicating that the tangent is the slope between two points as the limit of the distance between the points approaches zero.

Target Gesture	Number of Uses During Lesson	Number of Students who Mimicked (no gestures lesson)	Number of Students who Mimicked (gestures lesson)
Secants and tangents	2	1	4
Horizontal asymptotes	12	2	5
Crossing a horizontal asymptote	3	3	4
Slant asymptotes	3	0	3

Table 1: Summary of the number and type of gestures used

A gesture indicating a curve approaching a horizontal asymptote was shown once during the non-target gestures lesson and two of the students showed a gesture similar to this during the interview. In comparison, this gesture was used twelve times during the lesson with target gestures and during the interview was used by five students. Only one of the students in the gesture lesson did not replicate the gesture; and in fact this student made no gestures whatsoever.

Three students in the non-gesture lesson and four students in the gesture lesson made a gesture depicting a curve crossing a horizontal asymptote and then approaching it. Of the seven, only three showed the curve approaching the asymptote with a flat palm; the other four drew the entire curve with an index finger. The three who used the flat palm were in the gestures lesson.

In response to the question about slant asymptotes many students responded that they were asymptotes that were not vertical or horizontal. Some of the students drew a diagonal line with the flat of their hand. None of the students actually mimicked the target gesture used in the lesson, but the three noted in the table used another gesture which was to show the slant asymptote with their left arm, the vertical with their right arm, and then to draw the curve between the two with their right index finger.

Other gestures made by students during the interviews were vertical, diagonal or horizontal motions with arms or hands to show asymptotes.

When shown the video excerpt of the teacher demonstrating a curve approaching a slant asymptote, nine out of ten students correctly identified the topic. These ten students were out of the twelve who were interviewed. One student asked to watch the clip twice. Six students made gestures while describing the video clip. These gestures either indicated a slant asymptote or a curve that would have a slant asymptote. Two of these students were in the non-gestures lesson and the other four were in the lesson with gestures. Of these six, two students from each lesson

did gestures showing a hyperbola bounded by an oblique and a vertical asymptote. They did this by drawing the hyperbola in the air with an index finger. The other two students (both from the gestures lesson) showed both the hyperbola with an index finger and the asymptotes using their arms held on a vertical and a diagonal.

PRELIMINARY CONCLUSIONS

The following sections discuss an analysis of the results of the preliminary study and contain some suggestions for researchers as the study progresses.

Student Use of Gestures in Interviews

It is clear from the majority of the interviews that the use of the gestures during the lesson does promote the use of gestures by students. This confirms the results of the studies by Cook and Goldin-Meadow (2006) but for older students. However, there are some results suggesting that the use of gesture during instruction had no effect. Some possible reasons for these results follow.

One reason why three students from the non-gesture class may have mimicked the gesture showing the crossing of the horizontal asymptote is because both classes were shown the sketch of the function, and the gesture was basically the same. In the future it will be more helpful to make the gesture more unique, or not show the sketch, though it should be noted that three of the four students in the gestures lesson did mimic the specific change from index finger to palm as the curve approached the asymptote.

One student who was in the with-gestures lesson did not do any gestures during the interview or after being shown the video clip. It may just be that this student does not use gestures.

Watching the Video Clip

The majority of the students were able to identify what was being demonstrated when shown the video clip without sound. The student who responded incorrectly was in the non-gestures lesson and stated that in the clip the teacher was showing that you could cross a horizontal asymptote but not a vertical.

Some students gave additional explanation or information after watching the clip.

Other Interesting Results

Some interesting misconceptions and gaps in understanding became apparent during the interview process. For example, a number of students did not voice a completely correct understanding of the secant line and the tangent line. One possible reason for this is that they have had much less exposure to the term secant than the tangent. Also, many of them still hold to the belief that the tangent line can only touch the curve at one point (and never intersect the curve again). Another

misconception voiced by one student was that you can cross a horizontal asymptote, but only once. This was likely due to the particular example given in the lesson as it only crossed the horizontal asymptote once.

In general, it was noticed that the students who used smaller gestures were those who were less confident in their understanding. As well, the students who used no gestures at all were either very weak or very strong students. This is in accord with the results of studies that show that students who use gestures most often are those who are coming to understand a concept. The weakest students do not have enough understanding of the idea to create a gesture, and the stronger students may not need to use a gesture to communicate the concept. Those students who had a deep understanding of the material and used gesture may have done so in order to help explain the concept to the interviewer, or in response to the particular question which prompted them to explain it to another student.

Notes for the Future

In the future, as mentioned above, it will be useful to make the target gestures more distinct from any sketches that might be shown. That is, the gestures should be more metaphoric than iconic (McNeill, 1992).

References

- Arzarello, F., Paola, D., Robutti, O., & Sabena, C. (2009). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, 70, 97-109.
- Cook, S., & Goldin-Meadow, S. (2006). The role of gesture in learning: Do students use their hands to change their minds? *Journal of Cognition and Development*, 7(2), 211-232.
- Goldin-Meadow, S. (2004). Gesture's role in the learning process. *Theory into Practice*, 43(4), 314-321.
- Lim, V., Wilson, A., Hamm, J., Phillips, N., Iwabuchi, S., Corballis, M., Arzarello, F., & Thomas, M. (2009). Semantic processing of mathematical gestures. *Brain and Cognition*, 71, 306-312.
- McNeill, D. (1992). *Hand and mind: What gestures reveal about thought*. Chicago: University of Chicago Press.
- Núñez, R. (2003). Do real numbers really move? Language, thought, and gesture: The embodied cognitive foundations of mathematics. In R. Hersh (Ed.), *18 unconventional essays on the nature of mathematics* (pp. 160-181). New York: Springer.
- Radford, L. (2009). Why do gestures matter? Sensuous cognition and the palpability of mathematical meanings. *Educational Studies in Mathematics*, 70, 111-126.
- Robutti, O. (2006). Motion, technology, gestures in interpreting graphs. *International Journal for Technology in Mathematics Education*, 13(3), 117-125.

- Sinclair, N., & Gol Tabaghi, S. (2010). Drawing space: Mathematicians' kinetic conceptions of eigenvectors. *Educational Studies in Mathematics*, 74(3), 223-240.
- Singer, M., & Goldin-Meadow, S. (2005). Children learn when their teacher's gestures and speech differ. *Psychological Science*, 16(2), 85-89.

USE OF THE PHENOMENOLOGY THEORY PERSPECTIVE IN THE ANALYSIS OF PROFESSIONAL ACTIVITIES FOR IN SERVICE TEACHERS

Melania Alvarez

This research paper considers how Phenomenology Theory can be used in the analysis of professional development workshops for in service teachers in order to gain a better understanding of the challenges faced in this endeavour by the professional developer (PD) in order to elicit change in teachers' practice. The goal of the professional development was to introduce teachers to a new math program, which is mathematically and pedagogically different from what most of the teachers involved in these sessions have been exposed to in the past: this is a problem-solving centered program which demands a deeper knowledge of mathematics from teachers and a constant awareness of students' understanding throughout the lessons.

INTRODUCTION

With some frequency we see changes in the math curriculum and consequently a change in math programs at schools is required; as a result there is a consistent need for professional development workshops to help teachers transition into new programs. Professional development of teachers in support of curriculum change is one of the main challenges school districts face; this is mainly due to lack of resources or adequate ways to address teacher concerns with the instruction of the new curriculum.

Even when research shows that not many teachers used much of what they 'learned' in professional development sessions (Guskey, 2002); professional development is viewed by policy makers, administrators and teachers as the main agent of change for improving teachers' practice and consequentially student achievement.

The purpose of this phenomenological study is to describe and explain the factors that affect teachers' practice as a result of a professional development experience and the attempts on the part of the professional developer to better engage teachers in the process. An important goal is to determine the factors and characteristics of teachers' learning and beliefs that are more likely to contribute to sustained engagement in acquiring new knowledge and pedagogical skills which will contribute to changes in teaching as a result of professional development sessions.

The researcher will systematically reflect on the live experiences and behavior by teachers reacting to what occurs during professional development sessions. She will also analyze the challenges that the professional developer (PD) faces in order

to engage teachers to develop a community of learning where they will support each other as they face the implementation of a new math program. How should the professional developer become aware of and deal with teachers' concerns, fears, indifference, or enthusiasm? What happens at particular professional development sessions? Who is engaged and how? And who is not and why? How are teachers' knowledge, belief and values exposed by certain behaviours, and how can these behaviours help or deter the purpose of the professional development session? What are the challenges from the PD perspective: barriers, self-doubt, etc.?

Fundamentally this research tries to answer the following question: what are the external and internal factors that influence teachers' engagement and change in a professional development experience?

FRAMEWORK

Phenomenology is the descriptive methodology of human science, which explores and describes lifeworld experiences, and whose main goal is to discover the emerging meanings of those experiences; it can be defined as "the study of structures of experience or consciousness" (Woodruff 2011). Phenomenology uncovers the meaning of a phenomenon by uncovering the many layers that socially and culturally influence a person's experience in their lifeworld; where lifeworld is defined by Van Manen (1997) as "the world of immediate experience", the world as "already there" (p.182). Phenomenological research studies things as they arise in our experience and for this reason the researcher must first "describe what is given to us in immediate experience without being obstructed by pre-conceptions and theoretical notions" (Van Manen 1997, p.184).

In order for things to present themselves in the lifeworld of an individual they have to be part of the consciousness of a person and for that person to acknowledge them; if this is not the case then they cannot be part of the lifeworld of a person. Manen (1997) and Giorgi (1997) also point out that consciousness is intentional, therefore in order to explore a given phenomenon in the lifeworld of a person one must research how it is manifested to the consciousness of that person; they view intentionality as the inseparable connectedness of the human being to the world. We must keep in mind that human agency is always oriented; however intentionality is not always conscious: "Intentionality is only available to consciousness upon retrospective reflection." (Mostert 2002)

Phenomenology seeks for the essence, the true being, of the "things for themselves" as opposed to how they are experienced in the lifeworld. Van Manen (1997) defines essence as that which makes a thing what it is before cultural and social meanings are attached to it.

To find the essence the researcher must examine a phenomenon by first asking, what is it like? And afterwards what is it like for me in my world? It is through reduction that the researchers look for the essence of the phenomenon. “As we then explore the lived experience within the lifeworld, bringing the phenomenon to consciousness and aware of intentionality, we attempt to reduce reflection beyond the immediate context and aim to discover the essence or essentialness of the phenomenon. This is the fibre of phenomenology” (Mostert 2002); for this reason the phenomenological researchers must provide rich descriptions of the phenomenon and within its context. (Kensit 2000, p.104)

Summarizing: the theoretical framework that guided this research is a phenomenological reflection on teachers’ behaviours while learning, and teachers’ change through learning in the social context of professional development sessions. The researcher’s goal is to do a phenomenological analysis of individual change and as well as group change as teachers experience learning with and from their peers. An objective is to determine how new knowledge is incorporated into an existing schema and how teachers can participate in a discourse that extends their knowledge and system of beliefs if the professional development is effective in addressing its goals.

METHODOLOGY

Participants and setting

The participants in this study were the teachers involved in the professional development and the professional developer. The professional developer is a participant because her reflections on teachers’ behaviours will be part of the study.

A new school-wide math program was being implemented at a school in the Lower Mainland in British Columbia, Canada. Thirty of its teachers teaching grades K to 6 participated in the research. The teachers were divided in seven groups corresponding to the grade they were teaching. Each professional development session tended to a particular grade. There was never a session where two or more grades worked together, except for the introductory session where the professional developer gave an introductory overview of the program and most of the teachers at the school came to this session.

This professional development was led by the researcher of this study, who took the role of the professional developer and who had experience with the program to be implemented. The main goal of this professional development was to help teachers become familiar with the math program to be implemented at the school, but another goal the professional developer had in mind was to get teachers to look at the materials in a critical way such that if in the future another program was to be implemented, they would look for key facts to get a better idea of its content

and pedagogical philosophy. The professional developer envisioned herself more as a coach than an instructor and wanted to support the development of a relatively “self-sufficient” community of practice for each of these groups of teachers participating in the sessions. The professional developer (PD) was a temporary coach that was there to point out the nuances in the new program and bring out interactions among teachers to help them learn from each other; and if it was the case to be able to tackle future changes in the curriculum as a group without necessarily having to have someone from outside to guide them through the whole process every time there was a change in the curriculum.

In phenomenological research, the researcher cannot be detached from the process and according to Mouton and Marais (1990, p. 2) in many cases the explicit beliefs of the researchers regarding the issued to be researched have to be part of the data. This makes it quite natural for the researcher to take on the role of a PD in a study like this.

Process

Each group of teachers met with the PD between 10 and 15 hours distributed between 3 to 6 sessions.

The PD’s main concerns were:

1. To be able to address the needs and concerns of the teachers in order to engage them in a pursuit to further develop their practice and math content knowledge.
2. Opportunities for teachers to take risks in sharing their beliefs among their peers.
3. To be able to have sufficient time for the teachers to learn and maintain their learning.
4. Accountability.

Throughout the professional development sessions the PD supported the teachers with both readings and explanations of particular math concepts and ideas that came up during the session.

During some sessions the PD asked teachers in a group to first explain how they would teach a particular mathematical concept; after the ensuing discussion, teachers were exposed to the books and workbooks used by the math program without the benefit of the manual. Teachers were asked to compare between their initial take on the concept and how the materials in the program dealt with it. Afterwards the manual was introduced with the goal of furthering discussions

regarding the teaching of the concept. Teachers were asked about what additional ideas were included in the manual, and how useful it was to better teach the ideas students found in the book and workbooks. For some groups the same procedure was done for several sessions, and for others the process changed according to what the PD thought would be a better way to elicit engagement on the part of the teachers.

It is important to point out that in this particular math program the books and workbooks are geared towards students and the manuals are different from many other manuals in that for most grades, except for kindergarten, the teachers will not see a reproduction of book pages in the manual. The manual presents the following information as they introduce a new concept: objectives, notes which inform teachers what ideas students have learned in the past which will help them introduce the new concept, how to introduce a new concept by using students' previous knowledge (many times with activities that are not in the student materials), how to use the book in class to elicit student discussion and a workbook for practice and assessment.

Data and Analytical Tools

Data regarding the actions of teachers in a professional development setting was gathered in order to find if they were engaged in the discussion and how. This was done by recording most of the professional development sessions and by notes made by the researcher during and after each session. After every session the PD made a few notes reflecting on the effectiveness of the session, which concerns raised by the teachers were addressed and the level of engagement. Most observations of teachers' behaviours were made during the professional development sessions; however, the researcher had the opportunity to visit classrooms to observe the teaching practice of a few of the teachers.

The PD gave an initial survey and a final survey but only about one third of the teachers filled these documents, this is an important sign regarding the issue of accountability.

For the data analysis the researchers used the following protocol which is a simplified version of Hycner's (1999) explicitation process used by Groenewald (2004) together with some steps delineated by Van Manen (1997) for the methodical structure of hermeneutic (interpretative) phenomenological inquiry:

1. Investigating experience as we live it rather than as we conceptualise it;

2. Bracketing: researcher must bracket presuppositions in order to remain true to the phenomenon
3. Phenomenological reduction: reflecting on the essential themes which characterise the phenomenon;
4. Delineating units of meaning by extracting those narratives which throw light on the researched phenomenon (Creswell 1998; Hycner 1999).
5. Clustering of units of meaning to form themes: units of significance are created by grouping units of meaning together. (Creswell, 1998; Moustakas, 1994; Sadala & Adorno, 2001)
6. Summary that brings together all the units of significance obtained from the data within the context from which these units arise. (Hycner, 1999) This is the moment when the researcher “transforms participants’ everyday expressions into expressions appropriate to the scientific discourse supporting the research”.
7. Create a composite: creating a balance by taking into account the part and the whole.

PRELIMINARY RESULTS

At this stage of the research we are able to provide some preliminary results for units of significance and some themes have emerged. We will present here the preliminary results on two themes.

Use of the manuals

During the sessions teachers were asked to look at the manuals and to discuss the mathematical and pedagogical content of a particular section and its lessons; they were given enough time to carefully read this material and correlate it with the students’ textbooks and workbooks. The PD noticed that some teachers would just browse the assigned section of the manual for a minute or two and wait for others to finish or try to cut short the reading time for the whole group to move on. Sometimes the whole group would just browse and be done within a few minutes, not the time one would expect for a careful analysis of the materials. When the researcher looked at the data to find out the reason for these behaviours and correlated it with other behaviours that occurred during the session and information she had from the teachers involved two interesting units of significance emerged:

1. Those teachers who had been using the program for several years and who didn’t use the manual had a very hard time incorporating this new resource into their practice. They believed that because they have been using the materials for a

while, in some cases years, they knew how to use those materials. They resisted a thorough reading and discussion of the manual even when the PD pointed out some important conceptual ideas that were missing which were important to bring out in a lesson.

2. If the materials in the manual looked ‘too familiar’ many teachers were not thorough in their reading of the manuals. However if the program was new to them and the PD pointed out some interesting ideas or key points that were new to them, they were more willing to take a second look at the materials than the group discussed above.

Looking at the new within the old

In the early primary years, most teachers are more interested on learning new activities or ways to teach a concept that in taking a second look at the mathematics that are taught because they think they have a good grasp of the mathematical ideas they are teaching. However, by making teachers aware of new ideas within the mathematics that they thought they knew, teachers became engaged in deeper mathematical discussions and were more enthusiastic about conveying these new ideas to their students.

To provide here an example: The researcher used some interesting insights from Ron Aharoni’s book *Arithmetic for Parents* and she gave the teachers some excerpts from his book to read. Here is Aharoni’s (2007, p. 69) discussion on the meaning of addition:

“The expression $3+2$ applies to the joining of two groups ... Joseph has 3 flowers, Reena has 2 flowers. How many flowers do they have altogether? ... However, before we go any further, we must discern a subtlety of meaning. There are actually two different forms of addition: dynamic and static. In dynamic addition, to join means to change the situation: 3 birds were sitting on a tree, 2 joined them. How many birds are there now? In static, joining signifies grouping of types: A vase contains 3 red flowers and 2 yellow flowers. How many flowers are there altogether?... I do emphasize difference, especially because of the link to subtraction...children find static subtraction difficult”.

This passage created quite a discussion among teachers in grades K to 2. Many of them could give examples of students who could add and subtract but had difficulty with problem solving. Some teachers discussed the possibility of what would change in their students’ understanding and problem solving abilities if

they emphasized the difference in class. They wondered about how they never thought about this difference in meaning before and how to introduce this new idea in their practice.

There are many other examples like this in the data, with concepts that teachers thought they knew well, whose subtleties like the one above make the concept “new” in some way and more engaging.

CONCLUSION

Phenomenology and the process of phenomenological inquiry will not provide all the answers to professional development since it cannot generate systemic change. What it can do however is provide access to a process and a way to analytically reflect on that process to envision ways to engage teachers in learning and change, and to provide professional developers with some insight into the professional development experience.

In the end we hope that research like the one initiated above will help program developers to create professional development workshops that are more meaningful and helpful to all teachers.

References:

- Aharoni, R. (2007). *Arithmetic for Parents*. Sumizdat.
- Creswell, J. W. (1998). *Qualitative inquiry and research design: Choosing among five traditions*. Thousand Oaks, CA: Sage.
- Giorgi, A. (1997). The Theory, Practice and Evaluation of the Phenomenological Method as a Qualitative Research Procedure, *Journal of Phenomenological Psychology* 28(2), 235- 261.
- Guskey, T. R. (2002). Professional development and teacher change, *Teachers & Teaching* 8(3/4), 381-391.
- Groenewald, T. (2004). A phenomenological research design illustrated', *International Journal of Qualitative Methods* 3(1), 1-26.
- Hycner, R. H. (1999). Some guidelines for the phenomenological analysis of interview data. In A. Bryman & R. G. Burgess (Eds.), *Qualitative research*, Vol. 3, pp. 143-164. London: Sage.
- Kensit, D. A. (2000) 'Rogerian theory: A critique of the effectiveness of pure client-centred therapy', *Counselling Psychology Quarterly* 13(4): 345-352.
- Moustakas, C. (1994). *Phenomenological research methods*. Thousand Oaks, CA: Sage.

- Mouton, J., & Marais, H.C. (1990). *Basic concepts in the methodology of the social sciences*. Pretoria: Human Sciences Research Council.
- Mostert, W. (2002). Phenomenology: discovering new meanings of pedagogy within the lived experience. <http://www.aare.edu.au/02pap/mos02453.htm> Retrieved October 20, 2011.
- Sadala, M. L. A., & Adorno, R. (2001). Phenomenology as a method to investigate the experiences lived: A perspective from Husserl and Merleau-Ponty's thought. *Journal of Advanced Nursing*, 37(3), 282-293.
- Smith, D. W. (2008). Phenomenology. In the Stanford Encyclopedia of Philosophy. <http://plato.stanford.edu/entries/phenomenology> .
- Van Manen M. (1997) *Researching the Lived Experience: Human Science for an Action Sensitive Pedagogy*. London, Ontario: University of Western Ontario Press.

LEARNERS' NOTIONS OF THE LAW OF LARGE NUMBERS

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Secondary school teachers' notion of the law of large numbers in a coin tossing context is discussed. The data suggests that there is an inadequate understanding of this key concept which leads to some contradictory arguments and incorrect judgment of probability of certain events. Then the participants are presented with a task that is aimed to help them develop a feel and an insight for the law of large numbers in an empirical sense through the results of many simulation trials. Computer simulation of a coin flipping experiment is used to aid the interviewees who couldn't comment on the probability questions otherwise (due to not remembering formulas) connect with the problem and to invite them to reflect on their beliefs by comparing their expectations with the simulation results.

LAW OF LARGE NUMBERS, HISTORICAL BACKGROUND

In *The Taming of Chance*, Hacking (1990) renders an interesting account of the aftermath of the emergence of probability and the evolutionary process it undertakes during 18th and 19th century. Central to the initiation of the law of large numbers the book offers: In 1835, in the course of his statistical jurisprudence, Poisson coined the phrase "law of large numbers" and proved an important limiting theorem. This provided a further rationale for applying the mathematics of probability to social matters. It also seemed to explain how there could be statistical stability in social affairs. The progressive stabilization of the relative frequency of a given outcome in a large number of trials, that has been observed for centuries and was translated by Bernoulli to a mathematical theorem, served as a justification for the frequentist definition of probability as we know it now. Modern generalizations of this theorem are known as Laws of Large Numbers. These laws lead to connections between probability and statistics and they give validity to statistics as a methodological tool in experimental sciences.

LAW OF LARGE NUMBERS IN MATHEMATICS EDUCATION

Freudenthal (1972) in one of the earliest works that attempts to identify and resolve learners' difficulties in understanding empirical probability discusses some issues related to teaching the empirical law of large numbers. He criticizes the common textbook practice of finding probabilities; probability of "heads" in a fair coin for example through repeated experiments in a classroom setting by tossing a coin 100 times. He argues that the chances of getting an empirical estimation that falls between 0.48 and 0.52 are only around 2/5 if we are to use 100 coin flips. A sample of 2500 trials is needed in order to feel 95% confident of getting between

48% and 52% of the coins showing up heads. Since this is not a feasible number of trials to be done in a classroom setting, he suggests using an averaging technique to reduce the dispersion of the data coming from coin flips or to use a table of random numbers imitating the outcome of the real coin. In a study of naïve and emergent knowledge of randomness, done by Pratt & Noss (2002), children of 11-12 ages in the study were interviewed before and after working with a computer chance environment. In initial interviews they never articulated an appreciation of the significance of aggregating results of repeated experiments with dice, spinners and other randomness generators over the long term and also there was no reference to the frequencies of a particular outcome over a period of time (contrary to what Piaget & Inhelder claimed in their 1951 work stating that by the age of 12 most children could reason probabilistically about a variety of randomizing devices and had developed sound statistical notions including an understanding of the Law of Large Numbers). Pratt & Noss (2002) describe the computer environment “chance maker” as a model in which stochastic knowledge is connected to a set of randomness resources abstracted from everyday life, none of which is associated with long term aggregated behaviour. However during the course of study through the interaction with carefully designed external resources, two new resources, the Large Numbers resource and Distribution resource, emerge and develop. They propose that the capabilities of computer simulation makes it “natural” for the students to start with small trials and quickly turn to large number of trials and at the same time a dynamic mode of representation of the outcomes such as a pie chart gradually imparts to them an insight of the Law of Large Numbers.

COMPUTER SIMULATION IN PROBABILITY EDUCATION

Philosophical Dimensions in Mathematics Education (Francois and van Bendgemen, 2007) is one of the resources that has taken to itself to reflect on research in statistics education around the use of technology and more particularly the use of simulation in teaching probability and statistics concepts such as understanding of sampling distributions, long run patterns, randomness, making inferences and so on. However it reports on studies indicating that even a well-designed simulation is unlikely to be an effective teaching tool unless students’ interaction with it is carefully structured, but for most part simulation has received a good evaluation as a pedagogical tool that can play a significant role in enhancing students’ ability to study random processes and statistical concepts: “Experiments and computer simulations performed in the classroom to facilitate learning of statistical concepts should be perceived as fundamental sources for the students and not simply as motivations for step-by-step teaching of the teacher’s intended goals”, p. 144. Simulation in interaction with theory is the perfect instrument to clarify and to enliven the basic concepts of probability, asserts Tijms

(2007). He pays great attention to the role of using simulation in teaching probability not only because of the practicality it offers for tackling complicated problems as well as to the hows and whys of the underlying probability theory related to those problems, but also for the didactic aid it provides. When using a computer for simulation, the learners are able to make the data and the result of the analysis graphically visible, put their ideas into test almost instantly and try more of the “what happens if ..” type of questions, adds Konold in his 1995 confessions where he discusses several critical considerations with regard to using simulations instructionally.

METHODOLOGY

The participants of this study are five secondary school in-service teachers (coded as MO, MS, MA, SH, and NE) holding a bachelor degree with a major in one of the biology, mathematics, and computer science subjects. All of the participants have received a formal (not advanced though) training in probability in their undergraduate courses. The interviews are done individually (one-on –one) and voice recorded. The interviews are designed in three parts: in part A the participants are asked to assign probabilities to certain coin flipping events and to express their thoughts about what those probabilities mean. In part B, by making use of the answers they have provided in the first part or by bringing to their attention the widely known facts and definitions of probability theory a conflict situation is brought to their attention. In part C, a task is proposed that when tried by the participants seem to shed some light on the conflict in the previous section.

PART A

Q1: If one hundred fair coins are flipped, what are the chances of getting half heads and half tails?

MO: It should be around $\frac{1}{2}$.

MS: Well, each one has a $\frac{1}{2}$ probability, let’s see, each has a $\frac{1}{2}$ so the overall probability is, mmmm, in 100 coins, half may turn up heads and half will show tails.

MA: The probability is $\frac{1}{2}$. But wait, let me consider four coins instead, ... (after 10 minutes of writing down all the possible outcomes and finding the answer for 4 coins and doing the same for 6 coins, extends the method and gives the answer for 100 coins) ok, here we go it comes to 0.08, this is the probability of half heads-half tails in 100 coins.

SH: It is very big, actually bigger than other probabilities. In small number of trials anything might happen but as the number of flips goes up, I expect that the number of heads and tails get closer to each other.

NE: Each coin has a $\frac{1}{2}$ probability of landing on heads so for the 100 coins we'll have one over 2^{100} .

PART B

At the beginning of this part some time is spent on binomial formula (the probability of obtaining exactly K heads in n coin toss is $\binom{n}{k} \times \frac{1}{2^n}$), and the type of questions it answers and then each participant is presented with a basic form of computer simulated game of flipping 100 coins as described below (excerpts from the interview with *MS* are presented as example).

The interviewer (coded as *S*) opens a worksheet in Excel and generates 100 zero-one random numbers. The sum of these 100 0-1's showing the total number of heads is displayed in a separate cell. Conditional formatting is used on the cell containing the sum so that whenever the sum is exactly 50, the cell turns blue. In another column of the worksheet a record of *MS*'s win-loss is kept by writing down "L" for each loss and "W" for each win (*MS* wins each time a 50 heads-50 tails occurs). The random numbers previously generated change and so does the game result. The rest of the conversation takes place as both *S* and *M* are looking at the numbers changing and what turns up as the result of each game.

S: Here we go, the sum says 64, which means the number of heads are 64 and the rest of coins (36) turned up tails. Click, click, ...

MS: That one was close; a 49! And yet another 49!

S: Yes, and this one is 52, now 51

MS: 52, 48, 49 again, 52, 45, 52, so many 52's, ok this one is far out.

S: 51, 53, 52

MS: All of them are in a range of very close numbers. Click, click, ...

MS: See, this is why I don't trust probability! I haven't won it even once. Click, click,...

After half a minute a 50 appears

MS: (cheerfully) one after all!

Another 50 appears soon

The game goes on with *S* and *MS* looking at numbers, saying them out loud and writing down the scores.

S: Ok, let's see, how are we doing, so far we have played this game for 245 times out of which you have won 16 times.

MS: Yeah, it is pretty rare; I'm even surprised that it happened 16 times.

By this time, the interviewees have agreed that the empirical probability under discussion (half heads-half tails) is less than 10% and it gets smaller if the number of coins increases.

Q: One of the interpretations/definitions of probability (as seen before) is the success ratio in long sequences of the repeated experiment. More specifically if a fair coin is tossed many times the ratio of heads to total gets very close to $\frac{1}{2}$. Now the question is: why the probability of getting half heads-half tails in the case of 100 coins is so low and as we saw gets lower for more coins?

MO: Huh, you got me! Yeah, it is contradictory. We should find out which one is wrong.

MS: I don't know (pause). May be we are taking two different things, we discussed the probability of "heads" and it is almost half (long pause) which means half of the times it comes up heads so the other half of the times it comes up tails. Doesn't it mean that all the times the number of heads and tails are equal then?

MA: Because 100 is not that big of a number. (she is reminded again that if the number of coins increase, the probability of half heads-half tails will decrease, in response she says) actually I think that this decreasing behavior should stop at some point and the probability of half-half starts to increase so that it eventually comes to $\frac{1}{2}$.

SH: (Silent for quite a while), seems funny, so the problem is that on one hand in any long sequence of heads-tails the ratio is very close to half but on the other hand the probability of getting half heads- half tails is very small, very funny!

NE: May be this is some kind of sampling error, mmmm, but you say that if n increases, the probability of half heads-half tails gets smaller, huh! It shouldn't be this way because even if there are some errors they should diminish when n is large.

PART C

Background

One of the well-known misconceptions about probability is the *illusion of linearity*; a term borrowed from Van Dooren et al. (2003). They present evidence of problematic judgment in probabilistic situations in which the proportional reasoning is applied to non-linear problems. The particular example they use is the birthday problem in which many incorrect answers are provided for questions such as these: How many people are needed in order for having a 50% chance of at least two birthday matches? If within a given number "n" of people there is a 20%

chance of birthday match, what will the probability change to if the number “n” is doubled? The authors propose that a curriculum in which proportional reasoning is over emphasized might very well create hindrances in the students’ probabilistic reasoning.

The Activity

The task presented in this paper is inspired by the illusion of linearity: if the learners expect the number of success to increase proportional to the number of trials, then perhaps they’ll get surprised to see it behave differently and reconsider their approach. The interviewees are invited to decide on a certain set of outcomes in the experiment of 100 coin flips such that they have even chances of winning. In other words the ultimate question is: what are the numbers of heads that provide a 50% chance of winning?

The question is presented to the interviewees in several steps:

Q1: Instead of betting on “50 heads exactly” you can bet on three different number of heads and I’ll bet on the rest. Which three numbers will you pick?

In answer to this question there was a strong consensus that 49, 50, and 51 should be chosen, here is an example of the answers:

SH: The numbers we saw were just oscillating between the numbers close to 50 but not 50 itself. I think if I add 49 and 51 to my bet, it will cover a lot more of the chance. It is reasonable, 50 is the most probable so the numbers close to it should be almost the same.

Q2: What would you suggest in order to calculate the chances of your win if you bet on 49 or 50 or 51 heads?

MO, MS, & SH: We can either use the formula or use the random numbers again, this time we need to format the sum cell so that it turns blue if the sum is between 49-51. By repeating the game so many times we can find out what are the chances like.

MA: Use the binomial formula for 49 and 51 separately; add the results to 8%.

NE: Since 49 and 51 are very close to 50, then we can estimate the probability of those just by ignoring a small error and use the same thing we got for 50. So if we use 11%, this time we have a 33% or so chance of winning and if we use 8%, it comes to around 24%, not sure which one is correct.

At this point after a short discussion, the participants agree to use the binomial formula for further calculations since it is faster.

Q3: So the chances of winning are improved but it is not even. Let's calculate the number of cases close and symmetric to 50 that needs to be added in order to get 50% chance of winning. By using binomial formula the following results are obtained:

Number of heads bet on:	Probability of winning
50 only:	0.079
49, 50, 51:	0.235
48, 49, 50, 51, 52:	0.383
47, 48, 49, 50, 51, 52, 52:	0.515

Q4: So the probability of 50 ± 3 comes to almost 50%. Let's now consider flipping 500 coins instead of 100, how many numbers should be added to the middle number so that we can have a 50% chance of winning? What would be the answer for 1000 coins?

Four out of five participants expressed that since the number of coins is multiplied by five, so we need to add 30 numbers to the middle bet, in other words 250 ± 15 has 50% probability of showing up. Likewise for 1000 coins the answer is: 500 ± 30 . The other interviewee believed that since 500 coins are too many and the number of possible heads-tails is huge, it is safe to grab an even bigger interval from the numbers around the middle case.

The correct probability is calculated from binomial formula (83%, 94% for 250 ± 15 and 500 ± 30 respectively) by the interviewees stirring the following reactions (two interviewees had to quit since they didn't seem to follow and the time limit wouldn't let us go back and reflect on previous steps):

MO: You sure we did it right? (pause) so, for 100 coins the deviation was 3%, but for 1000 coins 3% of deviation from the middle number covers 94%? It is very strange! It is not how a normal curve behaves. This is like several normal curves that become sharper and sharper around the average, until it gets too steep.

MS: These numbers say that if I flip 1000 coins and bet on either of 500 ± 30 cases, in 94% of the times I'll win? It is almost like winning every time! Can I try it with random numbers? (He proceeds to play the game with 1000 random 0-1's with the sum cell turning blue whenever the number of heads is between 470, 530). It's all blue! No kidding!

SH: Wow, so it doesn't behave that way, how to put it, non-linear? (long pause) so, as the number gets larger and larger, fewer middle numbers are needed to collect a probability of 50%, may be not fewer numbers, I mean less in ratio, compared to how big the number itself is. Could you tell me 500±what number has a 50% chance of occurring for 1000 coins? (he is provided with the answer which is 500±10), mmm, ok, here it goes: for 100 coins we needed 6% of the middle numbers, for 1000 coins all we need is 2%. For 10000 coins perhaps it is less than 1%, or even less, very close to zero. Is this the answer to the paradox thing you asked (referring to the question in part B)?

At the end, the three interviewees commented that now they have more trust in the 50% probability of either sides of a coin and they now perceive it differently. *MO* and *SH* also mentioned that they had learned about the law of large numbers before, but never tried it numerically and now they are seeing it as a law that guarantees that the number of unwanted and outlier outcomes compared to the number of trials, diminishes to zero.

DISCUSSION

Most of the participants expressed a belief at first that the probability of a half tails-half heads situation with 100 coins is $\frac{1}{2}$. One explanation for this faulty judgment could be sought for in the representativeness heuristic; first introduced by Tversky & Kahneman, later discussed and elaborated on by many researchers particularly by Konold, Pollatsek, et al. (1993). The explanation this heuristic provides is that the participants assign the same ratio of head to tails as is the case for the whole population. Since the hundred coins tossed is a representative of the whole coin toss population, it is reasonable to assign the same probability. Another alternative way to explain the answer could also be well interpreted through the "outcome approach" proposed by Konold (1989) that accounts for incorrect probabilities assigned to events because the answer is regarding the outcome of each of individual trials only. Also one might suggest that lack of tangible experience with limit-based notions (in this case The Law of Large Numbers), contributes to development of inadequate or non-coherent ideas about probability.

This study indicates that developing even an average understanding of the frequency-based probability heavily relies on the learner's experience of large number of stochastic trials. When teaching empirical probability (unknown to some interviewees) a teacher could very well use what computer simulation offers as a tool to help learners construct a valid understanding of a random behaviour through repeated trials. As the data suggests, Law of Large Numbers is perceived as an overall tendency of the outcomes toward a certain behaviour. The proportional reasoning of learners and the linearity illusion they hold around

probability could be employed to make a more accurate and realistic sense of The Law of Large Numbers.

References

- Francois, K., & van Bendegem, J. (2007). *Philosophical Dimensions in Mathematics Education*. New York: Springer.
- Freudenthal, H. (1972). The Empirical Law of Large Numbers or The Stability of Frequencies. *Educational Studies in Mathematics* 4, 484-490.
- Hacking, I. (1990). *The Taming of Chance*. Cambridge: Cambridge University Press.
- Konold, C. (1989). Informal conceptions of probability. *Cognition and Instruction* 6, 59-98.
- Konold, C., Pollatsek, A., Well, A., & Lipson, A. (1993). Inconsistencies in Students' Reasoning about Probability. *Journal for Research in Mathematics Education* 24(5). 392-414.
- Konold, C. (1995), Confessions of a Coin flipper and a would-be instructor. *The American Statistician* 49(2), 203-209.
- Piaget, J., & Inhelder, B. (1951/1975). *Origin of the Idea of Chance in Children*. (L. Leake, P. Burrell, & Fishbein, H., Trans.). New York: W. W. Norton & Company.
- Pratt, D. & Noss, R. (2002). The Microevolution of Mathematical Knowledge, The case of Randomness, *Journal of the Learning Sciences* 11(4), 453-488
- Tijms, H. (2007). *Understanding Probability*. New York: Cambridge University Press.
- Van Dooren, W., De Bock, D., Depaepe, F., Janssens, D., & Verschaffel, L. (2003). The illusion of Linearity: Expanding the Evidence towards Probabilistic Reasoning. *Educational Studies in Mathematics* 53(2), 113-138.

INTERACTING AGENCIES AND THEIR INFLUENCE ON EMOTIONAL EXPERIENCES

Sean Chorney

This research report adopts a theoretical perspective of interacting agencies when a student engages with a mathematical digital tool. Adopting the notion that external activity is analogous to inner activity originally posited by Vygotsky, students emotional experiences outlined by Leont'ev are analysed. Using the tools of discourse analysis as well as the implementation of written reflections outlined by Sannino is used to access the inner workings of the student while they engage with an exploration activity based within a dynamic geometry environment.

INTRODUCTION

This paper is about the role of digital tools in the classroom and their relevance and effect in mathematical learning. Often the computer is viewed as an amplified tool but computers, in general, are different from other tools in that they offer numerous ways of engagement. They can function in different ways at different times with different users. This versatility allows users to be expressive and creative: to express their agency. Given this freedom of expression mathematical learning can be enhanced or it can be affected negatively. When the mathematical learning is affected negatively, often the student is to blame. Since the student is the central figure in an interaction with a tool, breakdown is often attributed to the student. I think that this perspective can have significantly harmful effects on student's personal sense in mathematical activity. Given the computer's power and versatility, I suggest that the computer imposes its own agency in any interactions. I grant agency to the computer in this paper particularly because it can de-center our perspective and offer new descriptions of interactions. Adopting that students have agency and granting agency to computers offers a new perspective for analysis.

Papert (1993) describes computer software as an expressive technology. Considering their range of possibilities and feedback, the material agency of computers is an important topic of study. Computers provide a way to see how both technological and social processes interact and present an environment where students can make choices.

I suggest that there are other aspects of tool use that are significant factors in students' engagements with tools: breakdown, surprise, dismay, shock and pleasure are all viable reactions and, I contend, relevant in working with tools. I suggest that the emotional dimension is often absent from the frameworks commonly employed in mathematics education. I also suggest that personal

attitudes, feelings, tactile experiences, inefficient use and non-appropriation are issues that researchers should consider, especially with regards to sophisticated digital tools. Leont'ev's (1978) definition of emotional experience is identified as the student's reflection of her motives and the possibility of fulfilling those motives. Saninno (2008) appeals to this definition and states: "This definition raises the important and complex issue of tangible manifestations of emotional experience and its materialization in action" (p.273).

My research goal is to understand the dialectical engagement of student agency and a dynamic geometry software's agency, namely The Geometer's Sketchpad, by attending to the interaction emergent in activity.

THEORETICAL FOUNDATION AND FRAMEWORK

Traditionally, agency has only been attributed to humans (Giddens, 1984; Dant, 2005) to indicate their ability to act independently, exercise choice or express themselves. This idea of agency is synonymous with intention or will. There are, however, recent extensions of agency such as those proposed by Pickering and Malafouris, both of whom include material agency. Material agency is the act of the material, or its influence upon our actions. According to Pickering (1995), there are things in the world, and we interact with them; these things provide resistances and obstacles and consequently impose limits upon us. Pickering considers obstacles a natural part of interacting with things, and accommodation the most appropriate response to those obstacles. He points out that scientists will accommodate their actions or even their intentions to overcome the obstacle. Since this model depends on the resistances afforded by the material, Pickering also argues that scientists' goals are often not known prior to the interaction. This issue of goal parallels Leont'ev's idea of motive. Pickering is an important individual for two major reasons. He proposes the construct of material agency and explains its relevance, and he presents a performative idiom of practice that entails a dialectic of agencies.

This imposition of an obstacle is material agency. "Agency" is a contentious term; the idea of material agency is a fairly new one that many will find challenging to identify. Agency, by many accounts (Malafouris, 2004; Coole, 2010), is an emergent product of activity; Coole states that we need to "rethink agency not as an essential characteristic of the rational subject, ... but as those contingent capacities for reflexivity, creative disclosure and transformation that emerge hazardously within the folds and reversals of material/meaningful flesh" (p. 113).

In this paper, "agency" will be defined quite broadly as "essentially who or what is the cause of the doing" (Malafouris, p. 23). It considers "any element which bends

space around itself, makes other elements dependent upon itself and translates their will into a language of its own” an actor (Malafouris, p. 33). It asserts that denying the agency of non-humans furthers the belief that scientists are the ones that give materials life.

I seek to start a dialogue that addresses some fundamental attributes of material influence on students’ experiences in handling and appropriating a mathematical tool. Agency focuses on organic attributes rather than describing rational development. Agency is a focus of this study for a number of reasons. Firstly, I contend, agency is a viable and relevant construct to approach “who or what is the cause of the doing.” In the absence of a theoretical account of tools, I suggest that agency is a good starting point that may reveal some generative ideas. Secondly, I propose that agency is accessible. This accessibility originates from the activity of interlocutors: through discursive analysis, the agency of both the individual and the program will be identified. Thirdly, the activity of agency is always existent and, I contend, all around us. Paraphrasing Pickering, there are things going on in the world all around us. This agency may initiate from humans, or it may not. Finally, agency does not focus on cognitive or affective aspects specifically; instead, it attends to all forms of behaviour. If the student is developing in a cognitive sense, that development may be accessible by attending to their choices and actions, their agency in action. If, however, they are unmotivated, this lack of motivation too will likely be identified by expressions of agency.

The purposes of this study are, first, to look at this interaction of agencies in this process of activity, and, second, to consider the end result of the activity and interpret the process in terms of the student’s experiences and the material influences on that experience.

Part of Pickering’s argument is that the scientist could not achieve much on their own; that is, they need materials to perform. Although Pickering does not elaborate, it seems that this point indicates that materials do not just resist but can enhance performances as well. His analysis offers an important aspect to consider in the formation of progress and growth. His simple model can be put simply as follows: materials can be understood as having agency when their structures, make-up and design restrict the subject within a context of activity.

The attitude shared by the aforementioned researchers lays a foundation for describing mathematics as a discipline of negotiation, of conjectures, and of exploration. It also posits the view that often what students say reveals more than just propositional conjecturing: it reveals a declaration of agency.

I align myself with the view that the learning of mathematics is an activity of negotiation and activity. Learning mathematics involves more than accepting or

memorizing specific facts. As opposed to an accommodation and transfer paradigm, mathematics is developed in the communication and practice of investigations and/or problem solving.

METHODOLOGY

The process by which I will access their reactions, interactions and thinking is by using discourse analysis tools. Discourse analysis is a methodology that emerges from a variety of different disciplines and can be used for a wide range of research pursuits. There are tools in discourse analysis, such as voice, deixis, and modality that are well suited for this study. As Sinclair and Jackiw (2010) state: “verbal representation better allows us to pursue social, rather than cognitive, implications in technology design” (p. 154).

Discourse analysis tools attempt to get at these specific attributes by looking at the subject, focusing on the semiotic events and interpreting the activity through language. It is an expressive and social perspective. It holds that the spoken or written word, or any semiotic event, provides a way to access certain aspects of an individual’s beliefs, attitudes and commitments. As Rowland (2000) states, discourse analysis provides “language for the explicit communication of thought, and... a code to express and point to concepts, meanings and attitudes” (p. 3).

In addition to discourse analysis I also consider a method of data collection implemented by Sannino. Sannino (2008) identifies a gap in discourse analysis and activity. In dealing with the aspect of Vygotsky’s (1978) notion of inner speech, she addresses the need to access both external and internal conversations. While Sannino primarily focuses on conversations with others or oneself, I extend the idea of conversation to one that addresses the interaction of agencies, including the agency of materials. I will implement her method of collecting of data as an investigation addressing “two ‘dialogues’ unfolding over time, one external (the interactive process) and the other internal (the intra-active process)” (p. 276). After the activity is complete, students will be asked to write a detailed description of their experience with the software program. This particular method has its limitations but to approach this writing with the intention of contextualizing the activity in relation to their own recollection is a method that goes beyond external expression and has the potential to access inner speech. Sannino makes the point that often such an autobiographical account will attend to critical features of the interaction that can be re-analyzed by the researcher in the context of the activity.

The computer expresses its agency in major ways: on the one hand, it changes practices; on the other, its potential for interaction, expression and connection with the reciprocating individual makes it a powerful agent in the mathematics classroom.

RESEARCH CONTEXT AND PARTICIPANTS

Data collection took place in a Vancouver high school with students working with The Geometer's Sketchpad (GSP). Students worked in pairs and were presented with the following problem:

You are given a quadrilateral ABCD. Construct the perpendicular bisectors of its sides: a of AB, b of BC, c of CD, d of DA. H is the intersection point of a and b, M of b and c, L of c and d, K of a and d. Investigate how HMLK changes in relation to ABCD. Prove your conjectures (Olivero & Robutti, 2007).

Data was collected by means of a software capturing software, SMRecorder. SMRecorder recorded all the activity on the screen as well as recorded the verbal utterances of the two students.

The students started from an initially blank screen, a new file. They constructed the initial quadrilateral and then were expected to make conjectures. They spent approximately 40 minutes working on the construction and conjecture component of the activity. After the activity, the students were asked three days later to write reflections of their experiences. In general, students were relatively successful in finding at least one correct conjecture. The following is an analysis of one particular interaction between two students, Chris and Adrian.

Chris and Adrian constructing and conjecturing

This data represents some short excerpts of students' work. Adrian had control of the mouse.

In the initial stages, Adrian constructed the quadrilateral, Chris watched as Adrian inadvertently added some extra labels to the diagram.

- 1 Adrian: No, what is this? Giggling, laughing
- 2 Chris: Get rid of these letters (pointing)...(after watching Adrian unsuccessfully attempt to delete the extra labels)...don't worry about it.
- 3 Adrian: Let's just move them. No, no, no, no!
- 4 Adrian: You want to start a new file?
- 5 Chris: Ya, let's start over.

While Adrian builds and labels the quadrilateral, a few extra clicks of the mouse bring some new labels that are difficult to delete or hide. He spends a few minutes attempting to delete these extra labels, in the process, he creates some more. This aspect of GSP is annoying as they both indicate later in their reflection. Mistakes like this should be easily rectified with <ctrl><z>; however, in this particular case it is not. The reactions here indicate that Adrian and Chris were both surprised as

well as experienced a challenge to their original motive. They decided to start a new file as indicated in line 5.

Further on in the activity Adrian is measuring each angle that he considers relevant and he uses the calculate menu item to add angles to identify which angles add to 180 degrees. In his attempt to clean up all the angle measurements he begins to delete the individual angle measures and keeps only the summation expressions. But when he deletes the individual angles the summation statements disappear as well.

6 Chris: Dude!

7 Adrian: Oh...wait, no...that disappears too.

8 Chris: Let's start over.

Adrian and Chris are challenged as they explore the image for relationships. Moving and dragging for a few minutes, Adrian is trying to see something. He pulls and drags the points from one end of the screen to the other in an attempt to identify some relation and but the diagram implodes and explodes and Adrian falls into a trance.

9 Chris: Come on stop it, put it back.

10 Adrian: That's very intense.

Chris gets a bit frustrated with Adrian having complete control of the mouse and starts to draw the image on paper. He cannot see the complete diagram on the screen and asks Adrian to manipulate the diagram so that he can see the whole image.

11 Chris : Can you change it so that I can see all the dots? ...Ya, okay.

Adrian expands his diagram so that one of the vertices moves off the screen.

12 Chris: Where's F?

13 Adrian: It's way out there. (pointing).

These episodes indicate evidence of both material and personal agency. The utterances "Let's start again" (line 5 and 8) or "Let's just move them" (line 3) both employ the "I" voice, indicating an expression of agency from the student's perspective. Questions or expressions such as "No, what is this?" (line 1) or "Where's F" (line 12) are examples of manifestations of material agency. The students are not in control, nor do they know why something is occurring the way it is, as in line 1. There are some references to the things on the computer screen as indicated by expressions such as "put it back" (line 9) or "Can you change it..." (line 11). The students are accepting the "it" of the computer to act. From a discourse analysis perspective, their voice clearly varies between an "I" voice and an "it" voice.

DISCUSSION

Comparing a static traditional diagram experience with the practices of interaction on a computer is radically different. Although Chris is not in contact with the mouse or has any control over the construction he is involved in the experience. Adrian and Chris are sharing the computer screen. They are both able to share the same visual experience. Just this fact alone speaks to the agency of the computer. Without the computer screen visible to both students, their shared experience would be different. The material in this context provides a common medium in which most activity takes place. This common media provides a shared experience but also a challenge of agency, for example, when the pair does not agree to the necessary action such as in lines 6 or line 9.

As indicated in lines 6 through 8, Adrian and Chris use the measurement tool along with the summation operation to determine if any of the angles they are working with are supplementary. This is a radically different practice than more traditional roles of deductive reasoning. While a static diagram may impose the expectation of “seeing” supplementary angles, GSP, in this case, supports a very different action. The students measured all the angles and then identified supplementary angles by way of combining numbers together. So noticing that angle $ABC=45.2$ degrees and angle $GHI=134.8$ degrees was the method of identifying supplementary angles. In a static diagram this method may not be as justified because the sum of the measurements of the angles may not remain 180 degrees. But in GSP, the students were able to alter the diagram by dragging and noticed that the angles stayed relatively the same.

Another observation of interest was observed in how the students chose to redo their constructions. This particular action is an important aspect of the computer as it allows for a new approach to constructed work. One can always get rid of a paper construction but to recreate the image would be as time-consuming as the original attempt. On the computer, the re-creation may happen much faster as the tools, the menu items for example, are known and are familiar, as was the case with Adrian and Chris. In line 5 and in line 8, there was no hesitation to recreate the diagram already created.

In the presented episodes there is also evidence of emotional experience. As Leont’ev indicates, emotional experience is a reflection upon one’s motives. There were examples in these episodes where motives were changed. In an effort to clean up the organization of the data presented on the screen, Adrian attempted to delete individual angle measurements. In recognizing that the summation statements were also deleted, Adrian stopped that action. Expressions of “Dude!”

(line 6) indicate that there is a challenge to one's motive and a corresponding feeling of frustration from Adrian who question his actions in line 7. These feelings of frustration were confirmed in Adrian's reflections written a few days later. Pickering identifies this as a resistance and identifies a need for accommodation.

CONCLUSION

The inner activity of the students is one of dealing with resistance and accommodation. These manifestations occurred in the form of starting over, getting excited, identifying with the points, getting frustrated and altering motives. The students are not only interacting with a tool, their emotional experiences are not just innate expressions but reactions to manifestations of agency from an agent that does not necessarily act predictably. The fluid dynamic activity with the technological tool allowing for instant feedback offered an environment of exploration and an opportunity for expressing agency.

References

- Coole, D., & Frost, S. (2010). Introducing the New Materialisms. In D. Coole, & S. Frost (Eds.), *New Materialisms: Ontology, Agency, and Politics*. Durham, NC: Duke University Press.
- Dant, T. (2005). *Materiality and society*. Maidenhead: Open University Press.
- Giddens, A. (1984). *The Constitution of Society: Outline of the Theory of Structuration*. Berkeley, CA: University of California Press.
- Leont'ev, A. N. (1978). *Activity, consciousness, and personality*. Hillsdale: Prentice-Hall.
- Malafouris, L., 2004. The cognitive basis of material engagement: where brain, body and culture conflate. In E. DeMarrais, C. Gosden & C. Renfrew (Eds.), *Rethinking Materiality: the Engagement of Mind with the Material World*, pp. 53-61. Cambridge: McDonald Institute for Archaeological Research.
- Olivero, F. & Robutti, O. (2007). Measuring in dynamic geometry environments as a tool for conjecturing and proving. *International Journal of Computers for Mathematical Learning*(12),135-156.
- Papert, S. (1993). *Mindstorms: Children, Computers, and Powerful Ideas*. New York: Basic Books.
- Pickering, A. (1995). *The Mangle of Practice: Time, agency, and Science*. Chicago: The University of Chicago Press.
- Rowland, T. (2000). *The pragmatics of mathematics education: vagueness in mathematical discourse*. London: Falmer Press.

- Sannino, A. (2008). Experiencing conversations: bridging the gap between discourse and activity. *Journal for the Theory of Social Behaviour* (38)3, 267-291.
- Sinclair, N. & Jackiw, N. (2010). Learning through teaching, when teaching machines: discursive interaction design in Sketchpad. In R. Leikin, & R. Zazkis (Eds.) *Learning through teaching mathematics: Development of teachers' knowledge and expertise in practice*, pp. 153-168. Dordrecht: Springer.
- Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.

A CASE STUDY: STUDYING, SELF-REPORTING, AND RESTUDYING BASIC CONCEPTS OF ELEMENTARY NUMBER THEORY

O. Arda Cimen & Stephen R. Campbell

The objective of this case study is to look in depth into personal factors affecting metacognitive monitoring and control in self-regulated study and restudy of basic concepts of elementary number theory. We incorporate a wide spectrum of observational methods enabling us to record overt behaviour, psychometric questionnaires, and covert behaviour related to various psychophysiological responses. All this is applied toward an attempt to gain deeper insights into personal factors implicated in motivation, metacognition, and beliefs, pertaining to self-regulated learning and mathematics anxiety. Ultimately, our aim is to provide “learner profiles” that can be used to better inform assessment and tailor instructional design and mathematics education research.

OBJECTIVES

The broader objective of this investigation is to look in depth into personal factors affecting metacognitive monitoring and control in self-regulated study and restudy of basic concepts of elementary number theory. The specific objective of this investigation is to begin doing so with a single case study. What is unique about this investigation in educational research is that we incorporate a wide spectrum of observational methods, enabling us to record overt behaviour using audiovisual techniques and psychometric questionnaires, and covert behaviour related to psychophysiological responses of various organs, including brain, heart, lung, and skin, along with muscle response and eye movement. We further augment our observational control by presenting our stimuli via computer and using screen and keyboard capture.

Our justification for this more rigorous approach to behavioural control stems from a theoretical framework that views cognition and learning as embodied. Accordingly, recording and integrating embodied behavioural responses should shed light on cognition and learning that would otherwise remain hidden using more limited traditional techniques such as field notes and audiovisual recordings. In essence, this observational approach is to audiovisual recordings as the latter is to studies relying solely on field notes (e.g., Campbell, 2010).

We focus here on study and restudy of basic concepts of elementary number theory that include the division theorem, divisibility, divisibility rules, factors, divisors, multiples, and prime decomposition (Campbell & Zazkis, 2002; Zazkis &

Campbell, 2006; Campbell, Cimen, & Handscomb, 2009). Our aim is to gain deeper insights into personal factors affecting study and restudy of this material, interjected with self-reports of judgments of learning (Nelson, Dunlosky, Graf, & Narens, 1994).

THEORETICAL FRAMEWORK

All methods of observation and measurement have intrinsic limitations. Hence, it is not the case that every nuance of learning and lived experience will be observable, measurable, or identifiable in brain and body behaviour. Rather than attempting to seek out and identify psychophysiological manifestations for various subtle aspects of thinking and learning, we will likely meet with greater success in applying our various means of observational control to matters involving experiences that are more intensely embodied, such as anxiety. Indeed, the very fact that anxiety is such a deeply embodied phenomenon, to the extent of being physically disabling, in itself warrants inclusion of psychophysiological methods into our repertoire of observational methods. We expect to see evidence of anxiety, for instance, with increases in heart rate and respiration (e.g., Kelly, 1980; Dew, Galassi, & Galassi, 1984). Accordingly, we incorporate methods from psychophysiology with more traditional observational methods in educational research.

METHODOLOGY

We have chosen to focus on some basic concepts of elementary number theory for a number of reasons. First, it is our views that concepts of elementary number theory, especially with regard to division and divisibility, have a natural role to play in helping elementary and middle school students make the transition from arithmetic to algebra (Campbell, 2001).

Our instrument for investigating metacognitive monitoring and control of study-restudy of basic concepts from elementary number theory, comprised six pages of subject matter content delivered using gStudy (Perry & Winne, 2006). This subject matter content for study-restudy was specifically designed to involve three levels of learning: the first involving computation (C), the second involving understanding (U), and the third involving reasoning (R). Our participant was allowed to study this material at her leisure. The study material was then presented to our participant in a manner that highlighted different parts thereof (Figure 1), enabling her to provide a judgment regarding her learning (JOL), i.e., whether she understood that content very well, well, or not well. Once this was done, she was given an opportunity to restudy the material in preparation for a test.

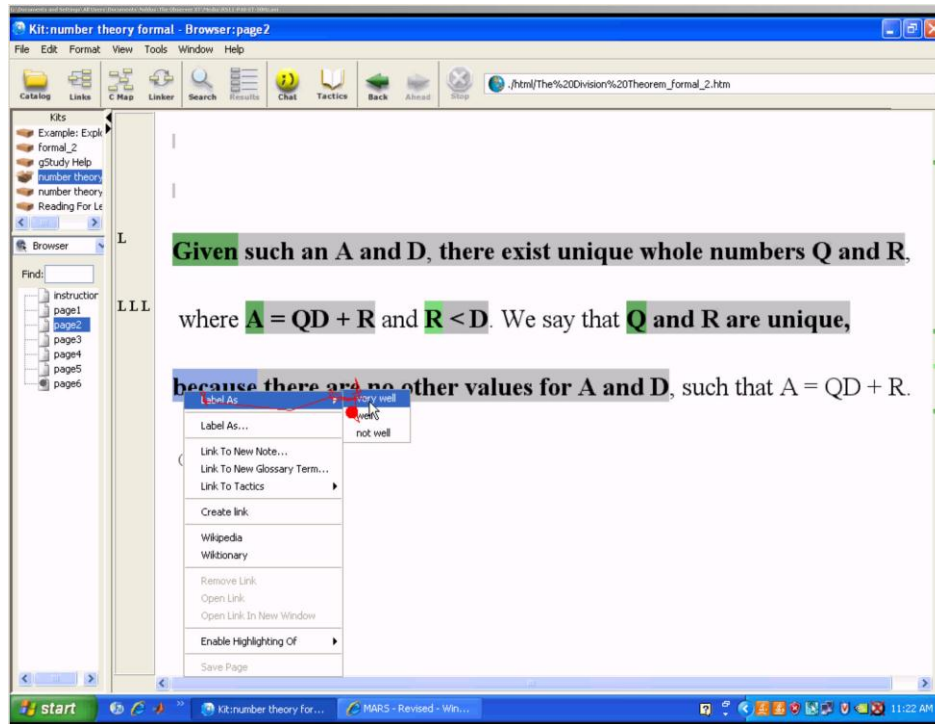


Figure 1: Screen capture of participant indicating JOL

Our model for interpreting our data on metacognitive monitoring and control is an adaptation of Elliot (1999) and Elliot and McGregor's (2001) motivational distinctions between mastery-performance and approach-avoidance, fused with Nelson et al.'s (1994) notion of self-reported judgments of learning (JOLs) resulting from metacognitive monitoring (Figure 2).

	Mastery / intrinsic motivation	Performance / extrinsic motivation
Approach / taking time	JOL: not well understood	JOL: very well understood
Avoidance / not taking time	JOL: very well understood	JOL: not well understood

Figure 2: Metacognitive monitoring and control of content and time allocation for study-restudy

We understand 'mastery-approach' to represent taking time in restudy to learn something for its own sake that was judged to be not well understood, whereas 'mastery-avoidance' represents not taking time for restudy of content judged to be well understood. 'Performance-approach' serves to better consolidate content judged as well understood, whereas 'performance-avoidance' represents not taking additional time to restudy content considered poorly understood. Accordingly, 'mastery' and 'performance' represent intrinsic and extrinsic motivation

respectively, and ‘approach’ and ‘avoidance’ represent time allocated to study-restudy.

DATA SOURCES AND EVIDENCE

Behavioural data

Using a 64-channel BioSemi EEG system enabled us to examine the average power in different frequency ranges generated by our participant during the study and restudy periods (Kahana, 2006). Our participant was also “wired up” to monitor fluctuations in heart and respiration rates. We presented the gStudy stimulus to our participant using a Tobii 1750 eye-tracking monitor, which detects reflections of infrared light pulses on a participant’s retina to precisely trace what is being looked at from moment to moment.

An ultra-sensitive microphone allowed for highly sensitive recordings of think-aloud narratives. Infrared video cameras recorded important aspects of the participant’s behaviour, such as facial expressions and body movements, from three vantage points. Several steps were taken to maximize the accuracy of eye-tracking data of the study-restudy material such as increasing font size and spacing of the study material. Data streams were integrated, time synchronized and analyzed using Noldus’ Observer XT (Figure 3). We relied on cross-calibrating audiovisual, eye-tracking, and other data to ensure we were selecting behavioural data for analysis at the appropriate times (Campbell & the ENL Group, 2007).

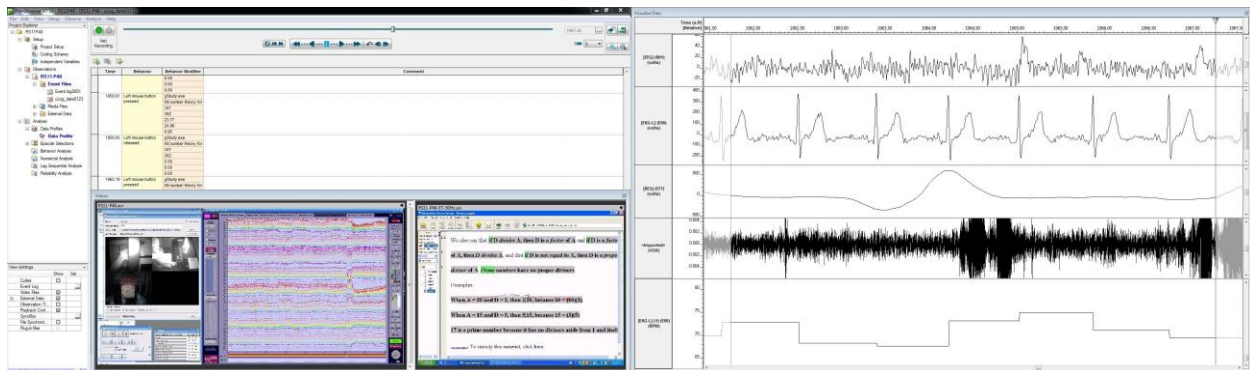


Figure 3: The integrated and synchronized data set on Noldus Observer XT

Self-report data

The participant gave informed consent. She filled out a demographic questionnaire. Pre- and post-questionnaires were used prior to and after engaging our participant in the study-restudy activity. Pre-questionnaires, we do not go into detail here, included the Motivated Strategies for Learning Questionnaire (MSLQ) (Duncan & McKeachie, 2005), the Epistemic Belief Inventory (EBI) (Schraw, Bendixen & Dunkle, 2002), the Metacognitive Awareness Inventory (MAI) (Schraw &

Dennison, 1994), and the Math Anxiety Rating Scales (MARS) (Derek, 2003). A pre-questionnaire was also given that was designed to gain insight into how comfortable the participant was with her abilities regarding calculation, reading, recall, comprehension, and reasoning.

After completing the pre-questionnaires, the participant engaged in the study component of the experiment. Following completion of this initial study period, the participant labelled their judgments of learning (JOLs) pertaining to how well she learned computational, conceptual, and inferential aspects of the study material. After labelling the JOLs, the participant was given a 10-question true/false test on the study material and asked to rate her confidence in her answers on a scale of 0-10. Following a short rest, the participant engaged in restudy of the material and then rewrote the same test. Finally, the participant filled out a metacognitive post-experiment questionnaire pertaining to her experiences in the experiment.

Participant

The participant was a 22-year-old female undergraduate student (in Molecular Biology) with a Vietnamese background. Her overall health was self-reported as good (no anxiety disorders or symptoms and no physical problems). After the observation she reported being “a little worried that it was going to be hardcore math theory that was being tested on the exam part” before the observation.

RESULTS

The participant’s average heart rate for the study period was ~75.1 beats per minute (bpm), and reduced to ~69.0 bpm for the self-report period, and reduced further to ~67.0 bpm for the restudy period of the same subject content material. Her respiration rates were ~20.3, ~18.0 and ~17.8 breaths per minute for the study, self-report and restudy periods, respectively, while her respective eye blink rate over those three time periods were 37.5, 16.0, and 34.3 blinks per minute. These values are summarized in Table 1.

	Time Spent (seconds)	Heart Rate (beats per minute)	Respiration rate (breaths per minute)	Eye Blink Rate (blinks per minute)
Study	608	~75.1	~20.3	37.5
Self-Report	278	~69.0	~18.0	16.0
Restudy	98	~67.0	~17.8	34.3

Table 1: Time and physiological data summary for study, self-report, and restudy periods

Considering that heart rate is a strong indicator for the level of stress and anxiety (Kelly, 1980; Dew, Galassi, & Galassi, 1984), the results clearly indicate that the participant was less anxious, i.e., more relaxed, for the restudy period, in comparison with the study period.

During the self-report period, the participant was re-shown the six pages of study material with items highlighted and she was asked to report her judgment of learning (JOL) regarding them (35 in total). She was asked to choose among three options per case for her self-reporting: not well, well and very well (Figure 1). We substituted scores of -1 for not well, 0 for well, and +1 for very well. We then tallied this scoring to give us a total JOL confidence indicator of +11.

All the JOLs labelled by the participant as “not well” learned involved calculations, and our data indicates she did not spend much time on these tasks. Hence, in accord with Table 1, the participant can be classified as having a ‘performance-avoidance’ orientation in this regard. The participant reported she learned most of the understanding tasks very well, while reporting most reasoning tasks she had learned well or very well.

Question	Question Type	Test 1 Results	Test 1 Confidence	Test 2 Results	Test 2 Confidence	Question Type	Number Theory pre-questionnaire
1	Calculation	Incorrect	7	Correct	8	Calculation	3
2	Calculation	Correct	10	Correct	10	Calculation	3
3	Understand	Correct	10	Incorrect	10	Understand	4
4	Understand	Correct	10	Correct	10	Understand	4
5	Reasoning	Incorrect	9	Correct	10	Reasoning	3
6	Reasoning	Incorrect	10	Incorrect	10	Reasoning	4

Table 2: Number theory test and questionnaire results

Table 2 summarizes our results from the test that was administered after the study period together with the results from the same test, which was administered once again after the restudy period. These results align well with results of her self-assessment from the Number Theory pre-questionnaire. She reports her level of comfort on a scale of 1 (not comfortable at all) to 5 (completely comfortable), with calculation tasks as 3, while reporting her level of comfort with understanding involving recall and comprehension as 4, and with aspects of reasoning as 3.5. Test results substantiate these reports, reiterating she is less confident with her answers with calculation tasks compared to understanding and reasoning tasks. Although she reports a higher confidence for reasoning tasks, she is less successful on this

type of task compared to understanding, which she self-reported prior to the study/restudy periods as being most comfortable with. Again, our JOL indicates she spent less time on the pages that involved calculation.

Another interesting result was the answers provided on the Number Theory Post-Questionnaire, which was directed to her after restudying the material. She stated that learning the task was not interesting for her (ranked 0 out of 7) and it was not challenging for her (ranked 2 out of 7). She indicated that, she restudied the items she found most difficult to understand. These answers indicate in this regard that she is a mastery-oriented learner when it comes to subject content involving understanding and reasoning.

DISCUSSION AND CONCLUSIONS

Based on the comparison of heart rate for the self-report and restudy periods, the results also indicate that reporting JOLs might help reduce the level of anxiety. The self-report data substantiates itself and the behavioural data substantiates itself. What about the EEG data? How might they further inform our learner profile for this participant? It is well established that various frequency ranges of electromagnetic energy, generated from various regional sources in the brain, correlate in statistically significant ways with various aspects of cognitive function (e.g., Kahana, 2006). The neural efficiency hypothesis suggests that experts exert less energy in performing cognitive tasks than do novices (Grabner, Neubauer, & Stern, 2006; Klimesch, 1999; Shipulina, Campbell, & Cimen, 2009). We have acquired similar data sets from other individuals and are in the process of expanding this study accordingly.

References

- Campbell, S. R. (2001). Number theory and the transition between arithmetic and algebra: Connecting history and psychology. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *Proceedings of the 12th International Conference in Mathematical Instruction: Volume 1* (pp. 147-154). Melbourne, Australia: University of Melbourne.
- Campbell, S. R. (2010). Embodied minds and dancing brains: New opportunities for research in mathematics education. In B. Sriraman & L. English (Eds.), *Theories of mathematics education: Seeking new frontiers* (pp. 359-404). Berlin: Springer.
- Campbell, S. R., Cimen, O. A., & Handscomb, K. (2009, April). *Learning and understanding division: A study in educational neuroscience*. Paper presented at the Brain, Neuroscience, and Education SIG at the annual meeting of the American Educational Research Association, San Diego, CA. Retrieved from the ERIC database. (ED505739)

- Campbell, S. R., & the ENL Group. (2007). The ENGRAMMETRON: Establishing an educational neuroscience laboratory. *Simon Fraser University Educational Review*, 1, 17-29.
- Campbell, S. R., & Zazkis, R. (2002). Toward number theory as a conceptual field. In S. R. Campbell & R. Zazkis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (pp. 1-14). Westport: Ablex.
- Dew, K. M. H., Galassi, J. P., & Galassi, M. D. (1984). Math anxiety: Relations with situational test anxiety, performance, physiological arousal, and math avoidance behavior. *Journal of Counseling Psychology*, 31, 580-583.
- Duncan, T. G., & McKeachie, W. J. (2005). The making of the Motivated Strategies for Learning questionnaire. *Educational Psychologist*, 40(2), 117-128.
- Elliot, A. J. (1999). Approach and avoidance motivation and achievement goals. *Educational Psychologist*, 34, 149-169.
- Elliot, A. J., & McGregor, H. A. (2001). A 2 × 2 achievement goal framework. *Journal of Personality and Social Psychology*, 80(3), 501-519.
- Grabner, R. H., Neubauer, A. C., & Stern, E. (2006). Superior performance and neural efficiency: The impact of intelligence and expertise. *Brain Research Bulletin*, 69, 422-439.
- Kahana, M. J. (2006). Mini-review: The cognitive correlates of human brain oscillations. *The Journal of Neuroscience*, 26(6), 1669-1672.
- Kelly, D. (1980). *Anxiety and emotions: Physiological basis and treatment*. Springfield: Charles C. Thomas.
- Klimesch, W. (1999). EEG alpha and theta oscillations reflect cognitive and memory performance: A review and analysis. *Brain Research Reviews*, 29, 169-195.
- Nelson, T. O., Dunlosky, J., Graf, A., & Narens, L. (1994). Utilization of metacognitive judgments in the allocation of study during multitrial learning. *Psychological Science*, 5(4), 207-213.
- Perry, N. E., & Winne, P. H. (2006). Learning from learning kits: gStudy traces of students' self-regulated engagements with computerized content. *Educational Psychology Review*, 18, 211-228.
- Schraw, G., & Dennison, R. S. (1994). Assessing metacognitive awareness. *Contemporary Educational Psychology*, 19, 460-475.
- Shipulina, O. V., Campbell, S. R., & Cimen, O. A. (2009). *Electrooculography: Connecting mind, brain, and behavior in mathematics education research*. Paper presented at the Brain, Neuroscience, and Education SIG at the annual meeting of the American Educational Research Association, San Diego, CA. Retrieved from the ERIC database. (ED505692)

Zazkis, R., & Campbell, S. R. (Eds.). (2006). *Number theory in mathematics education: Perspectives and prospects*. Mahwah, NJ: Lawrence Erlbaum Associates.

TOWARD UNDERGRADUATE STATISTICS STUDENTS' CONCEPTIONS OF VARIABILITY IN A DYNAMIC COMPUTER ENVIRONMENT

George Ekol

In an empirical study, students were asked to describe five key terms used in the measurement of variability. The interviews were videotaped and analysed using a "thinking as communicating" framework (Sfard, 2007). Results confirm earlier findings that students have quite sophisticated metaphors when describing terms not directly related to statistics. With regard to statistical terms such as 'the mean' and 'standard deviation', they were more likely to state a formula or procedure than provide a conceptual description. I propose that having fewer conceptual models and metaphors is one of the major challenges students face in understanding measures of variability. These findings have led me to design and test dynamic computer models (reported in another paper), to contribute to students' conceptual understandings of variability.

INTRODUCTION

Research in statistics education over the last two decades has drawn attention to the importance of statistical thinking, reasoning, and literacy (STRL) (see for example Garfield & Ben-Zvi, 2008). The focus on STRL is partly motivated by the need to prepare students to cope, and make rational decisions, with the increasing amount of data and information that they encounter on a daily basis. Many mathematics educators, statisticians, and statistics educators agree that the understanding of the concept of variability is central to developing and sustaining STRL in statistical learning. However, studies have shown that undergraduate students have serious challenges with the concept of variability (del Mas & Liu, 2005; Garfield & Ben-Zvi, 2008; Mathews & Clark, 2003). Mathews and Clark (2003, cited in Garfield & Ben-Zvi, 2008), interviewed undergraduate students who achieved an A-grade in their college statistics course and found that they had difficulties with the concept of standard deviation. I speculate that part of the challenge is because many students tend to rely more on statistical symbols and formulae, rather than on what the symbols actually mean. Moreover, based on Mathews and Clark (2003), I propose that another difficulty is due to a lack of models and metaphors to support students' understandings of the basic ideas underlying the symbols.

Using Java software, del Mas and Liu (2005) designed a computer model to examine students' understandings of standard deviation. In this model, pairs of different bar graphs were used with their mean values given. However, only one of the pairs of bar graphs had its value of standard deviation shown. The researchers'

objective was to gauge students' abilities to coordinate characteristics of variation about the mean with the size of the standard deviation as a measure of that variability. The study revealed that most of the students used a rule-based approach to compare variability across distributions instead of reasoning from a conceptual representation of the standard deviation. Moreover, their explanations were often based on finding a single value between the two graphs, rather than reasoning about the size of standard deviation by focussing on the spread of data around the mean.

The study recommended more research in developing models that might help students move from a rule-based approach to a more integrated understanding of variability, which is generalizable to a variety of situations. In addition to that, I suggest that more research is needed to expand the theoretical framework for analyzing data obtained from dynamic computer-based models. The current study extends del Mas and Liu's (2005) work by modifying the computer environment and making it more interactive and dynamic. The study extends the theoretical perspectives to include the role of movement, including gestures, in understanding human thinking and the strong connection between gesture and language (Seitz, 2000; Sfard, 2007).

The study is motivated partly by my personal interest in data analysis and by my work experience as a teaching assistant at a Canadian university. As a teaching assistant in the Department of Statistics and Actuarial Science, I noted that students had difficulties stating concepts of introductory statistics in their own words. Related studies indicate that part of the difficulty is because students do not have a firm understanding of the prerequisite statistical concepts such as 'centre', 'mean', 'distribution', and 'standard deviation' (del Mas & Liu, 2005; Garfield & Ben-Zvi, 2008). The purpose of this study is to investigate how undergraduate students talk about variability. Of relevance are the role of gestures in understanding human thinking and the strong connection between gestures and language. I will therefore use gestures in understanding how undergraduate students conceptualize variability. Using Sfard's (2007) proposition that thinking and communicating are one and the same thing, I examine what roles gestures and language play in communicating statistical ideas.

A second motivation relates to the growing research in statistics education on the use of computer-based technologies offering dynamic tools for learning statistics. These learning tools could be designed in dynamic geometry and dynamic statistics environments, or they could be sourced from the many web-based graphics tools and applets. I am particularly interested in understanding the connection between motion and human intelligence in light of Seitz's (2000) claim that motor activity is the basis of human intelligence. A similar argument, although not directly

related to mathematics education, is credited to Talmy (1996), who from the analysis of written and spoken words, proposed that humans often describe static objects as if they were in motion. For example, a student describing the centre of a data set as “a gathering place” indirectly imposes motion on the data, by moving the data points to a central collecting point or the gathering place. Using Talmy’s notion of “fictive motion”, a situation in which otherwise static objects are thought of as if they were moving, Núñez (2006) analyzed mathematical speech and found many instances of fictive motion, even if the objects were defined as static and having no sign of movement. Given the links between gestures and dynamic aspects of human thinking, another goal of this study is to examine what roles gestures play in students’ understandings of variability. Importantly, I am also interested in the contribution of dynamic models to students’ conceptual understanding of variability.

Conception of ‘variation’ and ‘variability’

The term ‘variation’ is closely associated with the concepts of ‘variable’ and ‘uncertainty’. ‘Variability’ is a quality of an entity to vary, including variation due to uncertainty or chance (Makar & Confrey, 2005). In this study, ‘variability’ is taken as observable characteristics of a variable or an entity and ‘variation’ is the quantification or measuring of the amount that data deviate or vary from the centre, specifically from the mean (Reading & Shaughnessy, 2004). Describing ‘variability’ in this way calls for an understanding of statistical concepts such as ‘distribution’, ‘arithmetic mean’, ‘deviation’ and ‘standard deviation’ (del Mas & Liu, 2005). However in this study, I expand the list to include six terms: ‘distance’, ‘centre’, ‘distribution’, ‘deviation’, ‘mean’, and ‘standard deviation’ and apply them to the measurement of variability. For the purpose of this paper I will use only five of them.

Specifically the study addresses itself to: What changes might occur in students’ thinking about the concept of variability, from static representations to more dynamic representations, afforded by dynamic computer-based models? I will use Sfard’s (2007) theory, which links thinking to communicating. I hypothesize that the dynamic models might contribute to students’ thinking and talking about variability in a more “dynamic” way. For example, students will impose fictive motion on otherwise static statistical objects. Furthermore, I speculate that dynamic models might provide students with a rich collection of conceptual metaphors to support their understanding of variability. Finally, I propose that after interacting with models, students will refer less to symbols and formulae when describing measures of variability such as the mean and standard deviation.

METHODS

Participants and setting

Participants were undergraduate students registered in three sections of an introductory statistics course at a university in British Columbia, Canada, where I was a graduate student and worked as a teaching assistant. The sections were taught by different instructors in the Department of Statistics & Actuarial Science. Selection for the interview was done randomly from students who attended classes in any of the three sections. Participants represented a wide range of interest in their major areas of study, including health science, engineering, business, computer science, criminology, psychology and education. To ensure uniformity, the interviews were conducted after students had already covered the relevant topics, including the mean and standard deviation, in their statistics classes. Each participant was interviewed only once for about one hour. The interview session comprised two segments: non-dynamic and dynamic.

Non-dynamic segment

In this segment, participants were asked to say what they thought of the terms ‘distance’, ‘centre’, ‘mean’, ‘deviation’, and ‘standard deviation’. The interview had only one open question: “What comes to [your] mind when you hear the word...?” All interviews were video recorded and transcribed.

Dynamic segment

The dynamic segment of the interview is not reported in this paper because of page limitations. However, an outline of the dynamic models is presented below in pictorial form. Dragging (moving data with the mouse from one point to another) was the main activity by participants during the dynamic segment of the interview.

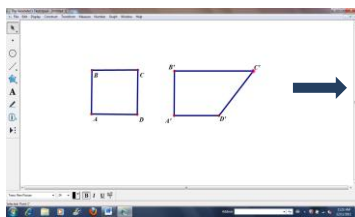


Figure 1: The GSP window showing point C on the square moved to C' by dragging, during the testing stage.

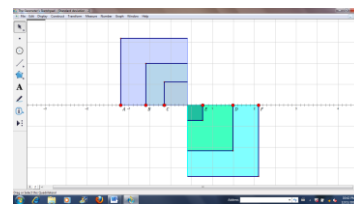


Figure 2: The GSP window showing the dynamic model for the mean and standard deviation before data points are moved on the horizontal axis.

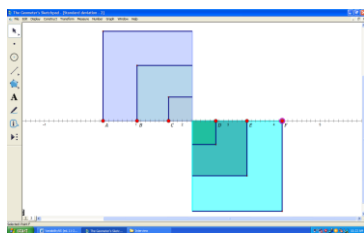


Figure 3: The GSP window showing much variability in the data before moving the data points

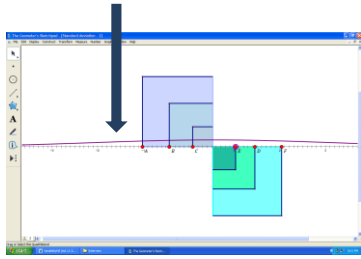


Figure 4: The GSP window showing least variability in the data after moving the data points.

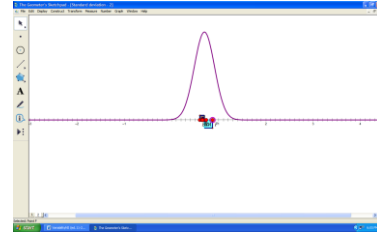


Figure 5: The GSP window showing high variability in the data distribution shown by the “flat” curve located just above the horizontal axis.

Figure 6: The GSP window showing almost no variability in the same data shown by the “smooth” bell curve.

RESULTS AND DISCUSSION

The transcript analysis of the non-dynamic interview focused on categorizing and describing how students explained the terms ‘distance’, ‘centre’, ‘mean’, ‘deviation’, and ‘standard deviation’. In the transcript below, the abbreviation ‘Int’ stands for the interviewer, while each student is given a pseudonym. Square brackets [...] indicate that a sentence is either incomplete or the student voice was not audible. Finally, curly brackets {...} indicate that a sentence has been shortened.

Students’ descriptions:

‘Distance’ as “one point to another”

Int: What comes to mind when you hear the word ‘distance’?

Kim: The points of, you mean the distance related to points or just physical distance? [*stretches her arm*].

Int: The word distance.

Kim: Just distance? Ok in physical size, distance just means two places [...] distance, I think for linear distance, is just one point to... another [*moves hand*], um I’m not sure the terminology is like it’s a vector [...]

Kim’s use of gestures makes distance to have some temporal quality. Her references to “physical distance” and “one point to another”, accompanied by hand movements, also carry the urgency of time and motion in space.

For Remi, distance is a number plot,

Remi: [Distance is] a number plot [...], ah like a plot of numbers [*moves his hands*], a distance between them {...}.

Like Kim, Remi's descriptions of distance are also accompanied by gestures, mainly hand movements.

'Centre' as "a gathering place" or "something popular"

Then students were asked to talk about 'centre'.

Int: What about the word 'centre'?

Kim: Centre just means a gathering place, the word like centre can mean something popular {...}

Kim's description of 'centre' as a "gathering place" or "something popular" was accompanied by hand movements which portrayed the activity of "gathering". The word 'gathering' itself implies an activity of moving oneself, or moving an object to a common point. It can, however, also be argued that something popular is bound to attract people's attention, resulting in people, either physically moving there, or turning their head to watch the popular thing. Either way, Kim's description imposes motion on the word 'centre'.

Remi, on the other hand, used the phrase "a bull's eye" as a metaphor to describe the word 'centre', and he also frequently moved his hands and body as he talked about 'centre'.

Int: What about the word 'centre'?

Remi: Centre is more like a bull's eye [*makes a small circle in space with his finger*] that's what comes to mind [...]

Int: Bull's eye?

Remi: Yeah, a bull's eye, like the targets, and they have like a, they have like a [...] [*moves his right hand*]

Remi proceeded to draw a circle on paper and marked a point in the centre of the circle to illustrate his idea of a "target" or "bull's eye". A "bull's eye" is a phrase commonly applied with regard to the game of darts. Although there are other contemporary uses of the phrase "bull's eye", I suspect that Remi had in mind the activity of playing a game of darts. The word 'centre' in that case is not just a static object in his mind, but takes some active usage as well, in a game of darts.

'Mean' as "a sum divided by the number of numbers"

Int: What comes to mind when you hear the word 'mean'?

Krista: {...} The mean is like the average, like um, um, yeah it's like you add [*moves two hands around to show adding together*] numbers there and divide by the digits there are and you just get it.

Although Krista focussed on describing the procedure for obtaining the mean rather than its conceptual meaning, her descriptions had lots of hand movements. Her use of the verbs such as: ‘add’, ‘divide’ and ‘get’, show emphasis on the procedure for obtaining the value of the mean. It seems that hand movements support both procedural and non-procedural thinking.

Like Krista, Remi also used similar descriptions such as “adding”, and “dividing”, which are procedural with regard to the concept of the mean,

Int: How about ‘the mean’?

Remi: Um, mean the same thing as the statistical thing, yeah pretty much *adding* all the numbers and *dividing* them by the number of numbers [*hand movement*].

Remi seems to suggest that as long as a concept is a “statistical” one, there is an established procedure for executing it. This is implied in his expression, the “same thing, the statistical thing, pretty much *adding*, and *dividing* the numbers”.

‘Standard deviation’ as “how far something deviates from the main point”

Int: How about ‘standard deviation’?

Anne: [Standard deviation is] how far something deviates from a point {...} kind of how far [*opens her hands*] something is from the main point.

Anne seemed to have referred to most of the previous concepts in her description of standard deviation. By using the phrase “how far”, Anne imposes some physical *distance* on the description of standard deviation. Her use of the phrase “is from the main point” clarifies her description even more. I think that by “main point”, Anne was referring to the *mean* or the *centre*, although she did not directly mention the mean. Her response showed some conceptual understanding of standard deviation.

Remi’s description of standard deviation was quite unlike Anne’s because he described the symbol for standard deviation, instead of the conceptual understanding of standard deviation.

Int: What pops up in your mind when you hear the word ‘standard deviation’?

Remi: I would think of a symbol, [I] forget the Greek letter, [*uses his finger to sign in space, the Greek letter sigma*(σ)], yeah or maybe there is another one, it’s like variance, I would think of that, yeah there is lots of formula, more abstract concept, that’s what comes to mind.

Int: Just abstract, no pictures, nothing graphic?

Remi: No, not quite, just [*moves his right hand as if to say he had nothing more to add*]

Remi's symbolic descriptions of standard deviation are quite consistent with his earlier descriptions of the mean. It is not obvious what Remi meant by saying that standard deviation is a "more abstract concept". In any case, he was not going to say any more about standard deviation during the interview. I can only speculate that Remi meant the concept of standard deviation is a challenge to describe in one's own words. This is not surprising because, as was mentioned at the beginning of this paper, studies have reported that students find the concept of standard deviation extremely challenging (del Mas & Liu, 2005; Garfield & Ben-Zvi, 2008). According to the data, Remi did not apply any of the terms he was asked to talk about in the interview, which might have given him some clue to describe standard deviation.

CONCLUDING REMARKS

Students' descriptions of the key words were full of fictive motion, as if they were describing moving objects. For instance, Kim talked of 'centre' as a "gathering place", implying an activity of moving data to the centre. Similarly, Remi described 'centre' as "a bull's eye", apparently linking centre to a game of darts. Based on Talmy's (1996) notion of fictive motion, an expression which labels a static object as though it was moving, I argue that the students' dynamic descriptions of static objects, as if they possessed mobile attributes, mirrored the dynamic nature of their thinking. However, dynamic thinking was observed both in situations where students only described procedures about the concepts and in situations where some students showed a clear understanding of the concepts. I propose that dynamic thinking supports both procedural and conceptual thinking. Students used well developed metaphors when describing terms which are typically more broadly used words. For instance 'centre' was variously described as: "a gathering place", "something popular", "a target", and "a bull's eye". On the other hand, purely statistical terms such as 'mean' and 'standard deviation', did not have such rich metaphors. Instead students used symbols and procedures for describing 'mean' and 'standard deviation'. Based on previous studies (e.g. del Mas & Liu, 2005), I propose that lack of conceptual models and metaphors is one of the challenges students have with measures of variability such as the mean and standard deviation. The implications of this finding are presented in my next study.

References

- del Mas, R., & Liu, R. J. (2005). Exploring students' conceptions of the standard deviation. *Statistics Education Research Journal*, 4(1), 55-82. Retrieved from www.stat.auckland.ac.nz/serj/
- Garfield, J. B., & Ben-Zvi, D. (2008). *Developing students' statistical reasoning:*

Connecting research and teaching practice. New York: Springer.

- Makar, K., & Confrey, J. (2005). "Variation-talk": Articulating meaning in statistics. *Statistics Education Research Journal*, 4(1), 27-54. Retrieved from www.stat.auckland.ac.nz/serj/
- Mathews, D., & Clark, J. (2003). *Successful students' conceptions of mean, standard deviation and the central limit theorem*. Retrieved from <http://www1.hollins.edu/faculty/clarkjm/stats1.pdf>
- Núñez, R. (2006). Do real numbers really move? Language, thought, and gesture: The embodied cognitive foundations of mathematics. In R. Hersh (Ed.), *18 unconventional essays on the nature of mathematics* (pp.160-181). New York: Springer.
- Reading, C., & Shaughnessy, J. M. (2004). Reasoning about variation. In D. Ben-Zvi & J. Garfield (Eds.), *The challenge of developing statistical literacy, reasoning and thinking* (pp. 201-226). Dordrecht, The Netherlands: Kluwer Academic Publishers.
- Seitz, J. A. (2000). The bodily basis of thought. *New Ideas in Psychology*, 18, 23-40.
- Sfard, A. (2007). When the rules of discourse change, but nobody tells you: Making sense of mathematics learning from a commognitive standpoint. *Journal of the Learning Sciences*, 16(4), 565-613.
- Talmy, L. (1996). Fictive motion in language and "caption". In P. Bloom, M. Peterson, L. Nadel, & M. Garrett (Eds.), *Language and space*, pp. 212-273. Cambridge, Mass.: MIT Press.

THE INTERPLAY BETWEEN DIAGRAMS AND GESTURES

Shiva Gol Tabaghi

This paper reports on two students' interactions with a dynamic diagram (dynagrams) that was designed to enable students to experiment with geometric representations of vectors and their image under different linear transformations. In describing their interactions, the students spontaneously used both gestures and diagrams. We show how the dynagrams gave rise to new gestures and, in turn, to new diagrams—thus confirming Châtelet's (2000) thesis about the interplay between gestures and diagrams, as well as their inventive potential.

DIAGRAMS IN MATHEMATICAL THINKING

Mathematics education researchers have highlighted the essential role of diagrams in problem solving (Diezmann & English, 2001; Polya, 1957; Nunokawa, 2006). These studies focus on the use of diagrams as illustrations or representations of problem situations and their role in helping students solve problems. Our research considers the diagram not as a mere representation of mathematical ideas and relationships but as means to work materially with mathematical objects in inventive and embodied ways.

Most static diagrams that are found in mathematical textbooks could be seen as a representation of final results of mathematical ideas. For example, the diagram in Figure 1 illustrates the end product of a transformation of two vectors, u and v , under matrix A . The diagram is meant to illustrate that v is an eigenvector since Av is a scalar multiple of v , but u is not. To correctly read the diagram, one must know the eigenvector (v) and its associated eigenvalue (scaling factor) before sketching v and Av , whereas u could be an arbitrary vector. This static diagram has a representational role since it depicts the geometric representation of the eigenvector v and the arbitrary vector u . It neither reveals the process of finding the eigenvector v nor does it draw attention either to the invariance property of v (i.e. its collinearity with Av) or to the existence of infinitely many eigenvectors collinear with v .

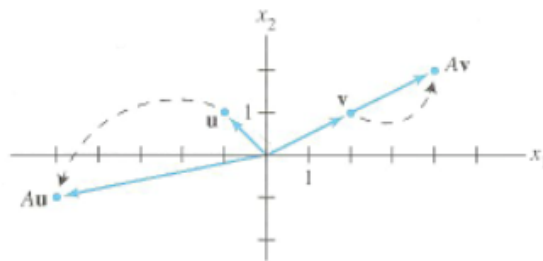


Figure 1: Static diagram of vectors and their transformations (Lay, 2006, p.303)

We are interested in the use of a diagram as an experiment in which students perform processes to develop an understanding of mathematical ideas. This use of a diagram is similar to the Ancient Greeks' use of diagram in which they performed their diagrams (Netz, 1999) rather than using them as representational tools, such as the one in Figure 1. We are interested in dynagrams for their potential to communicate the mobile and temporal aspects of mathematics that are often absent from the static diagram. In particular, we examine the way in which a dynagram, designed to enable students to find eigenvectors and eigenvalues, affects these students' ways of communicating about these concepts. This interest is fuelled by the recognition that visual images—and dynamic imagery in particular—are a key component of mathematical understanding (Sinclair & Gol Tabaghi, 2010).

THEORETICAL FRAMEWORK

Diagrams have played a central role in the development of new mathematical ideas, as Châtelet's (2000) historical investigation shows. These diagrams do not simply illustrate or translate an already available content; they invent new spaces and new ways of conceptualizing that emerge from the mobile, material acts of experimenting on the page. For Châtelet, the diagram is a mid-station from the embodied gesture to the more formal mathematics; he writes that diagrams “can transfix a gesture, bring it to rest, long before it curls into a sign” (p.10). For Châtelet, the gesture is an impulse in a sense that “one is infused with the gesture before knowing it” (p.10).

Gestures have received much attention in mathematics education (Edwards, Radford, & Arzarello, 2009), with many researchers drawing on the pioneering work of McNeill (1992; 2005), who has shown that gestures are integral to thinking and communicating. Within a psychological perspective, McNeill identifies different types of gesture: deictic, iconic, metaphoric, beat. In mathematics education, the use of metaphoric gestures, which can illustrate abstract ideas, has received more attention since they can be used to communicate the developmental process of mathematical ideas. Unlike McNeil, Châtelet is less interested in any sort of classification or description of gestures—than in the implications of the gesture on the diagram. He draws our attention “to the dynasties of gestures of cutting out, to diagrams that capture them mid-flight, to thought experiments” (p.10). It is helpful to think about how Châtelet could have seen the diagram in Figure 1 in light of his theory. The curved dashed arrows are the dynasties of cutting out gestures. Those gestures are meant to indicate linear transformation (or mapping); one suggests dilation and other rotation although they both have a similar construct. The diagram captures or transfixes the cutting out gestures, thus creating a new fold on the surface. It is born with the cutting out gestures. No longer are the tips of the

curved dashed arrows cleaving to the points Av or Au ; the arrows embody the effort of abstraction (that is linear transformation of v and u under matrix A) by participating in the concrete process of constituting a system of linear transformation. It is in this sense that Châtelet calls the diagram (in Figure 1) a diagrammatic (or thought) experiment. He writes:

The thought experiment taken to its conclusion is a diagrammatic experiment in which it becomes clear that a diagram is for itself its own experiment. The gestures that it captures and particularly those that it arouses are no longer directed towards things, but take their place in a line of diagrams, within a technical development (p.12).

His philosophical view has given rise to more recent hypothesis that working systematically with dynamic imagery could increase students' material interaction (see de Freitas and Sinclair, 2011). This is in contrast to the static and confining aspects of textbook diagrams that take away students' inventive acts by which mathematics can be grasped through using gestures and embodiment.

de Freitas and Sinclair point out that Châtelet emphasizes the relationship *between* gestures and diagrams, while also insisting on the diagram's capacity to give rise to new gestures. In this study our goal is to examine the participants' interactions with the dynamogram to learn more about the diagram/gesture relationships.

METHODOLOGY OF RESEARCH

This study draws on data collected during a larger study on the effect of dynamic geometry software on students' modes of thinking. Data was collected using one-on-one semi-structured clinical interviews. Each interview lasted about 30 minutes and was videotaped. The participants were two students; Jack was pursuing his undergraduate degree in computer science and Mike was completing his Master's of Science degree in secondary mathematics education at a medium-sized North American University. They were given a sketch and a worksheet (that included the formal definition of the concepts of eigenvector and eigenvalue and the task). The *eigen* sketch enabled them to explore linear transformations of a vector named x ; by dragging vector x , its image vector, Ax , is being up-dated as the result of transformation under matrix A . As shown in Figure 2, the sketch includes a draggable vector x , its image vector Ax , and vectors u and v that are representative of the column vectors of matrix A . The sketch also includes numeric values of the matrix-vector multiplication (Ax).

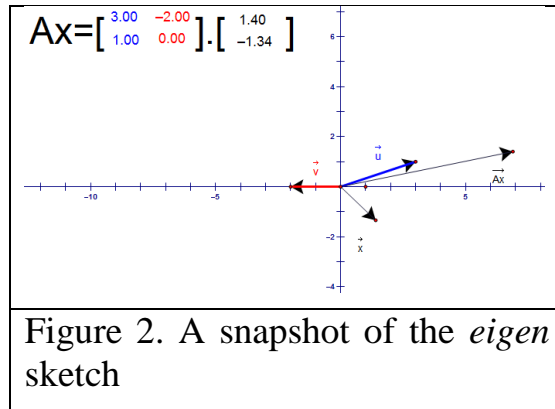


Figure 2. A snapshot of the *eigen* sketch

The sketch design allowed them to change the values of the matrix A to create the given matrices on the task shown in Figure 3.

Given a sketch that represents matrix A and an arbitrary vector x . Double click on entries of A to change their values to the givens below, then drag x to find eigenvector(s) and associated eigenvalue(s), if they exist.

(a) $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, (b) $A = \begin{bmatrix} 4 & -1 \\ 1 & 2 \end{bmatrix}$, (c) $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, and (d) $A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$

Figure 3. The interview task used to collect data

ANALYSIS OF DATA

We provide excerpts of the interviews where participants' discourse accompanied by gestures or diagrams. The experts enabled us to respond to our research question: what is the effect of dynamagrams on the participants' ways of communicating (through gesture and diagram) the concepts of eigenvectors and eigenvalues?

We identify instances of diagram/gesture relationships in participants' interaction with the dynamic diagram. We prefer to use Châtelet's theory in analyzing the participants' gesture rather than classifying their gesture. This is because it is hard to identify a metaphoric gesture from an iconic gesture. According to McNeill (1992), an iconic gesture depicts a concrete event or picture, whereas a metaphoric gesture depicts an abstract idea. But, we could not determine whether a gesture's referent is concrete or abstract.

Mike's interaction with the dynamic diagram: gesturing

Mike had completed a linear algebra course during his bachelor's degree and he could not recall the concepts of eigenvector and eigenvalue at the beginning of the interview. He looked at the formal definition of the concepts on the worksheet and

had difficulties making sense of them. He then used the mouse pointer several times to indicate the geometric representation of the vectors x and Ax on the dynagram as he tried to match the symbolic representations used in the definition with the ones used on the dynagram. After putting aside the worksheet, he used his right index finger to indicate the vectors x and Ax and also moved his right index finger along the vectors u and v shown on the screen as he was tracing the vectors from tail to tip.

Shortly after his interaction with the dynagram, Mike started using the verb “to line up”. While he interacted with the *eigen* sketch to find a set of eigenvectors of the third given matrix, he was prompted by the interviewer (1st author) to describe his method:

Mike: Since I want the x to line up with the Ax [drags x in the first quadrant where the two vectors overlapped] I am gonna have to find some spot [...] that they go to the same direction so if I go around 360 degrees I am interested in the spots like there [drags x in an anti-clockwise direction into the third quadrant where x overlaps with Ax and then uses line dragging].

His use of the verbs “to line up” and “to go” suggests evidence of time and motion in his description of the processes that he went through in finding eigenvectors. His statement suggests that he concretized the concept of eigenvectors as a special vector that lines up with its image vector. This is very different from the static symbolic equation, $Ax=\lambda x$, that is emphasized in the formal definition of the concepts of eigenvectors and eigenvalues.

Towards the end of the interview, Mike used his hands as vectors to illustrate the behaviour of the vectors x and Ax . He placed his right hand extended fingers on his left hand extended fingers as he said “[...] x and Ax to line up in the same direction, [...]”, shown in Figure 4. He then placed his right hand extended fingers pointing to right direction on his left hand pointing to left direction, as shown in Figure 5, to illustrate collinear eigenvectors that have opposite directions. He also rotated his right index finger around, as shown in Figure 6 while he said “I would move x around 360 degrees to see if these two cases showed up”.

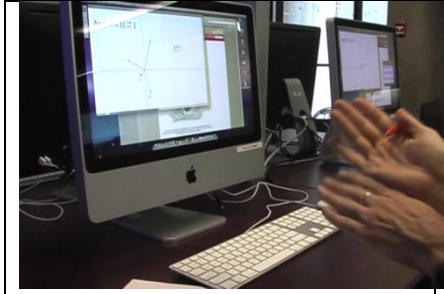


Figure 4. Mike's hands point to the same direction



Figure 5. Mike's hands point to the opposite directions



Figure 6. Mikes' index finger rotates around a circle

As evidenced, Mike's gestures were triggered from his interaction with the *eigen* sketch. This confirms Châtelet's (2000) theory on the diagram/gesture relationships in that the diagram—a dynamic one in this case—gave rise to gesture. In fact, his hands have become vectors that he moved and positioned them to depict the collinear property of eigenvectors. Mike's gestures and speech suggest that he provided an embodied geometric description of finding eigenvector as he used his hands as vectors to gesture the geometric representation of x and Ax where x is an eigenvector.

Jack's interaction with the dynamic diagram: gesturing and diagramming

Jack had taken a linear algebra course recently (about two months prior to the interview). He did not have a difficulty in making sense of the definition, but did not remember it completely. Like Mike, he used his right index finger to point to the given symbols on the worksheet as he read the definition. He then used the mouse pointer several times to indicate the matrix A and the geometric representation of the vectors x and Ax on the sketch as he tried to match the symbolic representations used in the definition with the ones used on the sketch.

Shortly after finding the eigenvectors of the first given matrix, I prompted him asking about the *eigen* sketch representation of eigenvectors associated with the eigenvalue equal to one. In response, Jack used his hands to represent the two vectors as shown in Figure 7. He then brought his hands together and placed them exactly on each other (as shown in Figure 8) to illustrate the geometric representation of vectors when the eigenvalue was one. This shows that the use of the dynagram enabled him to use his hands as vectors to gesture about eigenvectors associated with eigenvalue equal to one. It is noteworthy that he did not use gestures when he was looking at the formal definition.

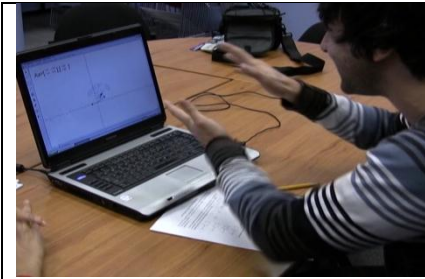


Figure 7. Jack's hands are positioned parallel to each other



Figure 8. Jack's hands are exactly placed on each other



Figure 9. Jack positions his hands in an angular shape

Jack also gestured after seeing the dynamogram of eigenvectors associated with a negative eigenvalue. He positioned his hands extended fingers in an angular shape as shown in Figure 9. This again suggests that the dynamogram gave rise to new gestures for Jack.

After completing the interview task, Jack was prompted by the interview to describe how he went about finding eigenvectors. Jack said: "I tried to make x touch Ax " as he dragged x in a spiral fashion beginning far from the origin, turning in an anti-clockwise direction, and ending at the origin. He then drew a diagram to illustrate eigenvectors for a negative eigenvalue as shown in Figure 10. He drew this after gesturing geometric representation of eigenvectors associated with a negative eigenvalue as shown in Figure 9. The interaction with the dynamogram enabled him to use his hands as vectors and to position them in order to depict the collinearity of two vectors. Then, he sketched the diagram of two vectors positioned collinearly with opposite directions on a sheet of paper as he tried to describe his understanding. This confirms Châtelet's ideas that "the gesture envelopes before grasping and sketches its unfolding long before denoting or exemplifying" (p.10, 2000).

Similar to Mike, Jack initially pointed to the symbols and the objects, but did not have much sense of what they are. By the end, his hands and arms have become the vectors communicating his understanding of eigenvectors. This implies that the gestures came right out of his interaction with the dynamogram.

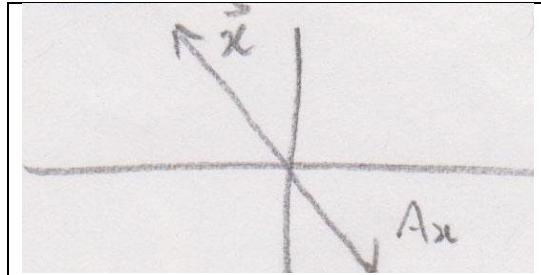


Figure 10. Jack's drawing of eigenvectors

DISCUSSION

At the beginning of the interview, Jack and Mike both used gestures to refer to the symbols and the objects on the dynamic diagram. By the end of the interview, their gesturing had evolved so that their hands have *become* the vectors that were positioned in specific ways. Incidentally, their use of hands as vectors in describing eigenvectors is similar to that of mathematicians (see Sinclair and Gol Tabaghi, 2010).

Jack's case is further interesting, in the context of our study, because he also sketched a diagram of an eigenvector associated with a negative eigenvalue on the worksheet. His diagram in Figure 10 immobilized his gesture in Figure 9 in order to produce on paper an eigenvector associated with an eigenvalue.

The gestures that came right out of their interaction with the dynamogram reveal that both Mike and Jack embodied the invariance property of eigenvectors that is the collinearity of eigenvectors. In contrast to Châtelet, who is interested in the new, inventive diagrams produced by mathematicians (and emerging from gestures), here we focus on the new gestures that arise from particular kinds of diagrams. With Jack, we also show how the gesture to diagram move can occur. Our findings confirm de Freitas and Sinclair's (2011) hypothesis that working systematically with dynamic imagery can increase students' mathematical understanding.

Unlike dynamograms, most diagrams that are found in textbooks tend to represent final results of mathematical ideas rather than illustrating processes that accompanied the development of ideas. The absence of time and motion makes a static diagram seem like a representation of a finished process. In contrast, the use of dynamograms such as the one described in this study offers students opportunities to perform mathematical actions. This perspective of diagrams echoes Bender's and Marrinan's view of diagrams as process-oriented tools that are situated within experience and map "the chain of thoughts and gestures of attention that give them meaning" (p.19).

We plan to enlarge our understanding of diagram/gesture relationships by studying closely the diagramming that occurs after gesturing, and not just diagramming that happened spontaneously in the case of Jack's interaction with the *eigen* sketch. We think that Châtelet's theory provides an insight of a participant's mathematical understanding and will continue to employ it in our future study to broaden our understanding of diagram/gesture relationships.

References

- Bender J. & Marrinan, M. (2010). *The culture of diagram*. Stanford, CA: Stanford University Press.
- Châtelet, J. (2000). *Figuring space: Philosophy, mathematics and physics*. Translated by Robert Shore and Muriel Zaghera. Dordrecht, The Netherlands: Kluwer Academic Publishers.
- de Freitas, E. & Sinclair, S. (2011). Diagram, gesture, agency: Theorizing embodiment in the mathematics classroom. *Educational Studies in Mathematics*. DOI: 10.1007/s10649-011-9364-8.
- Diezmann, C, M. & English, L. D. (2001). Promoting the use of diagrams as tools for thinking. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics* (pp. 77-89). Reston, VA: National Council of Teachers of Mathematics.
- Edwards, L., Radford, L. & Arzarello, F. (2009). Gestures and multimodality in the teaching and learning of mathematics. *Educational Studies in Mathematics*, 70(2), 91-215.
- Kendon, A. (2000). Language and gesture: unity or duality? In D. McNeill (Ed.), *Language and gesture* (pp. 47-63). Cambridge: Cambridge University Press.
- Lay, D. C. (2006). *Linear algebra and its application*. USA: Pearson Addison Wesley.
- McNeill, D. (1992). *Hand and Mind: What Gestures Reveal about Thought*. Chicago: University of Chicago Press.
- McNeill, D. (2005). *Gesture and Thought*. Chicago: University of Chicago Press.
- Netz, R. (1999) *The Shaping of Deduction in Greek Mathematics: a Study in Cognitive History*, Cambridge: Cambridge University Press.
- Núñez, R. (2006). Do real numbers really move? Language, thought, and gesture: The embodied cognitive foundations of mathematics. Reprinted in R. Hersh (Ed.), *18 Unconventional Essays on the Nature of Mathematics* (pp. 160-181). New York: Springer.
- Nunokawa, K. (2006). Using drawings and generating information in mathematical problem solving processes. *Journal of Mathematics, Science and Technology Education*, 2(3), 33-54.
- Polya, G. (1957). *How to solve it*, 2nd ed. Princeton, NJ: Princeton University Press.

Sinclair, N. and Gol Tabaghi, S. (2010). Drawing space: mathematicians' kinetic conceptions of eigenvectors. *Educational Studies in Mathematics*, DOI# s10649-010-9235-8.

THE EFFECT OF DYNAMIC GEOMETRY ENVIRONMENTS ON CHILDREN'S UNDERSTANDING OF THE CONCEPT OF ANGLE

Harpreet Kaur

This paper reviews literature on children's understanding of the concept of angle and proposes some ideas for a future study which aims at investigating the effect of computer-based Dynamic Geometry Environments (DGEs) on young learners' understanding of angle.

INTRODUCTION

The concept of angle is a multifaceted concept that can pose challenges to learners even into secondary school (Close, 1982; Mitchelmore & White, 1995). Despite these difficulties, children show sensitivity to the concept of angle from very early years. For example, Spelke, Gilmore, and McCarthy (2011) found that 4- to 6-year-olds display use of distance and angle information in map navigation tasks, leading to the possibility of formation of holistic representations of the geometric displays (such that distance and angle are encoded and retrieved together). There has been an increasing interest in children's mathematical understanding acquired prior to formal instruction due to increasing stress on linking school mathematics to children's everyday experiences. Angles are normally introduced to children quite late in formal school settings. The curriculum document, *Mathematics K to 7: Integrated Resource Package* (British Columbia Ministry of Education, 2007) introduces the concept of angle at grade 6, while the students are expected to describe, compare, and construct 2-D shapes, including triangles, squares, rectangles and circles etc., at the grade 2 level. The strong capacity of young children to attend to and identify angles in various physical contexts motivated us to investigate their angle conceptualisation in early school years through formal teaching with the use of dynamic geometry environments (DGEs).

The use of computers might be appropriate in teaching the concept of angle in early school years. We have been investigating other geometry-related concepts at this age too. Previous research has shown that it is possible to develop mathematical concepts such as shape identification and symmetry, even at the kindergarten level, with the use of DGEs (Sinclair, Moss, & Jones, 2010; Sinclair & Kaur, 2011). In the research literature, there is evidence of using a computer micro world like LOGO for teaching the concept of angle (Clements, Battista, Sarama, & Swaminathan, 1996; Simmons & Cope, 1990, 1993). The students showed the notion of embodied conception of angle by linking the angle turns with

the turns of their bodies. We believe that DGE might be helpful in visualizing the angles in the form of turns and rotations more effectively.

CHILDREN'S UNDERSTANDING OF ANGLE

In the research literature, the concept of angle is shown to have different perspectives, namely: angle as a geometric shape; union of two rays with a common end point (static); angle as movement; angle as rotation (dynamic); angle as measure; and, amount of turning (Close, 1982; Henderson & Taimina, 2005). Due to different prevalent definitions of the term angle, teachers often face difficulty in knowing what definition of angle to use (Close, 1982). There are always difficulties in showing “where the angle is” in cases of traditional geometric shapes of angle.

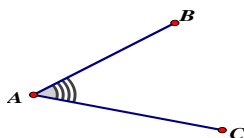


Figure 1

In the research literature, Mitchelmore (and colleagues) and Clements (and colleagues) have done the most prolific research in the area of angle concept over the past twenty years. Much research has been conducted on the development of the concept of angles, focusing at the grades 3, 4, and higher levels. Mitchelmore and White (1995) suggest that angles occur in a wide variety of physical situations that are not easily correlated. Mitchelmore (1997, 1998) conducted an investigation of children's informal knowledge of a variety of physical angle situations. He found that children had an excellent knowledge of all situations presented, but that specific features of each situation strongly hindered recognition of the common features required for defining the angle concept. Mitchelmore and White (1995) proposed a framework for research on the development of the angle concept, based on the theories of abstraction (Skemp, 1986) and situated knowledge (Brown, Collins, & Duguid, 1989). It is suggested that children initially acquire a body of disconnected angle knowledge situated in a large number of everyday experiences; they then group situations to form angle contexts such as turns and corners; and finally they form an abstract angle concept by recognizing similarities across several angle contexts. In subsequent studies, Mitchelmore uses the idea of stages or levels of understanding. Mitchelmore and White (2000) identify three stages of knowledge about angles:

- *Situated angle knowledge.* Knowledge of specific situations where angles are observed by the child, for example, a steep hill or a corner.

- *Contextual angle knowledge.* More general knowledge that integrates related situations; for example, slope of hill with slope of roof, or turn of a doorknob with turn of a jar lid.
- *Abstract angle knowledge.* More general knowledge that integrates different angle contexts, for example slopes and turns.

Later works of Mitchelmore with other colleagues involve teaching experiments (Prescott, Mitchelmore, & White, 2002; White & Mitchelmore, 2003) in which they divide angle situations into three clusters—2 line angles (corners of room, intersecting roads, pairs of scissors, body joints), 1 line angles (doors, windshield wipers, clock hands, and slopes), and 0 line angles (the turning of a doorknob or a wheel). The researchers wrote a sequence of 15 lessons which initially explored 2-line angles and then moved on to 1-line angles. There were other interesting theoretical perspectives offered by Fyhn (2006, 2008) including the angle concept embedded in physical activity and real life experiences such as jumping.

Some misconceptions related to the concept of angle are found in the research literature. Students tend to develop the misconception that “longer sides imply larger angles” as a result of the intuitive rule “More A–More B” (Stavy & Tirosh, 2000). The other misconceptions related to angles found in the literature include cases like: the size of an angle varies with the length of the arms, the size of an angle varies with the size of arc made with the angle vertex as centre, a right angle only exists between vertical and horizontal lines, and different orientations of an angle is a source of confusion.

RELEVANCE OF DGES FOR TEACHING OF ANGLE

DGEs such as *The Geometer’s Sketchpad* enable the creation of *dynamic* representations of various mathematical shapes such as triangles, rectangles, circles, etc. They also provide additional features like *continuous dragging* of these shapes to different sizes and orientations, while retaining the fundamental properties of the shapes. DGEs enable learners to build a rich set of examples of a particular concept, as opposed to the limited number of examples available in more static environments, thus enabling them to generalize (Battista & Borrow, 1997). Sinclair (2010) has proposed that use of DGEs at the early primary grades may have a lasting impact in developing imagery and sustaining further learning. Young learners show a propensity to reason in terms of motion and transformation, but this is not supported by the static environment of the classroom (Lehrer, Jenkins, & Osana, 1998). DGEs enable morphing and visualization of mathematical objects in reality, where it can be shared with others also. Thus, a single shape describing an angle can be morphed to see different types of angles by turning it in different directions and changing the size of its arms. Thus, DGEs help

enable children's thinking to move from general to particular, leading to the immediate mastery of broad classes of examples and problems.

Mitchelmore (1997, 1998) found that children demonstrate situated knowledge of turns, slopes, crossings, bends, corners, etc., but have difficulty in recognizing the common features which define the angle concept. So, this indicates that they already have some experiential and embodied understandings of angle, which can be used to develop more formal understandings. DGEs support some of the particular embodied notions of angle, such as turning (movement) and rotation. DGEs would be good at developing the static as well as dynamic perspective of angle. DGEs can be very helpful in overcoming the misconceptions related to angles. For example, students can drag the arms of the angle and notice the resulting change in angle simultaneously, by using the measure angle feature in *Sketchpad*.

Research using *Logo*, which is a programming based micro world in which students have to write commands for the movement of a turtle, shows that students tend to visualize the turns of turtle as turns of their body, but making these turns involves writing numerical commands (Clements et al., 1996). Students tend to develop confusion between external and internal angles while using *Logo*. For example, Simmons and Cope (1990, 1993) found that some students labelled a 60 degree angle as 120 degrees and the corresponding 120 degree angle was described as 60 degrees. In each case, they used the number of degrees of turtle rotation required to produce the resulting angle rather than the actual angle which was presented. A dynamic geometry environment such as *Sketchpad* does not involve the writing of the commands and can thus be used at an earlier age to develop more qualitative understandings of angle. The 'trace' feature of *Sketchpad* might be very helpful in visualizing the process of turning along with the final position. Thus, the dynamic environment of *Sketchpad* enables one to see the process along with the product of a phenomenon. Furthermore, a strong understanding of angle at early primary grades can be very helpful in developing other ideas, which are typically done at this level. For example, usually at kindergarten level students are taught to identify shapes such as triangle, square, rectangle, etc. The development of the concept of angles at an early school level may provide a whole new set of possibilities for better understanding of different geometric shapes and their properties.

THEORETICAL PERSPECTIVE

The central role of tools in learning and their lasting effect on the way learners think is acknowledged by Vygotsky in cultural-historical activity theory. Davydov's curriculum, based on cultural-historical activity theory, emphasizes the

presence of three properties in conceptual thinking of students (Moxhay, 2008). First, their thinking should be *object-oriented*, it must be grounded in transformational actions with real objects and students should be able to transform the conditions of the tasks to find the solution. Second, it should be *generalized*; it must relate to a whole system of related tasks. Third, the students' must show *reflective* thinking; they must be able to understand the basis of their own actions. What is important for concept development is the creativity that results from the students' transforming the conditions as a result of their collective discussion and trial-and-error work. DGEs provide a good basis for the first two properties by providing ways for transformational actions and a whole system of related tasks. The third component, *reflective* thinking, can be developed by providing appropriate guidance by the teacher.

The van Hiele model (1986), which is considered to be the best-known theoretical account of students' learning about geometric concepts, does not adequately help to explain the mediating role of symbolic tools (including language and computers) in children's mathematical thinking (Sinclair, 2010). Sfard's (2008) communication-based framework provides a good way to explain the underlying basis of children's understanding and the causes for changes in understanding. For Sfard, thinking is a type of discursive activity. Sfard's approach is based on a participationist vision of learning, in which learning mathematics involves initiation into the well-defined discourse of the mathematical community. The mathematical discourse has four characteristic features: word use, routines, narratives and visual mediators. Learning geometry can thus be defined as the process through which a learner changes her ways of communicating through these four characteristic features. Embodied cognition also informs the theoretical orientation of this research. Núñez (2006) investigates the role of motion in mathematical concept development. She shows that mathematicians think about mathematical objects in highly dynamic, temporal and embodied ways, even though their formal written work implies a highly static conceptualization.

METHODOLOGY

The objectives for the proposed research are: (1) to develop and design productive sets of dynamic images and sketches related to angles using *Sketchpad* that can be helpful to introduce the concept of angles at elementary levels. We will draw on the literature to adapt some tasks related to angles to the *Sketchpad* environment. (2) To implement the designed sketches in K-2 classrooms for teaching the concept of angles with the active participation of the students in the lesson. (3) To study the ways in which students perceive and interpret dynamic imagery to develop the concept of angles. In this research, we will try to answer the following questions: How does dynamic imagery affect mathematical conceptualization of angles? Can

young children use their everyday experience of angle to develop a more formal understanding of angle by interacting with DGEs? How does a robust dynamic conception of angle affect the development of other more static conceptions of angle (such as two rays with a common end point or measure)? How do children with a dynamic conception of angle perform on the more static angle tasks proposed in the literature?

Some Sketchpad based designs:

Students can be introduced to the concept from real life situations such as talking about the corners in the classroom. Then, later they can be shown some *Sketchpad* figures (Figure 2) and be asked to play with those figures to create different situations; where there is only no angle, one angle or more angles.

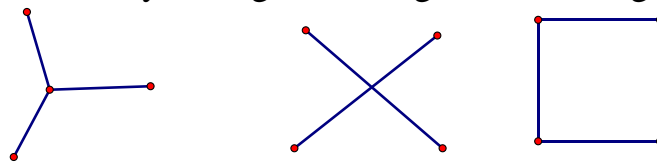


Figure 2

From the research literature, it emerged that children have difficulty seeing a static angle as a turn. The situation is more problematic where the two arms (of the angle) are not clearly visible. Sinclair (2010) has developed a model named ‘Driving Angle’ using *Sketchpad*, which enables the student to see the static as well as dynamic perspectives of the angle at the same time (Figure 3). It includes a car that can move forward as well as turn around a point. The turning is controlled by a little dial (which has two arms and a centre). This will enable the student to see the static turn and the dynamic turn at the same time. Students can do open playing with it. They might regulate motion and turns around different paths to create different shapes like random paths, squares, rectangles, etc.

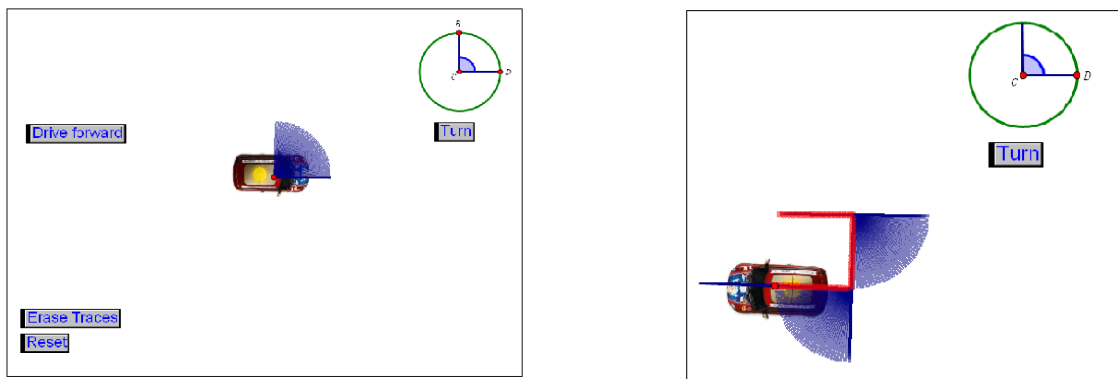


Figure 3

Furthermore, we might be able to talk about the benchmark angles as half turns or quarter turns. The idea of right angle can be developed as being the symmetry

where there is presence of as much angle on one side as on the other. The other ideas for the *Sketchpad* based designs are to create some sketches, where the real life activities of humans such as walking, jumping, or weight lifting can be visualized as the change in angles between various body parts such as legs, feet, hands, etc. These changes can be traced to see the actual process of turning of various body parts. Similarly, we could try to create some models for one line angle situations such as doors, where the turning of the door can be traced to see the rotation of the door and the actual angle. The traces would enable learners to overcome the difficulties that they face in visualizing such angles in physical situations.

We plan to investigate the research questions mentioned above through (1) videotaping the classroom episodes; (2) tests requiring the use of *Sketchpad* to solve the tasks related to angles; (3) paper pencil tests with tasks related to angles. Data of the videotaped classroom episodes will be reviewed and analysed by using Sfard's (2008) framework to analyze the discourse of students during the instruction on the concept of angle in a dynamic geometry environment. It will help to see the effect of dynamic imagery on mathematical conceptualization of angles. For example, at lower levels students can use the terms like 'corners', 'pointy', 'turn', 'twist', 'tilted lines', etc. to describe the angles, and at higher level they would be able to use formal terminology for the concept. *Sketchpad* experiences enriched with appropriate activities and discussions might help children become cognizant of their mathematical intuitions about angles and move to higher levels of geometric thinking of angle concepts.

References

- Battista, M., & Borrow, C. (1997). Shape Makers: A computer microworld for promoting dynamic imagery in support of geometric reasoning. In *Proceedings of the 19th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education: Volume 2* (pp. 571-578). Columbus, OH: Ohio State University.
- British Columbia Ministry of Education (2007). *Mathematics K to 7: Integrated resource package* (rev. ed.). Province of British Columbia.
- Brown, J. S., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher*, 18(1), 32-42.
- Clements, D. H., Battista, M. T., Sarama, J., & Swaminathan, S. (1996). Development of turn and turn measurement concepts in a computer-based instructional unit. *Educational Studies in Mathematics*, 30, 313-337.
- Close, G. S. (1982). *Children's understanding of angle at the primary/secondary transfer stage*. London: Polytechnic of the South Bank.

- Fyhn, A. (2006). A climbing girl's reflection about angles. *Journal of Mathematical Behavior*, 25, 91-102.
- Fyhn, A. (2008). A climbing class' reinvention of angles. *Educational Studies in Mathematics*, 67, 19-35.
- Henderson, D. W., & Taimina, D. (2005). *Experiencing geometry: Euclidean and non-Euclidean with history*. Ithaca, NY: Cornell University.
- Lehrer, R., Jenkins, M., & Osana, H. (1998). Longitudinal study of children's reasoning about space and geometry. In R. Lehrer & D. Chazan (Eds.), *Designing learning environments for developing understanding of geometry and space* (pp. 137-167). Mahwah, NJ: Lawrence Erlbaum Associates.
- Mitchelmore, M. C. (1997). Children's informal knowledge of physical angle situations. *Learning and Instruction*, 7(1), 1-19.
- Mitchelmore, M. C. (1998). Young students' concepts of turning and angle. *Cognition and Instruction*, 16(3), 265-284.
- Mitchelmore, M. C. & White, P. (1995, April). *Development of the angle concept by abstraction from situated knowledge*. Paper presented at the annual meeting of the American Educational Research Association, San Francisco, CA.
- Mitchelmore, M.C. & White, P. (2000). Development of angle concepts by progressive abstraction and generalization. *Educational Studies in Mathematics*, 41, 209-238.
- Moxhay, P. (2008, September). *Assessing the scientific concept of number in primary school children*. Paper presented at ISCAR, San Diego.
- Núñez, R. (2006). Do real numbers really move? Language, thought, and gesture: The embodied cognitive foundations of mathematics. Reprinted in R. Hersh (Ed.), 18 *unconventional essays on the nature of mathematics* (pp. 160-181). New York: Springer.
- Prescott, A., Mitchelmore, M. C., & White, P. (2002). Student difficulties in abstracting angle concepts from physical activities with concrete materials. In *Mathematics education in the South Pacific: Proceedings of the 25th Annual Conference of the Mathematics Research Group of Australasia Incorporated*. Auckland, New Zealand.
- Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses and mathematizing*. Cambridge, UK: Cambridge University Press.
- Simmons, M. & Cope, P. (1990). Fragile knowledge of angle in Turtle Geometry. *Educational Studies in Mathematics*, 21, 375-382.
- Simmons, M., & Cope, P. (1993). Angle and rotation: Effects of different types of feedback on the quality of response. *Educational Studies in Mathematics*, 24, 163-176.
- Sinclair, N. (2010). *Developing powerful geometric thinking in young children through dynamic visualisation software: Program of Research*. SSHRC proposal.

- Sinclair, N., & Kaur, H. (2011). Young children's understanding of reflectional symmetry in a dynamic geometry environment. In Ubuz, B. (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol.4, pp.193-200). Ankara, Turkey: PME.
- Sinclair, N., Moss, J., & Jones, K. (2010). Developing geometric discourse using DGS in K-3. In M. M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 185-192). Belo Horizonte, Brazil: PME.
- Skemp, R. R. (1986). *The psychology of learning mathematics*. Harmondsworth: Penguin.
- Spelke, E. S., Gilmore, C. K., & McCarthy, S. (2011). Kindergarten children's sensitivity to geometry in maps. *Developmental Science*, 14(4), 809-821.
- Stavy, R., & Tirosh, D. (2000). *How students (mis-)understand science and mathematics: Intuitive rules*. New York: Teachers College Press.
- Van Hiele, P. M. (1986). *Structure and insight: A theory of mathematics education*. Orlando, FL: Academic Press.
- White, P., & Mitchelmore, M. (2003). Teaching angles by abstraction from physical activities with concrete materials. In *Proceedings of the 27th International Group for the Psychology of Mathematics Education Conference held jointly with the 25th PME-NA Conference* (Vol. 4, pp. 403-410). Honolulu, HI.

TENSIONS IN TEACHING MATH FOR TEACHERS: MANAGING AFFECTIVE AND COGNITIVE GOALS

Susan Oesterle

This paper presents partial results of a study which investigated the experience of teaching mathematics content courses to preservice elementary teachers. Interviews with ten mathematics instructors who teach these courses revealed several major tensions, including one that arises as instructors strive to set priorities and balance their affective and cognitive goals for their students. An analysis of two of the instructors' expressions of this particular tension will provide insight into the factors that contribute to it and how it is managed. Implications for practice are considered.

BACKGROUND

Concern over the mathematics preparation of elementary school teachers has led to increasing calls for prospective teachers to take specialised mathematics content courses, i.e. Math for Teachers (MFT) courses, during their undergraduate programs (Greenburg & Walsh, 2008; Conference Board, 2010). These courses are usually taught by instructors in mathematics departments, and some recent studies have begun to call into question whether these instructors are equipped to meet the needs of MFT students, particularly with respect to affect (Hart & Swars, 2009).

Although there has been some research done into teaching styles of post-secondary mathematics instructors generally (Strickland, 2008), and into the difficulties they face in implementing reform approaches (Wagner, Speer, & Rossa, 2007), there seems to be little information about mathematics instructors in the context of teaching MFT courses. The original study upon which this paper is based sought to address this gap in the literature. Interviews with ten mathematics instructors who teach MFT courses at various post-secondary institutions in British Columbia were analysed in order to answer questions, including: What are the major tensions they experience? What factors contribute to these tensions and how are they managed?

Given space limitations, this paper will discuss only one of six major tensions revealed in the full study, specifically, the tension related to instructors' efforts to balance affective and cognitive goals for their students.

SUPPORTING LITERATURE

The research reported in this paper is informed by prior research into the cognitive and affective needs of prospective elementary school teachers (with respect to mathematics), as well as literature on tensions.

Cognitive and affective needs

With respect to the students in MFT courses, there is evidence to support concerns that they have poor understanding of the elementary school mathematics topics; Ball's (1990) study of 252 preservice teachers revealed "understandings that tended to be rule-bound and thin" (p. 449). Regarding their beliefs, the elementary preservice teachers in the group "tended to see mathematics as a body of rules and facts, a set of procedures to be followed step by step, and they considered rules as explanations" (p. 464). Preservice elementary teachers often suffer from mathematics anxiety (Hembree, 1990), and some are only enrolled to "fulfil a requirement rather [sic] to learn more mathematics" (Kessel & Ma, 2001, p. 477).

Although it is clear that MFT students have much mathematics to learn, and often come in with negative attitudes and beliefs, the literature does not provide specific advice on whether cognitive skills or affect should take precedence in teacher preparation. In fact, there is considerable literature engaged in debate over this issue. While some researchers make a case for the priority of strong mathematics knowledge, pointing out that such knowledge can both boost confidence and make teacher practice (i.e. the implementation of teachers' pedagogical beliefs) more effective (Schwartz & Riedesel, 1994; Goulding, Rowland, & Barber, 2002), a large number advocate for an emphasis on teachers' beliefs in specialised mathematics content (and methods) courses (Kessel & Ma, 2001; Liljedahl, Rolka, & Roesken, 2007), observing that beliefs will affect both students' learning in preservice mathematics courses and their later teaching. Still others promote the view that students' knowledge *and* beliefs need to be challenged in teacher education programs (Borko et al., 1992). This debate in the literature is reflected in the tension experienced by the MFT instructors described in this study.

Tensions

Tensions, often expressed as "dilemmas", have been recognised as an integral part of teaching practice, dating back at least to the early 1980s. In their seminal work, Berlak and Berlak (1981) examined the complex and sometimes contradictory behaviours of teachers in responding to the curriculum within socio-cultural contexts. Their use of the language of dilemmas was taken further by Lampert (1985), who emphasised the personal and practical aspects of dilemmas.

Lampert (1985) proposes the view that tensions in teaching are often “managed” rather than resolved. She characterises teachers as “dilemma managers” who find ways to cope with conflict between equally undesirable (or desirable but incompatible) options without necessarily coming to a resolution. In the face of a teaching dilemma, the teacher must take action, finding a way to respond to the particular situation, even while the “argument with oneself” (p. 182) that characterises the dilemma remains. For Lampert, the ongoing internal struggles presented by the tensions arise from and contribute to the developing identity of the teacher, and as such have value in themselves. Furthermore, she comments: “Our understanding of the work of teaching might be enhanced if we explored what teachers do when they choose to endure and make use of conflict” (p. 194).

METHODOLOGY

Data for this study was gathered through interviews with ten participants, five male and five female, all instructors in mathematics departments at post-secondary institutions who teach the MFT course. Theoretical sampling (Creswell, 2008) was used to achieve a variety in type of post-secondary institution represented, as well as varying degrees of experience in teaching MFT. The ten instructors represented nine different institutions, and their experience teaching the MFT course ranged from novice to 20 years.

The one-hour long interviews were semi-structured, beginning with a set of core questions but allowing for variations and additional questions to be asked as needed. Such an open-ended (“clinical”) approach is advocated by Ginsburg (1981) in situations where discovery or identification/description of a phenomenon is the objective. The questions sought to elicit the instructors’ conceptions of the MFT course, by asking them to examine their goals, describe the approaches they take, compare the teaching of MFT with teaching of other mathematics courses, and reflect on the challenges and the successes they experience.

The interviews were audio-recorded, transcribed, and analysed using constant comparative analysis (Creswell, 2008). An iterative coding process (Charmaz, 2006) was employed in order to allow concept codes and themes to be identified. Very few new codes emerged after the tenth interview, suggesting saturation of the data. Specific concept codes, including “priorities”, “wishes”, “doubts”, “barriers”, and “resistance”, helped to locate instances of instructor tensions in the transcripts.

Analysis of the tensions was further facilitated by techniques of discourse analysis and considerations of positioning (Harré & van Langenhove, 1999).

RESULTS

One of the major tensions that emerged through the coding and thematic analysis involved instructors' struggles with managing their cognitive and affective goals for their students in the MFT course. Two perspectives on how this particular tension is experienced will be presented here through brief analyses of the cases of Bob and Maria (both pseudonyms).

Bob

Bob's experience of this tension emerges in the contrast between his comments at the beginning of his interview with respect to his primary goals in his MFT course, and his later reflections on the ultimate outcomes for his students.

Bob's reported emphasis is on building deep conceptual understanding in his students, although affective considerations are also very important to him. His course "focuses on a very sound fundamental ability to appreciate [mathematics], in a theoretical way, why things work", along with having "a secondary by-product of what you do in the classroom is to get the students to enjoy it".

The cognitive and affective are closely related for Bob. His students' anxieties are at least in part caused by, and at the same time the cause of, their lack of arithmetic skills, and so helping his students learn about the structure of mathematics will solidify their understanding, giving them confidence, competence, and enjoyment.

At the end of his course, Bob believes that his students "have improved most in their technical abilities", along with having gained some problem-solving skills, although these need to continue to be developed. But he is ultimately disappointed, both in his hopes to build deep theoretical understanding, and in his hopes to increase his students' appreciation for, and love of, mathematics.

In terms of appreciating some of the more subtle aspects of the theory, I think that's another thing that they could do better, if they had better basic arithmetic skills, coming in. So...yeah, in terms of what I produce, I guess, in terms of the other goal, for love of math? Unfortunately, the course is so packed, that in some ways, I think they do get a little bit beaten by the end, and they're just tired.

He does see some success with improving their technical skills, but admits that he is less than successful (by his own standards) in terms of affective gains. He is trying to cover too much, to the extent that his students are overwhelmed.

A closer look at this passage, with particular attention to pronoun use, offers some further insights. In the first sentence, he ostensibly places the responsibility on the students, "they could do better, if they had better basic arithmetic skills". However, as Bob is aware, the prerequisites for the course are not controlled by the students, or set by him, but are negotiated by the larger community. Whether it is

the fault of this community or the students themselves, the lack of student skills coming into the course is an impediment to his ability to realise his goals for his students.

He then switches to consider what he (“I”) produces. Having already mentioned that students have increased their technical abilities, he moves to his “secondary” goal, improving affect. The results here are “unfortunate”; he describes his students as “beaten” and “tired”—not at all what he desired. The phrase, “the course is so packed”, is telling. It is offered as an explanation for the students’ states of exhaustion; there is too much material in too little time. But it appears from the use of the passive voice that Bob is not in control of the course content; with it he positions himself as unable to remedy this “unfortunate” situation. The course, as he is expected to deliver it by his institution, demands too much of the students.

Bob is not a new instructor of the MFT course, and so has likely lived with this problem for some time. He is stuck in this dilemma. On one side he has students who are unprepared for the level of mathematics he believes they need in order to “appreciate” the mathematics (both in a cognitive and in an affective sense). On the other side, he has a prescribed curriculum he is expected to “cover”. He feels a strong responsibility as a mathematics instructor, seeing himself as being charged with “delivering the content” (Bob’s words).

From Bob’s perspective, the situation could be improved if the students were stronger coming in. This is not within his immediate power to change, so he manages the best he can, trying to meet the needs of the students and the demands of the institution, never completely satisfied with the outcome.

Maria

Similar to Bob, Maria expressed a strong intention to improve students’ mathematical understanding, emphasising cognitive goals. However, Maria’s use of the past tense in describing these goals in her interview, even though she was teaching the course at the time, suggested she was having second thoughts about her priorities.

Maria was a first-time instructor of the course at the time of the interview, and was surprised by the needs of her students, not only their weak mathematics skills, but their mathematics anxiety and the barrier to learning it presented.

So my goal was, primarily, sort of more content, and I [...] knew that there would be some issues of, let’s describe it as “math phobia” or anxiety, with math. I just [was] still surprised to see it so strong at this level, that it overrides their learning, that it blocks their learning! That’s what I discovered, and it surprised me that it would be this strong.

She went into the course expecting that she would be teaching mathematics and would need to deal with some math anxiety, but at some point she realised that, at least for some of her students, the affective issues would need to be addressed before they could learn the mathematics. Maria commented that she believed she had lost about a third of her students, and was not sure how to get them back on track.

For this group of students at this point, content? Forget it. I need an attitude change. I need [their] perception of math to change. And I can't reach it anymore. It was very high, you know, it was a good high in the beginning of the course, because of what I did, free, sort of, problem-solving, open discussion, everybody let's just... [there was a] fuzzy, cosy atmosphere. But the topic does get difficult, yeah?

Maria feels as if she has missed an opportunity. For this particular group of students, she does not believe it will be possible for them to make progress learning the content without an attitude change, and this change is not possible to attain "anymore". She speaks nostalgically about a time at the beginning of her course where her approach was different: there was "open discussion", "free" problem solving, and a friendly atmosphere. She takes responsibility for the positive feelings at the beginning of the course; it was good because of "what [she] did", but something changed; her approach changed, and in this excerpt the reason for the change is the "topic", i.e. the mathematics, which gets more difficult as the course progresses.

Again, this raises the spectre of the course syllabus. Maria is torn between what she feels she should be delivering—the mathematics content—and what her students need. Like Bob, she feels a responsibility to cover the listed topics. But there is an additional consideration for Maria that adds to her tension; it is a perception that the MFT course has the potential, if not the responsibility to act as a filter. Earlier in her interview, her comments with respect to the importance of deep mathematics content knowledge for mathematics teachers reveal a strong commitment to ensuring that she does her part in the preparation of future elementary teachers. She does not want to let them move on to become teachers if their mathematics skills are too weak.

As well, Maria's tension between her desire to "cover" the content, as well as to build conceptual understanding and address her students' affective needs, becomes a tension with respect to teaching methods. Her efforts to complete the course content compel her to reduce in-class activities, such as open discussions of readings and problem-solving sessions, methods that she believes are effective. Covering the material is important to her, but it troubles her that she is leaving students behind. Those students still suffer from negative attitudes to math and continue to have weak skills.

Maria is far from resigned to living with this tension. She is still seeking to understand her students better and find methods that will be more effective for them, to find a way to change their attitudes so that the mathematics can be learned.

DISCUSSION

Bob and Maria's cases are similar in that, although they both hope to foster positive attitudes, their priority is on developing students' understanding of mathematics. At the same time there are differences in their experiences. Bob hopes to improve affect through building cognitive skills (one of the views reflected in the literature), but his affective aims are sabotaged by the volume of content. There is too much for his students to absorb given their skills coming into the course. Maria's comments reveal a growing awareness that her cognitive aims *cannot* be attained, at least for some of her students, until affective barriers have been removed (reflecting the other side of the affective/cognitive debate).

Both Bob and Maria seem to manage this tension between cognitive and affective aims by staying true to the course syllabus and "covering the material", even though they are unhappy that this means overwhelming the students and leaving some behind. There are indications within the broader study that this commitment to the prescribed course content is a prevalent norm amongst post-secondary mathematics instructors, carried over from teaching mathematics courses that tend to be sequential in nature and whose content is agreed upon across institutions within the system. It is unclear whether instructors are consciously aware of this norm or have considered its appropriateness in the context of MFT courses.

Two additional factors that contribute to this tension include the weak mathematics skills of students coming into the MFT course, as articulated by Bob, and the perception that one of the roles of the course is to act as a filter to prevent those with poor mathematics skills from becoming elementary school teachers, as indicated by Maria. Both of these concerns point to larger problems within the system of teacher preparation, problems with defining the level of mathematics proficiency elementary teachers need, and with clearly defining the role of MFT courses in their preparation.

CONCLUSION

This tension is not easily resolved. It is certainly not simply a matter of refocussing priorities on affective rather than cognitive goals. One of the instructors in the study whose priority was on affective goals also experienced this tension, expressing concern that despite her efforts, what her course provides may not be enough to meet either one of her students' affective or cognitive needs

(Oesterle, 2010). Rather than attempt to resolve the tension, in the spirit of Lampert (1985), we consider instead what can be learned from it.

The study by Hart and Swars (2009) suggests that approaches of MFT instructors may negatively impact student affect. This study counters that even when MFT instructors are concerned about students' attitudes and beliefs, their ability to respond to the students' affective needs may be constrained by normative commitments to course syllabi, beliefs about the level of mathematics proficiency needed by future teachers, and understandings of the role of the MFT course. Maria's comments about reducing in-class activities in order to get through the material suggest that these may also be barriers to instructors' adoption of more reform-oriented approaches. Both Bob and Maria are dissatisfied with the outcomes of their MFT courses, but there are indications (especially in the case of Bob) that they do not believe they have the power to make the necessary changes.

One further observation is that although the debate in the literature is exemplified within the cases of the two instructors, the mathematics education research literature does not play a direct role in informing these instructors' efforts to deal with their tensions. This is even more evident in the larger study. Although with respect to the cognitive/affective debate, the literature to date offers no clear resolution, closer contact with the mathematics education community might offer these instructors new strategies or alternate perspectives as they strive to manage their tensions generally. Further research into ways to support these instructors would also be beneficial.

References

- Ball, D. L. (1990). The mathematical understandings that prospective teachers bring to teacher education. *The Elementary School Journal*, 90(4), 449-466.
- Berlak, A., & Berlak, H. (1981). *Dilemmas of schooling: Teaching and social change*. London: Methuen.
- Borko, H., Eisenhart, M., Brown, C. A., Underhill, R. G., Jones, D., & Agard, P. C. (1992). Learning to teach hard mathematics: Do novice teachers and their instructors give up too easily? *Journal for Research in Mathematics Education*, 23(3), 194-222.
- Charmaz, K. (2006). *Constructing grounded theory*. London; Thousand Oaks, CA: Sage Publications.
- Conference Board of the Mathematical Sciences. (2010). *The mathematical education of teachers*. Providence RI; Washington DC: American Mathematical Society and Mathematical Association of America.
- Creswell, J. W. (2008). *Educational research: Planning, conducting, and evaluating quantitative and qualitative research* (3rd ed.). New Jersey: Pearson Education.

- Ginsburg, H. (1981). The clinical interview in psychological research on mathematical thinking: Aims, rationales, techniques. *For the Learning of Mathematics*, 1(3), 4-11.
- Goulding, M., Rowland, T., & Barber, P. (2002). Does it matter? Primary teacher trainees' subject knowledge in mathematics. *British Educational Research Journal*, 28(5, Subject Knowledge and Application), 689-704.
- Greenberg, J., & Walsh, K. (2008). *No common denominator: The preparation of elementary teachers in mathematics by America's education schools*. Washington, DC: National Council on Teacher Quality.
- Harré, R., & van Lagenhove, L. (1999). *Positioning theory: Moral contexts of intentional action*. Oxford; Malden, MA: Blackwell.
- Hart, L. C., & Swars, S. (2009). The lived experiences of elementary prospective teachers in mathematics content coursework. *Journal of Teacher Development*, 13(2), 159-172.
- Hembree, R. (1990). The nature, effects, and relief of mathematics anxiety. *Journal for Research in Mathematics Education*, 21(1), 33-46.
- Kessel, C., & Ma, L. (2001). Mathematicians and the preparation of elementary teachers. In D. Holton (Ed.), *The teaching and learning of mathematics at university level: An ICMI study* (pp. 467-480). New York: Kluwer Academic Publishers.
- Lampert, M. (1985). How do teachers manage to teach? Perspectives on problems in practice. *Harvard Educational Review*, 55(2), 178-194.
- Liljedahl, P., Rolka, K., & Roesken, B. (2007). Affecting affect: The re-education of preservice teachers' beliefs about mathematics and mathematics learning and teaching. In M. Strutchens & W. Martin (Eds.), *69th NCTM yearbook – The learning of mathematics* (pp. 319-330). Reston, VA: National Council of Teachers of Mathematics.
- Oesterle, S. (2010). Tensions related to course content in teaching Math for Teachers: The case of Alice. In M. M. F. Pinto & T. F. Kawasaki (Eds.), *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education: Volume 3* (pp. 369-376). Belo Horizonte, Brazil: PME.
- Schwartz, J. E., & Riedesel, C. A. (1994). *The relationship between teachers' knowledge and beliefs and the teaching of elementary mathematics*. Paper presented at the annual meeting of the American Association of Colleges for Teacher Education, Chicago.
- Strickland, S. (2008). *Rhyme and reason: A rhetorical, genealogical examination of undergraduate mathematics*. Unpublished doctoral dissertation, Michigan State University.
- Wagner, J. F., Speer, N. M., & Rossa, B. (2007). Beyond mathematical content knowledge: A mathematician's knowledge needed for teaching an inquiry-oriented differential equations course. *Journal of Mathematical Behavior*, 26, 247-266.

CONCEPTUALIZING ROLE AND POSITION IN INTERACTIONS AMONG TEACHERS ENGAGED IN COLLABORATIVE DESIGN OF MATHEMATICS LEARNING ARTIFACTS

Preciado Babb, A. Paulino

The collaborative design of mathematics teaching and learning artifact by teachers and other educators has proved to be effective as both developing curricular material and teacher professional development. Teachers' collaborative design, in this paper, refers to the design of these artifact that includes: (1) the collaborative design of the artifact based on negotiated goals or purposes, (2) its implementation in the classroom, and (3) the debriefing of the results. The purpose of the paper is to conceptualize role and position of participants in teachers' collaborative design from a social perspective framed in embodied cognition. Such conceptualization would help to understand the dynamics and interactions—co-determinations—of teachers, and other educators, engaged in this mode of collaborative work.

INTRODUCTION

The purpose of this paper is to theorize on the interactions among participants, including teachers and educators, in the collaborative design of mathematics teaching and learning artifact. These artifacts include lesson plans, assessment rubrics, class projects, and mathematical tasks. I called the type of work considered in this paper 'teachers' collaborative design,' which includes the following steps: (1) creating, or adapting, the artifact, (2) implementing it with the students, and (3) reflecting on the results of its implementation. Examples of teachers' collaborative design have been well documented in the literature (Ponte & Chapman, 2006; Slavit & Nelson, 2009). A team of collaborative design can be considered as a learning system having an emerging structure determined by the interactions among its members. Understating these interactions provides insight on the team as a unit. In this paper, I use empirical data as a means to exemplify two, highly interrelated, concepts that serve to understand interaction in this context: *role* and *position*. These empirical data were based initially on symbolic interactionism (Blumer, 1969). However, for the discussion presented in the paper, embodied cognition will be used as a framework (Maturana & Varela, 1987).

ROLE AND POSITION

Socially, the *role* and *position* of members within a group—e.g. a team of teachers' collaborative design—include a pre-giving set of expected behaviours according to some specific rules or determined duties. 'Role' originally referred to "the roll of

paper on which the actor's part was written" (Role, 2010). 'Position' referred to physical space relative to another objects: "a place where someone or something is located or has been put ... a particular way in which someone or something is placed or arranged" (Position, 2010). Both concepts have evolved and expanded from their original meanings to a variety of contexts including social situations. For instance, 'role' has been defined as "the function assumed or part played by a person or thing in a particular situation" (Role, 2010) and "the position or purpose that someone or something has in a situation, organization, society or relationship" (Role, 2011). The latter definition links role with position, which, in a social context, also means "a person's point of view or attitude towards something" (Position, 2010). Thus, role and position can be defined beyond a set of pre-determined, or expected, behaviour within a group.

In the case of mathematics teacher education, Kaasila and Lauriala (2010), attempting to create an interactionist framework for students teacher learning, used the notion of role in a team of pre-service mathematics teachers participating in collaborative design. They considered that "in social situations a person must adopt a social role, which refers to a set of expectations of how a member of a special group or community is expected to act in his/her position" (p. 855). Blumer (1969), however, argued that "social interaction is obviously an interaction between people and not between roles; the needs of the participants are to interpret and handle what confronts them ... and not to give expression to their roles" (p. 75). Embodied cognition serves as a framework to describe roles without dismissing the individual perceptions of and meaning about such roles.

Both symbolic interactionism and embodied cognition have similar perspectives on the nature of objects and their relationships with human beings. For instance, these two perspectives denied the existence of a pre-given world in which people are immersed. However, symbolic interactionism focuses on a particular type of interaction among human beings, whereas embodied cognition includes other learning structures (Maturana & Varela, 1987). Moreover, in the latter perspective individuals not only make meaning of their world, but they also shape the environment, including other individuals or learning entities. The world is not pre-established, but enacted by each person. That is, we have multiple world and multiple realities according to the way each person co-evolves in a group in a specific situation. Interactions by means of language and action in a team of collaborative design are the means by which participants co-evolve. Language plays an important role: "Every human act takes place in language. Every act in language brings forth a world created with others in the act of coexistence which gives rise to what is human" (p. 247).

In this paper, the concepts of role and position are considered as shaping, and being shaped, by the interactions among the members of a team of collaborative design. Contextual factors, the environment, as participate in this *coupling* and will be taken into account for the discussion of these concepts. The concept of position (Langenhove & Harré 1999) serves to extend the notion of role. The descriptions of the roles generated by the data in the Loughheed project share some features with this concept: (1) the ongoing storyline developed in time, (2) the individual interests of each member to participate in the project, and (3) the varied perceptions of the role of some team members.

The Visual Thesaurus online dictionary (Thinkmap, 2011) includes several definitions of role and position, as well as the connection with other concepts. Table 1 shows definitions of these concepts that are embedded in social contexts. Note that last definitions of role and position in the table coincide. Among the definitions presented in the table, three attributes can be identified that I have identified empirically in teachers' collaborative design: (1) assumption; (2) opinion and attitude; (3) customary or specified activities within a group.

Role	Position
<p>the actions and activities assigned to or required or expected of a person or group</p> <p>an actor's portrayal of someone in a play</p> <p>what something is used for</p> <p>any specific behaviour</p> <p>normal or customary activity of a person in a particular social setting</p>	<p>the act of positing; an assumption taken as a postulate or axiom</p> <p>a rationalized mental attitude</p> <p>the act of assuming or taking for granted</p> <p>an opinion that is held in opposition to another in an argument or dispute</p> <p>the post or function properly or customarily occupied or served by another</p> <p>the relative position or standing of things or especially persons in a society</p> <p>a job in an organization</p> <p>(in team sports) the role assigned to an individual player</p> <p>normal or customary activity of a person in a particular social setting</p>

Table 1: Definitions of role and position in social contexts (Thinkmap, 2011).

Role is also defined as a type of position. Thus, role and position can be often considered, within a social context, as synonymous. However, position, as “the act of positing; an assumption taken as a postulate or axiom” can be considered as an extension of the notion of role (Langenhove, Harré, 1999).

EXAMPLES FROM THE DATA

In order to show some examples of role and position I will use data generated from a case of teachers' collaborative design taking place from September 2008 to April 2009. In this project three high school mathematics teachers engaged in weekly one-hour meetings with the purpose of designing and implementing two mathematics lessons. All the meetings took place at the school where teachers were working. I participated as a researcher and member of the team. In addition to the meetings for collaborative design, two meetings were dedicated to group interviews, one in December, 2008, and the other in April, 2009. Individual interviews with each teacher were conducted in June, 2009. All the meetings were video recorded and analyzed using a grounded theory approach which gave as a result two emerging themes: (1) the focus of the conversation while designing the artifacts, and (2) the roles and positions of each participant during the conversations held by the team. I use part of the data of this study as examples of roles and positions from the perspective of embodied cognition.

An important moment during the research which had an impact on my decision of focusing on roles was the first group interview. Before the interview teachers read preliminary findings of the research. During the interview, one of the teachers commented on my role of researcher in the team.

Arnold: Because I think you want to be an insider, but, can you? Cause you are doing this ethnography and it's really hard to become fully immersed.

Although I participated actively in the design of the artifacts, I was an outsider from the community of the teachers. Arnold's perception of my role as a member of the design team was based on my position in the university as an instructor and researcher. This comment triggered a short discussion which reflects different perceptions about published research work.

Arnold: But, I think though there is a very special place as a researcher and as you become published; that always will set [you] outside of this community.

Sofia: I don't think publishing gives any more respect or any more trust to what you are saying—just because you are published. Just because it is written doesn't mean is any more true.

Arnold: But it does exist in a professional literature, and so is a privileged location.

For Arnold, this privileged position represented an authority within the team. Such position was not perceived by Sofia in the same way.

Arnold: You clearly have to have more authority on what you say. ... I would perhaps give more weight to what you say just because in theory you have more background knowledge. ... You are becoming a professional in this area. So, in theory you should know more.

Brad: Like you are the supervisor. You have your own supervisor and you are the supervisor of us—kind of.

In this excerpt, Brad positioned Armando's role as a supervisor, which is a different role from the one Arnold stressed as a professional, or an authority, in the area.

In the second part of the interview, members of the team described the role that each participant played during the project. Everyone wrote descriptions of the roles played by each member before taking turns explaining such descriptions. The video recordings were useful to contrast such descriptions with the conversations held by the team.

Although there were similarities in the descriptions of these roles, particular individual roles were perceived slightly different among the team. Arnold always brought resources such as books, papers, or details of websites.

Brad: [Arnold], as you guys mentioned, is sort of the data-base expert. We come up with something, [Arnold] will have something in the locker or in the cabinet or somewhere. She pulls it out of the air: all this background and research so rich in terms of information.

For me and Sofia, Arnold also contributed in an important way by suggesting better words for the students' worksheets and teacher's instructions.

Brad was often wondering whether students would learn what was intended through the designed lesson, forcing the team to think about the real impact of the lesson on the students' learning. He also proposed the goal that the team pursued in the first round of the project.

Sofia: [Brad] focused on difficulties students might have questioning. And came up with the question we were working on now.

Arnold: Building on what [Sofia] said, I think it is true: that excellent understanding of class dynamics, problems of how weaker students are going to express on that topic. Clear understanding of what needs to be taught and how this might be done.

Arnold considered Brad as an experienced teacher with a good understanding of what has to be taught, and knowledge about weak students' troubles in learning the mathematical content.

Brad also played an important role in asking questions related to the research project and contributed to it by commenting on the teachers' activities outside of the meetings. He often commented on how the impact of the activities on my research. Brad described himself as being focused on the original goal for the lesson which was for students to translate word problems into algebraic expressions.

Sofia's role included contributing to mathematical tasks for the lesson, and refocusing the team on-task when the conversation deviated from designing the lesson. Additionally, Arnold and Brad perceived in her a level of expertise in mathematics and mathematics learning.

Brad: I saw [Sofia], in this context, as the mathematics expert. She was coming with all this terminology So, I was learning new things from you. And you [Sofia] are always doing the puzzles. I would be sitting and watching you actually figuring out the patterns and coming up with the expressions. So, you were taking a much more active role in the sense that you were trying out and I just sit and watch.

Brad saw Sofia as an expert in mathematics, and at the same time, he positioned himself as a learner. In the individual interview at the end of the project Brad mentioned that he learnt mathematics content from Sofia. During the project, he often made questions related to some topics which were not necessary related to the lessons that were designed.

When teachers were asked, in the final group interview, about their motivations for participating in this particular research project, they agreed that Sofia had played a crucial role in their decisions. Brad's choice of participating in the project was based on the fact that Sofia invited them and that Arnold also had accepted.

Brad: Like you were saying, if it is a colleague you tend to believe more. If I just had a letter "[Armando] from SFU wants teachers to volunteer in a lesson study project,"... I don't know this guy, I don't know anything about lesson study. I probably put it aside somewhere. But, because it was [Sofia]'s initiative, and then [Arnold] was on board ...

The teachers mentioned that they usually trust their colleagues, sometimes even more than an external expert. This was a factor that made Arnold and Brad accept Sofia's invitation to join the project.

Arnold mentioned that the first group interview was a remarkable moment, when teachers read the paper with some preliminary findings, to be the most interesting part because of the reaction of the teachers after the interview.

Arnold: So, for me, that was the most interesting part, because then people were aware: “Oh we are participating in a study group, how what would you say or do is being commented on.” whether I disagree or agree with you is different, but just how people responded, it was sort of amusing.

This last excerpt reflects teachers' awareness of how what they had said would be commented on as part of the research project. Teachers were influenced by the fact that they were participants in a research study. This suggests that the interaction of the team might be not the same in a different scenario where they would not be participating in a research study.

The descriptions of the roles presented here go beyond a pre-established set of duties or responsibilities for the team. These descriptions were based mostly on the type of contributions that each member made during the design process in the first round of the project. Roles such as the mathematics expert, Sofia, or the database expert, Arnold, reflect the type of contributions made by particular members. Additionally, my role as a participant-researcher was perceived differently by each member of the team: as an expert, as an authority, and as a supervisor.

I found that these roles, and the positions taken by the team members, were influenced by three factors: (1) the collegiality of the team, (2) the fact that this was a research project focusing on participant teachers, and (3) the interests that each teacher had to participate in the project. On the team there were some fixed roles, such as the role of Arnold and Sofia as teachers who implemented the lessons in their classrooms. My role as a researcher was not perceived the same by the each of the teachers: authority, supervisor, and support and validation of their ideas. The fact that people perceived others' roles in different ways suggests that the role is made by all the perceptions—possible different—of how a member is expected to act in a particular situation. The concept of position (Langenhove & Harré 1999) serves to extend the notion of role. The descriptions of the roles generated by the data in the project share some features with this concept: (1) the ongoing storyline developed in time, (2) the individual interests of each member to participate in the project, and (3) the varied perceptions of the role of some team members.

CONCLUSION

The concepts of role and position explored in this paper entail constant change and a variety of perceptions. Although some roles were determined from the beginning, such as my role as researcher, the perceptions of each participants were not the

same. Other roles were developed by the interactions of the participants during and before the project. The three participant teachers had developed a collegiality which was reflected not only in the interactions during the project, but also in their decision of participating in the research project. Sofia's role as a mathematics expert, as perceived by Brad, might be developed before the project. Arnold constantly brought resources such as books and articles to the meetings. This was not a pre-established duty or expected role: It was a customary performance.

The role, as perceived by each member of a team of collaborative design, forms part of the world that each person *enact* within the team (Maturana & Varela, 1987). Such roles and positions influence, and are influenced, during the interactions of the team. The decisions made during the design of the lessons during the project are the result of the conversations and actions of the team members. Considering the team as a unit, or as a learning system, the interactions among their members represented the constantly evolving structure of the team. The conceptualization of role and position described in this paper served to understand such structure.

References

- Blumer, H. (1969). *Symbolic interactionism: Perspective and method*. Berkeley, CA: University of California Press.
- Kaasila, R., & Lauriala, A. (2010). Towards a collaborative, interactionist model of teacher change. *Teaching and Teacher Education*, 26(2), 854-862.
- Langenhove, L., & Harré, R. (1999). Introducing position theory. In R. Harré, & L. van Langenhove (Eds.), *Positioning Theory*. Oxford, UK: Blackwell Publishers.
- Maturana, H. R., & Varela, F. J. (1987). *The tree of knowledge: The biological roots of human understanding*. Boston: Shambhala.
- Position. (2010). In *Oxford Dictionaries*. Oxford University Press. Retrieved October 8, 2011 from <http://oxforddictionaries.com/definition/position>
- Ponte, J. P., & Chapman, O. (2006). Mathematics teachers' knowledge and practices. In A. Guitierrez, & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 461-494). Rotterdam, UK: Sense Publishers.
- Role. (2010). In *Oxford Dictionaries*. Oxford University Press. Retrieved October 8, 2011 from <http://oxforddictionaries.com/definition/role>
- Role. (2011). In *Cambridge Advanced Learner's Dictionary & Thesaurus*. Cambridge University Press. Retrieved August 8, 2011 from http://dictionary.cambridge.org/dictionary/british/role_1?q=role

Thinkmap. (2011). *Visual Thesaurus*. Retrieved October 8, 2011 from <http://www.visualthesaurus.com>

Slavit, D., Nelson, T., H., & Kennedy A. (Eds). (2009). *Perspectives on Supported Collaborative Teacher Inquiry*. New York: Routledge.

CALCULUS BEYOND THE CLASSROOM: APPLICATION TO A REAL-LIFE PROBLEM SIMULATED IN A VIRTUAL ENVIRONMENT

Olga V. Shipulina

This study concerns the correlation of mathematical knowledge with a corresponding real life object within the theoretical framework of Realistic Mathematics Education. By simulating an interactive milieu in the Second Life Virtual Environment (VE), this study explores how students find a ‘real-life’ optimal path ‘practically’, and how they then re-invent the corresponding calculus task. The instructional design, based on simulation in VE, allowed students to explore mathematical solutions relative to their intuitive findings in VE. By mathematizing their own ‘real-life’ activities, students connected them with corresponding mathematics at an intuitive level.

INTRODUCTION

A troubling problem with current education is in the practical application of knowledge to life. Graduates do not know how to apply knowledge to many problems that arise outside the walls of school (Ilyenkov, 2009). There is a common recognition among mathematics educators that a serious mismatch exists and is growing between the skills obtained at schools and the kinds of understanding and abilities that are needed for success beyond school (Lesh & Zawojewski, 2007).

The attempts of some instructional theories to solve the problem by creating systems of rules of ‘how to apply knowledge to life’ impede rather than help things (Ilyenkov, 2009). The decisive part of cognition, going from the object to the abstract, remains outside of student activity. A special kind of activity related to correlating knowledge and its object should be implemented in contemporary classrooms. “Here, what is needed is activity of a different order—activity oriented directly at the object. Activity that changes the object, rather than an image of it” (p. 223).

The problem of ‘the practical application of knowledge to life’ is especially significant for calculus, which was developed from real world applications and has a real world context. In the late 1980s the ‘Calculus Reform Movement’ began in the USA. The Calculus Consortium at Harvard (CCH) was funded by the National Science Foundation to redesign the curriculum with a view to making calculus more applied, relevant, and more understandable for a wider range of students.

This paper’s purpose is to set out an instructional design based on students’ exploration of their ‘real-life’ activity and their primary intuitions in connection

with mathematical formalities. Simulated in the *Second Life* VE, the interactive milieu encouraged students to find an optimal path on the basis of their primary intuitions; and then with the help of a specially designed journal, to re-invent the corresponding calculus task. Such an instructional design allowed the students to learn to analyse and formalize their primary intuitive acquisitions, to psychologically connect mathematical concepts with ‘real-life’ intuitions, and to connect mathematical formalities with a real-life situation.

THEORETICAL BACKGROUND

More than forty years ago Freudenthal (1968) posed the problem of lack of connection between knowledge and its real-life object. The Freudenthal Institute has developed a theoretical framework, now referred to as Realistic Mathematics Education (RME) (Freudenthal, 1968, 1973, 1991; Gravemeijer, 1994).

The RME instructional theory is based on Freudenthal’s idea that mathematics must be connected to reality. The use of realistic contexts became one of the determining concepts of RME. The most general characteristic of RME is mathematizing; the realistic contexts must be used as a source for mathematizing. The role of mathematizing in mathematics education is also stressed by a number of authors (Wheeler, 1982; Treffer, 1986; De Lange, 1996; Presmeg 2003; Mason, 2004; Russmussen, Zandieh, King, & Terro, 2005; Liljedahl, 2007;). Particularly, Treffer (1986) formulated the idea of ‘progressive mathematizing’ as a sequence of two types of mathematical activity—horizontal mathematizing and vertical mathematizing. The process of extracting the appropriate concept from a concrete situation is denoted by De Lange (1996) as ‘conceptual mathematization’. This process forces the students to explore the situation, find and identify the relevant mathematics, schematize, visualize, and develop a corresponding mathematical concept.

The RME theory has been accepted and adopted by some educational institutions in England, Germany, Denmark, Spain, Portugal, South Africa, Brazil, Japan, and Malaysia (de Lange, 1996). In America, RME was adopted in the “Mathematics in Context” project for US middle schools. In spite of such wide acceptance and adoption of RME, recent research shows that there is still a wide gap between the world of knowledge obtained at school and the world of conceptions found in everyday experience (Lesh & Zawojewski, 2007; Ilyenkov, 2009).

The major idea of this paper is to point out that the reason why students do not connect the mathematical world with reality is because they continue to mathematize ‘word problems’ with ‘ready-made’ images instead of active real-life situations. Moreover, students do not involve their intuitive cognition while mathematizing ‘word problems’ and ‘ready-made’ images. Intuition, intuitive

cognition, intuitive understanding, and intuitive solutions form some of the basic components of mathematical activity along with formal aspects such as axioms, definitions, and algorithmic components (Fischbein, 1994). Furthermore, intuition gives the behavioural meaningfulness of a mathematical notion (Fischbein, 1987).

Although a number of authors have stressed the important role of intuition in mathematics education (e.g., Fischbein, 1987, 1989, 1994; Tall, 1991, 1996, 2000; Burton, 1999), they do not point out its role in RME instructional design theory, which is purported to connect mathematics with reality.

METHODOLOGY: MATERIALS, METHODS, AND PARTICIPANTS

The *Second Life* VE was used to program an interactive setting for a real-life optimal navigation task. The simulated setting includes a pond with shallow water, surrounded by bushes and trees (Fig. 1). It was programmed so that walking/running speed on land is sufficiently larger than walking/running speed in water.

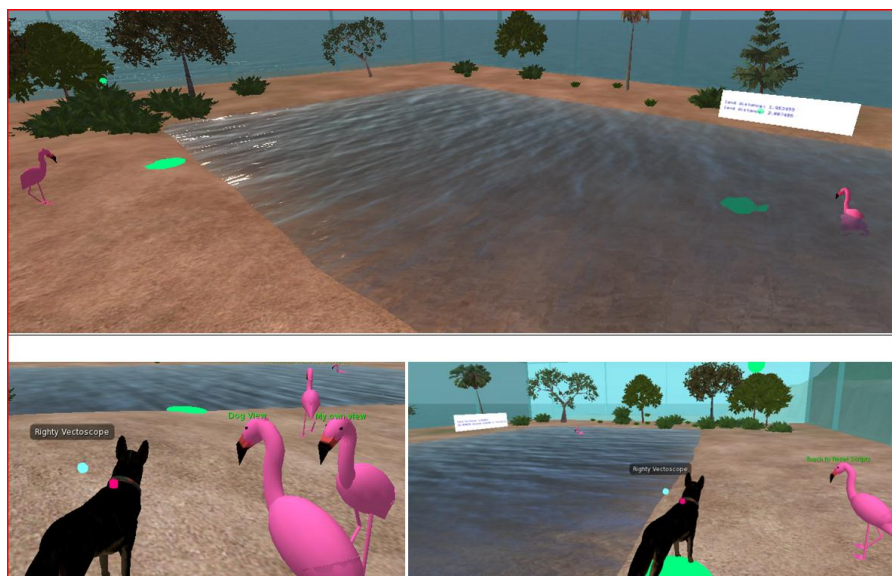


Figure 1: Simulation in the *Second Life* VE interactive milieu for finding an optimal path.

The task for the student in this VE was to travel between the two green platforms (see Figure 1), trying to minimize the total time of travel for the trip. One platform is located on land near the water's edge; another is located in the water. The environment is programmed to record time spent for each trip and the distance traveled by land. After each trip the student had to transfer this data into a specially designed guiding–reflecting journal, which is an integral methodological part of the instructional design.

The aim of the guiding–reflecting journal is to connect the student’s optimal navigation practice in the VE with the calculus optimal-path-finding task. The journal contains instructions, tables for transferring data collected from every trip in the VE, reflecting questions, guiding instructions, and questions initiating the student’s reasoning. The journal guides the student’s mathematical reasoning. It contains schematization of the problem and areas for independent reasoning. The back page of the journal offers some formula tips and a detailed solution of the calculus problem. The geometrical schematization and solution provided in the journal were adapted from Pennings’s (2003) work. Figure 2 demonstrates a schematization of the task.

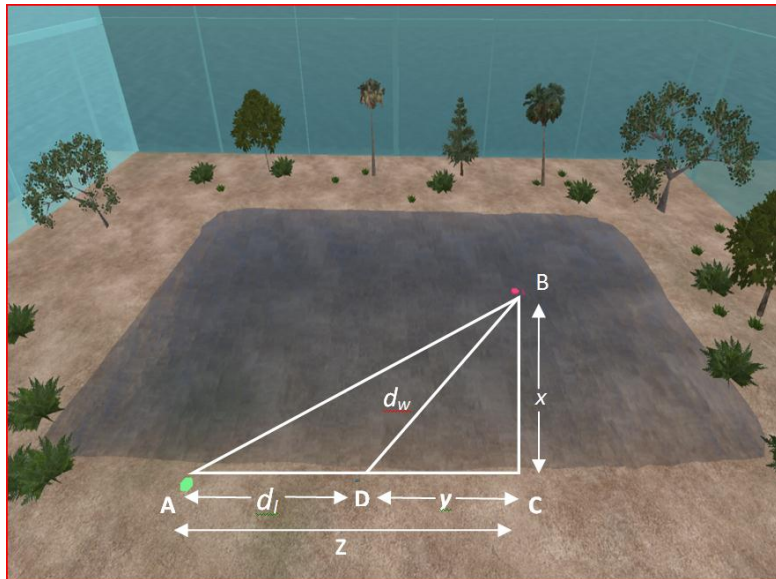


Figure 2: Schematization of some possible paths provided in the guiding-reflecting journal.

According to the schematization, A is a land platform, B is a water platform. The shortest path from A to B is the most direct path AB. Since the speed in water is slower than on land, students can choose the path with the shortest distance traveled in water, path AC and then CB, where ACB is a right angle. Finally, there is the option of using a portion of the land path, up to D, and then entering into the water at D and moving diagonally to the water platform. In this diagram x represents the distance between B and C; d_l and d_w are distances traveled by land and by water, respectively. The distance between A and C is z , and $y = z - d_l$.

The solution of the minimal path finding task includes the following reasoning:

According to the schematization above, $T = T_l + T_w$. Since $T_l = \frac{d_l}{s_l}$, and $T_w = \frac{d_w}{s_w}$,

then $T = \frac{d_l}{s_l} + \frac{d_w}{s_w}$, which gives $T = \frac{z-y}{s_l} + \frac{\sqrt{x^2+y^2}}{s_w}$, where s_l and s_w are speeds on

land and in water respectively; T is the trip time; T_l is time spent for the land portion; and, T_w is time spent for the water portion of the trip.

The condition of minimal time is $T'(y) = 0$, or $(\frac{z-y}{s_l} + \frac{\sqrt{x^2+y^2}}{s_w})' = 0$

Following the journal instructions, the student obtains the final formula:

$$y = \frac{x}{\sqrt{\frac{s_l}{s_w} + 1} \sqrt{\frac{s_l}{s_w} - 1}}$$

The journal provides the exact values of virtual distances and speeds in the VE. The student is instructed to use these values to calculate the optimal distance traveled by land with corresponding minimal time using the formulas above, and to compare the mathematically obtained values with his/her best finding in the VE.

Ten students, ranging in age from 17 to 18 years, who had almost completed the AP calculus course at a secondary school, participated in the research study. Each participant provided a signed Parent Consent Form. They also read and signed the Assent Form before participating. The experiments were conducted in the school's Teachers' Room. Each session of 60–90 minutes included an exploration trial, followed by the main task which consisted of the participants' work with both the computer and the guiding-reflecting journal. The mathematical part was devoted to the participant working solely with the journal. The participants' exploration of the computer environment was screen recorded using SMR software. Their work with the journals was video-recorded. During one session the computer lost the SMR data while automatically updating its basic software. Therefore, the collected data from 10 sessions included screen recordings of exploration trials and VE tasks from only 9 sessions, video-recordings of students' working with journals, and the completed journals from all 10 sessions.

RESULTS

The first part of the data analysis was devoted to students' finding the optimal path in the VE based on their primary intuitions. The students' first trips in the VE demonstrated that students have different life experiences connected with optimal navigation, therefore different intuitive solutions. Figure 3 shows four different choices of students' first paths.

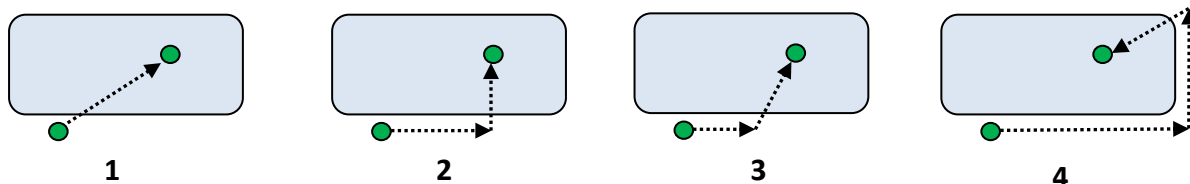


Figure 3: The diagrams of four different choices of first trips with which students started their optimal-path-finding task in the VE.

Analysis of nine computer screen recordings showed that the first trips of four students were the shortest distances as shown in diagram 1. Three students started finding the optimal path by trying to minimize the water portion as shown in diagram 2. One participant also tried to minimize the water portion in his first trip by running around the pool as shown in diagram 4. Only one out of nine students intuitively chose the first path corresponding to the exact mathematical solution of the problem (see diagram 3).

The second part of the data analysis was devoted to students' capacity to mathematize their activity, utilizing newly obtained knowledge of calculus. One of the ten students mathematized the problem horizontally and vertically without any guidance. He started to mathematize the problem during the completion of the optimal navigation task in the VE. After two trips in the VE this participant asked "Actually can I do math?" After another two trips he started drawing diagrams, schematizing his activity in the VE. He then completed two final trips (totalling six out of ten offered in the journal) and then switched to developing a mathematical solution of the problem. Although he did not find the final formula due to an initial mistake in writing connections between time, distance, and speed (which he later corrected), he showed an excellent example of horizontal and vertical mathematizing of a real-life situation. A second participant was able to mathematize the problem following the guidance of his journal. He was good with differentiation and used the tips in the journal effectively. The remaining participants required both journal guidance and oral explanation of the mathematical solution to the problem.

CONCLUSIONS AND IMPLICATIONS FOR MATHEMATICS EDUCATION

The conducted experiments confirmed that the problem of applying calculus knowledge to a task beyond the classroom still exists. It was shown that only one student out of 10 participants was able to convert his real-life activity into mathematical symbolic problem solving without any guidance. The chief outcome of this research is a new approach to RME instructional design. Particularly, we demonstrate that instead of situations described in 'word problems' with ready-made images to be mathematized, the real-life activity can be simulated in a VE. We showed a particular example from calculus which allowed students to try to

solve the optimal-path-finding problem ‘physically’ on the basis of their primary intuitions and then mathematize the problem with guidance of a specially designed journal. This ‘real life’ activity in the simulated VE helps students to become aware of tacit conflicts between their intuitions and the formal mathematical solution. Such awareness helps to shape ‘right’ intuitions, which in turn gives the behavioural meaningfulness of a mathematical notion (Fischbein, 1987). Practical ‘real-life’ activity in a simulated VE and its further mathematizing connects the particular activity with corresponding mathematical formalities. Implications of the offered instructional design can bring real-life problems from outside the school into the classroom.

References

- Burton, L. (1999). Why is intuition so important to mathematicians but missing from mathematics education? *For the Learning of Mathematics*, 19(3), 27-32.
- de Lange, J. (1996). Using and applying mathematics in education. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education: Part 1* (pp. 49-97). Dordrecht, The Netherlands: Kluwer Academic.
- Fischbein, E. (1987). *Intuition in science and mathematics*. Dordrecht, The Netherlands: Reidel.
- Fischbein, E. (1989). Tacit models and mathematical reasoning. *For the Learning of Mathematics*, 9, 9-14.
- Fischbein, E. (1994). The interaction between the formal, the algorithmic, and the intuitive components in a mathematical activity. In R. Biehler, R. W. Scholz, R. Strässer, & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 231-245). Dordrecht: Kluwer Academic.
- Freudenthal, H. (1968). Why to teach mathematics so as to be useful? *Educational Studies in Mathematics*, 1, 3-8.
- Freudenthal, H. (1973). *Mathematics as an educational task*. Dordrecht, The Netherlands: Riedel Publishing Company.
- Freudenthal, H. (1991). *Revising mathematics education*. Dordrecht, The Netherlands: Kluwer Academic.
- Gravemeijer, K. P. E. (1994). *Developing Realistic Mathematics Education*. Utrecht: CD-b Press / Freudenthal Institute, Utrecht University.
- Ilyenkov, E.V. (2009). *The ideal in human activity*. New York: Lawrence & Wishart.
- Lesh, R., & Zawojewski, J. S. (2007). Problem solving and modeling. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 763-805). Greenwich, CT: Information Age Publishing.

- Liljedahl, P. (2007). Mathematics and the imagination: Thoughts on ‘doing’ mathematics. In K. Egan, M. Stout, & K. Takaya (Eds.), *Teaching and learning outside the box* (pp. 61-74). New York, NY: Teachers College Press.
- Mason, J. (2004). *Fundamental constructs in mathematics education*. New York: RoutledgeFalmer.
- Pennings, T. J. (2003). Do dogs know calculus? *The College Mathematics Journal*, 34(3), 178-182.
- Presmeg, N. (2003). Creativity, mathematizing, and didactizing: Leen Streefland's work continues. *Educational Studies in Mathematics*, 54(1), 127-137.
- Russmussen, C., Zandieh, M., King, K., & Terro, A. (2005). Advancing mathematical activity: A practice-oriented view of advanced mathematical thinking. *Mathematical Thinking and Learning*, 7(1), 51-73.
- Tall, D. O. (1991). Intuition and rigour: The role of visualization in the calculus. In S. Cunningham & W. S. Zimmermann (Eds.), *Visualization in teaching and learning mathematics* (pp.105-119). Washington DC: MAA.
- Tall, D. O. (1996) Functions and calculus. In A. J. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education: Part 1* (pp. 289–325). Dordrecht: Kluwer.
- Tall, D. O. (2000). Technology and versatile thinking in mathematics. In M. O. J. Thomas (Ed.), *Proceedings of TIME 2000: An international conference on technology in mathematics education* (pp. 33-50). Auckland, New Zealand: The University of Auckland & AUT.
- Treffer, A. (1986). *Three dimensions. A model of goal and theory description in mathematics instruction—the Wiskobas Project*. Dordrecht, The Netherlands: Kluwer Academic.
- Wheeler, D. (1982). Mathematization matters. *For the Learning of Mathematics*, 3(1), 45-47.

MATHEMATICS, ABSTRACTION AND TEACHING: REVISITING TIMMS 1999 VIDEO LESSONS

Krishna Subedi

Mathematics is an abstract subject. When teachers plan, one of their most important challenges is to figure out ways of translating abstract concepts into understandable ideas. This paper explores the notion of mathematical abstraction from teaching view point and proposes a theoretical framework of Reducing Abstraction in Teaching (RAT). By analysing mathematics classroom practices from the public release video lessons of TIMMS 1999, this paper illustrates various tendencies of teachers dealing with mathematical abstraction. It also exemplifies some instances where 'reducing abstraction' seems to be an effective teaching strategy while in other cases it may go unsupportive for the development of student's mathematical understanding.

INTRODUCTION

Abstraction is often seen as the fundamental characteristic of mathematics; and it "has been recognized as one of the most important features of mathematics from a cognitive viewpoint as well as one of the main reasons for failure in mathematics learning" (Ferrari, 2003, p. 1225). As such, in the recent years, abstraction has received a growing interest in research community among psychologists and mathematics educators. In fact, when teachers plan, one of their most important challenges is to figure out ways of translating abstract concepts into understandable ideas. If teachers understand more clearly what mental process their students go through while coping with mathematical abstraction they are attempting to teach, they may be able to teach more effectively.

Reducing Abstraction (Hazzan, 1999) is one of the theoretical frameworks that examine the mental process of learners while coping with abstraction of new mathematical concepts. It has been used to examine the mental process of learners in different areas of mathematics and computer science (Hazzan & Zazkis, 2005). I am however, not aware of any study that specifically looked at how teachers deal with mathematical abstraction in teaching. Hence, this paper aims to examine the teacher's behaviours in regard to dealing with abstraction in teaching. Because of the space limitation, detailed discussion of the study is not possible here. I, however, provide a brief overview of the study and the theoretical framework followed by the methodology. Finally the results and discussion followed by some concluding remarks.

THEORETICAL FRAMEWORK

Hazzan's (1999) research on how undergraduate students learn abstract algebra is an important work that provides a window to look at the mental process of students while learning new mathematical concepts. Her finding is that learners usually do not have the mental construct or resources '*to hang on to*' to the same level of abstraction as introduced by the authorities (textbook, teachers etc.) and hence they tend to reduce the level of abstraction in order to make the concept mentally accessible. This usually happens *unconsciously*. In other words, when a student sees a mathematical object, he or she will try to make sense of it based on his or her past experiences with other mathematical objects.

What does this tell us about teaching? This clearly points to the idea that while introducing new mathematical concept (often abstract), teachers should make an effort to 'concretize' them by using students previously acquired knowledge, experience and level of thinking as well as their familiar contexts. In so doing, a "right relationship" in the sense of Wilensky (1991) can be established between the learners and the new mathematical concept so that the abstractness of the concept may be reduced. According to Wilensky:

"Concreteness is not a property of an object but rather a property of a person's relationship to an object. Concepts that were hopelessly abstract at one time can become concrete for us if we get into the "right relationship" with them. ..." (p.198).

Along the same lines, Cornu (1991) states, "for most mathematical concepts, teaching does not begin on virgin territory" (p.154), all students come with certain ideas, intuition, and knowledge already formed in their mind on the basis of their previous experience. Therefore, he says, it is important for teachers to become explicitly aware of this difficulty of their students and attempt to reconstruct their knowledge structure to accommodate the new concepts. Safuanov (2004) suggests:

"Strict and abstract reasoning should be preceded by intuitive or heuristic considerations; construction of theories and concepts of a high level of abstraction can be properly carried out only after accumulation of sufficient supply of examples and facts at a lower level of abstraction" (p.154).

This idea is in line with many other psychologists and educators (see Piaget, 1970; Vygotsky, 1996; Hershkowitz, Schwarz, and Dreyfus, 2001). For example, Piaget's idea of developmental psychology and genetic epistemology tells that children develop abstract thinking slowly, starting as concrete thinkers with little ability to create or understand abstractions. Based on this idea, genetic approach to teaching mathematics is widespread.

From this perspective, effective teaching should involve with the process of introducing new abstractions; concretising or semi-concretising them; then

repeating at a slightly higher level. That is, the concept are concretized and presented to the students in a lower level of abstraction temporarily. The goal is however to go to the higher level of abstraction using the lower level as stepping stone. This activity, I argue, is an attempt to reduce the level of abstraction of the concept on the teachers' part in order to make the concept mentally accessible to the students. Hence, the notion of Reducing Abstraction in Teaching (RAT) comes into play.

METHOD

The initial questions that guided this work are: how do teachers deal with abstraction in teaching? Do teacher reduce abstraction? If so, what is nature of reducing abstraction?

Hazzan's (1999) *Reducing Abstraction* framework examines how learner's deal with mathematical abstraction while learning new mathematical concept. My aim in this study however, is to look at how teacher deal with abstraction in teaching. This is a shift in perspectives because a learner's goal is to learn mathematics for themselves whereas teachers are the mediators and their goals are to help their students to learn mathematics. This shift in perspective necessitates a modified version of reducing abstraction framework. To this ends, a comprehensive literature review was conducted and categorised the findings according to themes. With the refinement of the themes, three interpretation of reducing abstraction have been identified and presented in three categories. In so doing, a new theoretical framework, which I call Reducing Abstraction in Teaching (RAT), has emerged. I used TIMSS 1999 public release video lessons in order to gather empirical data.

TIMSS 1999 video lessons were already analysed by TIMSS videos study team itself and others using different theoretical frameworks (Hiebert et al., 2003) with a focus on various aspects of teaching and learning. I am however, not aware of any study with its focus on teachers' dealing with abstraction in teaching. Hence, my present study took place. My analysis is based on transcripts of the public release video lessons that have been translated into English by TIMSS 1999 video study team. Keeping the three interpretations of reducing abstraction in mind, I read the transcripts repeatedly with an eye towards identifying key teachers' actions and searching and developing the meaning for each of their action. For lessons from English speaking countries, I also watched videos repeatedly.

RESULTS AND DISCUSSION

Building on the work of Hazzan (1999), Wilensky (1991) and Sfard (1991), three interpretations for abstraction level have been identified, all of which interpret teacher's action as some way of reducing abstraction of the concept. From the

analysis of the empirical data, various tendencies of teachers reducing abstraction have been emerged in each category and listed them as subcategory. Because of the space limitation, only few examples have been presented here.

Category 1: Abstraction Level as the Quality of the Relationships between the Mathematical Concept and the Learner

It is based on the Wilensky's (1991) assertion that whether something is abstract or concrete is not an inherent property of the thing, "but rather a property of a person's relationship to an object" (p.198). On the basis of this perspective, the level of abstraction is measured by the relationship between the learners and the concept (mathematical object). Reducing abstraction in this category was coded if there is a situation where an attempt has been made to make unfamiliar (therefore abstract) concept more familiar (therefore concrete) to the students by any of the following ways:

FamRw : Reducing abstraction by connecting mathematical concept to real- world situations .1.1

FamLang: Reducing abstraction by using familiar but informal language rather than formal mathematical language .1.2

- 00:04:33 T We know that the edges of a triangle- or any figure- are called "sides".
- 00:04:38 T In a right-angled triangle, this side is attached to a right angle. So what should we call this side? A right-angled side.
- 00:04:47 T Yes? Because this side is attached to a right angle so you call that a right-angled side.
- 00:05:00 T Do we have any other right-angled side in there?
- 00:05:02 SN Yes.
- 00:05:03 T Yes, all the way on the other side. That one is attached to the right angle as well, therefore you call that a right-angled side as well.
- 00:05:19 T Then I still have one side left. It isn't so obvious because it is lying flat. But if you see this triangle, what can we call that side?
- 00:05:28 SN The long side.
- 00:05:29 T The long side. That is correct. Or in a different way?
- 00:05:33 SN The right side?
- 00:05:34 T It is actually at an angle. If you see it in such an- like a diagonal- so you call this the sloped side or the hypotenuse, is what you call

this one.

00:05:45 T These are just names, you know, you may also keep calling this "the long side", no problem. (NL 02)

FamRep: Reducing abstraction by connecting new mathematical concept to familiar representations (that includes use of pedagogical tools such as graphs, diagrams, tables, metaphors, gestures, manipulative etc.) .1.3

00:38:57 T what do we have that's tricky? We have four X is equal to four X... minus one. So how are we going to see that?

00:39:04 SS It's not possible!

00:39:09 T It's impossible. Valentine?

00:39:11 SN Because it need a place, for example four X is equal to three X minus one. Otherwise we can't take away the X's. We can't take away one.

00:39:21 T Let's try to take it away, let's try to be as methodical as we can, then let's see what happens. Let's see, what do we find, so-

Dialogue continues...

00:39:35 T Therefore $0 = -1$. Is that possible?

00:39:42 SS No!

At this stage, some of the students seem to be struggling to make sense of what it means when they get $0 = -1$ while solving the equation. At this time, the teacher refers to the graphical method in which case students saw clearly that the lines are parallel and so there is no point of intersection. That is, there is no solution. The dialogue continues:

00:41:33 S It is parallel.

00:41:33 T It is, well, parallel. Therefore when do they meet each other, lines?

00:41:37 SS Never.

00:41:37 T Never. Therefore the- what we're looking for as a solution, it's the points that meet each other. Therefore there's none.

Category 2: Abstraction Level as Reflection of the Process-Object Duality

Reducing abstraction in this category is based on Sfard (1991) theory of 'process-object duality' which states, "abstract notation such as a number, function etc. can be conceived in two fundamentally different ways: structurally- as objects and operationally- as processes" (p 1). According to this theory, the process conception is less abstract than an object conception. The following tendencies of teacher presenting the mathematics task have been identified under this category.

2.1. DuProc: Teacher reducing abstraction by shifting the focus on procedure even though the problem or discussion implies a focus on concepts, meaning, or understanding

Example :

Earlier in the lesson, students worked on finding volumes of beams (rectangular prism) and cylinders and the solid formed by combining them. They also worked on problems that required students to find the volume of such solids when one or two dimensions had been doubled. This problem consists of situation where students are to find how the volume is changed when all the three dimensions are doubled.

17:32 T Well, what do you think will happen then?

17:34 SN But...but how is that possible - or can the height be done also?

17:36 T Yes, yes, so and the length, and the width and the height.

17:40 SN Yes, but it doesn't say (in the book).

17:41 T No, but we will just add those together. Because then we have all the possibilities together.
Well,

what happens then?

17:48 SN You get two times two times two.

17:49 T Yes, two times two times two. You have - this is not new to you, right?

17:52 SN That is twelve...

17:55 SN (...)

17:56 T This is for a beam. And this is actually also what they mean for assignment thirty-nine.

(in Birky, 2007)

Although there were mathematically important opportunities to lead the discussion of how doubling three dimension will results the solid equivalent to eight of the original solid with a diagram and reference to the solid, the teacher shifted the focus on arithmetical calculation (procedural knowledge) rather than conceptual understanding (Birky, 2007). According to Skemp and Sfard, procedural knowledge (process conception) is less abstract than conceptual knowledge (object conception). Hence this act can be interpreted as reducing abstraction in this level.

2.2. DuAns: Reducing abstraction by shifting the focus on answer (end-product) even though the problem or discussion implies a focus on concepts, meaning, or understanding

Category 3: Degree of Complexity of Mathematical Concepts

In this category, abstraction level is determined by the degree of complexity. The working assumption here is that “the more complex a problem or concept is the more abstract it is” (Hazzan, 1999). Reducing abstraction in this category involves the following situation.

3.1. Comp_xPG: Reducing abstraction by shifting focus on particular rather than general (thus making the problem less complex for their students. It may however, provide a partial picture of the concept rather than the complete one.)

3.2. Comp_xRO: Reducing abstraction by routinizing the problems (that is, by taking over the challenging aspects of the problems either by telling student how to solve or by solving the problem for students. In so doing, the complexity of the concepts may be reduced, but takes away the opportunities for students from doing mathematics on their own.)

3.3. Comp_xSC: Reducing abstraction by stating the concepts rather than developing it.

3.4. Comp_xGA: Reducing abstraction by giving away the answer in the question or provide more hints than necessary (Topaze effect- See Brousseau, 1987)

The name of the ‘Topaze effect’ comes from a play by Marcel Pagnol written in 1928 in Paris. In the play, Topaze is a school teacher. When the student cannot find the answer easily, the teacher gives away the answer within the question itself in a slightly indirect way thereby lowering the intellectual demandingness of the tasks (cf. Brousseau, 1997).

Example:

- 00:07:32 T What are they? What is meant by equiangular polygon? And what is meant by equilateral polygon?
- 00:07:50 T That means- okay, I- I- I- I- I- give you some time to think. What does it mean by equilat- equilateral tri- polygon?
- 00:07:58 T Equiangular polygon?
- 00:08:00 T Do you still remember it?
- 00:08:02 S? Yes.
- 00:08:03 T Yes. What does it mean? For- for- for the first type. I'll give you some hints. All the sides...?
- 00:08:10 SN Are equal.
- 00:08:10 T All the sides are equal. Okay? All the sides of the polygons are equal.

- 00:08:16 T How about equiangular polygon?
00:08:17 SS(inaudible)
00:08:18 T All the sizes of the angle?
00:08:21 SN Are equal.
00:08:21 T Equal. Get it?

(HK 02)

At 6:40, and at 7:32, teacher seems to open up the dialogue on the concept of concave and convex polygon, and equilateral and equiangular polygon respectively. He however, took over the challenging aspects of the problem by telling them the meaning of concave polygon (6:45) or by asking product questions (8:03, 8:18), rather than allowing students to discover the meaning by themselves. This act of teacher reduced the complexity of the problem for the students but it took away the opportunity for them to progress on their own.

CONCLUSION

One of the challenges all mathematics teachers face in teaching is to deal with mathematical abstraction and find ways to translate abstract (unfamiliar) concepts into understandable ideas. There is however no framework, I am aware of, that specifically looked at how teacher deal with abstraction in teaching. The framework of Reducing Abstraction in Teaching (RAT) is the result of this necessity which I may be helpful to explore the actions of teachers and sources of teaching activities in regard to dealing with mathematical abstraction in teaching. I believe that the framework has “the potential to provide insight into one of the central aspects of learning mathematics and inform instructional practice” (Dreyfus & Gray, 2002, p. 113). I have exemplified various tendencies of reducing abstraction in teaching - in some cases, it seems to be pedagogically effective and in other cases, it may be not be supportive for student mathematical knowledge development. Further research is needed to find the impact of each tendencies of reducing abstraction on students understanding of mathematics.

References

- Birky, G. (2007). *Implementing Eighth Grade Mathematics Problems in Six Countries: A Secondary Analysis of the TIMSS 1999 Video Data*. Unpublished Doctoral Thesis, Department of Curriculum and Instruction, University of Maryland, USA.
- Cornu, B. (1991). Limits. In D. Tall (Ed.), *Advanced Mathematical Thinking*, pp. 153-166. Dordrecht: Kluwer Academic Publishers.
- Dreyfus, T. & Gray, E. (2002) Research Forum 1?Abstraction: Theories about the emergence of knowledge structures. In A.D. Cockburn and E. Nardi (Eds.),

Proceedings of the 26th Conference of International Group for the Psychology,
Norwich, UK: PME.

Ferrari P.L. (2003). Abstraction in mathematics. *Philosophical Transactions of the Royal Society London B* 358, 1225-1230.

Hazzan, O. (1999). Reducing abstraction level when learning abstract algebra concepts. *Educational Studies in Mathematics* 40(1), 71–90.

Hazzan, O. & Zazkis, R. (2005). Reducing abstraction: the case of school mathematics. *Educational Studies in Mathematics* 58, 101–119.

Hershkowitz, R., Schwarz, B.B. and Dreyfus, T. (2001). Abstraction in context: Epistemic Actions. *Journal for Research in Mathematics Education* 32, 195–222.

Hiebert, J., Galimore, R., Garnier, H., Givvin, K. B., Hollingsworth, H., Jacobs, J., Chui, A. M., Wearne, D., Smith, M., Kersting, N., Manaster, A., Tseng, E., Etterbeek, W., Manaster, C., Gonzales, P., & Stigler, J. (2003). *Teaching mathematics in seven countries: Results from the TIMSS 1999 Video Study*. Washington, DC: U.S. Department of Education, National Center for Education Statistics.

Noss, R. & Hoyles, C. (1996). *Windows on Mathematical Meanings: Learning Cultures and Computers*. Dordrecht, The Netherlands: Kluwer Academic Publishers.

Piaget, J. (1970). *Genetic epistemology*. New York: W. W. Norton and Company.

Safuanov, I., (2004). Psychological aspects of genetic approach to teaching mathematics, *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, 153–160. Bergen, Norway: PME.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics* 22, 1–36.

Van Oers, B. (2001). Contextualisation for abstraction. *Cognitive Science Quarterly* 1, 279- 305.

Vygotsky, L.S. (1926/1997). *Pedagogical psychology*. Boca Raton, Florida: CRC Press.

Wilensky, U. (1991). Abstract meditations on the concrete and concrete implications for mathematical education. In I. Harel and S. Papert (Eds.), *Constructionism*, pp. 193-203. Norwood, NJ: Ablex Publishing Corporation.

WORD PROBLEMS IN MATHEMATICS EDUCATION: IMPACT ON STUDENTS' MATHEMATICAL ACHIEVEMENT

Ike Udevi-Aruevoru

Mathematical word problems impact students' mathematical achievement because they require advanced language skills and knowledge of problem context and content. Each of these adds to the degree of difficulty of the problem. This paper investigates how these categories of difficulties may impact students' mathematical achievement when they solve word problems.

INTRODUCTION

The use of mathematical word problems for student assessment is intended to promote the application of mathematical ideas and principles taught in the classroom to everyday problems (Reed, 1999). However, students dread word problems because it impacts their mathematical achievement. Factors like advanced language skills and knowledge of the problem context, while not directly part of the test, add to the degree of difficulty of the word problem as they impact a student's ability to understand and correctly interpret the problem statements. Research on the impact of word problems on a student's mathematical achievement may be classified into three categories: impact of language skills, knowledge of problem context, and knowledge of problem contents.

Bernardo and Calleja (2005), Bernardo (2002), and Renninger, Ewen, and Lasher (2002), all show that some of the difficulties students have in relation to solving word problems are more intensified for foreign students than for students learning mathematics in their native language. Also, Renninger, Ewen, and Lasher (2002) show that even when a student has the language skills and vocabulary to understand the word problem, the arrangement and/or sequencing of the problem elements also add to its degree of difficulty and also impact a student's ability to solve the problem. Another language factor impacting understanding and the solution of word problems is the complexity of the problem statements (Wheeler & McNutt, 1983). Other studies show that the context in which mathematical ideas are tested significantly determines the degree of difficulty perceived by the students and impacts their ability to solve the problem (Kaizer & Shore, 1995). Voyer (2010) reports that as the distance between a student's knowledge/life experience and the problem context increases, the more difficult it is for that student to correctly solve the word problem. The contents of word problems may be divided into two types: surface content and deep content. The surface content of word problems includes problem events and objects used in describing the problem, while the deep content is the mathematical ideas and principles that are useful for solving the problem. Blessing and Ross (1996) show that changing the

surface content of a word problem with respect to its deep content changes its degree of difficulty even with experienced problem solvers. Also, Leon (1994) shows that adding extraneous information to word problems increases their degree of difficulty for many students, especially when the students are not sure of how to solve the problem.

THEORETICAL FRAMEWORK

To correctly translate a mathematical word problem, a student needs to understand and follow the mathematical statements in the text description of the problem. Thus, while a student's language skills are not explicitly part of the test, research results show that the language used for teaching mathematics affects the learning of mathematics for many students, and thus impacts their mathematical achievement. Gooding (2009) argues that student difficulties with mathematical word problems may be grouped into five categories, namely:

1. Reading and understanding the mathematical word problem.
2. Imagining and contextualizing the mathematical word problem.
3. Correctly translating the language description of the word problem into an algebraic equation.
4. Carrying out the required mathematical operations.
5. Correctly interpreting the results.

This study assumes that the student has good reading and comprehension skills in the language of instruction, and thus will focus specifically on only four of the five categories of student difficulties listed above (categories 2 to 5). Where any of these categories of difficulty is identified, this study will further investigate its basis and any process or method that the student employed to overcome it.

METHOD AND DATA COLLECTION

Five mathematical word problems were selected for this study (see Appendix 1). The questions were selected with the intent of investigating students' attitudes when solving mathematical word problems that are cast in familiar and unfamiliar contexts, the types of difficulties they encounter when the problem context is unfamiliar to them, and what steps they take to overcome these difficulties.

Question 1 was chosen because it has a universal and thus very familiar context and, like the handshake problem in Buerk (1982), every student is expected to have access to this problem. A student may not get the right answer to this question, but he/she can always take a stab at it. Also, the text narrative is substantial and may act as a distraction to some students, possibly affecting the method the student chooses for solving this problem.

Question 2 requires the student to make some assumptions and assume facts not explicitly stated in order to solve the problem. As a result, this question is expected to reveal students' difficulties with the deep content of word problems and how they deal with them.

Question 3 is another universal and familiar real-world problem. It is expected that almost every grade eleven or twelve student would relate to this problem. Also, its text narrative is substantial, like that of Question 1.

Question 4 has a small text description and its solution requires that the word problem be reduced to a pair of simultaneous linear equations. The problem is, however, embedded in a chemistry context and many students, especially those not very comfortable with chemistry or science, may find this problem very confusing, and perhaps be intimidated by it.

Question 5 is a good algebra question. It requires good thinking and good algebraic skills to translate the text description into the correct algebraic equation.

RESULTS

This study was conducted with a Grade 12 student, here referred to as IV. She was presented with five word problems to solve (see Appendix 1). Also, she was asked to rank the five questions in terms of her perception of their degree of difficulty using a scale of 1 to 5 (1 = easiest; 5 = most difficult). In addition to ranking the questions, she was also asked to make comments, observations, statements, etc. about each question, including why she sees a question as easy or difficult, and any other thoughts she deemed useful when thinking about each problem. The table below shows her ranking of the questions, her performance for each question (marked as correct, incorrect, or incomplete) and finally, her comments on each question. (Her comments were hand-written comments on paper, reproduced verbatim below.)

Question#	Difficulty	Answer	IV's written comments on this question
1	2	Correct	This question was difficult at first but after I made a chart and was able to organize all the info given in the question, it was easy to solve. I had to reread the question several times because the information was confusing at times.
2	4	Incomplete	I didn't really solve this question, and I don't know if it was a trick question or not. I found it a little bit confusing because the wordings and scenarios for each person were similar yet different. I tried to visualize the problem – with no luck.
3	1	Correct	This question was easiest to solve because the question gave a lot of information, and was easy to solve.
4	5	Incomplete	This question was the hardest. I didn't understand the question; the wording was kind of confusing. I was not able to solve it.
5	3	Correct	This question was pretty easy. At first it seems daunting but when I organized the info given, I was able to solve it by guessing and checking. I did this at first but then I noticed that with the numbers I used, a pattern started to form and when I finally used 7 it worked.

IV had identified Question 4 as the hardest of the five questions. She did not attempt it, saying that she did not know how to begin answering the question. The next day, I asked her to solve the pair of simultaneous equations (below) as a 6th question. I also asked her to rate its degree of difficulty in relation to Questions 1-5 that she had attempted the previous day.

Question 6:

$$x + y = 10$$

$$0.3x + 0.5y = 4.2$$

She solved the simultaneous equations above in less than two minutes, and rated Question 6 as the easiest of all the six questions.

DISCUSSION

IV identified Questions 1 and 3 as the easiest questions because of her familiarity with the problem context. In both cases, she did not seem to care about the length of the text narrative of either of the problem statements. Her familiarity with the problem context allowed her to focus on the important problem elements. Her statements during the interview, “I had all the information that I needed”, followed by “all I had to do”, sums up her attitude and thinking about Question 3. In fact,

her use of the phrase “all I had to do”, with respect to this question, almost trivializes the question and makes the case for the effect of familiarity with the problem elements. She not only understood the problem, but she was also able to describe her solution strategy and interpret her results. She did not have any difficulty with this question.

IV rated Question 5 as the third most difficult question. Her various attempts to translate the problem into an algebraic equation failed (Category 3 difficulty) as can be seen from the following transcripts from her written work (see Figure 1):

Q5:	
20 + than son	$28 = 5 + 2x$
8 yrs $5 + 2x$	$23 = 2x$
$20 + x = y$	$11.5 = x$
$5 + 2x = y$	$31.5 = \text{father}$
	$+ 8 = 39.5$

Figure 1

However her familiarity and understanding of the problem elements allowed her to “reason out” the solution, as she put it. Below is a transcript of how she did it (see Figure 2).

Q5 continued:	
$6 + 8 = 14 \times 2 = 28$	$26 + 8 = 34$
$7 + 8 = 15 \times 2 = 30$	$27 + 8 = 35$
son = 7	father = 27
+ 8 yrs = 15	+ 8 yrs = 35
$\times 2 = 30$	

Figure 2

IV’s struggle with this question and her final solution demonstrate the impact of familiarity and knowledge of the problem elements and context in solving mathematical word problems. Here, her real-world knowledge of this problem allowed her more than one way of thinking about this problem, and thus more than one solution option. She was unable to correctly translate the word problem into the correct algebraic equation; however, her knowledge of the problem context and

familiarity with the mathematical ideas allowed her to improvise a solution. It is through this multiplicity of solution options, when the school taught algorithm is not accessible, that the context of a mathematical word problem impacts a student's mathematical achievement.

IV rated Question 2 as the fourth most difficult question. Her attempt to solve it failed because she did not make any assumptions about the length of the course covered by both Anna and Maria, which was not given (Category 3 difficulty). Her difficulty here is a deep content problem, but it is masked by a surface content problem. Because of her lack of the mathematical knowledge required to solve this problem (deep content problem), she was unable to identify the missing information that would be necessary for solving the problem (surface content problem), namely the length of the course.

The transformation of Question 4, from a word problem embedded in a chemistry context to a pair of simultaneous linear equations (Question 6), transformed this question for IV from the most difficult to the easiest of the six questions. She was unable to solve this problem, insisting that she had no idea of how to begin to attempt it (Question 4), saying that she does not know how to "make 42% with 30% and 50%". However, following the transformation, she solved Question 6 easily and rated it as the easiest of the six questions. When I asked her to compare Questions 4 and 6, she stared at both questions for awhile and then shouted, "They are the same!", as she immediately recognized all the problem elements that hitherto she had failed to recognize, probably because they were masked by the problem context. This is a Category 2 difficulty and demonstrates again the effect of problem context in solving word problems.

The results of this study suggest that some of the widespread difficulties many students experience with mathematical word problems may not necessarily be because the word problems are intrinsically difficult. Some of these difficulties may be because many students are unable to make any connection between the text descriptions of the word problems with the mathematics they know. It may be for this reason that mathematical word problems remain a source of anxiety and consternation for many students.

CONCLUSION

Three of the five categories of difficulties identified by Gooding (2009) were observed in this study. The first was difficulty with translating the word problem into an algebraic equation. IV overcame this difficulty through her familiarity with the problem context. The second difficulty observed was the inability to carry out the required mathematical operations (deep content problem masked by surface content problems). IV was unable to overcome this difficulty. The third difficulty

observed was that of imagining and contextualizing the mathematical problem. This is a problem-context problem; she was unable to overcome this difficulty, but when the same problem was cast in a familiar context (as a pair of simultaneous equations), she was able to solve the problem.

This study confirms the results of other studies that show when students have a good understanding of a mathematical word problem, but cannot recall the school taught algorithm for solving it, they usually attempt to solve the problem using their everyday knowledge and life experience. This is not necessarily a bad thing. In fact, I would argue that it is a good thing because it also shows that with good pedagogy, school taught algorithms may become the algorithm of choice. The challenge for mathematics educators is to teach these algorithms with such relevance and anchors that they will be remembered when needed. This study also shows that some of the categories of difficulties identified by Gooding (2009), namely difficulty with translating the word problem into an algebraic equation and difficulty with deep mathematical content, may sometimes be masked by difficulty with the problem context and also difficulty with the surface content. Thus the difficulty with the deep problem content might not be investigated in isolation from the knowledge of problem context and surface problem content.

APPENDIX 1

Question 1: Dylan is meeting his sister and four of her women friends for lunch. The five women are called Alicia, Rachel, Lani, Donna, and Casey. Three of the women are under 30 years old and two are over 30. Two of the women are lawyers and three are doctors. Alicia and Lani are in the same age group. Donna and Casey are in different age groups. Rachel and Casey have the same profession. Lani and Donna have different professions. Dylan's sister is a lawyer and is over 30. Who is Dylan's sister?

Question 2: Anna and Maria enter a race. Anna walks half the time and jogs half the time. Maria walks half the course and jogs half the course. If both girls walk and jog at the same rate, then which girl will complete the course first?

Question 3: Rajan, Eric, and Lucy are all sales representatives, but they are paid in different ways. Rajan is paid a straight commission of 5% of his total sales. Eric is paid a base salary of \$250 a week, plus 2% commission on sales above his sales quota of \$5000. Lucy is paid a graduated commission of 2.5% on sales up to \$3000, plus 6% on sales in excess of \$3000. If each person's sales for one week are \$12 000, what is each person's gross income?

Question 4: A chemistry teacher needs to make 10 L of 42%-sulphuric acid solution. The acid solutions available are 30%-sulphuric acid and 50%-sulphuric

acid, by volume. How many litres of each solution must be mixed to make the 42% solution?

Question 5: A father is now 20 years older than his son. In 8 years, the father's age will be 5 years more than twice the son's age. Find their present ages.

Feedback: Please rate the above questions from 1 to 5 in order of difficulty to you, with 1 being the easiest and 5 the most difficult, and give reasons for your ordering. Any comments, statements, etc. that you can think of would be very useful.

References

- Bernardo, A. B. I. (2002). Language and mathematical problem solving among bilinguals. *The Journal of Psychology*, 136(3), 283-297.
- Bernardo, A. B. I., & Calleja, M. O. (2005). The effects of stating problems in bilingual students' first and second languages on solving mathematical word problems. *The Journal of Genetic Psychology*, 166(1), 117-128.
- Blessing, S. B., & Ross, B., H. (1996). Content effects in problem categorization and problem solving. *Journal of Experimental Psychology: Learning, Memory, and Cognition*, 22(3), 792-810.
- Buerk, D. (1982). An experience with some able women who avoid mathematics. *For the Learning of Mathematics*, 3(2), 19-24.
- Gooding, S. (2009). Children's difficulties with mathematical word problems. *Proceedings of the British Society for Research into Learning Mathematics*, 29(3), 31-36.
- Kaizer, C., & Shore, B. M. (1995). Strategy flexibility in more and less competent students on mathematical word problems. *Creativity Research Journal*, 8(1), 77-82.
- Leon, R. E. (1994). The effects of the presence of extraneous information in mathematical word problems on the performance of Hispanic learning disabled students. *New York State Association for Bilingual Education*, 9, 15-26.
- Reed, S. K. (1999). *Word problems, research and curriculum reform*. Mahwah, NJ: Lawrence Erlbaum Associates.
- Renninger, K. A., Ewen, L., & Lasher, A. K. (2002). Individual interest as context in expository text and mathematical word problems. *Learning and Instruction*, 12, 467-491.
- Voyer, D. (2010). Performance in mathematical problem solving as a function of comprehension and arithmetic skills. *International Journal of Science and Mathematics Education*, 9(5), 1073-1092.

Wheeler, L. J., & McNutt, G. (1983). The effect of syntax on low-achieving students' ability to solve mathematical word problems. *The Journal of Special Education, 17*(3), 309-315.

USING A CONVERSATIONAL APPROACH: CAN THIS INFORM A TEACHER ABOUT STUDENTS' 'UNDERSTANDING'?

Kevin Wells

In the classroom, students may engage in several casual conversations with their teacher or peers regarding mathematics. Typically, these conversations are informal and may not be consciously used by the teacher as a means of formative assessment. This paper investigates the possibility of analysing the structure of casual classroom conversations to question if a student's ability to hold a conversation reflects on their conceptual understanding of the topic in hand.

INTRODUCTION

In the initial phases of learning, a student's knowledge may support only partial understanding. Misconceptions that arise in earlier grades can be strongly held and hard to change, but can play an important role in the development of students' conceptual understanding (Smith, diSessa, & Roschelle, 1993). Tapping into these partial understandings and misconceptions would seem to be an important starting point for educators in determining a course of instruction. Diagnostic tools are available that purport to allow the teacher to gain some insight into the students' knowledge, but these are often in the form of multiple choice or short-answer questions, which may not be sufficient to give a clear picture of the students' thinking or understanding. Reform-based mathematics teaching encourages the development of questioning and eliciting student responses. A number of research studies focus on supporting this process by illustrating how a teacher can develop better questioning techniques to expose students' thinking (e.g. Herbel-Eisenmann & Breyfogle, 2005). In addition, as an alternate way of probing student understanding, classroom discourses have also been used (e.g. Williams & Baxter, 1996). In this paper, I examine the notion of paying more attention to the informal conversations held by students while discussing an idea. In particular, by examining the structure of the conversation, I examine the ability of a student to hold a conversation and ask if this reflects on their understanding of the concepts discussed.

FRAMEWORK

Rorty (1979) viewed conversation as the ultimate context within which knowledge is to be understood, while a basis of social constructivism is that conversation is central to learning (Ernest, 1998). In a classroom where group work and collaborative problem solving is a focus, conversation between students is central to the process. Such conversations could be extended beyond problem solving to

discussion of the concepts and ideas. Paying attention to these conversations may be a productive way to gain fresh insight into a student's thinking. While these conversations have been studied by researchers (e.g. Walshaw & Anthony, 2008), the goal has not been about individual assessment of particular concepts or the students' ability to participate in the conversation.

There are, however, inherent difficulties when working with discourse which need to be considered. Morgan (1998), in her discussion of the "Myth of Transparency" (p. 197), highlights a fundamental problem with assessment that focuses on students' writing. At issue is the assumption that the thinking of a student is accurately reflected by what they write or say. This 'naïve transmission' view of communication is demonstrated as flawed when Morgan notes that different teachers interpret the same piece of student writing in different ways. Further, the notion of 'understanding' is given an objective reality during this form of assessment, even by teachers who may reject this positivist paradigm in their teaching. A postmodernist viewpoint suggests that there is an illusion of meaning in language and what we refer to as 'truth'. The teacher's own understandings of the mathematics, as well as the role the teacher adopts in assessment, mean that there is no simple correspondence between student intent and teacher interpretation. As such, paying attention to the content of a student's utterances is subject to interpretation which, as Morgan further points out, may also be a reflection of the student's poor understanding of the teacher's expectations. Indeed, Morgan suggests that teachers themselves do not have an adequate explicit vocabulary to provide guidance to students in producing better texts. Analysing the casual conversation of students about a mathematical topic may be a way to help bridge this gap.

Ernest (1998) develops the premise that mathematics is "inescapably conversational" (p. 169). This view suggests that conversation is not just a tool for outcomes; rather, language shapes and constrains our experiences as much (or more so) than it reflects them. In diagramming a model of the social construction of mathematical knowledge, Ernest places conversation at the centre of a cyclic process through which collective mathematical knowledge and personal knowledge of mathematics recreate each other. Leaning on the work of Wittgenstein, Ernest stresses that personal learning of mathematics is acquired through socially situated conversations. Teachers structure conversations in the classroom based on their own understanding but, as Ernest points out, "sustained two-way participation in such conversations is necessary to generate...mathematical knowledge and competencies, and not some partial or distorted version" (p. 221). If conversations form what we perceive to be a student's mathematical knowledge, then paying attention to the way a student

participates in conversations may help inform the teacher about the structure of students' understanding. However, the same concerns must be raised with the conversational approach as with the written text, namely the teacher's ability to interpret the conversation and the students' expectations of their role in classroom conversations.

First, however, if we are interested in using conversation to shed light on understanding, there is the question of what 'understanding' is. Wittgenstein considers that when someone appears to do something correctly then they do it "as we would do it" (Wittgenstein, 1967, p. 145). This places 'understanding' as a relative term, as not everyone will agree on the 'correctness'. Wittgenstein earlier notes that the possibility of getting a student to 'understand' will depend on the student going on to write it down independently, while getting someone to understand requires changing their way of looking at things. If the student is capable of imagining what is being taught, then Wittgenstein considers that there is "a capacity to learn" (p. 144). Importantly, understanding is considered a 'source' of correct usage, not an application. As a result, understanding must be more than knowledge of a formula; there must also be an accomplishment, or manifestation, of that understanding. Wittgenstein suggests not thinking of understanding as a 'mental process' at all, but as the set of circumstances in which a student is able to 'go on'. Mental processes are characteristic of understanding, but understanding is not a mental process. If we relate this to a conversation, then understanding could manifest itself in terms of the ability of a student to carry on with the particular exchange to its conclusion. In addition, if a student demonstrates the ability to continue with an exchange, it may suggest they are ready to build on their understanding. Notwithstanding a student's willingness to participate in a conversation, Halliday (2009) observes that:

For communication to take place at all it is necessary for those who are interacting to be able to make intelligent and informed guesses about what kinds of meanings are likely to be exchanged. They do this on the basis of their interpretation of the meanings. (p. 2)

Pask (1975), in taking a phenomenological stance while researching cognition, builds a more formal structure. He considers that if a student is able to explain a topic, then this is evidence for a concept. If the explanation is agreed upon by the interlocutor, then that is evidence for a concept equivalent to (not necessarily identical with) a concept entertained by the other person. If the person can further explain how the concept is constructed, and if the explanation is agreed upon by the interlocutor, then that is evidence for an equivalent (not necessarily identical) memory. This condition is called 'understanding'. This is in keeping with Wittgenstein's notion in that understanding is relative between interlocutors—a

sense of shared memory. This understanding occurs in conversation, but Pask (1975) notes an external observer can change the characteristics of the conversational environment. Understanding may now not be shared by the participants and the observer.

Modern mathematics teaching characterises learning as a “generative process of meaning-making that is personally constructed” (Pimm, 1993). It would seem, therefore, important to foster dialogue as a means of both generating and observing understanding. A number of researchers, including Pimm (1987) and Rowland (2000) have built on a linguistic framework developed by Halliday (1973, 2009). Halliday develops the notion of ‘acts of meaning’, communicative acts that are intentional and symbolic. Acts of meaning are social acts that are only meaningful when another party joins in to give them value (a conversational process). Wooffitt (2005), for example, suggests it is a mistake to think of language as a means of representation when it should be regarded as a medium of social action.

Conversational Analysis, developed by Sacks (1992), takes a more detailed look at the structure of a conversation. Sacks realized that, far from being just a means to pass on thoughts, conversation has a social structure. This is seen in conversational turn-taking where responses between interlocutors often occur in pairs and where there is an expectation of a certain response. Subsequent studies by Sacks, Schegloff and Jefferson (1974) indicate that all aspects of speech are significant. Even minor or irrelevant aspects such as ‘umm’ are important as a way of indicating ongoing thought. The onset of simultaneous speech, overlapping, and the timing of gaps between turns, also indicate important information. A feature of conversation is that participants understand and respond to one another in their turns at talk. Phrases which are an indication of conversation are adjacency pairs, in which a second utterance of an exchange is functionally dependent on the first. A conversation is also indicated when the participants attempt to engage one another in an informative and relevant way. An aspect of this is seen in the cooperative overlap in which one speaker talks at the same time as another. Intonation also plays a role in a conversation as it expresses both textual and interpersonal meanings. Intonation and rhythm, especially the pitch contour of speech, figure prominently in the information system (Halliday, 2009). Tone sequences create textual structures for the interlocutor. For example, a non-falling tone creates an expectation that some further information is coming. A level tone can be informational, while fall-rise seems determinate but is not, and rise-fall seems not determinate but is.

METHODOLOGY

As a test case I conducted research on a class of grade 11 ($n = 30$) students during a routine lesson for which I was the teacher. The students self-selected themselves into groups of three and were asked to record a discussion based around questions related to curriculum content from grade 10. One student acted as camera operator while the other two discussed prompts. The students could not use any reference source. The camera operator could also join in the conversation. After the students felt they had exhausted their ideas on one question they changed the camera operator and moved onto the next question. Specifically the students were asked: 1) What do you think a function is? 2) What does a function do? 3) What is meant by the notation $f(x)$?

If the students in this study possess a working concept image of the function it may not be coherent at this introductory stage (Tall & Vinner, 1981). The students may simply provide a concept definition, what Tall and Vinner refer to as “a form of words used to specify that concept” (p. 152), but the purpose of the prompts is to evoke their concept image. The videos were analysed first for the spoken word, and then viewed to code the rise and fall of the voices.

ANALYSIS OF A SAMPLE OF DATA

Note that: (.) represents a hesitation, (..) represents a longer hesitation, (n) represents a pause of n seconds, and underlined text represents students talking over each other.

Transcript: (Jane and Jill are sitting facing each other. Mike operates the camera.)

- 1 Jane: What is a function Jill?
- 2 Jill: I don't know, you tell me Jane.
- 3 Jane: Umm, I think (..) you can write it in $y = mx + b$ format.
- 4 Mike: Yes, like in the equation for a graph, right?
- 5 Jane: Yeah. Oh, another one, you put a number in and it comes out with a different number.
- 6 Jill: Yeah, and you have to do (unintelligible)
- 7 Mike: You have to leave what?
- 8 Jill: Never mind.
- 9 Mike: No, it's okay, tell us what you're thinking...
- 10 Jane: Like the input is the x and the outcome, the output, is the y, right?
- 11 Mike: Yeah, so you have f of x and then you have your little equation, right?
- 12 Jill: And you know that f of x, you have to leave that at the ending, you have to leave that alone, don't you? With the number it equals... does that make sense at all?
- 13 Mike: Like you have f of x right...

- 14 Jill: equals and then ...
- 15 Jane: ..oh how you write it..
- 16 Jill: yeah, you do that.
- 17 Mike: So, like, but in the end what f of x equals is the same as what y would?
- 18 -21 *The students agree*
- 22 Jane: Oh! I know that in the function m is the slope and b, b is the y-intercept.
- 23 Mike: Yeah, just like in, umm, $y = mx + b$ right?..
- 24 Jane: Yeah.
- 25 Jill: What are you talking about?
- 26 Mike: What are the other questions?
- 27 Jill: What does a function do, can you describe it in another way?
- 29 Jane: Yeah, $y = mx + b$.
- 30 Mike: And you have the standard form right? Like you have $Ax + By$ right?
- 33 Jane: It's like a special box, you put something in and something comes out.
- 34 Mike: Yeah. Didn't we talk at the beginning of the year it's like a machine, like you put in a 2 and a 4 comes out?
- 35 Jane: Yeah. So I think that's about it.
- 36 Jill: It's all we know.
- 37 Mike: Isn't, isn't it like, the fu.. it's like.
- 38 Jane: There are many different forms of a function (.) like the linear one (..)and the V one and the weird ones (.)
- 39 Mike: yeah and there's the restrictions, like there can only be one x value...
- 40 Jane: oh yeah
- 41 Mike: Like you can only use one x value once
- 43 Mike: Like you can't stack the points or it's not a proper function?
- 45 Jane: Oh if you put a number in (.) but you can't have all different (.) like (.)outcomes.
- 46 *The students agree and the clip ends. (Note: missing lines were edited for space)*

Jane and Mike participate in an exchange that has clear indicators of a conversation as there are many adjacency pairs and overlapping interruptions. The students are responding to each other in a way that suggests they share a level of understanding. It is immediately clear that Jane is confident in her knowledge about the function. She faces Jill and gestures towards her. Jane's expression is open and smiling but her language is firm. She begins with a personal pronoun (line 3) to establish agency and uses 'you' to generalize the concept. She may also be using a deictic gesture in line 3 indicating that she views her interpretation of a function as an object, although this gesture may also be interpreted as Jane emphasising that Jill could perform the action. It is noticeable that Jane is the only one of the three to use 'I' (line 3). She then uses a modal auxiliary verb 'can'. Her tone is rise-fall

indicating that she is confident in her words while not wanting to assert them. She is being conversational. In line 5 she uses the interjection 'oh', in direct access to attract attention. She also refers to the function as 'it', indicating that she thinks of the function as an object. In line 10 she finishes with 'right?', used as a verb and hedged performative with the likely purpose of ensuring the others are with her train of thought. Jane's gesturing in line 38 is more deictic than beat oriented, indicating a reference to a mental object she has formed.

Mike, whose tone carries equal confidence, is more inclusive in his talk than Jane. He uses 'we' in line 34 and, with rising tone, uses 'right?' frequently, but in his case it seems more likely he is checking the others are understanding him and are included. He commonly uses a general form of 'you' to indicate the way a function would be used by other people in a wider mathematical community. Mike is off camera so there is no indication of his use of gestures but he is clearly engaging with Jane in a conversation and also attempting to draw Jill into the conversation more (line 9).

Jill offers contributions to the conversation, as in line 9, but does so tentatively, either withdrawing the remark immediately or seeking confirmation, as in line 12. In addition, her tone in line 12 is fall-rise which Halliday (2009) considers to seem determinate but is not. She relies on Jane and Mike to complete her thoughts (lines 14-18). As Jane and Mike engage in a deeper conceptual conversation, Jill's attention drifts away. When Mike's comment (line 44) returns closer to her image of a function as a line on a graph, it renews her interest and she questions with a rising tone. From this brief exchange it can be suggested that Jill does not share the same concept image that Jane and Mike do.

SUMMARY

The sample case illustrates the potential for using a conversational approach to examine the ability of students to participate in a discourse. Two of the three students appear to share an understanding by agreeing on their explanations. The third participant is either unwilling or unable to participate on an equal footing with her peers. While this may be due to a less developed concept, causing her to be unable to produce an explanation for her peers, caution must be made in making this assumption. Alternative explanations may explain this lack of participation; she may feel socially intimidated, or simply disinterested in participating. Analysis of further conversations, or an alternative approach to investigating the student's understanding should then be used with this student. The sample also illustrates the potential to use a toolkit of conversation and discourse analysis tools to study the form and meaning of the exchange. In further analysis of the conversations between other students in the class, degrees of participation in conversation could

be similarly identified. Participation ranged from non-existent to very involved and animated. Correlating the students' performance on a post-test based on functions indicated a match between performance and the ability to converse on the topic in most cases. Of further interest was the performance of Jill in formal testing, who was successful in questions of a procedural nature, but less so in questions of a conceptual nature. A further case of interest involved a student who was able to converse well but whose language was vague. In the post-test the student did not perform well on the procedural style of questions but was able to start questions of a conceptual nature well before losing his way in the mechanics of the question. Any conclusions based on this study are premature, but it suggests a potential area in which to delve further.

A suggestion to develop this process further would be to couple this process with gesture analysis to give the potential to examine a further dimension of the discourse. Studies indicate that gesturing is used to lighten the cognitive load while thinking (e.g. Goldin-Meadow, 2003), not only allowing the speaker to use more resources to access memory but also playing a role in shaping the speaker's cognitive state. Goldin-Meadow suggests that by not gesturing, the student is not using their full cognitive capacity and does not perform as well as when they do. If, as Goldin-Meadow suggests, "gesture and speech form an integrated...and synergistic system" (p. 521) and that "gesture has the potential to display thoughts that are not conveyed in the speech", then a consideration of how conversation and gesture interlink might prove to be an interesting further study in this area.

References

- Ernest, P. (1998). *Social constructivism as a philosophy of mathematics*. New York: State University of New York Press.
- Goldin-Meadow, S. (2003). *Hearing gesture*. Cambridge, MA: The Belnap Press of Harvard University Press.
- Halliday, M. A. K. (1973). *Explorations in the functions of language*. London: Edward Arnold.
- Halliday, M. A. K. (2009). *The essential Halliday*. New York: Continuum.
- Herbel-Eisenmann, B., & Breyfogle, L. (2005). Questioning our patterns of questioning. *Mathematics Teaching in the Middle School*, 10(9), 484-489.
- Morgan, C. (1998). *Writing mathematically: The discourse of investigation*. London: Routledge Falmer.
- Pask, G. (1975). *Conversation, cognition and learning*. New York: Elsevier.
- Pimm, D. (1987). *Speaking mathematically: Communication in mathematics classrooms*. London: Kegan Paul.

- Pimm, D. (1993). Just a matter of definition. *Educational Studies in Mathematics*, 25(3), 261-277.
- Rorty, R. (1979). *Philosophy and the mirror of nature*. Princeton, NJ: Princeton University Press.
- Rowland, T. (2000). *The pragmatics of mathematics education: Vagueness in mathematical discourse*. London: Falmer Press.
- Sacks, H. (1992). *Lectures on conversation* (Edited by G. Jefferson). Oxford: Basil Blackwells.
- Sacks, H., Schegloff, E. A., & Jefferson, G. (1974). A simplest systematics for the organisation of turn-taking for conversation. *Language*, 50, 696-735.
- Smith, J. P., diSessa, A. A., & Roschelle, J. (1993). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *Journal of the Learning Sciences*, 3(2), 115-163.
- Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, 12(2), 151-169.
- Walshaw, M., & Anthony, G. (2008). The teacher's role in classroom discourse: A review of recent research into mathematics classrooms. *Review of Educational Research*, 78(3), 516-551.
- Williams, S. R., & Baxter, J. A. (1996). Dilemmas of discourse-oriented teaching in one middle-school mathematics classroom. *The Elementary School Journal*, 97(1), 21-38.
- Wittgenstein, L. (1967). *Philosophical investigations*. Oxford, UK: Basil Blackwell.
- Wooffitt, R. (2005). *Conversation analysis and discourse analysis*. London: Sage.