Faculty of Education
Simon Fraser University

MEDS-C 2010 Proceedings

PROCEEDINGS OF THE 5TH ANNUAL MATHEMATICS EDUCATION DOCTORAL STUDENTS CONFERENCE

SEPTEMBER 25, 2010

8:40 – 8:55	**Welcome and coffee**	
9:00 – 9:40	**Shiva Gol Tabaghi** Shifts of attention in DGS to learn eigen theory	**Susan Oesterle** An activity theory perspective on teaching math for teachers
9:45 – 10:25	**Elena Halmaghi** Undergraduate students' conceptions of inequalities: the missed-before	**Melania Alvarez** How a community of practice is created out of an assessment project
10:30 – 10:40	**Break**	
10:45 – 11:25	**Christian J. Bernèche** Teachers and resources: How might one track their interactions?	**Simin Chavoshi Jolfaee** On exemplification of probability zero events
11:30 – 12:30	**Plenary Speaker - Robin Barrow with Q & A**	
12:30 – 2:00	**Lunch: Himalayan Peak Restaurant**	
2:05 – 2:45	**Olga Shipulina** Calculus in navigation/ bodily calculus	**O. Arda Cimen** How do children multiply: commuted pairs
2:50 – 3:30	**George Ekol** Operations with negative integers in a dynamic geometry environment	**Krishna Subedi** Dealing with mathematical abstraction in teaching
3:30 – 3:40	**Break**	
3:45 – 4:15	**Kevin Wells** The pragmatics of mathematical dialogue through email	**Veda Abu-Bakare** Making the familiar strange: An analysis of language in postsecondary calculus textbooks then and now
4:20 – 5:00	**Paulino Preciado** Different roles in teachers' collaborative design of mathematics teaching artefacts	**Sean Chorney** Material agency: Questioning its mediational significance in mathematics learning

Plenary Session:

SOME FUNDAMENTAL QUESTIONS AND EGREGIOUS ERRORS IN EDUCATIONAL THOUGHT

Robin Barrow
Simon Fraser University

Abstract

This paper argues for the crucial importance of knowing what we are trying to achieve in education, and hence in pursuing the questions of what it is to be human and well-educated. The widespread contemporary fascination with relativism and generic skills is criticized, as is our unwillingness to discriminate in education. In conclusion a knowledge-based curriculum is resolutely defended.

Research Reports:

MAKING THE FAMILIAR STRANGE: AN ANALYSIS OF LANGUAGE IN POSTSECONDARY CALCULUS TEXTBOOKS THEN AND NOW

Veda Abu-Bakare

Simon Fraser University

Three calculus textbooks covering a span of about 40 years were examined to determine whether and how the language used has changed given the reform movement and the impetus to make mathematics accessible to all. Placed in a discourse analytic framework using Halliday's (1978) theory of functional components –ideational, interpersonal and textual, and using the exposition of the concept of a function as a unit of comparison, the study showed that language is an integral indicator of the author's view of mathematics and an important factor for textbook adoption in the pursuit of student success.

HOW A COMMUNITY OF PRACTICE IS CREATED OUT OF AN ASSESSMENT PROJECT

Melania Alvarez

Simon Fraser University

An assessment to look into the learning of mathematics among third grade classes throughout the year is being developed by a group of teachers and a team of assessment researchers. The goal of this group is to create an assessment tool that is summative and formative where teachers are able to track throughout the year the mathematical development of their students. Throughout the process of creating this tool a community of learning was created among the teachers. This paper shows how this community is created among teachers by a desire to acquire a better mathematical understanding for themselves and of their students' knowledge and the need for communicating with other teachers about their practice.

TEACHERS AND RESOURCES: HOW MIGHT ONE TRACK THEIR INTERACTIONS?

Christian Berneche

Simon Fraser University

This paper aims to identify and clarify some interactions between teachers and the resources they use while teaching or while planning their teaching. A list of possible conceptions of resources is discussed as well as a model to study how teachers pull material together to generate resources via a documentation system. The goal is to draw attention to likely artefacts and re-sourcing strategies that could illustrate teacher capacity.

ON EXEMPLIFICATION OF PROBABILITY ZERO EVENTS

Simin Chavoshi Jolfaee

Simon Fraser University

In this paper the example space of pre-service secondary teachers on probability zero events is examined and different aspects of such events as perceived by the respondents are discussed. Meanwhile the participants' understanding of "more complicated" is explored.

MATERIAL AGENCY: QUESTIONING ITS MEDIATIONAL SIGNIFICANCE IN MATHEMATICS LEARNING

Sean Chorney
Simon Fraser University

Tools in the mathematics classroom are often not given the credence or the attention they warrant. Considering Vygotsky's view of mediation, tools may play a larger role in mathematics than originally thought. This study presents a framework for looking at tools in student mathematical learning. Using Pickering's analytic framework (1995) distinguishing individual, disciplinary and material agencies, I analyze two students in grade 12 and their interactions with a dynamic geometric software, specifically Geometer's Sketchpad. In the process of solving a problem I will analyze the students' engagement with the tool in terms of the different types of agencies, based on their spoken words and their actions in using the program.

HOW DO CHILDREN MULTIPLY: COMMUTED PAIRS

Arda Cimen
Simon Fraser University

Multiplication is one of the most important abilities gained through school life. Because more advanced topics in the curriculum depend on previously gained arithmetical abilities, teaching of multiplication is crucial. Some recently discussed methods for the teaching of multiplication and multiplication table are claimed to be more efficient, more easily learned and applied faster by students. This study includes interview transcriptions of eight 5th grade students and summarizes different techniques they use for multiplication in terms of efficiency, accuracy and responsiveness.

OPERATIONS WITH NEGATIVE INTEGERS IN A DYNAMIC GEOMETRY ENVIRONMENT

George Ekol
Simon Fraser University

We review difficulties elementary students (and teachers) face with the concepts of, and operations with negative numbers and zero (Davidson1992; Streefland 1996; Lincheveski & Williams, 1999).We then examine how the dynamic geometry environment (DGE), through use of the Geometer's Sketchpad software contributes to the understanding of the integer operations in general, but negative

numbers, and zero in particular. We extend Davidson's (1992) object oriented framework, and Lincheveski & Williams'(1999) semiotic activity framework in a dynamic geometry environment and hypothesize that use of dynamic learning activities changes the way integers, in particular, negative numbers are perceived, and that a dynamic learning environment contributes to an action oriented thinking.

SHIFTS OF ATTENTION IN DGE TO LEARN EIGEN THEORY

Shiva Gol Tabaghi
Simon Fraser University

Dynamic geometry software has been shown to facilitate students' construction of their own mathematical objects in the linear algebra context (Sierpinska, Dreyfus, and Hillel, 1999). However, its use in the teaching and learning of university level mathematics has received less attention. This study offers a refined look at the development of explicit awarenesses of the concepts of eigenvectors and eigenvalues through the use of dynamic (and not just geometric-visual) 2d representations of the concepts. Mason's theory of awareness is used as a theoretical framework to analyze participants' developmental process of mathematical understanding.

UNDERGRADUATE STUDENTS' CONCEPTIONS OF INEQUALITIES: THE MISSED-BEFORE

Elena Halmaghi
Simon Fraser University

This report comes from a broader study that investigates undergraduate students' conceptions of inequalities. It presents a discussion regarding undergraduate conceptions of inequalities through the theoretical framework of the Three Worlds of Mathematics. The CONCEPTIONS OF INEQUALITIES, as they emerged from the data, are first introduced and a background on the Three Worlds of Mathematics is given. The CONCEPTIONS OF INEQUALITIES are then projected on the Three Worlds of Mathematics. The projection reveals that the conceptions of inequalities occupy lower or improper levels on the Three Worlds of Mathematics. The speculation is that students have plenty of missed-before experiences that prevent their understanding and manipulating inequalities at the expected level.

AN ACTIVITY THEORY PERSPECTIVE ON TEACHING MATH FOR TEACHERS

Susan Oesterle
Simon Fraser University

This theoretical paper considers how Engström's (1999) model for activity systems could be applied in the analysis of the activity of preparing prospective teachers of mathematics. It suggests that consideration of this activity as a 'nested activity system' opens possibilities for gaining a better understanding of the challenges faced in this endeavour. The approach is illustrated via application to data gathered in an interview with an instructor of a preservice mathematics content course

DIFFERENT ROLES IN TEACHERS' COLLABORATIVE DESIGN OF MATHEMATICS TEACHING ARTIFACTS

A. Paulino Preciado
Simon Fraser University

The collaboration among teachers and educators in the design of mathematical teaching instruments—such as lessons, activities, or assessment instruments—has been widely used as a means for both mathematical learning improvement and teacher professional development. This study focuses on the interaction among participants in this type of collaborative design, in particular the roles they play during the designing of such artefacts. The data were obtained from three different sources: (1) the video recordings of one collaborative design team over eight months, including group and individual interviews; (2) interviews with participants of another three different cases of collaborative design; and (3) related literature that includes other cases of collaborative design. Domains of variability and similarity were identified resulting in a categorization of the different roles that participants hold during collaborative design.

CALCULUS IN NAVIGATION/ BODILY CALCULUS

Olga Shipulina
Simon Fraser University

The current study is devoted to investigation of the computer simulated optimal path navigation related to the calculus problem of optimal path finding. My hypothesis is that tacit dynamics modeling of optimal path navigation involves the allocentric frame of reference. The virtual environment paradigm, designed in

Second Life, contains two different mediums and provides voluntary choice between allocentric and egocentric views. Reinventing the calculus problem of optimal path finding from the virtual navigation and its mathematizing should give a powerful intuitive link between the everyday real world problem and its symbolic arithmetic. The designed paradigm belongs to the framework of Realistic Mathematics Education (RME). Analysis of the voluntary choice between egocentric and allocentric views should give an indirect indication of what frame of reference is utilized and, as such, should provide better understanding of mental processes a particular calculus problem solving situation

DEALING WITH MATHEMATICAL ABSTRACTION IN TEACHING

Krishna Subedi
Simon Fraser University

When teachers plan, one of their most important challenges is to deal with abstract mathematical concept and figure out ways of translating them into understandable ideas. By analyzing mathematics classroom interaction through the lens of reducing abstraction, this paper discusses how teachers deal with mathematical abstraction in teaching.

THE PRAGMATICS OF MATHEMATICAL DIALOGUE THROUGH EMAIL

Kevin Wells
Simon Fraser University

This paper investigates the nature of the mathematical dialogue carried out between pairs of high school students using email as a problem solving medium. A set of discourse tools are applied to study the pragmatics of the dialogue and comparisons made to the traditional formats for dialogue. Differences which appear in the use and choice of personal pronouns are examined along with the interactions between the students in their virtual shared space.

LESSON STUDY IN CIRCLE GEOMETRY: THE EFFECTS OF TEACHER'S PEDAGOGICAL CHOICES IN THE DEVELOPMENT OF STUDENTS' GEOMETRIC REASONING

Natasa Sirotic
Simon Fraser University

Geometric reasoning both in its classical axiomatic approach as well as in its empirical-intuitive approach is one of the hallmarks of mathematical reasoning, and a fundamental aspect of mathematics - in the former, through the development of a deductive proof, and in the latter through experimentation and visualization of the dynamically changing objects of analysis (angles, lengths, shapes, as well as their properties and relationships). As such, it is also one of the major goals of mathematics education. Yet, how to assist students in the development of these skills remains elusive. In this paper we examine the interaction between teacher and student actions in the development of a proof scheme for a theorem in circle geometry. Lesson study as an on-site professional development process acts as a window for our exploration.

AN ACTIVITY THEORY PERSPECTIVE ON TEACHING MATH FOR TEACHERS

Susan Oesterle
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This theoretical paper considers how Engström's (1999) model for activity systems could be applied in the analysis of the activity of preparing prospective teachers of mathematics. It suggests that consideration of this activity as a 'nested activity system' opens possibilities for gaining a better understanding of the challenges faced in this endeavour. The approach is illustrated via application to data gathered in an interview with an instructor of a preservice mathematics content course.

INTRODUCTION

In recent years there have been a number of research studies, reports and policy documents whose recommendations are aimed at improving the preparation of school mathematics teachers (Ball & Even, 2009; Greenburg & Walsh, 2008; National Mathematics Advisory Panel, 2008; Conference Board of the Mathematical Sciences, 2010). This focus of interest has come in the wake of long-existing concerns about school teachers' mathematics knowledge and the effect this has on their students' learning of and appreciation for mathematics (Ma, 1999; Ball, Lubienski, & Mewborn, 2004). Despite the proliferation of recommendations, it is notable that the extent to which these documents are research-based varies considerably. In their report, Greenberg and Walsh (2008) freely admit that many of their recommendations are not based on research, but maintain that the need to improve the mathematics preparation of teachers is too urgent to wait for definitive research results. Recent studies that report on the effects of implementing some of the suggested reforms (i.e. increasing the number of required content courses) show the results of these reforms may not be as was intended (Hart & Swars, 2009). The problem of how to effectively prepare school mathematics teachers is still an open one.

With this theoretical paper I hope to contribute to the on-going conversation by exploring what an analysis from an activity theory perspective may have to offer. After a brief description of the relevant tenets of activity theory, I provide a sketch of how the activity system around the preparation of school mathematics teachers might look. I make a case for its consideration in terms of what I will call *nested activity systems*, and offer some examples of how this may be applied using interview data from an instructor of a preservice mathematics content course.

THEORETICAL FRAMEWORK: ACTIVITY THEORY

Activity theory has its origins in the cultural-historical approach to psychology of Vygotsky (1978) and his students Leont'ev (1981) and Luria (1976). The theory has continued to be developed, and has been increasingly referenced in Western research literature over the last 30 years (Roth & Lee, 2007). A detailed exposition of the theory is beyond the scope of this paper, however I will address the characteristics that suggest its suitability as a framework for studying the preparation of mathematics teachers, and describe Leont'ev's (1981) three-level model of activity and Engström's (1999) activity triangle, both central to this discussion.

Why activity theory?

Activity theory takes "*object-oriented, collective,* and *culturally mediated human activity*" (Engström, 1999, p. 9) as its central unit of analysis. The preparation of mathematics teachers is such an activity. An examination of it through the lens of activity theory allows for consideration of the complex inter-relationships between those engaged in the activity, the community within which they operate, the resources at their disposal, the rules (both explicit and implicit) that they operate under, and how the work they do is divided. It embraces the social, cultural and historical contexts of the activity and finds value in the inconsistencies that may be revealed. "The internal tensions and contradictions of such a system are the motive force of change and development" (p. 9).

Some preliminary studies that have examined the work of post-secondary instructors of preservice mathematics content courses (just one aspect of the activity under discussion here) have shown the wide variety of influences that affect instructors' content decisions and approaches for these courses (Oesterle & Liljedahl, 2007; Oesterle, 2010). The dialectical reasoning endorsed by activity theory (Roth & Lee, 2007) provides a tool for understanding the relationships between the individual and the community, and facilitates study of a unit of analysis that is in flux, changing even while (and also because) it is being studied.

Leont'ev's three-level model of activity

An important aspect of activity theory is the distinction it makes between three levels of activity: activity (motive-driven), action (goal-driven) and automatic operations (resource-driven). At the highest level, the activity is defined by an over-arching motive that seeks to meet the needs of a collective. Actions are specific tasks with their own goals that are consciously performed by individuals or groups to achieve the object of the activity. Automatic operations involve the unconscious use of tools in the execution of actions. The relationship between activities, actions and operations is dynamic (Jonassen & Rohrer-Murphy, 1999).

Activities can become actions, actions operations, and the reverse, depending on the frame of reference and the level of consciousness of the actors.

Engström's (1999) activity triangle

A fundamental schema in modern activity theory analyses is Engström's model of an activity system (see Figure 1). The basic components of the activity system are: the subject of the activity, the object (related to a specific outcome), the community within which the activity takes place, the mediating artifacts (signs and tools) employed in the activity, the explicit and implicit rules that govern what occurs, and the division of labour amongst the participants. The system is described from the perspective of the subject. It is seen to be in a state of constant transformation, with each component influencing and being influenced by every other.

The activity system does not sit in isolation, but is considered to be part of a complex network of activity systems in which different aspects of a component play the same or different roles in other activity systems. For example, a textbook may be a mediating artifact in one activity system, while at the same time it is an object of another; a teacher may be the subject of an activity system whose motive is the education of school children, while at the same time participating as a member of the community of parents.

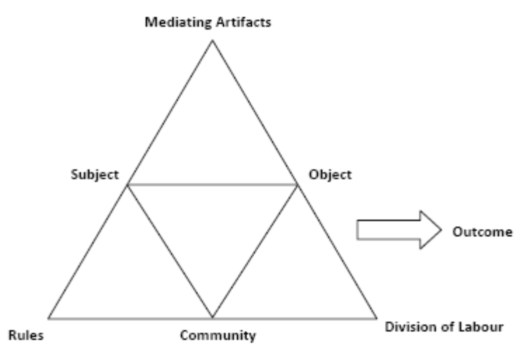

Figure 1

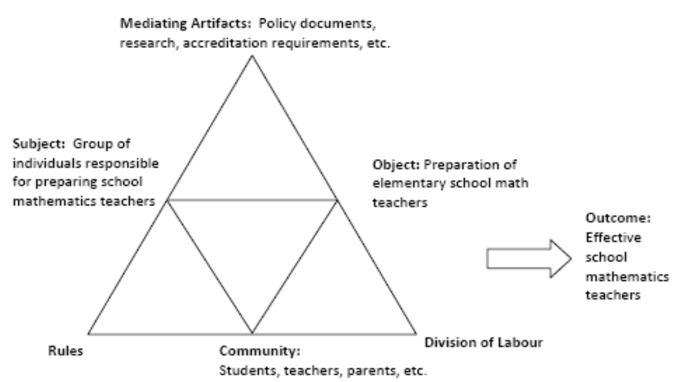

Figure 2

AN APPLICATION OF THE MODEL

Figure 2 illustrates how the activity of preparing prospective elementary teachers to teach mathematics might look under Engström's model.

Within the frame of reference I have chosen, the *subject* includes all of the people who have a role to play in the mathematical preparation of elementary teachers, including teacher educators, teachers of mathematics, mathematics education researchers, mathematicians, policy makers, post-secondary institutions and accreditation agencies.

Their *object* is the preparation of elementary school mathematics teachers, with the intended outcome of producing elementary school teachers who can teach mathematics effectively. Some of the goals/actions that are enacted in this system are: providing future teachers with adequate mathematics content knowledge, pedagogical content knowledge, and curricular knowledge (Shulman, 1986), as well as equipping them with pedagogical skills, classroom experiences, and appropriate dispositions. In fact much of the current research and debate in this domain can be viewed as efforts to delineate exactly what goals/actions are needed.

The *community* includes all of those listed above who constitute the subject, but in addition includes most importantly the students (i.e. the future teachers), and also the secondary players who are involved in shaping and defining both the rules and the vision of the outcome of the activity: school teachers, elementary school administrators, parents and children. This list is not necessarily exhaustive.

Some of the *mediating artifacts* available to facilitate this activity include: policy documents, institutional/accreditation requirements, mathematics education research, school curricula, course/program descriptions, mathematics texts designed for elementary teachers, and mathematics teaching resources. These are some of the tools created by the community (in other related activity systems), that mediate the actions, object, rules and division of labour within this system.

The explicit *rules* are set out in many of the above-listed documents, but other tacit *rules* and expectations exist, including such things as societal norms regarding who can or should do mathematics. Finally, the *division of labour* delineates the roles of the community members within the activity system.

Tensions and contradictions within and between networked activity systems represent opportunities for change and development as participants work to resolve them within a constantly transforming system (Engström, 1999). Having a deeper understanding of the inter-relationships between components and the dynamics of the system and the networks it forms part of should facilitate positive transformation.

DISCUSSION

The challenge lies in how to achieve this deeper understanding. Engström (1999) writes:

> Activity system as a unit of analysis calls for complementarity of the system view and the subject's view. The analyst constructs the activity system as if looking at it from above. At the same time, the analyst must select a subject, a member […] of the local activity, through whose eyes and interpretations the activity is constructed. This dialectic between the systemic and subjective-partisan views brings the

researcher into a dialogical relationship with the local activity under investigation (p. 10).

In the remainder of this paper I attempt to follow Engström's recommendation, taking examples from data collected during an interview with an instructor of a mathematics content course for preservice elementary teachers, and considering how the instructor's perspective (as mediated by the interviewer) can shed light on the dynamics within this activity system.[1]

Setting the frame of reference: nested activity systems

Maria is an instructor in a mathematics department at a post-secondary institution who teaches a course entitled "Math for Teachers" (MFT). This is a mathematics content course that is required for all prospective elementary teachers within her local community. Maria is an individual participant within the "subject" of the above activity system (Figure 2); she is a member of the collective engaged in the activity of preparing future elementary mathematics teachers.

Taking advantage of the fluidity of the definitions of activity and action, we can take a specific action from the larger activity system, say that of teaching a math for teachers course, and consider this as an activity in its own right. Taking the collective of instructors and others who play a part in the development and delivery of these courses as *subject*, we see a new activity system, whose subject and object are subsets of their eponymous components on the larger scale, as are the rules, mediating artifacts and division of labour. In effect by shifting the frame of reference, we reveal an activity system that is nested within larger activity systems. This zooming-in can continue down to the level of the individual instructor, Maria in this case, whose motive is nominally to develop the mathematical content knowledge of a particular group of prospective elementary teachers.

Considering these nested activity systems offers an opportunity to explore not only the inter-relationships between components within any specific system, but to see how actions at one level influence what happens at other levels. Some tensions and contradictions that arise may be horizontal, occurring predominantly only on one level, while others may extend vertically across levels. The type of analysis this offers may offer insight into where positive change may more easily be introduced.

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[1] This approach was inspired by the PME 34 Discussion Group entitled "Reinterpreting previously collected data through activity theory" (Araújo, et al., 2010).

The case of Maria

Maria participated in a one-hour semi-structured interview that was part of a larger study that sought to understand the goals, aspirations and frustrations of instructors of preservice mathematics content courses. She was asked questions about her educational and teaching background, about what she tries to accomplish in the course, and whether or not she felt she was successful. The interview was transcribed and initial coding was done using constant comparative analysis and Charmaz's (2006) recommended technique of coding with gerunds. For the purpose of applying an activity theory perspective, the next stage of focussed coding classified the initially coded fragments as either identifying actions or informing one or a combination of the major components of the activity system.

A full analysis of her interview from an activity theory perspective is not possible here. Two examples will be discussed that will attempt to illustrate some of the potentials of this type of activity theory analysis.

The object of Maria's activity

Beginning with the interview data it is possible to build up a clearer understanding of each of the components within both the local and the more global activity system. As an example we can try to understand what Maria's *object* in her activity system consists of. Within the larger system of preparing school mathematics teachers there is a division of labour negotiated amongst the subject group and the community that places certain expectations on instructors of math for teachers courses. A tentative suggestion for Maria's object, as inferred from the global activity system might be the development of mathematical content knowledge in her students.

An analysis of Maria's interview produces a list of actions that she engages in, revealing at very least some of what Maria believes is expected of her. In this case, some of the actions she explicitly mentions include: understanding her students (prior knowledge, beliefs, attitudes), delivering lectures on mathematics topics, evaluating her students (setting assignments and grading), establishing clear criteria, teaching problem-solving, introducing mathematical thinking including logic and proof, teaching mathematical terminology, improving her students ability to write mathematically, strengthening their mathematical backgrounds (beyond the level of the specific mathematics they will have to teach), addressing mathematics anxiety, modelling what mathematicians do, facilitating group work, initiating students into mathematical norms (use of appropriate notation, concise writing), revealing connections between mathematical ideas, building confidence, relating content to their future practice as teachers, encouraging her students to reflect on their own learning, providing opportunities for students to engage with manipulatives, diffusing misconceptions about arithmetic, changing her students' conceptions of mathematics, sharing her love of the subject, as well as reflecting on and revising her own practice.

Although this list is not complete, it is already evident that her motives go beyond merely building the mathematical content knowledge of her students. She is actively involved in addressing issues of affect (anxiety and confidence), trying to influence their belief systems (regarding mathematicians and the nature of mathematics), and moving them towards seeing themselves as teachers. All of these actions/goals are consistent with her role as a member of the *subject* group of the larger-frame activity system. But when considered in light of some of the tensions Maria experiences at the local level (e.g. frustrations over how little time she has with the students, surprise at the degree of their mathematics anxiety) certain questions arise. Locally: Is Maria's object (which includes far more than content preparation) attainable given the rules (time constraints), the community (her students) and the mediating artifacts at her disposal? At the global level: Has the division of labour been adequately defined by the community? Are instructors of these courses being asked to take on too much? The notion of a nested activity system then allows us to simultaneously take into consideration how the local system both reflects and contributes to the overall activity of preparing future mathematics teachers, and vice versa.

Maria's perceptions of her students

Within Engström's model, the students in Maria's class are members of the community in which she practices her activity. Indeed they are privileged members, essential to the activity, and in some senses are the raw material for the intended outcome. Figure 3 offers a slight modification of Engström's model that brings out this privileged status, identifying the students as co-participants in the activity, and illustrating their interaction with all other components within the system.

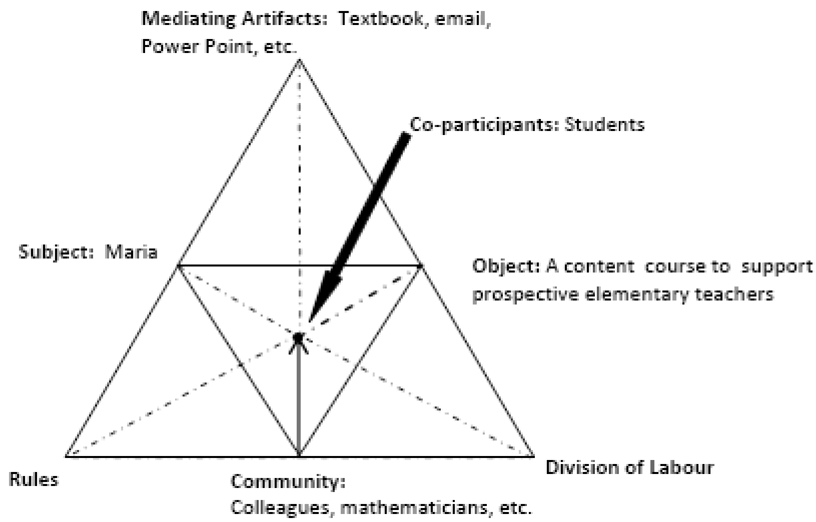

Figure 3

When the model is constructed with Maria as subject, using Maria's interview as its basis, what emerges is a picture of the students as Maria perceives them. Within this local system it is her perception that matters. It is what informs her choice of mediating artifacts (textbook, email, supplementary materials) as well as her actions (lecturing, providing opportunities for group work). From the interview data it can be shown that Maria situates her students in the context of being post-secondary students, which affects her perceptions of what they need from her and what they are capable of, which in turn has an influence on the division of labour that is negotiated between Maria and her students. She also situates her students as potential members of the teaching community and of the mathematics community, affecting the specific actions she feels she needs to engage in. She describes the severe math anxiety that many of her students experience that she notes (with surprise) is severe enough in some of them to prevent learning. This has a mediating effect on her overall motive: she needs to address the affective barriers before she can move on to address content. Interviews with other instructors of this course can help reveal to what extent this phenomena permeates the layers of the nested activity systems.

CONCLUDING REMARKS

This preliminary discussion offers no more than a cursory look into how activity theory might be used to contribute to building a better understanding of the task of preparing school mathematics teachers. It shows how an examination of activity systems that are nested within the larger system may provide insight into the tensions that exist both at the local and more global levels. It offers a method for interpreting data that is based on individual experience, but that captures the reality that individual action and experience takes place within a historical, cultural and social environment, and that the situation under study is constantly undergoing transformation.

It also suggests a vast project. Maria's voice was but one of many in the collective of instructors of content courses for preservice teachers, which in turn is just one group among many involved in the mathematics preparation of school teachers. Analysing the many other activity systems nested within the larger will allow us to build a more complete understanding of potentials and challenges within the system. Other relevant activity systems that are linked to this one include the activities of the students themselves, and of other members of the community, including mathematicians and mathematics educators.

There will be no simple solutions. Consistent with the philosophy of activity theory, as tensions are identified and changes are made, these changes will have an impact and be impacted on by other components within not only the system itself, but by others in the complex network of activity systems. However, the method opens the door to identification of directions for positive transformation and offers the hope of research-based recommendations for the improved preparation of school mathematics teachers.

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SHIFTS OF ATTENTION IN DGE TO LEARN EIGEN THEORY

Shiva Gol Tabaghi
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Dynamic geometry software has been shown to facilitate students' construction of their own mathematical objects in the linear algebra context (Sierpinska, Dreyfus, and Hillel, 1999). However, its use in the teaching and learning of university level mathematics has received less attention. This study offers a refined look at the development of explicit awarenesses of the concepts of eigenvectors and eigenvalues through the use of dynamic (and not just geometric-visual) 2d representations of the concepts. Mason's theory of awareness is used as a theoretical framework to analyze participants' developmental process of mathematical understanding.

BACKGROUND

Studies reveal that students develop procedural ways of computing eigenvalues and eigenvectors with little understanding of their geometric meaning (Stewart and Thomas, 2007; Meel and Hern, 2005). The algebraic procedure of finding eigenvalues and eigenvectors does not necessarily reveal these two important features of eigen theory: (1) that an eigenvector, x, is a nonzero special vector collinear with Ax, where A is an $n \times n$ matrix; (2) that an eigenvalue is a scalar indicating that the associated eigenvector is dilated as a result of the transformation under A.

Although students often encounter a strictly algebraic approach to eigen theory, several researchers promote the use of geometric representations; Meel and Hern (2005) even argue that an excessive algebraic focus can hinder the development of geometric meanings. Further, in his article titled *About Geometry*, Tahta (1980) points out that students' failure to master elementary algebra may be due to their ignorance of underlying geometry. He refers to Gattegno's theory of awareness—that only awareness in educable—and offers a definition of geometry as an awareness of imagery and of algebra as an awareness of dynamics, that is, "an awareness of the mind at work on whatever content" (p. 6). According to Tahta, "there cannot be an adequate awareness of dynamics if there is nothing to act dynamically on" (p. 6). This suggests the importance of geometrical imageries and representations in educating awareness and thus, understanding algebra.

Mason (2008) extends Gattegno's theory of awareness and discerns different degrees and forms of awareness. He uses the construct of "shifts in attention" to help identify the different forms of awareness involved in mathematical activity, and to relate these to students' mathematical understanding. Tahta's distinction between algebraic awareness and geometric awareness has motivated us to study the shifts of attention between algebraic and geometric representations involved in understanding and using the concepts of eigenvectors and eigenvalues.

THEORETICAL PERSPECTIVES

Attention, awareness and attitude are three aspects of human psyche that Mason uses in his theory to highlight the developmental process of mathematical being. Awareness refers to our conscious and unconscious powers, and sensitivities to detect changes and to choose proper actions in certain situations (Gattegno, 1987; Mason, 2008). To educate awareness is to draw attention to actions which are being carried out with lesser or greater awareness. The actions that are chosen will depend on specific mathematical topics and themes, on the basis of past experiences and even without any conscious knowledge. Gattegno (1987) believes that we have many awarenesses. Among these many awarenesses, Mason (2008) distinguishes explicit awareness from implicit awareness. The former refers to awareness which can be articulated whereas the latter denotes awareness that is not ready to articulate or no longer articulable. For example, students can solve the characteristic equation $\det(A-\lambda I)=0$ and then find the associated eigenvectors without being explicitly aware that they are identifying a special vector that is collinear with its image.

Education of awareness comes about when implicit awareness is refined through sufficiently rich experiences in which new states of attention can be developed. In such situations, attention can be drawn not only to mathematical objects, relationships and properties, but also to manifestations of mathematical themes, and to heuristic forms of mathematical thinking (Mason, 2008). According to Mason (2008), the refinement and development of awareness can be seen through changes to the structure of one's attention.

The structure of attention comprises macro and micro levels; *what* is being attended to is as important as *how* it is being attended to. At the macro level, Mason describes the nature of attention as follows: "attention can vary in multiplicity, locus, focus and sharpness" (p.5). At the micro level, he distinguishes five different states of attending: holding wholes, discerning details, recognizing relationships, perceiving properties and reasoning on the basis of agreed properties. Holding wholes is when a student gazes at a definition, collection of symbols and/or diagram. The student may not focus on anything in particular, yet 'waiting for things to come to mind'. Looking at the wholes, the student may discern and identify useful sub-wholes or details. Discerning details is a process that participates in and contributes to subsequent attending. As the student discerns details, she may recognize relationships between symbolic and geometric representations of mathematical concepts. When she becomes aware of possible relationships, she may perceive a property. As she continues attending, she can use the perceived properties as a basis for mathematical reasoning. It is noteworthy that the described states of attention are not leveled or ordered. They often last for a few micro-seconds and alternate among other states. Those that become stable and robust against alteration for varying periods of time may block further development of awareness (Molina and Mason, in press).

In this study, we analyse a student's actions with DGE sketches in terms of the macro and micro-level states of attention described by Mason. The shifts in attention that occur, say, in discerning detail (micro-level), or in moving the locus of gaze (macro-level), enable us to characterize the structure of a student's attention, and thus the education of the student's awareness.

RESEARCH CONTEXT

This study is the first instalment of a broader project which investigates the effect of the use of dynamic geometric representations on students' conceptualizations of the concepts of eigenvectors and eigenvalues. In this paper, we describe our findings from a clinical interview (conducted by the first author) with one participant. Jack was pursuing his undergraduate degree, majoring in computer science, at a large North America university. He had taken a linear algebra course recently (about two months prior to the interview) and achieved a high grade. He was relatively familiar with *The Geometer's Sketchpad* software because of a previous geometry course.

METHODOLOGY

Jack was given a worksheet with a formal definition of eigenvectors and eigenvalues. He was then given a sketch designed to enable exploration of eigenvectors and eigenvalues for given 2×2 matrices. As shown in Figure 1, the matrix A was represented on the coordinate system in terms of its column vectors *u* and *v*. The sketch includes a draggable vector x and the vector Ax. The sketch also includes numeric values of the matrix-vector multiplication (Ax). The user can change the values of matrix A.

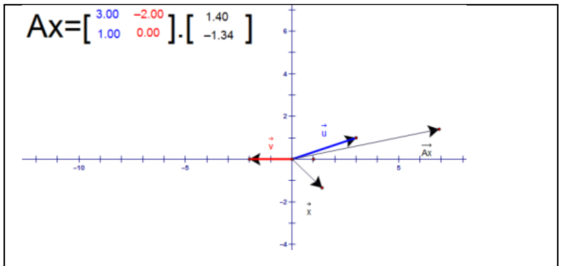

Figure 1. A snapshot of eigen sketch

The first question asks the participant to find, if possible, the eigenvector(s) and associated real eigenvalue(s) for each of the following matrices: (a) $A=\begin{bmatrix}3 & -2\\ 1 & 0\end{bmatrix}$, (b) $A=\begin{bmatrix}4 & -1\\ 1 & 2\end{bmatrix}$, (c) $A=\begin{bmatrix}1 & 6\\ 5 & 2\end{bmatrix}$, and (d) $A=\begin{bmatrix}1 & -2\\ 3 & 1\end{bmatrix}$. To find the eigenvector of a A, the user can drag x until it overlaps Ax.

Jack was asked to talk aloud while he interacted with the sketch. The interview, which lasted about 30 minutes, was videotaped and transcribed. The videotaped was analysed in terms of Mason's theory of attention. Specifically, we identified occurrences of shifts of attention at both the micro- and macro-levels.

Jack's Shift in Attention

Jack begins by reading aloud the formal definition of eigenvector from the given worksheet. He immediately points to each symbols given in the definition (using his right index finger) saying "this is a matrix, this is a vector, this is a scalar, and this is a vector". After looking at the sketch, then back to the definition, reading it over again to himself, he drags vector x a very small amount in the 4th quadrant (using the default matrix (a)). He then stops, says "I see," returns to the definition, says "now I'm confused," and looks back to the definition again. Then he starts dragging, this time moving x into the other quadrants somewhat randomly.

In this initial segment, Jack first focuses his attention on the definition, discerning details as he articulates every symbol one by one. He then shifts his locus of attention to the sketch and to the draggable vector x. His initial tentative dragging suggests he doesn't quite know what to expect and, indeed, he looks back at the definition again. Then the interviewer prompts him to drag x, so he begins to do so asking "yes, but to what end?" As he engages in wandering dragging, Jack's attention is holding wholes as he watches and waits for feedback, not quite knowing what he's looking for.

As evident in this interaction with the interviewer, Jack then begins to recognise relationships:

[9] Jack: Oh. I see. I see. So by dragging it, it is maintaining the eigenvectors or but, um, it doesn't output lambda […] should it be outputting lambda?

[10] Int: Yes, it doesn't show the lambda on the sketch, but you might be able to see it as you drag.

[11] Jack: I see, so I guess I line them up [drags vector x directly into the first quadrant until it overlaps with Ax]. I guess I could have lambda there. And then should I change this value [pointing with mouse to the matrix].

[12] Int: Could you tell me how you got into that if you line them up it's going to be what you looking for?

[13] Jack: Because I looked at this [pointing to the definition] and I realized that there was a scalar transformation so the vectors have to be co-linear.

In turn [9] Jack remarks that changing the position of x results in a changing of the vector Ax on the screen, suggesting that he is attending to the relationship between the two vectors. He then shifts to focusing on lambda, and, more particularly, on its absence in the sketch. After being told that lambda does not appear on the sketch, Jack seems to infer that he needs to make the two vectors collinear, thereby shifting his attention to perceiving properties. In turn [13], Jack responds to the interviewer by focusing his attention back to the definition and now reasons in terms of the properties of the definition, stating that the algebraic condition that $Ax = \lambda x$ implies that Ax is a vector that is a scalar multiple of x. In going back to the definition, Jack infers the collinearity from the scalar transformation whereas in his actions with the sketch, the collinearity precedes the identification of lambda.

The interviewer prompts Jack for the value of lamba, which he states immediately as "It looks like it's 2," without explaining how he is looking at. We infer that his attention shifted to the geometric relation between x and Ax, that is, the relative lengths of the vectors. Jack also writes the eigenvector down on his worksheet. While the interviewer invites him to seek other eigenvectors, Jack is focused on dragging vector x along its collinear path with Ax (see Figure 2), making it longer—as he engages in this "guided" dragging (see Arzarello et al, 2002), he says

[17] Jack: Lambda still looks like two, x has changed, should I write that down?"

[18] Int: Okay. But how does this x [pointing to the longer vector on the sketch] relate to this one [pointing to the original eigenvector written on the worksheet]?

[19] Jack: [Writing on this worksheet.] Um. I guess it's a linear transformation of this [pointing to vector on worksheet] because of the definition. It looks like it's a linear transformation of this.

[20] Int: Can you continue dragging x?

[21] Jack: You mean to here [dragging counter clockwise to another position that overlaps with Ax in the third quadrant]?

[22] Int: So what's the lamba here? And what the x?

[23] Jack: Um. [Looks back at definition, then to sketch]. So […] are you trying to hint that all values of x are linear transformations of each other?

---

[29] Jack: So I guess I would at this point I would probably realise that they look very [dragging x toward origin along the line where the two vectors overlap], that they are all on the same axis [dragging x away from origin into first quadrant] I guess and I would deduce [now dragging x more quickly back and forth along the straight path] that the value of lamba wouldn't change but that there are infinitely many eigenvectors.

In this interaction Jack encounters a new position where the two vectors line up and shifts his attention to the relationship between the two vectors he has now found. After shifting focus to the definition, Jack identifies a property of the eigenvectors (they are "linear transformations of each other"—more precisely, that they are scalar multiple). Then, as he engages in guided dragging of the vector along the path of invariant collinearity, he moves from tentative statement about visual perception "they look very" to one that seems more certain, "they are all on the same axis." At this point, Jack's attention is involved in reasoning on the basis of the properties.

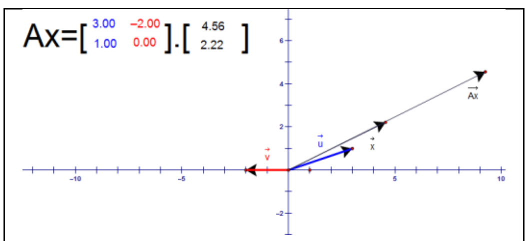

Figure 2. Vectors x and Ax are lined up in the same direction.

Jack is about to proceed to the next question when the interviewer prompts him to look for another set of eigenvectors. After dragging in a counter-clockwise circular fashion, Jack cannot identify another eigenvector, and says, "it makes sense since there should only be one." The interviewer asks him to drag x to (1,1). At this point, Jack sees this as an eigenvector, drags x again in the region of (1,1) and after some time, during which he writes on the worksheet, he eventually identifies lambda as 1.

Given his haste to move to the next question, and his statement that there should only be one eigenvector, we infer that Jack's attention is blocked to the possibility of finding another. This is exacerbated by the difficulty he has in seeing the second eigenvector, probably because of the value of lambda—the two vectors coincide, which makes them difficult to see.

Having completed the first question, Jack turns his attention to each of the three others. He changes A to correspond to matrix (b), randomly drags vector x into all the quadrants, and asks "is the eigenvector non-existent?" He comments "they don't exist." After some more circular dragging further from the origin, he hits upon a vector in the third quadrant "so they're lined up so there's a lambda. I don't know why I said non-existent. Whatever this length is [dragging mouse along the vector Ax] divided by that length [dragging mouse along x]" (in other words, the ratio of the vector lengths). When asked whether there might be another eigenvector, Jack responds, "just because it is a linear equation it should only be one. I guess, but assuming that I do not know that if I drag that [vector x] around a circle I could find out". He now uses an explicitly circular dragging strategy and finds no other eigenvector.

Jack's new dragging strategy suggests a shift in attention that involves two components: first, an awareness that there can be more than one eigenvector, and second, a use of dragging that is intended not only to locate one eigenvector (as was the case in his wandering dragging), but to identify all possible eigenvectors. He seems only implicitly aware of how this new strategy works in that he does not articulate why dragging in a circle will allow him to identify all possible eigenvectors.

Next, Jack changes the matrix to (c). Using his circular dragging strategy, he immediately finds an eigenvector and approximates the associated eigenvalue. He is about to change the values of the matrix when the interviewer prompts him to find other possible eigenvectors. He drags x in an anti-clockwise direction, speedy fashion, focusing only on the position where vector x lines up with vector Ax. Finding nothing, the interviewer invites him to drag vector x slowly into the second quadrant. When the two vectors are collinear (but not overlapping) Jack says "it's the opposite eigenvector… is that right? […] I'm trying to recall […]. I guess it's just. Um." He looks back at the definition and then when the interviewer asks whether he has other eigenvalue and eigenvectors, Jack immediately says "oh yeah, yeah. I guess it would be -8" (the actual value is -4).

Until now, Jack's attention has been focused on looking for positions where vectors x and Ax line up. Again, he has a stable and robust state of attention that seems to block other states. The interviewer's intervention helps re-direct his attention to the existence of another interpretation of 'lining up.'

When he moves on to the fourth matrix (d), Jack uses his circular dragging strategy to establish that there are no eigenvectors. When asked (in the following question) how he went about trying to find the eigenvectors, Jack says "I tried to make x touch Ax [dragging vector x in a spiral fashion beginning far from the origin, turning in an anti-clockwise direction, and ending at the origin] and I guess for the third one we tried to make that happen [drawing two vectors that are collinear but not touching]." In this final reflection on his interaction with the sketch, Jack introduces a new dragging strategy that varies both the angle and the distance from the origin of the vector.

DISCUSSION

Jack has taken a linear algebra course recently and recalls the meaning of the symbols involved in the algebraic definition (including the fact that Ax is a vector). As he shifts his attention to the sketch (and the dependence relationship between the two vectors), then to the definition, then to lambda (and its absence in the sketch) and then to the need to "line up" the vectors, we see the structure of his attention change: in particular, we see a geometric awareness of the equality between Ax and λx develop. While initially hesitant about the absence of lambda, Jack eventually sees lambda in the relationship between the lengths of Ax and x. Further, as he drags x in a guided way—keeping it collinear to Ax—Jack seems to realize for the first time that there are many eigenvectors associated with the same eigenvalue.

The geometric representations and actions cannot be seen as merely providing Jack with a visualization of the concepts of eigenvectors and eigenvalues. In fact, we've identified three ways in which the dynamic interaction changes the concepts involved. First, the geometric approach distinguishes the negative from the positive eigenvalue, unlike with the algebraic approach. This gives Jack some problems at first, and his shift in attention to the possibility of a negative dilation makes explicit his previous awareness that eigenvalues can take on integer values.

Second, unlike with the algebraic approach, where one first computes an eigenvalue, and then solves for the eigenvector, the order is reversed in this sketch: one identifies the eigenvector first. In terms of the historical development of eigen theory, the eigenvector is the central object of attention—one wants to find the direction of the flow or the swing—and it is not surprising that the geometric approach shares this epistemological precedence. The secondary status of the eigenvalue is emphasized in the sketch by the fact that it is only approximated by the user.

Thirdly, while the eigenvalues of a matrix emerge simultaneously in the algebraic approach, and then lead to eigenvectors, the geometric approach requires a more sequential process of identification. This may explain Jack's repeatedly omitting to find a second eigenvector. However, this very difficulty also leads him to develop particular dragging strategies that increasingly are more successful in exhaustively identifying the eigenvectors of a matrix. These dragging strategies—circular, clockwise, anti-clockwise and spiral paths—are interesting to consider; in particular, they make explicit the "implicit dynamism of thinking" (Leung, 2008). At this very point then, we see Jack acting algebraically, in Tahta's sense of the term, as his mind (and his hand) is at work on the geometric objects. The changes in dragging strategies provide evidence for changes in the structure of Jack's awareness.

Mason's theory of shifts in attention has helped us to analyse Jack's interactions with the sketch and to describe his mathematical awareness. These shifts were made evident both only by the Jack's changing focus of attention from the definition to the sketch, but also in his different dragging strategies which he used to identify eigenvectors and explore the relationship between eigenvectors and eigenvalues. Despite his strong algebraic awareness—developed in his course—Jack's awareness continues to grow as he coordinates his emerging geometric with his algebraic awareness. The geometric awareness emphasises especially the invariant collinearity of infinitely many eigenvectors for a given eigenvalue, the centrality of the eigenvector and the possibility of having more than one eigenvector.

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UNDERGRADUATE STUDENTS' CONCEPTIONS OF INEQUALITIES: THE MISSED-BEFORE

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This report comes from a broader study that investigates undergraduate students' conceptions of inequalities. It presents a discussion regarding undergraduate conceptions of inequalities through the theoretical framework of the Three Worlds of Mathematics. The CONCEPTIONS OF INEQUALITIES, as they emerged from the data, are first introduced and a background on the Three Worlds of Mathematics is given. The CONCEPTIONS OF INEQUALITIES are then projected on the Three Worlds of Mathematics. The projection reveals that the conceptions of inequalities occupy lower or improper levels on the Three Worlds of Mathematics. The speculation is that students have plenty of missed-before experiences that prevent their understanding and manipulating inequalities at the expected level.

INTRODUCTION

Inequalities have been under the lens of some prominent researchers in mathematics education (Linchevski & Sfard, 1991; Boero & Bazzini, 2004; Kieran, 2004; Sackur, 2004; Dreyfus & Hoch, 2004; Tsamir, Tirosh, & Tiano, 2004; Tsamir & Reshef, 2006; Abramovich and Ehrlich 2007). The studies focused mainly on common errors when solving inequalities, possible sources of students' incorrect solutions, and identifying promising ways of teaching inequalities. No special attention was given to the conceptions of inequalities. My purpose here is to present undergraduate students' CONCEPTIONS OF INEQUALITIES, which emerged from a longitudinal study at Simon Fraser University, to project the conceptions on the theoretical framework of the Three Words of Mathematics (Tall, 2007), and to discuss the *met-befores* that may have prevented as well as the *missed-before* that could have ensured a desired placement of undergraduate students' conceptions of inequalities on the Three Words of Mathematics.

THEORETICAL FRAMEWORK

The three worlds of mathematics

Three worlds of mathematics is a theoretical framework of long-term learning that presents three ways in which mathematical thinking develops. It incorporates three different but intertwined worlds of mathematics: *conceptual-embodied*, *proceptual- symbolic* and *axiomatic-formal*. This framework explains the cognitive development of mathematics of individuals from childhood to the stage of working on and appreciating pure mathematics (Tall, 2007).

Conceptual embodiment refers to the thinking about objects' properties after individual's perception of or physical interaction with the objects. Euclidean geometry is one example of an area corresponding to the conceptual embodied world of mathematics. Actions on physical objects followed by actions on mental objects, such as counting, sharing, adding, subtracting, or multiplying give another aspect of the conceptual-embodied world (Tall & Lima, 2010). The *proceptual symbolic world* is characterized by the compression of actions on physical objects into procepts such as number, sum, product, fraction, equation or other concepts to think about. Flexible thinking and fluent work with symbols are expected from someone operating in the proceptual symbolic world (Tall & Lima, 2010). The conceptual embodiment domain and the procedural symbolic world intertwine throughout school mathematics. The *axiomatic formal world* of mathematics refers to the axiomatic systems, formal definitions and mathematical proofs that are the object of pure mathematics at university (Tall, 2008).

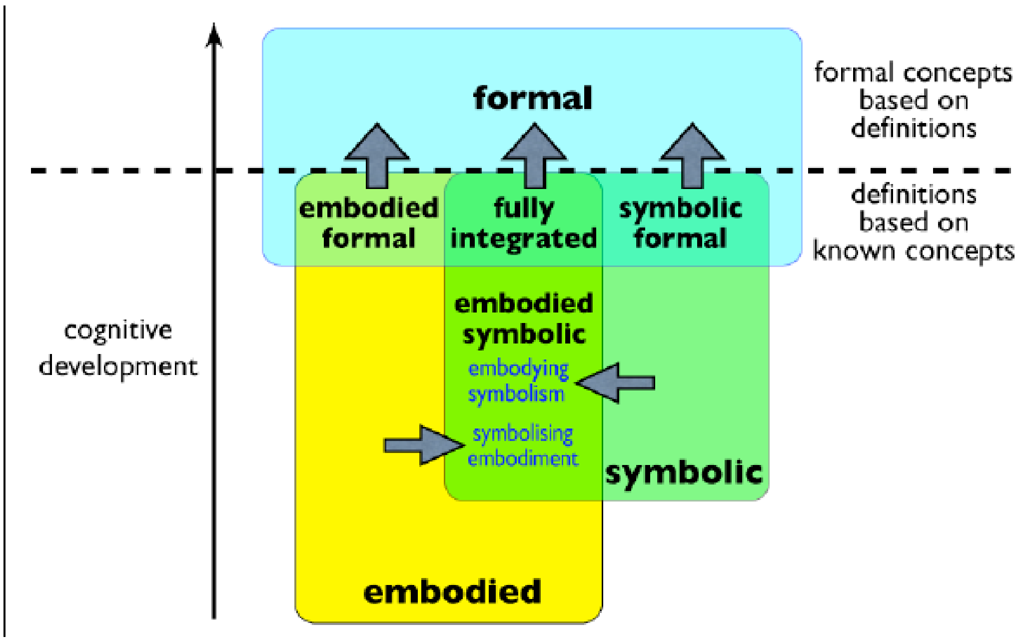


Figure 1 - The cognitive growth of three mental worlds of mathematics (from Tall, 2007, p.4)

The diagram in Figure 1 shows the three worlds of mathematics, their connections and interactions, and the direction of the cognitive growth, starting with the embodied world – of manipulating counters, for example – and moving up to functioning in the formal world where the concepts are based on definitions.

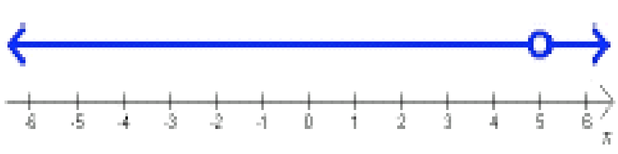
THE RESEARCH STUDY

The setting for this study is Simon Fraser University and the participants are three classes of students: two FAN X99 classes and a Math 100 class.

FAN X99 (Foundation of Analytical and Quantitative Reasoning) is a non-credit mathematics course, designed for students who need to upgrade their mathematical background in preparation for quantitative courses. The class met twice a week for a two-hour seminar and I was the instructor of my subjects. The survey was administered 8 weeks into the course, after three classes on inequalities. In all there were 57 participants.

MATH 100 (Precalculus) is a course designed to study functions in preparation for first year Calculus. The class met twice a week for a one-hour-and-a-half lecture and I was again the instructor of my subjects. Data was collected as an item in the final exam, after an entire semester of intense work on functions with inequalities omnipresent in every topic. The first two weeks of classes were devoted to recalling and improving algebra skills from which inequalities were one of the most emphasized concepts. Starting with linear inequalities and the convention of writing the solutions in interval form as well as to show them graphically, then doing work with rational inequalities, polynomial and absolute value inequalities, as well as inequalities having irrational components. The methods for solving all those types of inequalities varied from reading from the graph the intervals describing the inequality, to making a sign chart from where the solution could emerge, or to analysing the effect of each component of an inequality and making logical connections. Also, inequality was the main tool for working out the domain of a function or describing the behaviour of a function on intervals. I do not exaggerate if I claim that there was at least one inequality to be set up and solved in every lecture, assignment, or term test for a period of thirteen weeks. In all there were 43 participants.

The Math 100 tasks

> In each case give an example (an equation, a picture or a description) of a mathematical inequality with solution a) $(-3, 5]$; b) $x = 2$; c) represented by the graph 
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> Explain briefly why your example meets the conditions.

The normative examples could be: a) $-3 < x \leq 5$, b) $(x-2)^2 \leq 0$, c) $x \neq 5$. However, given that the work on inequalities was extensive, when the students reached the final exam it was safe to assume that their all-term exposure to inequalities in all imaginable settings and registers will ensure a set of data with maximum variation and room for interpretation.

The FAN X99 tasks

a) Create a worked example that will show someone how to solve linear inequalities.

b) Explain in short terms the concept of *mathematical inequality*. Please use symbols, pictures and words. Even if your image/belief about inequality is vague, I appreciate if you as clear as possible will try to give an explanation. Try to recall or reconstruct the provenance of your images of inequality.

RESULTS AND DISCUSSION

The data was analysed with the help of concept image-concept definition (Tall and Vinner, 1981), instrumental-relational as well as procedural-conceptual frameworks for understanding mathematics (Skemp, 1976; Hiebert & Lefevre, 1986), the unit of description in phenomenography (Marton & Pong, 2005) and the structures of the APOS Theory (Dubinsky & McDonald, 2001). As a result of seeing the data through all these lenses, the CONCEPTIONS OF INEQUALITIES were coined, backed-up with evidence from the data and endorsed by theory.

CONCEPTIONS OF INEQUALITIES

Five solid conceptions of inequalities were identified:

Conception 0 *Inequality as a miscellanea of images or symbols encountered in a mathematics setting*

Miscellanea

SF1: $\neq$ - when two answers don't agree

SF2: The concept of mathematical inequality can be a set of numbers express on a number line. It is a collection of symbols (i.e. R, ϕ, no solution) that illustrate the mathematical solution of an equation.

SF3: Mathematical inequality is when 2 numbers or variables do not match up as an final answer. You may have equations linking the same system but the product of the equation, rather the solutions do not equal to each other as they are supposed to. For example, the given equation is $A = B$ but the solution for A is 5 and the solution B is 6 therefore coming and inequality as $A \leq B$.

SF4: Something to do with graphing. Solving an equation with a certain formula. Different formulas are applied to different situations. This can create images on a graph like parabola.

SM1: $1 < x < 3$ is an inequality with solution $x = 2$.

SM2: [Solution to c) is the same as] $4 < x < 6$.

Groping for symbols, images, or words to describe the concept of inequality is visible in all of the above quotes. **Conception 0** is associated with fumbling in a foreign region of mathematics worlds and incoherently trying to describe the object some people call inequality. I am tempted to use the word ***Miscellanea*** to name this conception. The students here seem to operate in the embodied world of counting, where there are no other numbers in between 4 and 6, except 5, even though the solution to the required inequality was presented as intervals on a real number line. The concept images of inequalities are vague – "when two answers don't agree", confusing – "inequality is when 2 numbers or variables do not match up as an final answer," or blurry – "something to do with graphing."

Conception 1 *Inequality as an instrument for comparing known quantities or a tool for expressing restrictions*

Tool

SF1: Mathematical inequalities are equations that do not have a real answer. It is more a comparison rather than an equation. Ex. $2 \neq 1$ but $2 > 1$.

SF2: It is when something compared to another. Then that one thing is either greater than the other one, smaller, greater than or equal and smaller than or equal. Example O > o, the larger circle is greater in size than the smaller circle.

SM1: By having a hole at +5 the graph $\frac{x^2}{x-5}$ therefore cannot have $x = 5$.

The conception of 'inequality as a comparison' is close to the formal definition of inequality. The first part of the quote coming from SF1 looks more like **Conception 0**; however the examples that accompany the definition use inequality symbols to compare given numbers which is definitely **Conception 1**. The designation ***Tool*** for this conception is more evident in the responses coming from the Math 100 students, which have seen the inequality at work when establishing the domain of a function, for example. The thinking process revealed by SM1 is the mental association of the domain of a rational function with a whole in the graph. This conception is a bit more solid than the previous one and it in grounded into the embodied experience of comparing quantities as well as in the uneven balance metaphor which was present as a picture in some papers. It also relies on the use of the symbols for mathematically expressing the comparison; therefore a natural placement on the world of mathematics will be at the intersection of the Embodied and the Symbolic Worlds.

Conception 2 *Inequality as a (strange) relative of an equation*

Equality

SF1: [Mathematical inequality] is an equation where the two sides aren't equal.

SF2: When one side does not equal the other side of the equation. Does not solve equally.

SF2:

$2x+8<20+4x$	- Sub '<' with '=' to turn it in a linear equation.
$2x+8=20+4x$	- Solve as you would a linear equation by adding 8 to both sides and subtracting 4*x* from both sides.
$-2x=12$	
$x=-6$	- Resub '= with '<' to get values for *x*. Answer: $x<-6$
$x<-6$	

The definition of 'inequality as equation with unequal components' is a veritable example of **Conception 2**. With this concept image in their head, when acting on inequalities, students – such as SF3 – replace the inequality symbol with the equal sign and solve an equation, which results very often in erroneous solution. What is also interesting about this conception is that the conception is not derived solely from looking at students' work and coding as in other groups of papers; it comes directly as student's declaration, his concept definition of inequality. It was documented that familiar procedures are performed on symbols that do not have natural conceptual embodiments (Tall, 2004). Here, the inequality is not encapsulated yet and the process of solving it is carried in a routinized way based on the met-befores procedures from equations; the familiar look of inequality invited not only the application of the procedure from equations, but a complete substitution of the new symbol with the symbol which was more familiar. This conception could be placed in the lower level of the Symbolic world.

Conception 3 *Inequality as a mental or algebraic process*

Process

SF1: Inequalities are the formula that shows greater than or less than some number. Ex: $a>x>b$ or $-5<x<2$. [The student also shows the inequality $-5<x<2$ graphically and indicates correct use of symbols and intervals.]

SM1: $x\neq 3, 5$ cannot be done since 5 has to be an asymptote in order for it to be an end point but that can't be done since 5] means that there is a value when $x=5$ and a point on the line contacts it so it's not an asymptote.

SM2: $\left|\frac{1}{x-5}\right|\geq 5$ it works because the domain is all real numbers except for 5.

SM1 mentally sees that $x\neq 3$ could come from a function having an asymptote at $x=3$. Other students used fractions, absolute value, radical or polynomial functions as examples for this inequality. SM2 embedded an absolute value inequality with a rational algebraic expression which is defined on a real domain except 5. The mental process is not fully accomplished; the student missed that using 5 on the right side of the inequality will restrict the solution to the intervals $\left[4\frac{4}{5}, 5\right)\cup\left(5, 5\frac{1}{5}\right]$. The inequality should have read: $\left|\frac{1}{x-5}\right|\geq 0$. In conclusion, the connection of inequality with the domain of a function is a good one and shows a higher level of thinking, which could be situated at a *process* level in the APOS

Theory. I consider some papers to be at this level of conception even though many of the provided examples did not fully resolve in the given solution. As for the algebraic process, this is not visible in the chosen transcripts from data; however, it is very well reflected in the "solve the inequality" tasks which was part of the survey but the limitations of this report does not allow for detailed samples from that.

Conception 4 *Object* *Inequality as seen by mathematicians – a complex mathematical concept that could be expressed in different registers – symbolic, interval, or graphic; and could perform different functions – compare quantities, express and resolve constrains or deduce equality.*

SM1: a) $\log_2(x+3)-3 \leq 0$ has a vertical asymptote at -3, stays below the *x*-axis for all values less than 5 and crosses the *x*-axis at 5.

b) $-\frac{1}{3}(x-2)^2 \geq 0$ Parabola opens down, *x*=2 is the only value where the function is greater or equal to 0.

c) $f(x)=7(x-5)^2>0$ when $x=5$, $f(x)=0$, 0 is not > 0.

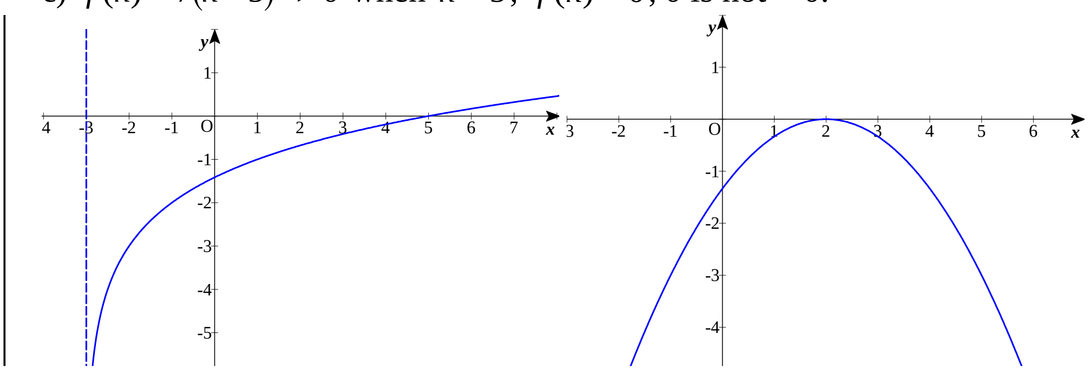

Figure 2 – Graphs from SM1's work

The examples here are coming from the same subject. They are sophisticated, correct and correctly explained. Different registers where used to represent the same inequality. **Figure 2** are the graphical representations of inequalities a) and b).

At this level of conception, the inequality composed of many other entities – such as functions with their domain and graphical representation, it involves several processes – such as mental algebraic transformation of inequalities as well as graphical transformations of functions, all of them perfectly coordinated in a *schema*.

The CONCEPTIONS OF INEQUALITIES on the Three Mental Worlds of Mathematics

The limitations of this paper do not permit for a detailed statistical analysis of the undergraduate students ‘conceptions of inequalities. However, it seems necessary to mention that the majority of the data falls on the lower levels of conceptions. Moreover, conception 4 is almost imaginary, or the ideal conception of undergraduate students: SM1 was the only student in Math 100 who produced work at this level. As for the FAN X99 students, there was a small number of students with conceptions identified at the level 3. No need to say that a good performance in Calculus correlates with at least Conception 3 at the beginning of the course.

Figure 3 shows the projection of the CONCEPTIONS OF INEQUALITIES on the Three Mental Worlds of Mathematics. As stated before, the concentration of the conceptions are in the lower levels while qualitative work in undergraduate mathematics require a solid conception of inequalities, situated in the Symbolic Formal or at the blending of all three worlds.

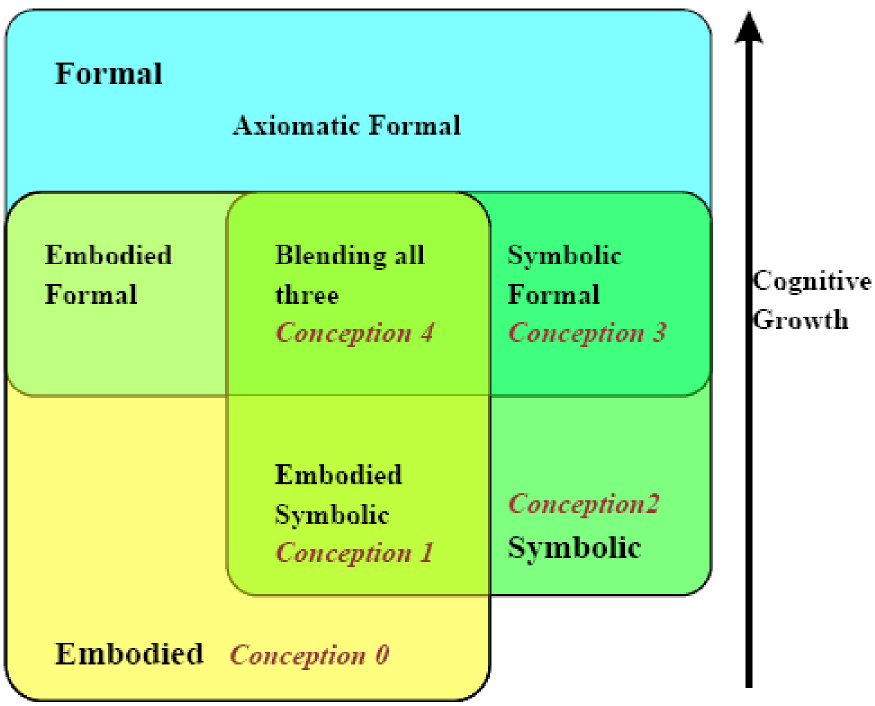

Figure 3 - The conceptions of inequalities on the three mental worlds of mathematics

The *met-before* and the *missed-before*

> “we are losing them somehow in the early days and the repercussions are felt later and severely!” (Nardi, 2007, p.123).

The results exhibit many of the patterns of (mis)understandings as well as misconceptions identified by research on inequalities. However, my framework for presenting the results reveals more than some local error or aspect misunderstood by my students: With inequalities well situated in the three worlds of mathematics, I can capture the *missed-befores,* if I am allowed to say so, which are at least as important as the *met-befores* in analysing the conceptions of inequalities. Tall (2008) defines *met-before* as “a personal mental structure in our brain *now* as a result of experiences met before” (p.4). I will define the *missed-befores* as “all experiences and embodiments that were not met before and, if met, they could have had the potential of helping *now* a relational understanding of a concept.” Paraphrasing Tall (2004), I hypothesize that it is precisely the *missed-befores* that prevent so many undergraduate students to have conceptions of inequalities situated higher on the three world of mathematics diagram.

Literature well documents that the analogy with equations gets in the way of understanding inequalities. It is possible that the students replace the inequality symbol < with = because there seem to be no invitation to action in the inequality symbol, but there is the ‘do something’ embodiment in the equal sign (Kieran, 2006). It could be the case that students are missing embodied experiences and have no access to manageable metaphors for inequalities. What they met-before related to the inequality symbol was merely ‘compare’ and the compare was used mostly in the static setting of comparing given numbers. The embodied ‘compare’ from geometry seems to be missing from their experience, and, together with that, the whole dynamic aspect of inequalities. Also, at the axiomatic-formal level the students were completely lost since they missed initial training in the game of mathematically proving something.

Lima (2010) finds that students’ difficulties in solving equations are not in the transition from arithmetic to algebra but in seeking meaning for the symbolic objects in the embodied world rather than the symbolic one. It seems that students, unable to find connections in the world of mathematics they are performing a task, are searching for meaning in neighbouring worlds. This could explain why undergraduate students performing on inequalities seek meaning for the symbolic manipulation of inequalities in their resemblance with equations and this act is very ineffective, to say the least.

Research claims that algebraic training starts long before introducing letters in calculations; it can be performed with numbers and word problems (Kieran, 2004; Tall, 2001). Similarly, training in inequalities can be introduced long before algebra, not only by comparing numbers, but also by exposing students to embodied geometrical inequalities, for example. Producing equality by means of inequalities, one of the major aspects of inequalities is completely missing from the school curriculum.

> The *equality indicates a boundary*, but we are really concerned with what lies *inside* and *outside*. The equality is like a fine ceremonial dress, beautiful for show; but you get into your shirt- sleeves for the real work. In fact that seems to be the keynote of the situation; we like to present our *finished mathematics*, mathematics for show to the public, as much as possible in *equality form*, but in the mathematical workshop inequalities are the standard tools. (Tanner, 1961, p.294)

It seems that Tanner (1961) observed that even mathematicians are not showing to the public the rough work done intimately with inequalities to produce results for the mathematics' show.

My paper seems to converge toward a conclusion: Revisiting the *met-befores* as well as the *missed-befores* in the framework of cognitive growth could inform curriculum designers about the embodiment, symbolism and formalism students have met-before and can build upon or have missed-before and must get in order to have the concept of inequality encapsulated as a solid mental object. It seems that the *met-befores* which are personal mental structures, as well as the *missed-befores* which are symbolic experiences and embodiments should be taken into consideration in preparation for exposing students to inequalities. Instead of losing them in the early days, we should empower our students with experiences that will help the long term development of mathematics concepts.

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HOW A COMMUNITY OF PRACTICE IS CREATED OUT OF AN ASSESSMENT PROJECT

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An assessment to look into the learning of mathematics among third grade classes throughout the year is being developed by a group of teachers and a team of assessment researchers. The goal of this group is to create an assessment tool that is summative and formative where teachers are able to track throughout the year the mathematical development of their students. Throughout the process of creating this tool a community of learning was created among the teachers. This paper shows how this community is created among teachers by a desire to acquire a better mathematical understanding for themselves and of their students' knowledge and the need for communicating with other teachers about their practice.

INTRODUCTION

An assessment to look into the learning of mathematics among third grade throughout the year is being developed by a group of teachers and a team of researchers. The goal of the group is to create an assessment that is summative and formative where third grade teachers are able to track throughout the year the mathematical development of their students. This assessment will be use by various schools which use different math programs. The aim is to find out which are the concepts and ideas that students have difficulty learning, if there are some programs orsections of programs that work better than others and if there are some resources that are needed to complement the materials that teachers have for their teaching. A key part of this assessment is that teachers are active participants in its creation.

This paper will describe how this project was developed and how it provided teachers with a great opportunity to become part of a community of practice where participants engaged in creating, developing and learning better teaching practices by means that make sense to them culturally. By working together in the development of an assessment tool, teachers become aware about their teaching, the effectiveness of their practice, as well as what other teachers are doing. This awareness, connection and collaboration with other teachers, math educators, and assessment professionals can help them make choices on improving the way they teach. "By working in groups to improve instruction, teachers are able to develop a shared language for describing and analysing classroom teaching, and to teach each other about teaching" (Stigler and Hiebert 1999, p.123.).

FRAMEWORK

In order to understand how working in an assessment group whosefinal goal is to assess students in order to improve teachers' practices and students' mathematical understanding, we need to ask: what is learning? Wenger (2007) states that learning is inherently human, it is a social endeavour that can provide us with a structure to interact locally and globally and can be a source of social energy. Learning can create and transform individuals' identity. Learning and knowledge are very much dependent on community and identity, on how community and identity interplay with each other and use learning and knowledge. While pursuing their interests individuals in a community engage in mutual activities, they learn from each other, and they share and produce information and knowledge. In one word *learning* is cultural: "living is a constant process of meaning" (Wenger 2007:53), and *practice* is a way of giving meaning to those experiences of everyday life in the way individuals engage with a community and the types of connections they are able to negotiate and participate in. As individuals participate and become involved in the world, they get a sense of who they are socially and personally. "Participation is a source of identity" (Wenger 2007: 57).

Wenger focuses on learning through participation, and he develops a *Theory of Practice* where he analyses the way individuals and communities produce and reproduce "specific ways of engaging with the world" (Wenger 2007: 13). From reading Wegner, it became clear to me that he provides the research framework that could help us analyse how these teachers working together on an assessment project formed a community of practice: through active participation in a join enterprise that matters to them. These teachers involved themselves in an experience where they shared and created resources for negotiating meaning and learning: "meaning arises out of a process of negotiation that combines both participation and reification" (Wenger 2007, p 135). Wenger studies *Communities of Practice* by looking at the interplay and balance among the four dualities mentioned bellow, which we were able to observe in our group as they formed a Community of Practice.

Participation/Reification: The process of acquiring or giving meaning occurred as members of the assessment team participate in the tool creation activities, however there is another component that complements participation and which would allow us to "congeal our experiences into a thingness" (Wenger 2007: 58), Wenger calls this element reification. Reification can be associated to a material object or a set of rules or a mathematical algorithm, an artefact.

Participation and reification are dual concepts since there needs to be a balance between them for effective learning to occur. If there is only participation in a community but no objects are developed to anchor the practice then the community will not have the benefit of the experience of previous engagements, it will have to restart over and over again: “Explicit knowledge is thus not freed from the tacit. Formal processes are not freed from the informal. In general, viewed as reification, a more abstract formulation will require a specific participation to remain meaningful” (Wenger 2007, p.67). Alignment is where there is a successful relation between participation and reification that will bring together individuals towards a common purpose.

In this project teachers are participating in the creation of an assessment tool which will allow them to reflect, evaluate and criticize instruments of evaluation. The learning that will be produced is reified through the production creation and revision of the assessment tools produced by the group.

Identification/Negotiability: An individual’s identification depends on an “investment of the self in relations of association and differentiation” and negotiability is determined by the level of “control over the meaning in which we are invested” (Wenger 2007:188). Identity can be a source of social power, however in order for this to happen the individual must be able to accomplish and negotiate meaningful experiences in the community of practice. “We accumulate skills and information not in the abstract as ends in themselves, but in the service of an identity. It is in that formation of an identity that learning can become a source of meaningfulness and of personal and social energy” (Wenger 2007: 215).

Teachers in a group are able to negotiate some power given that they are the ones who know what is happening in the classroom and are in charge of teaching. Their expertise is invaluable in this project, even if sometimes they are not aware of it. Feeling confident about producing assessment tools adds to the identity of a teacher as a source of knowledge and this confidence can evolve into a source of social energy given that teachers are empowered to act on this new knowledge.

Local/Global: In a *Community of Practice* meaningful learning happens when there is a solid connection between communal competence and a profound respect for particular experiences. According to Wenger if a community is able to give his members through mutual engagement access to competence, in particular newcomers, and at the same time to integrate that competence into a personal identity of participation, then this community of practice fulfils optimal conditions required for acquiring knowledge. Wenger tells us that meaning is a process of negotiation between participation and reification, but “knowing in practice involves and interaction between the local and the global” (Wenger 2007:141)

One way to enrich a practice is by visiting other practices, by looking at products of reification produced by other practices. Crossing boundaries to learn and/or experience other practices is a process that could allow us to increase our learning capability. Knowledge depends on how we are also able to use these practices within broader perspectives. (Wenger 2007: 141).

In the case of this group, one of the first steps was to introduce them to several articles related to several assessment practices and the group discussed how these practices could relate to the tool we wanted to create, with this activity we hope to allow them to develop a deeper understanding of specific assessment goals. These articles connected them with other assessment practices and the point of view from other teachers participating in this exercise. Then materials were created which teachers who were not on the team used and criticized. These processes of participation and reification between the local and the global bring individual meaning, social power and global knowledge to the practice.

Designed/Emergent: Practice is not immune to the influence of a theory of education, and practice is not an unreflective process, however as Wenger points out "even when it (practice) produces theory, practice is practice" (2007, p 48-49), and we cannot have full control of outcomes. Assessment is a process; to create a tool takes time and several tries to get it to be good enough. This is a reality many teachers in our team became aware of as the project developed. Some results surprised us; some questions and materials were not useful and were discarded; other questions we did not think much of in the beginning ended up working beautifully to give us some insight into the children's understanding.

A *Community of Practice* where learning happens and is meaningful should provide its members with various possible ways of evolving out of this history of practice. This set of trajectories represent and include a history where as the individuals participate in this practice their identity is transformed as their identities react, engage and choose among the multiple possible convergent and divergent trajectories. This is how teaching practices are transformed. In addition, teachers can be or can become connected to various communities: local ones like their classrooms, their schools, the community in which the school is located; and global communities like the relations they have with schools in other districts, and cities, as well as with education experts and mathematicians in various universities, school administrators and politicians.

METHODOLOGY

There was an initial meeting where 15 teachers participated in a two day gathering. On the first day teachers were given a series of assessment articles to read and to discuss with others. The articles came from the 1993 NCTM Yearbook: Assessment in the Mathematics Classroom.

Ten chapters were selected from this book (Chapters 3,5,6,7, 8, 9, 10, 14, and 15).

The teachers were divided into 5 different groups and each group read 2 articles. After a discussion, teachers presented the most important points in the articles they just read and they had to tell us if and how these articles had brought new knowledge into what they knew about teaching and assessment. A discussion ensued with the rest of the group.

On the second day teachers discussed the different programs used in the schools, the curriculum and what were the skills they thought students needed to have in order to be successful in third grade.

After the meeting teachers were given a questionnaire which was used by the researchers to find out what they learned from the meeting, what they thought it was missing and what they thought should be the next steps.

With all the information that was obtained our research team produced the following materials:

- A ten lesson review with formative assessment for the first two weeks of class containing what kids entering third grade should know.
- A summative to give third graders after the two week review.
- A summative assessment to give thirds graders at the end of the year.

All these materials were sent to the teachers in the team. Six of the teachers made comments on the materials that were sent to them and the materials were changed accordingly. A test was given to the students at the end of the school year.

From the results of the end of the year assessment, some changes were made to the assessment tools. In addition, a pamphlet with activities emphasising learning and understanding of place value and various ideas around our number system was produced, given that students showed some misunderstandings in these areas. A test for students starting grade 4 was also produced, which was very similar to the end of the year assessment for third graders. The aim of this test was to find out how much students forget during the summer and how prepared they are when starting grade 4.

Currently we are waiting for the results of the assessments happening at the beginning of the year and we will continue the process of refining the assessment tool at least for another year.

We also gave teachers the contact information of all the teachers in the team and encouraged communication among all the members, which they have been actively doing. A meeting is scheduled for the group this fall.

Our method has been to bring together assessment materials, disseminate them among the teachers, ask for their input and also look at the results from the testing. We share the results and make changes to the tools given the teachers’ comments and as it turns out to be their own research until we are all satisfy that what has been produced is a reasonable product.

RESULTS

Here are some of the comments made by the teachers at the end of our two days of meetings. It is important to point out that most of them sincerely appreciated the opportunity to get together with other teachers teaching at the same level, and to be able to exchange ideas and learn more about assessment:

- It got me excited to think about Math assessment. There are a lot of possibilities/resources, etc and I'm hopeful that I cannot only improve my own math assessments, but also that this new tool will be developed, tried, tested and helpful for me in the future to have assessment to match the curriculum and be easy to use (and other educators as well!) It's exciting!
- Sharing ideas with each other-amazing how many similarities in our schools.
- The best about the last 2 days was talking about assessment and how you can change it to be more encompassing of the students. Also, just discussing ideas with each other. It was great to see the goals for what's next and the discussion around it.
- The chance to improve my understanding of math assessment. The chance to network with other same grade teachers and share and learn. Exposure to new ideas and renewal of commitment. The opportunity to be a part of something like this!
- Collaboration time, sharing info on strategies and resources. Reconnecting the Generations. Brainstorming/communicating with others in similar situations. Common concerns. Not alone. How other schools, teachers do things.
- The extra support by our experts answered a lot of questions, gave me many ideas.
- I was hoping that we would be leaving with assessments. But I understand more really what our plans are and look forward to working with everyone.

These comments are significant in that most of the teachers in the team kept in communication after the meeting; mainly to exchange resources for their teaching. As I mentioned before 6 teachers made comments on the materials which were developed, however the rest of the teachers made comments on their comments. Except for two teachers, there was full active participation of the group.

It became clear that this group of teachers had become a community of practice when the results from the first assessment came back and some of the comments from the researchers' team made quite an impact among the teachers.

We asked students to: Add or Subtract as shown:

	7 4 5 + 1 3 4		4 0 4 + 1 2 3		4 6 5 - 1 3 4		3 9 8 - 3 2 4
	4 8 2 + 4 8		$4.65 +$2.85		$5.35 - $2.75		6 0 0 - 2 7 6

Most students did the first 6 items correctly, but it was almost universal to see the last two done incorrectly. One of the main assumptions about these kinds of mistakes is that these students saw no meaning in the "borrowing" algorithm they were attempting to use. The way students will usually do the subtractions show that there is also a problem with dealing with zero. Students will usually give the following results:

6 0 0 - 2 7 6 4 7 6		6 0 0 - 2 7 6 2 3 4

However, another part of the assessment which was puzzling was that a substantial number of students made quite a singular mistake when asked the following questions:

(a) Count up by 3's from 400.

400, 403, ______, ______, ______,______, 418

(b) Count down by 3's from 266.

266, ______,______, ______, _______,251

Many students had errors when they had to "go across zero" -- for (a), after 406 and 409 we would see 411 or 413 instead of 412, and for (b), after 260 we would see 258 or 256. Other errors were also seen, but these were far more common. This mistake together with the mistake in the subtraction problem showed that there was a serious issue concerning zero and how students have difficulty understanding "zero as a place holder".

When this connection was brought up to the teachers, a great discussion started among them. The teachers who had students who had made the mistake became interested in finding out from their students how they counted and why they were skipping zero in these problems. They found out that they saw zero in most instances as “nothing” and therefore when they were counting up or down zero was skipped. However this was tricky because they could not ask, why did you counted wrong? Some teachers admitted they did this the first time they approached their students with this question, they had to learn how to learn to ask without leading the answer.

Teachers were amazed of what they could learn from connecting mistakes in two problems, and learning how assessment can work to acquire a deeper understanding of mathematics not just for the students but for themselves as well. Most of them confessed that they just looked at problems in a test as indicators of “individual concepts” and don’t see connections. From this discussion a pamphlet was developed to address this issue with zero and the teachers have been giving feedback on this by doing their own research. They are also distributing this pamphlet and talking about these results to teachers who were not in the original team. Somehow out team seems to be growing.

CONCLUSIONS

American and Canadian teachers work very much in isolation, and for this reason their teaching scripts do not evolve much after they become teachers (Wilms 2003). In addition many of them are removed from any kind of research that is done about teaching and learning mathematics. In other cultures which seemed to be more successful in teaching mathematics, teachers have time to work together and develop their practice and assessment tools (Ma 1999) and for example, in the case of Japanese teachers, they are contributors to the development of teaching theories (Stigler and Hiebert 1999:125, Takahashi 2000).

This paper showes that teachers have great need to work with other teachers, to exchange ideas and they can even be interested in doing some detective work on their own to improve their knowledge as well as their students’ knowledge. Teacher acquired a social energy as they felt validated through the work that was done and the materials that were produced with their help.

What is incomprehensible to me is why if research shows that these kinds of communities are needed if we are to improve teaching practices in this country, there is not a consistent effort to do so. We should look for way to bring teachers together working in projects like the one described in this papers. Not much was needed to create this community of learning and much was gain..

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TEACHERS AND RESOURCES: HOW MIGHT ONE TRACK THEIR INTERACTIONS?

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This paper aims to identify and clarify some interactions between teachers and the resources they use while teaching or while planning their teaching. A list of possible conceptions of resources is discussed as well as a model to study how teachers pull material together to generate resources via a documentation system. The goal is to draw attention to likely artefacts and re-sourcing strategies that could illustrate teacher capacity.

Firstly, I would like to discuss what resources are in a broad context and in my area of interest. Secondly, I wish to describe a system of analysis of interactions between teachers and resources during teacher planning. Lastly, I would like to explore some ideas regarding links between teacher capacity and the material they use in class.

There seems to be a wide range of resources that can be used in mathematics classrooms from a pencil to a blackboard, textbooks, manipulatives, activity sheets, evaluation tools, Blackline masters, digital tools and computer software, etc. For example, in the Western and Northern Canadian Protocol, the common Mathematics Curricula have established lists of resources recommended by this provincial and territorial consortium. The learning resources, therein, could fall into a category of curricular resources, which encompass the following elements, as per government publications: Print, Graphics, Illustrations, Video, Digital Resources (software, interactive media, on-line resources) and Manipulatives. The British Columbia Ministry of Education specifically provides evaluation criteria to assist teachers, school principals and school district authorities to select appropriate resources. The B.C. Ministry of Education also defines "recommended resources" as *resources that have undergone the provincial evaluation process* …(Integrated Resource Package, 2007) Resources such as assessment models are part of the curriculum packages. They include assessment rubrics and check lists for the expected learning outcomes, which are standard-based objectives. Furthermore, these approved resources include textbooks, software such as Geometer's Sketchpad and Graphers, Blackline masters (producibles), geoboards, videos for student on mathematics applications in daily life. Again all of the above resources are curricular in nature.

Other types of resources may not seem so obvious but are also important. These resources are sometimes so basic that they are taken for granted but are necessary for schools to function or to function better. Some are physical in nature such as buildings and services, others are of a human nature such as qualified staff, teacher-pupil ratios (Adler, 2000). Adler argues that a wider view of resources is required and should include cultural components such as language. One could also add to the list class composition, money, scheduling and school traditions and culture (school grammar) (Cuban & Tyack, XXXX) to the list but eventually it seems that we are moving away from resources and into required conditions for schooling to occur. Adler (2000) tabulated what different scholars consider resources to be. Her table illustrate a summary of her findings based on what various scholars understand by resources (see table 1). Adler's list is exhaustive and seems to reflect our customary view of resources. Adler's most salient point is perhaps her interpretation or resources as the verb re-source; teachers re-source themselves in finding, using, modifying resources. This definition of resource is very appealing, has a lot of potential for the professional growth of teachers and stresses the importance of the work teachers do when using new or re-visited resources.

TABLE I

Categorisation of Resources in School Mathematics

Resources	Exemplars	Issues
Basic Resources – For the Maintenance of Schooling		
Material	school buildings, water, electricity, fence, desks, chairs, paper, pens, etc.	absence makes call for "more" obvious and necessary
Human	teacher-pupil ratios, class size, teacher qualificationsagreed as basic, but scope and content of qualification, and what constitutes optimal class size, are contested	
Other Resources and Their Transparency		
Human		
Persons	teacher's knowledge-base; parents	scope, content, weightings, orientations all contested
Processes	collegiality	for maintenance of the practice as well as change

Material		
Technologies	chalkboard, calculator, computer, copier	need for invisibility to see through technology to practice
School mathematics materials	textbooks, other texts, cuisenaire rods, geoboards, computer software	mathematical meaning not obvious; mathematical meaning and pedagogical possibility is built-in; in use "learner centred" pedagogy – can become too visible
Mathematical objects	proofs, number lines, magic squares	specifically mathematical, but with social history, also need to be visible and invisible
Everyday objects	money, newspapers, stories, calculators, rulers	uses outside of maths, so need to be visible and invisible
Social & cultural		
Language	L1, L2, code-switching (CS), verbalisation, communication	assumptions that CS and talk are enabling; need to be visible and invisible
Time	time-table; length of periods; homework	structuring of time needs to be visible and invisible; with new pedagogies or when schooling breaks down, can become too visible

From Adler, 2000

Contrastively, in France, different views of resources are emerging. Gueudet and Trouche (2008) consider resources such as textbooks, software and websites to be *des ressources-artefacts* (*potential resources*, my translation and clarification). These resources are inert but are what teachers could draw from to generate a document, teaching material that they will use. Rabadel (in Gueudet & Trouche, 2008) claims that an artefact requires a scheme of utilisation in order to become an instrument (diagram 1). It is the instrument that a teacher uses in a classroom. The scheme of utilisation (delivery parameter, my interpretation) is the invariant in the multiple uses that the teacher makes of the instrument. Gueudet and Trouche employ different terms for similar conceptions; they place resources and a scheme of utilisation together to form a document (diagram 2). This document is the task that a teacher has generated for his students. Accordingly, one could argue that resources could be viewed as having *potential*, when they are not used with specific delivery parameters or not used at all, but they become *active* resources when they are employed deliberately and effectively with students. Consequently, no matter the resources, there would be only two categories and they are dependent on the teacher's actions onto them: *potential resources* and *active ones*. Potential resources are the ones without a scheme of utilisation and active resources (documents) are the ones with a teacher-designed scheme of utilisation. The significance of the teacher is clearly marked in this perspective.

$$\textit{Instrument} = \textit{Artifact} + \textit{Scheme of Utilization}$$

Diagram 1 (Rabadel's view)

$$\textit{Document} = \textit{Resources} + \textit{Scheme of Utilization}$$

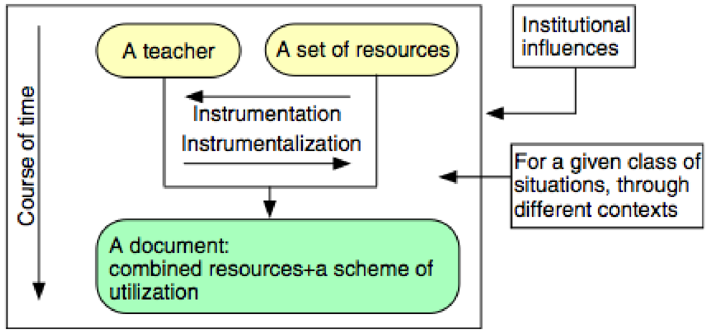
Diagram 2 (Gueudet's & Trouche's view)

Other researchers, without clearly defining resources, have measured the impact of additional or available resources within a school on student success (Cohen, Raudenbush & Ball, 2003). Results are encouraging in so far as there is a connection between resources and student success. This being said, what is one to do? Purchase more resources without knowing the varied types of resources available? Such a situation seems to have taken place in Alberta schools in regards to Internet technology, when significant expenditures were dedicated to computer resources (Gibson & Oberg, 2004). This study showed that resources alone are not sufficient to guarantee or demonstrate effective use of resource affordances in a manner that may positively influence learning and teaching. More professional development seems to be required to bring about effective use of resources by teachers. One possible approach may be to increase teachers' capacity to use resources or to re-source themselves.

As I mentioned earlier, resources are varied and are viewed differently. I am specifically interested in all curricular resources as described at the beginning of this paper, and in persons as resources such as experts or colleagues. I am not interested in time constraints and other contextual elements, though I recognise their pertinence to the teaching environment.

In this second part, I wish to discuss the Gueudet and Trouche's mathematics teachers' documentation systems. They state that *the intrumentalization dimension conceptualizes the appropriation and reshaping processes* and that *the intrumentation dimension conceptualizes the influence on the teacher's activity of the resources she draws on* (Gueudet & Trouche, 2008).

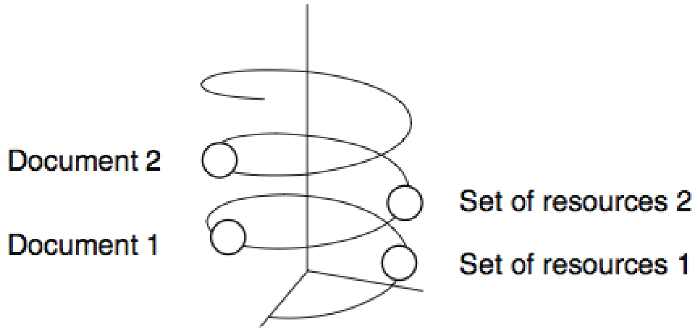
Fig. 1 Schematic representation of a documentational genesis



The interactions between the resources and the teacher shape both the teacher and the resources. In other words a teacher can become more adept at using resources in creative ways and can generate resources or modify existing ones for set purposes. According to Gueudet and Trouche there is an evolution in the way teachers create documents for teaching. This evolution manifests itself in the way teachers appropriate and transform resources as they work alone or with other teachers (Gueudet & Trouche, 2008). The increased ability that teachers have to generate and to modify resources, I would suggest, is an increase in capacity.

In their research Gueudet and Trouche analysed the professional evolution of teachers by attending to the integration of new material, to changes of practice, knowledge or beliefs. They studied teachers' usage of software over a three-year period, followed on-line communication between teachers who were coached in integrating technology into their teaching, and visited nine teachers in their homes to learn about their documentation systems. A list of items judged important to the teacher's documentation was drafted, three documents were presented and explained by teachers. -It's important to mention that French teacher typically go home to plan when they don't have courses. -Teachers were asked in interviews about past evolutions for the last ten years and about expected ones for the next ten years (Gueudet & Trouche, 2008). The researchers noticed that the resource and document have a dialectical relationship. *We consider here accordingly that a document developed from a set of resources provides new resources, which can be involved in a new set of resources, which will lead to a new document, etc* (Gueudet & Trouche, 2008).

Fig. 2 The resource/document dialectical relationship



Gueudet and Trouche focus on three aspects of the teacher generated document:

a) material component such as paper, computer, etc
b) mathematical content
c) didactical component such as organisation

The researchers conclude that *the analysis of the documentation system permits a better understanding of professional development*.

Lastly, I would like to point to some artefacts that need attention for they may be indicative of increased teacher capacity. As stated by Gueudet and Trouche, there is an evolution in the way teachers generate documents and these can be used to track professional development. Based on this conclusion, though Gueudet and Trouche were mainly dealing with digital resources, any resources could be used by teachers to generate documents. Researchers ought to record what resources employed by the teachers as this could indicate teachers' migration towards a variety of resources, teachers' combination of many resources into a single document or teachers' ability to be more creative in their documents. It would also be important to track how documents are being revisited for improvements or as resources in their own right.

Teachers' descriptions of their most effective documents with students, the documents themselves and the resources that inspired the document, may shed some light on task design. Teacher's favourite documents should be analysed based on Gueudet and Trouche's three points:

a) material component such as paper, computer, etc
b) mathematical content
c) didactical component such as organisation

Unlike Gueudet and Trouche's question regarding the past ten years and the next ten years to come, I would focus on the evolution within the group of colleagues and possibly into the close future for predictions. I feel that this would be more closely linked to the artefacts produced and documented.

Another significant aspect of teacher capacity is connected to the classroom practice and it would therefore be essential to gather some data in this regard. This could be accomplished by having classroom observations or by sharing student work with colleagues and by reflecting on the documents and their delivery in the classroom.

Finally, teachers' testimonies about their own professional development is important to gage their beliefs about the process of documentation.

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ON EXEMPLIFICATION OF PROBABILITY ZERO EVENTS

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In this paper the example space of pre-service secondary teachers on probability zero events is examined, different aspects of such events as perceived by the respondents are discussed. Meanwhile the participants' understanding of "more complicated" is explored.

MOTIVE (INTRODUCTION)

A question posted on an online discussion forum gave rise to this research: "Give an example of an event with probability zero". The answers (10-12 answers) provided to the question, suggested that the common perception of a zero probability event is that of an event that cannot happen. The examples given were all finite and except for a few, all of the examples were describing logically impossible events such as rolling a 7 with an ordinary die.

RESEARCH BACKGROUND

Despite the plentifulness of probability language literature found in mathematics education (Lester, 2007), little or no attention is paid to zero probability events.

THEORETICAL FRAMEWORK

The importance of experiencing with examples has always been dealt with in theories and frameworks for describing the learning of mathematics. Watson & Mason (2005) define a concept as being aware of dimensions of possible variation and with each dimension, a range of permissible change within which an object remains an example of the concept. They also develop the idea of example space as collections of certain types of examples and suggest this idea as central in teaching and learning. Another study highlights that when invited to construct their own examples, learners both extend and enrich their personal example space, but also reveal something of the sophistication of their awareness of the concept or technique (Bills, 2006).

Goldenberg and Mason shed more light on the construct of example space and on how it can inform research and practice in the teaching and learning of mathematical concepts (Goldenberg, 2008).

Building on the works of Watson and Mason and Goldenberg we examine the example space of the participants of the study on some certain aspects of probability.

METHODOLOGY

The participants were pre-service secondary teachers (n=30), most of them with a minor in mathematics, some with biology or history background.

The task: They were asked to:

Give an example of an event with probability zero.

Give an example of a more complicated event with probability zero.

They were supposed to submit their answers in writing within a week, so that they could work on the task when it was the best for them. The task was followed by classroom discussion around the general notion of probability zero events and the given examples.

The research questions were:

How pre-service teachers interpret and exemplify probability zero events in variety of situations?

What is their personal example space with regard to events with probability zero?

In addition to these questions, their answers were used to locate and elaborate significant aspects of their perception of probability zero and their understanding of ‘more complicated’.

DATA ANALYSIS

1. Definitions

1.1 Finite and infinite

Among all possible ways to look at the data we chose to look at them first in terms of finite and infinite.

We mean by finite, an experiment with a finitely many events in its sample space. Flipping a coin twice, rolling a fair die once or picking a random integer from the set of all 10 positive integers are examples of finite random experiments. Evidently all finite sample spaces are countable.

We call an experiment infinite if the sample space has infinitely many events in it. Two types of infinite are distinguished: countable and uncountable: flipping a fair coin infinitely many times is an example of having a countably infinite sample space, while picking a random number from the interval $[1,2]$ is an example of an uncountably infinite experiment.

1.2 The type of zero probability

After the first examination of data, other categories emerged. This time the examples were classified in terms of the type of probability zero, or in other words how or why their probability is zero. Four categories are distinguished from the data:

Logically impossible zero probability: Rolling a 7 with a standard die for instance.

Probability estimated to be zero: Flipping 10 coins all sitting in heads, an event with a probability 0.00097, which is estimated to be zero by some.

Probability is converging to zero in limit: tossing a fair coin infinitely many times, all of them sitting in head.

Measure-theoretically explainable zero probability: picking a certain number from [1,2].

According to the classic definition, probability is a fraction. There are two different ways that a fraction would be equal to (or perceived to be equal to) zero:

1) Zero divided by a non-zero number. The logically impossible examples are of this type. The number of plausible events within the sample space is zero resulting in a probability zero event.

2) A nonzero number divided by infinity: those examples in which probability is converging to zero in limit are from this type.

It is worth noticing that the few measure-theoretically explainable zero probability examples could fall into both of the above-mentioned categories. The classroom discussion revealed that some perceive the probability of picking a number from interval [1,2] (e.g. 1.456) to be zero in terms of one plausible case out of infinitely many possible cases. However it is also explainable as zero divided by one, the measure assigned to points versus the measure assigned to real intervals.

2. The summary of data analysis

In the following tables a summary of data with regard to two categorizations is presented:

	Finite	Infinite
Countable	53	5
Uncountable	-----	2

Table 1: The data summary in terms of Finite or Infinite.

Logically impossible	Estimated to be zero	Converging to zero in limit	Measure-theoretically explainable
50 examples	3 examples	5 examples	2 examples

Table 2: The data summary in terms of type of probability zero

3. Probability generators

The examples were examined in terms of the probability generators used to make a random experiment. The impact of classical textbook objects for teaching probability on the example space of the teachers is conspicuous.

From 60 examples, 32 use dice, 14 use coins, 8 use marbles in a bag (or equivalent variations of it), one uses a spinner, one uses a deck of cards, 2 use picking random numbers and 2 use real life objects such as vending machine and street crossway.

4. The analysis of the second example:

Watson and Mason discuss the “give another example” thoroughly in their 2005 book. From the examination of second examples in this study it turned out that in 24 out of 30 the first and second examples are of the same category (both in terms of finite-infinite and the type of probability zero).

5. Adding more complexity

Another interesting thing to look into was to note how the participants have made their second example “more complicated”:

It turned out that combining is quite a popular technique to get more complicated events. A total of 20 examples out of 30 were combining two events in order to give an example of a more complicated event.

Three different types of combination are recognizable from the data:

The impossible-possible combination:

In this type of example the impossible event described in first example is frequently used as the impossible component; first example is rolling a 7 with a fair die while second example is asking for rolling a 5 and then rolling a 7 with a fair die.

The impossible-impossible combination:

Some have conceived "more complicated" as an event even less likely to happen than their first impossible event. The second example is a combination of two probability zero events.

First: Getting infinitely many 1's when rolling a fair die infinitely many times.

Second: Getting all faces when flipping a coin infinitely many times while getting infinitely many 1's when rolling a fair die at the same time.

The possible-possible combination.

Another way to get a "complicated" event happens to combining the possible events in the sample space such that their intersection is empty, which at the same time makes the event a logically impossible example. The frequent example of this type is getting both 3 and 4 at the same time when rolling a fair die once.

As a second technique to add more complexity, some have used generalization; the second example is a generalized form of the first, so it is both perceived to be a zero probability event and more complicated. First example: rolling two dice and getting (6,7), second example: Rolling two dice and getting (i,j) such that i+j=13, for example. As Watson & Mason state leading the learners toward generalization is one of the merits of asking for another or more complicated example.

6. Number treatment

Any task designed for different research questions that deals in a way with numbers could reveal some by-product facts about people's perceptions on numbers and part of their real number sense. The task described in this paper is no exception. One of such interesting by-products is the different treatment of numbers found in two of examples: in both examples the random experiment is to pick a random number from a real interval and the probability zero event is to pick a certain pre-determined number, 4.3275 and 1.0000097 respectively. It could not be helped but notice that the examples are of the same nature: they give us "safe" examples of numbers that are not likely to be picked. However both respondents are aware of the fact that any number has the same probability zero, but they might feel that may be numbers like 0,1,2 or $\frac{1}{3}$ are not safe enough to mention. Our unsupported speculation is that it could be because they have been frequently asked as students to locate integers and simple fractions like $\frac{1}{3}$ on the number line during their education, but never have been asked or needed to find 1.0000097. The first numbers are then analogous to big bold dots or thick dashes on the number line; they are exposed numbers as opposed to anonymous numbers living safely in the oblivion of atom-size inseparable habitants of real line. In short, lack of encounter with integers as embedded in the context of real numbers is suggested as justification of this number treatment.

7. A word game or an interesting twist?

As mentioned before the data collecting process was followed by classroom discussions of the examples. Most of the examples as seen in the results fall into the category of finite and of type logically impossible. A minor group that have carefully avoided both these type of examples, argued that rolling a 7 is not a good example of a zero probability event, since getting a 7 is not an event. They expressed their definition of sample space to be the set of all possible outcomes provided that they are equally likely. Meanwhile their agreed upon definition of event was: any subset of the sample space. While 7 is not among possible outcomes, it is not in sample space and not being in sample space makes it not eligible for being an event! Accordingly in no finite random experiment a probability zero event could be found except for the empty set, or in other words the event that none of the possible outcomes would happen.

In discussions followed this word game, a second word game was brought up. That the definition of the sample space might create other awkward situations like this: When flipping a fair coin twice the sample space is {HH, HT, TH, TT}, a set of four (2^2)equally likely members each having a probability of $\frac{1}{2^2}$. By the same token, tossing a coin infinitely many times leaves us with an infinite sample space, each event being an infinite string of heads and tails. Since the probability assigned to each of these events are zero ($\frac{1}{2^\infty}$), they are impossible by definition. The sample space then should be empty since it can only include all possible outcomes!

The above-mentioned arguments give evidence of vagueness in definitions of sample space and event and the confusing affect that the word “impossible” (that seems prudent to be replaced with “improbable” in text books) puts on the understanding of these concepts.

SUMMARY

The example space of 30 pre-service secondary teachers on probability zero events was studied through their examples. Their example space was found to be rather limited and dominated by the standard probability teaching examples. Also the expert example space seemed to be completely missing. ‘Complicatedness’ was mostly embodied in combination and generalization.

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SOME FUNDAMENTAL QUESTIONS AND EGREGIOUS ERRORS IN EDUCATIONAL THOUGHT

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This paper argues for the crucial importance of knowing what we are trying to achieve in education, and hence in pursuing the questions of what it is to be human and well-educated. The widespread contemporary fascination with relativism and generic skills is criticized, as is our unwillingness to discriminate in education. In conclusion a knowledge-based curriculum is resolutely defended.

I am truly grateful for this invitation, because I admire the math program, and I can't say that of all our programs. There are of course some other programs I admire: for instance, our programs in psychology and the arts, and I admire them not necessarily because of any affinity with the subject matter – indeed I am skeptical of at least some of what my colleagues in psychology get up to, just as, I'm sorry to say, I am all too ignorant about math. But I admire these program areas nonetheless, because they seem to me to offer coherent, systematic and academic organization and teaching, which frankly cannot be said of everything that goes on in this or any other faculty of education.

Sparing their blushes, I also admire your professors. Again, I say this not out of idle or conventional politesse, but for good reason. I have known Rina Zazkis for a long time and wish there were more like her. I don't know Nathalie Sinclair and Peter Liljedahl very well personally, but I am familiar with the high quality of their work and their teaching, from committees on which I have sat. And Sen Campbell, though frankly I can't understand a word he says or what his research is about, except when he is plainly wrong, somehow has nonetheless always had my affection and respect. Again I do not have a similar respect for all of our colleagues.

You may say: why start off with an implicit criticism, not to say rebuke, of some of my colleagues, some of our programs, and some of our work here. Isn't this very politically incorrect, very un-Canadian?

Perhaps it's just because I was born and brought up in England. It's been said of the English, and I'm quoting the novelist, A. N. Wilson, that "if [they] are rude about a man, it implies a level of intimacy with him. If you enter a room in England which is full of men who know each other well, the likelihood is that they will be insulting each other, though with smiles on their faces. The banter may be real or unreal, or a bit of both. But rudeness on this level is a signal of friendship.

Englishmen are rude to their friends and polite to strangers or enemies" (*Penfriends from Porlock*). So now you know that I count Sen among my friends.

Well, there may be some truth in Wilson's claim, but I also have other reasons for my seemingly aggressive start. I think that in general, it's high time we stopped calling a spade 'an interactive device for transmitting ideas relating to earth movement into practice', and called it a 'bloody shovel'. I think that it is time we stop being relentlessly polite, focusing on the sincerity and authenticity that people exhibit, and worrying about their self-esteem. I think it is time we point out when the emperor is wearing no clothes, and put an end to pretension and jargon. It's time to put a bit more emphasis on clear and plain speaking in academia, and on the quality of people's argument and the sense, or lack of it, in what they say.

It's high time, in other words, to take a good critical look at what is going on in universities generally, and faculties of education in particular, and to ask what exactly some of it has to do with serious scholarship and a genuine attempt to improve and transmit a reasonable understanding of our world.

But what am I going to talk about? As I've already suggested, math is not my strong suit, being by upbringing a classicist, historian and philosopher, i.e., steeped in the humanities, which I am very grateful for, but woefully lacking in terms of math and science, which I deeply regret. So I really can't talk to you about math per se.

And not much about math teaching either, because, and this point is central to both my thinking generally and to my remarks today, I do not believe that you can reasonably talk about the teaching of X when you are not competent in the field of X. In fact this seems to me so obvious that I'm almost embarrassed to state it. But the sad fact is that it is at a variance with what is possibly the majority view in the faculty and the view enshrined in the PDP program. Namely that teaching is a matter of mastering a number of generic skills and strategies, such that it doesn't matter what you are asked to teach provided that you are a good teacher. A good teacher can teach anything.

This, it seems to me, is palpable nonsense, though it may partially explain why we still accept the notion of generalist teachers for students up to twelve years of age, and why we have a Teacher on Call system that often requires people to teach things about which they know little or literally nothing.

While of course one concedes that some individuals are in broad and general terms better teachers than others, regardless to at least some degree of what is being taught and to whom, the general point remains that you can only be a good teacher of something that you understand. Teaching, like most things, is context specific. Just as, while there do appear to be some serially bad husbands, and may for all I know be some serially good husbands, nonetheless, in most cases the fact that I am a good husband to one person doesn't mean that I necessarily will be to another – just so, the fact that I am a good teacher of philosophy to eager undergraduates tells you nothing about my capacity to teach math to reluctant fourteen-year-olds.

So, what am I going to say that may nonetheless be of interest to math educators? I am going to air some of my concerns, convictions and hopes about education and teacher education generally, and try to link this where I can with a few no doubt rather banal remarks about math and math education.

The most important question in education for anyone – parent, professor, principal, teacher of math or art – is: "What should we teach?", "What should the content of the curriculum be?"

Incidentally, though I am an advocate of a subject-based curriculum, I don't want to beg that question. Even if you believe in a curriculum based on student choice or the development of various skills, that is still a view on the nature of curriculum; it's still a position on the issue of the content of curriculum. So the crucial question remains "What should we teach?"

My concern here is partly that it is unfashionable to focus on this question. People are not by and large very interested in justifying what they teach, still less in considering what else should be taught or wrestling with priorities. And partly that the question is de facto by-passed by people who focus on questions such as how to teach, or, worse, how to instil self-esteem, or even how to make people happy. But how we should teach must be at least partly dependent on what we are teaching (and to whom); and whether it really is the prime or indeed any concern of educators to develop self-esteem or try to make people happy is highly debatable – and as such ought to be debated. Similarly, the question is by-passed by a society that, generally speaking, takes it for granted that what should be taught is whatever contributes directly to a sound economy, getting a job, and a docile non-obese electorate who believe passionately in recycling and climate change.

So I am assuming that all in this room should at some time give or have given serious thought to the question of what the school curriculum should ideally include and, by extension, why we should be teaching math.

But if that is the central question, it surely invites two further fundamental questions, one looking backwards, so to speak, and the other looking forward: if we are to have a reasoned position on what we should teach, we surely need to have some clear idea of what we are trying to achieve in the name of educating people. What is our teaching of math supposed to contribute to? What kind of a person are we trying to develop? In other words, what is an educated person?

A brief aside on the nature of this question, which again is a rather unfashionable question. Critics have tended to assume that those who say that this question is crucial presume that there is some definitive answer to it, such that the philosopher will spell out the criteria that define education with the sort of certainty with which the mathematician will spell out Pythagoras' theorem. But this, the critics say, is simply not so.

Well, the critics are quite right that there is no definitive answer to the question of what it is to be educated. It is what has been termed an 'essentially contested concept', at least partly because it is value loaded, and what we regard as being well educated will necessarily fluctuate with our views of what is important, desirable and admirable.

This whole business of philosophical analysis and some of the arguments surrounding it goes right back to Plato's theory of Ideas or Forms. In outline Plato observed, first, that if you are going to make claims about whether people are well or badly educated you had better be able to provide a clear and coherent account of the concept.

Secondly, that if both you and I are relatively well-educated there must be something in common between us.

And thirdly, that arriving at these clear, coherent common conceptions, though itself an abstract activity, is of extreme practical importance.

But it is generally thought that Plato also thought that there was a definitive Form of everything, such that in principle, if we all search diligently we will all come to see the same essential properties of squareness, education, health, mud or anything else. And that, as I say, is generally regarded as plain wrong. Education isn't out there waiting to be discovered by the inquiring mind: we create a conception of education, and of course this may differ from culture to culture or person to person.

I've made this digression because, although I think the issue is more complex and subtle than this, and that Plato is not necessarily as wrong as is supposed, I also think the observation that there are essentially contested concepts, though correct, should not be taken to somehow invalidate the business of analysis.

Yes, the ancient Greeks had a different conception of education from us, at least in some respects, and yes, you and I can argue about whether having some historical understanding or some mathematical ability is or isn't an essential part of being educated. But that does not make the business of analysis otiose. On the contrary, it makes it even more important, partly so that we understand each other, partly so that we don't argue at cross-purposes, and partly because for each of us it is the journey rather than the destination that matters. What matters is that each of us should be able to articulate a clear, coherent, consistent conception of education that is compatible with our various other views and values. Only then can we have a serious debate about whether this or that is or is not important in relation to the school curriculum. As things stand, there seem to be about as many definitions of education as there are professors; but what's really worrying is that the majority of them are not clearly articulated and in many cases are barely coherent.

So, as I say, one further question that the question "What should the school curriculum comprise" invites is "What is it to be well-educated?" And it is a tragedy that this once staple question in the philosophy of education is now generally ignored even by so-called philosophers of education, and certainly by most other researchers and scholars.

But to say that we consider some characteristic or characteristics essential to being a well-educated person itself invites this second question: What is a person? Who are we? What are we? What is what Aristotle would call our "*telos*" or end, our essential excellence? What distinguishes us from other sentient beings? What is it to be human?

I have no hesitation in answering this question by saying that it is our minds that are the crucial criterion of our humanness. Of course we have other distinguishing characteristics such as a prehensile thumb, walking upright on two legs, size of brain and so forth. But 'human' is another partly evaluative term, or if you prefer, it has an evaluative sense as well as purely scientific or biological one; and it is the notion of humans as something to be proud of that should interest us. For education is clearly, in any age or culture, formally a matter of trying to make the best of people, enabling them to reach their full potential.

Now let us clearly distinguish between mind and brain here. A massive amount of confusion is caused by people failing to distinguish them. The brain is the physical object housed in the skull. The mind is an epiphenomenon. It is non-material and consists essentially in our understanding. In so far as one lacks understanding, so far one has an underdeveloped mind. Whatever we may have to say about the brain, whatever mechanical facts we are able to adduce, it seems clear that the truly significant distinctive feature of humans is a capacity for a unique kind of thinking, specifically, our capacity to think propositionally which allows us to hypothesize, speculate, calculate, introspect and so forth. Particularly worth noting is our self-awareness: we are as far as we know the only species aware of its own mortality This is of course only possible because of the brains we have, but it is not the same thing as the functioning of the brain: the brain, which can be described in purely mechanistic terms gives rise to thinking which has to be described on its own non-mechanistic terms. Thus man can be defined, as Aristotle observed, as a thinking animal; and it is our ability to understand that that constitutes the mind that is our essential characteristic. Once our mind is developed, we are able to proceed in accordance with reason and logic as well as in response to various stimuli. And this is our most remarkable characteristic: we can act in this way or that, not relying on instinct or stimulus/response mechanisms, but by calculating and recognizing the various consequences of different courses of action. We can be impelled by logic.

A couple of points: other animals do not seem to have this capacity, despite to me unconvincing attempts to teach them and even to claim success. Chomsky many years ago hypothesized that we are programmed uniquely to speak the type of language we do. That may be so, or it may be that one day another species will be able to engage in discourse with us. If that were to happen, it would not upset my point. It would mean that we would have to either include that species in the class of 'humans' or acknowledge that it was no longer a unique defining characteristic. But it wouldn't make it less significant to our notion of our humanness. And if they ever devise a computer or robot that truly thinks in the autonomous way that humans do, which I believe to be logically inconceivable by the way, but, if they do, the conclusion to draw is not that we are merely machines, but that computers too need to be sent to school and educated [cf. Margaret Somerville, Vancouver Sun, Aug 24 and response by Mark Mercer, Aug 27].

I should also note that I accept the implications of my view to the effect that newly born infants, and those who are, sadly, seriously brain-damaged or, for whatever cause, unable to reason, are to that extent less than fully developed as humans. This is always uncomfortable – many people here will have new-born babies or relatives with degenerative diseases of one sort or another. But I have not suggested that our mind is the only thing about us worthy of consideration, I have not said that such people are not human, and I have not said that only humans deserve love and respect. As a matter of fact I think all sentient beings should be treated with love and respect. I would no more be cruel to my dog than my mother. But I do think that ideally the human mind should be developed. And it is entirely compatible with love and respect for the newly born or the senile to acknowledge that they are operating at less than full human capacity.

So I can now answer all three of my fundamental questions succinctly: I believe that the school curriculum should provide the understanding that is the hallmark of the developed mind, which in turn is the distinguishing characteristic of the well-educated and the fully-developed human being.

So my positive view is that all educators are in the business of contributing to the development of mind or enlarging understanding. This obviously raises the question of what type or range of understanding is required. Does the educated person have to understand certain things rather than others, or does it not matter provided there is the capacity to think? I shall return to this in a moment.

But first some concerns about the current state of play. I have to say that simple as the view I am outlining is, and notwithstanding the fact that perhaps few would actually explicitly disagree with it, most of our theorizing and at least some of our practice proceeds in ways that implicitly and effectively reject or ignore it. This brings me to the egregious errors referred to in my title, of which I wish to draw your attention to three.

First, any notion of providing understanding, and still more of selecting worthwhile or important content or subject matter, is to some extent threatened both in theory and practice by the widespread contemporary fascination with relativism of one sort or another.

I like to think that you may be less plagued in math by this idea that everything is as you see it, or that nothing is true in itself, but only true for me or for you, than we are in the humanities. But perhaps you're not. You may be aware of a recent case in physics. A celebrated physics professor, Alan Sokal, became so irritated by so-called postmodern physics papers that he regarded as variously unscientific and well-nigh incomprehensible, that he collected a large number of passages at random from such papers and pasted them together in a meaningless order and forwarded it as a contribution to the postmodern journal of physics. Needless to say it was accepted. But when he admitted his hoax and pointed out that the journal was accepting complete rubbish, the editors replied "it may be rubbish to you…" (Alan Sokal and Jean Bricmont, 1997/*Impostures Intellectuelles*,1998*, Fashionable Nonsense*).

Anyway, on relativism: let us not confuse the fact that *some* things are subjective with claim that *all* things are. Whether coffee is nicer with sugar in or not, is fairly clearly a subjective matter or a matter of taste. On the face of it the law of gravity is not.

Let us not confuse the fact that *some* things seem to be more or less entirely culturally-based with the suggestion that *everything* is. Our notion of marriage is essentially nothing more than a cultural product; our capacity to fall in love may be partly but not entirely cultural but there is nothing cultural about the square root of 83.

Above all, let us not confuse the fact that most things, conceivably everything, have a *degree* of subjectivity or cultural preference about them, with the claim that everything is *entirely* the product of subjective and/or cultural preference. Some things are necessarily true: for example, nothing can be red and white all over. This has been challenged by those who say its truth is merely a function of our language. But that is absurd: what we mean in our language by the words "nothing can be red and while all over" is and must be true. Of course, if the words "red" and "white" had different meanings, then the statement might not be true. Or if you suggest that "red and white all over" could be taken to mean "orange".

But these are trivial and irrelevant points.

I personally am also very irritated by the common assumption in postmodern circles that embracing relativism is something new and up to date, no doubt a consequence of our great advance in thought over previous generations. Whereas in fact, relativism is as old as philosophy. Plato indeed could be said to have formulated his theory of Ideas in response to such relativistic notions as Protagoras' claim that man is the measure of all things, Heraclitus' observation that everything is in flux and you cannot step into the same river twice, or Gorgias' thesis to the effect that there is no truth; if there were, we couldn't know it; and, if we did know it, we couldn't communicate it.

Finally, I have to add that I am also intensely irritated by the bad faith that such colleagues display, for while they profess to their students that there is no truth and no knowledge, only personal perspective, we note that they do not eat arsenic for breakfast or trying walking in front of buses – more's the pity, perhaps.

Forgive me for dwelling on this – as mathematicians you may not waste as much time on this issue as we do in the humanities. But take it from me, we have a large number of colleagues who truly seem to believe, at any rate they say to their students, that thinking makes it so or that sincerity is the important thing, because there is no truth and no such thing as objective knowledge. One wonders of course what they imagine the status of that claim to be: is it true that there is no truth, or is that just a matter of opinion? But this isn't just a silly theoretical squabble. Those who hold this view have been known as a consequence to announce at the beginning of a course that all students will receive an A grade, since it is impossible to establish that one is objectively superior to another; or to argue that we should not praise or condemn, or in any other way evaluate beliefs, ideas or performances.

The second egregious error is commitment to the idea of generic skills, which I have already referred to. Just to be clear, what I mean by a generic skill is a skill that, if one possesses it, one can put to use in a wide variety if not all contexts. Thus, if you can juggle, you can juggle apples, oranges or billiard balls. If you can wiggle your ears, you can do it on Monday or Friday, in Vancouver or Moscow. So there are of course some generic skills. The mistake is to regard every skill as generic and in particular to see crucial mental qualities such as creativity, critical thinking or imagination as generic. You may notice that I have also introduced the word “quality” there: because it is not really apposite to refer to these mental abilities as skills; at any rate it is very misleading to think of mental activity as analogous to the world of physical activity where the notion of skills more happily has a place.

So what I am concerned about here is the widespread notion that various qualities such as the ability to think critically, creatively or imaginatively can be developed in themselves by practice in any context or on any material, and then, so to speak, let loose and utilized in any other context. If you can lift heavier weights than me, then you will be able to do so whether we are lifting lead, iron or brick, and more or less wherever we are. But it is plainly nonsense to suggest that because you are a more imaginative mathematician than I am you are necessarily a more imaginative philosopher than I am, or worse, tout court, more imaginative than I am - that is to say, a more imaginative person, period.

There are of course such things as dispositions, such as for example a disposition towards kindness or cowardice, which *are* general tendencies; and there are such things as work habits such as, say, concentration, patience or diligence. These are characteristics of a person and we do not attribute the characteristic in question unless it is generally displayed. And indeed there are a few people who are creative in many if not all fields. But such cases are rare. Leonardo da Vinci, for example, is an iconic figure precisely because of the range of his interests and abilities. But in the case of most of us, it is all too clear that we are not generally creative, imaginative or critical – that is to say we are not capable of imaginative, creative or critical thought across the spectrum. That’s an empirical fact I should have thought, readily observed in any classroom or gathering of people. But it’s hardly surprising that we can observe it, because it is logically inconceivable that you should be creative or critical except in fields where you understand or have knowledge of what is going on. It is impossible for me to be a creative, imaginative or critical mathematician, not because I am uncreative, unimaginative or uncritical, but because I don’t have sufficient grasp of mathematics.

So here again we see a need to give thought to what kinds of thing, what subject matter, we want people to be critical, creative or imaginative about. And of course, we must conclude that classes, exercises, programs and tests in creativity, critical thinking or imagination are largely beside the point. One does not teach imagination, one tries to teach people to think imaginatively in relation to one or more subjects.

It’s of real concern that we so often tend to disvalue specialist knowledge for teachers. More generally, it is worrying that we seem to accept the idea that one can talk intelligibly about things about which one is not insider. A great deal of writing on popular music of the fifties, for example, is written by people who were not there. The result is an inevitable focus on generalized claims of a sociological kind: hence, the often repeated points that rock' n' roll took off around 1956 because teenagers for the first time had money to spend, and the new 45 rpm plastic disc transformed the nature of recorded music. Both claims are obviously correct – and indeed, without the money and the new technology there would not, could not have been quite the growth in pop music that there was. But this still leaves out the crucial factor: the music itself, the songs.

Elvis Presley, Jerry Lee Lewis, Ricky Nelson, Roy Orbison and the Everly Brothers successfully blazed the trail they did because of their charisma, their personae and, above all, the songs they sang. How do I know this? I was there, that's how. And nobody who wasn't there can or should attempt to write the story. In much the same way, nobody should be expected to teach what they have not studied in depth. And there's a degenerative danger here. When I started in this business, to be a philosopher of education you had to have degrees in philosophy and some qualification to talk about some aspect of education. With the growth of education faculties, the next generation of professors was often people who had only studied philosophy of education in an education department. And now, not least when we get rid of departmental structures and allow more or less anyone to claim that they are doing some kind of philosophy, we have the inevitable result that many of those who profess to teach philosophy of education have no true philosophical training whatsoever.

My next, third and final concern is the lack of discrimination that we display. We must have more discrimination. I always introduce this word "discrimination" in my introductory undergraduate philosophy of education class and say, as I have now, that I am in favour of it. I do it because it wakes people up, since to many people "discrimination" is a bad word and something that we ought not to engage in.

The confusion arises from the fact that phrases such as "racial discrimination" or "gender discrimination" are indeed designed to carry negative connotations. These are bad practices. But that is because we mean by "racial or gender discrimination" discriminating between people on grounds of race or gender when it is inappropriate to do so. "Discrimination" itself simply means noting differences that are there to be noticed. And my point is that in education we seem afraid to make any distinctions, and yet you would have thought, given that it is about achievement, mastery, quality, correctness, moral character and so forth, that is to say loaded with ideals, standards and value judgments by definition, you would have thought that it would be obvious that discrimination is logically built into the idea of a sound education system.

So why are we so afraid of it? Well first, I should perhaps qualify what I've said, because it seems to me that we are in fact inconsistent on this issue. For example, on the one hand we adhere rigorously to the fundamental idea that our classes shall be age-related rather than anything else; on the other hand, we have in some jurisdictions so-called gifted classes. Conversely, we accept the principle that some children have special needs that require separate schooling, but in practice strive to keep all manner of children in mainstream classes who are not obviously benefiting from it and who on the face of it are making things harder for fellow pupils and teachers. I have no quick-fix solution to such practical problems and I am aware of various arguments on both sides. But I will note in passing that the social argument seems to me overplayed. Schools are primarily for educating people (although they undoubtedly serve many other purposes), that is to say for developing the mind. They should not be a branch of the social services; they are not there to right the wrongs and inequities of society. Education should not be confused with therapy. But my point here is to note that we are a long way from consistent in out theorizing and practice on this issue.

What I do think is that it is time to give serious consideration to the idea of more discrimination between students in respect of their current aptitude so that, in particular, we can make more use of setting, by which I mean dividing a given class up into different sets for different subjects according to their current aptitude for various subjects. In places where this is done, the criticism is sometimes made that it leads to stigmatization of those in less advanced sets. The answer to that is to stress that to say that in education what matters is understanding, is not to say that education is the only thing that matters. Better to be a good person than a well-educated person, for example, but you go to school primarily to be well-educated. Furthermore, not being the most proficient of mathematicians, or historians or whatever, is quite compatible with being relatively well-educated – education having more to do with breadth of understanding than depth.

Let me return to the positive. The mind is not just the brain, just as love is not simply a chemical reaction, even if pheromones are the basis of attraction. Love between humans is only possible because, amongst other things, of the way we think. So the refined question about curriculum becomes: understanding of what? How are we going to justify teaching math rather than Elvis Presley studies?

There are many approaches to this question. Those who lament our lack of common cultural references, have a point. But we certainly don't want a curriculum simply based upon such reference points, nor do we want one based on information, if that is all it is.

We must avoid the groundless presumption that it should be topical or already of interest to students. It is our job to interest the student in what is worth studying not to study what already holds their interest.

Nor does it make sense to design the curriculum around various mental skills, for reasons I have given.

P. H. Hirst's Forms of Knowledge thesis comes nearest to the mark. Hirst argued that there were certain fundamental methods of inquiry, each defined in terms of three criteria: first, a set of unique concepts; secondly, a unique logical structure; and thirdly a unique methodology, as, for example, scientific method differs from method in math.

(Incidentally, we have to be careful to distinguish the logical forms that Hirst is positing from their instantiation in the curriculum. Science in schools and universities, strictly speaking, is more than science, involving for a start, math. Hirst is pointing to the idea of pure scientific method, pure mathematical activity).

At different times he changed the thesis slightly, at one point positing four criteria, then reducing them to the three I've mentioned; at one point positing 8 such forms, science, math, history, religion, moral knowledge, social sciences, philosophy, and literature and the fine arts, the last incidentally being but one form; at another reducing these to 7, with some slight changes of nomenclature.

The details do not interest me: I think in many ways the argument is flawed. I do not see that religion can be accounted a form of knowledge, for example, although I do see that it could be accounted a system of thought; but then so could witchcraft. And I do not understand what he is trying to say about literature and the fine arts.

But, details aside, the broad thesis is surely correct: there are certain bodies of knowledge, or as I would prefer to say in order to stress the need for openness, bodies of understanding, that not only yield answers to a unique set of questions of relatively great importance, but that also have great power in helping us to generate further understanding of our world. This is surely true of science, math and philosophy, and it would both be difficult to make sense of anything, including the study of Elvis Presley, if you couldn't understand the differences between, and the limits as well as the powers of these three disciplines. And that is why all educated people should study these subjects. I don't think the argument for the study of history, literature and aesthetics can be successfully made in quite the same way; but I would say instead that these are the great repositories of human thinking and achievement, that in their own way shed tremendous light on the world and our place in it. So I would add that any educated person should have a decent historical understanding, a familiarity and ease with great literature, and an understanding and awareness of the aesthetic. (Perhaps something similar could be said about religion, but that is too big a topic for today.)

The school curriculum should continue to be based on this subject matter insofar as it already is, and we should resist the tendency to attempt to develop such understanding through project work or by integration or by reference to various skills. Part of the point is to recognize the different ways in which these fundamental bodies of understanding work. The integration should derive ultimately from the individual, not from the instruction. Development of so-called skills should take place in the context of these important bodies of understanding. And we should resist the temptation to create a curriculum consisting of short-term socially and economically useful skills, habits and information.

I am arguing in defence of a subject based curriculum taught with concern for truth, accuracy and understanding, making more demands on students than we currently do and recognizing differences in aptitude; all this rather than pursuing a social worker's guide to self-esteem.

Of course this argument happens to give math a significant place. But math doesn't need my help; its place is secure. The sadness to me is that it is often so for the wrong reasons.

The most common reason given for the importance of math is its utility. But while mathematical knowledge and understanding is undoubtedly crucial to our world and way of life, it is not clear that there is any great utility in making all students study it much beyond basic arithmetic and geometry. Just as we need medical knowledge but it doesn't follow that everybody should study medicine.

Mathematical ability is sometimes seen as a sign of intrinsic cleverness, as is the ability to speak a foreign language. I see no warrant for such claims. A good mathematician clearly has a talent but I know of no convincing evidence that it betokens a greater intelligence – if there is such a thing as a greater intelligence, as distinct from a greater I.Q, which seems to me highly debatable. In the same way I would not defend the teaching of math in the way that the teaching of classics used to be defended, namely that it sharpens the mind.

Plato, of course, thought that studying math was important as a step towards the ultimate goal of studying philosophy, because it involves dealing in abstraction. There may be something in that, but it is evidently not the only way to develop a capacity for abstract thought.

For me, the reason we should teach math is to be found in another point hinted at by Plato, and that is, that while there are serious questions that need to be examined relating to what level and by whom math should be studied, and whether we need to make math teaching more interesting or to teach people in such a way that we interest them more in mathematics, subtly different – despite the importance of such further questions, the important thing about math, educationally speaking, is that it is indeed a unique and fascinating type of inquiry and understanding that in its own way sheds as much light on our understanding of our world as do science and literature. People need to recognize, understand and appreciate the nature and domain of math rather than, generally speaking, to be mathematicians of any great standing. I found this point echoed this week in a review of a book by Alex Bellos, entitled, *Alex's Adventures in Numberland*. The reviewer, Jonathan Beckman, wrote: "There are undoubtedly physicists who write novels and biochemists who philosophize, but I don't know any arty types who spend their evenings curled up with some critically acclaimed number theory or multidimensional calculus. Unless you choose to study a maths-related discipline at university, then the subject - bar some frantic mental arithmetic at the supermarket till - is pretty much dead to you. This sparky new book demonstrates quite how much we're missing. Maths has the remarkable capacity to give birth to wonder: the pleasurable incredulity that occurs when the mind's conceptual limits are breached by the compulsion of logical proof…..I doubt this book will spawn a nation of avid hyperbolic geometers, but it ought to prove that mathematics can take your breath away" (Literary Review, August, 2010).

To which I would add Plato's observation that the beginning of true philosophy is a sense of awe or wonder – having your breath taken away.

I have simply tried to suggest that all educators should give serious thought to the fundamental questions of what is involved in being human, and what constitutes being well-educated, and following that, what is most worth studying. My answer to the last question has been certain basic types of understanding, including mathematics. But the more important point is that these are questions which should surely form a part of any doctoral program in education. I hope that you are able to pursue them with enthusiasm and profit in your time here – and that you are not sidetracked by shallow arguments about relativism, confused and mistaken assumptions about the nature of mental abilities, and a fear of sharp discrimination.

CALCULUS IN NAVIGATION/ BODILY CALCULUS

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The current study is devoted to investigation of the computer simulated optimal path navigation related to the calculus problem of optimal path finding. My hypothesis is that tacit dynamics modeling of optimal path navigation involves the allocentric frame of reference. The virtual environment paradigm, designed in Second Life, contains two different mediums and provides voluntary choice between allocentric and egocentric views. Reinventing the calculus problem of optimal path finding from the virtual navigation and its mathematizing should give a powerful intuitive link between the everyday real world problem and its symbolic arithmetic. The designed paradigm belongs to the framework of Realistic Mathematics Education (RME). Analysis of the voluntary choice between egocentric and allocentric views should give an indirect indication of what frame of reference is utilized and, as such, should provide better understanding of mental processes a particular calculus problem solving situation.

INTRODUCTION

Mathematics is present invisibly in our everyday lives. The term, "un-earthing" the hidden mathematics was first introduced by Ole Skovsmose in 1994 (Torkildsen, 2006). Fyhn (2010) applied this term for vector learning from climbing situations. She illustrates the connection between climbing and vector algebra notions via digital videos. In Fyhn's (2007) report, learning angles are based on students' experiences with physical activities, so that the students are able to un-earth the concepts of angles from the certain body movements.

Talking about calculus, it should be mentioned that calculus appeared from the real world application: Aristotle was one of the first who formulated laws on motion. Calculus has a real world context and is applied widely to description of and prediction the real world processes. Started in 1980s the 'Calculus Reform Movement' in the USA had aimed to make Calculus more applied, relevant, and more understandable for a wider range of students. Since calculus is a fundamentally a dynamic conception, I would like to demonstrate that it is present invisibly in some everyday activity and dynamic processes, and particularly, in navigation and its special case, optimal path navigation. For example, when we are driving in a big city, we choose optimal way: let it be from the viewpoint of minimizing the time of travel, or minimizing the length of the path. For the time minimization case, we usually take into account that driving along a highway may save time even if the total path length is longer. Tacit assessment of speeds and path length differences based on previous knowledge/ experiences helps to simulate mentally the optimal route.

The didactical goal of the present study is to help learners to ‘unearth’ a calculus optimal path problem from a real world navigation simulated in the Second Life (SL) virtual environment. This is the case of two mediums environment, water and land; the path must transverse both of them and the number of possible paths is not limited, as in the case of driving in a big city. The learners would reinvent the calculus problem after a few trials which are based on controlling computer-simulated body movements with either egocentric or allocentric views. The egocentric view provides the perception of ‘being’ within the virtual environment and seeing objects from the ‘first person’ view. The allocentric view is provided when the learner’s avatar is present in the environment and the learner controls the avatar navigation: in this case the virtual reality objects are spatially related to the avatar. This enactive computer paradigm simulating a real world navigation problem would allow the learners to explore mathematical ideas being engaged immediately into the task. Since the designed virtual environment contains two different mediums, the task of optimal path finding should involve the intuitive anticipation of speed difference in different mediums, on land and in water. Thus, the intuitively planned strategy will be based on this speed difference anticipation. After a few trials of virtual navigating, the learner should reinvent the calculus optimal path finding problem and should try to mathematize it. When the problem has been mathematized, the learner would be able to connect and compare the intuitive understanding of the problem with its symbolic arithmetic.

The research goal of the study is to explore how egocentric and allocentric frames of references relate to different phases of optimal path finding problem solving, which, in turn, would provide better understanding of mental processes during the particular calculus problem solving situation.

Navigation, optimal path navigation, and space

The notion of navigation is understood as the ability to find one’s way at sea, on land, and in the air; it has been studied in connection with the flights of birds, the voyages Micronesian sailors, who traveled to far islands without instruments, using only cognitive maps and techniques for updating their positions in relation to virtual landmarks (Besthoz, 2000). Navigation consists of two aspects: (1) topographic aspect, which allows constructing a cognitive map and intuitive modeling an optimal path within it; (2) procedural aspect, involving procedural memory that represents the trip itself (ibid.). These two aspects closely relate to each other. The topographic aspect is connected with a construction of cognitive map; the procedural aspect is connected with actual movements. Both topographical and procedural navigations include spatial orientation which is an aptitude of an organism to locate the position of objects and to relate this position to themselves and to other objects (ibid). This means that spatial orientation is an integral part of topographic and procedural navigation. The role of the environmental geometry in orientation is pointed out by Burgess (2006). It should be mentioned also that the virtual navigation sufficiently differs from the real navigation: the virtual navigation doesn't involve vestibular, translation, or locomotor memory which, according to Berthoz (2000), are inherent to real space body navigation. In the virtual environment the visual system plays the main and crucial role. The optimal path navigation is a particular case of navigation with all characteristics described above, and with additional one of choice a path, traveling along which, would require minimum time (time minimization) if to compare with all the other possible paths.

Any motion implies being in space: let it be real life body motion or arithmetically and geometrically conceptualized motion describing change. The real life body motion takes place in naturally continuous space which is our normal conceptualization, and which we can't avoid ("It arises because we have a body and a brain and we function in the everyday world" (Lakoff & Núñez, 2000, p. 265)). The geometrically conceptualized motion takes place in a Space-As Set-of Points; even professional mathematicians think in naturally continuous space when they are functioning in their everyday lives (ibid). Berthelot and Salin (1998) structured the naturally continuous space with respect to sizes: "microspace" which corresponds to grasping spatial relations, "mesospace" which corresponds to spatial experiences from daily life situations with domestic spatial interactions, and "macrospace" which corresponds to the distant unknown objects such as mountains, or the unknown city and rural spaces. The optimal path navigation, and its computer simulation with corresponding virtual meters, takes place in mesospace.

Frames of references: egocentric or allocentric

According to Berthoz (2000), the brain uses two frames of reference for representing the position of objects: egocentric and allocentric. For example, the relationships between objects in a room for estimation the distances and angles can be encoded either 'egocentrically' or 'allocentrically. In the first case, everything is related to yourself; in the second, spatial relationships between the objects themselves are encoded in relationship to a frame of reference external to your body.

Most animals are capable of egocentric encoding; only primates and humans are genuinely capable of allocentric encoding (ibid.). The power of allocentric encoding is that it enables mental manipulation of the relationships between objects including geometric relationships. It also is constant with respect to person's own movement, so it is well suited to internal mental simulation of displacements (ibid.).

When moving and navigating in real life and in the naturally continuous space we see the environment egocentrically within our personal space. But our brain is able to encode the objects' locations in the space with both egocentric and allocentric frames of references. As Burgess (2006) asserts, egocentric and allocentric representations exist in parallel, combining to support behaviour according to the task. It could mean that that particular task stage may require either egocentric or allocentric brain encoding, or, probably, that the encoding takes place in both of them. The aim of this research is to investigate whether construction of cognitive maps involves allocentric encoding, when the environment is viewed egocentrically. The SL virtual environment provides both egocentric and allocentric views. There are some important questions which could be clarified by corresponding questions in a post-experimental survey. The first one is that if the subject/learner has chosen the allocentric view for navigation after the cognitive map has been constructed, does this mean that the cognitive map was encoded allocentrically in the learner's brain? How does the choice of view reflect the type of the brain encoding? Does the voluntary chosen egocentric or allocentric type of view lead to a better understanding of calculus or geometry?

THEORETICAL FRAMEWORKS

The new virtual paradigm of optimal path navigation is intrinsically of an enactive nature. Tall (1997) asserted that "the calculus concepts are starting from enactive experiences as an intuitive basis" (p.4). It means that the new virtual paradigm is in accordance with his schematization of the growth of representations and the building of the concepts of calculus. On the other hand, the computer simulation of body movements expressed either by an egocentric view of 'being' in the environment or by an allocentric view through controlling the avatar navigation, provides an explicit perception of 'bodily' navigation which can be expressed in terms of embodiment. Tall (2007) categorizes mathematical thinking into three intertwined worlds: the conceptual-embodied, the proceptual-symbolic and the axiomatic-formal. He considers such categorization particularly appropriate in the calculus. According to Tall (2007), conceptual-embodied world of mathematics is based on perception of and reflection on properties of objects. For the particular dynamic tasks of optimal path navigation and taking into account the dynamic nature of calculus on the whole, I extend a conceptual-embodied world into procedural-conceptual-embodied world, reflecting embodied dynamism of body movement. This extended world is based not only on perception of and reflection on properties of objects, but also on an active experience in its dynamism such as change of body position, speed, and acceleration. For the navigational type of tasks, we first, mentally simulate the trajectory, and then we compare the actual movement with the predicted movement (Berthoz, 2000). For this mental simulation stage of navigation the conception of 'tacit intuitive model' introduced by Fischbein (1989) is adaptable, but with some modifications. The common characteristics of the tacit intuitive models are that they have structural entity, they are of practical and behavioral nature, they are mental, intuitive, and primitive, they are representable in terms of action, they are autonomous entity with their own rules, and they are not perceived consciously by an individual. The last characteristic of the intuitive mental model is its robustness and its capacity to survive long after it no longer corresponds to the formal knowledge (ibid.). For the case of optimal path navigation the last characteristic should be omitted and the tacit intuitive model should be modified. As Cazzato, Basso, Cutini, & Bisiacchi (2010) pointed out: people produce incomplete plans at the beginning of a route and continuously make decisions along the trajectory of navigation. So, for the case of optimal navigation, the tacit intuitive model should be modified into a more flexible concept, reflecting dynamism and procedural nature of continuous adjustment according to the model's effectiveness. The term of *'tacit dynamics modeling/simulation'* would reflect both the procedural embodied world, on the basis of which the kind of tacit model is constructed, and flexibility and procedural character of such intuitive modeling. Since the computer simulated optimal path paradigm provides a link between the real world situation and its symbolic formal representation, it belongs to the theoretical framework of Realistic Mathematics Education (RME) (Freudenthal, 1991; Freudenthal, 1973; Freudenthal, 1968). The hypotheses of the research are the following: 1) the *tacit dynamics simulation* of the optimal path engages allocentric frame of reference even if the virtual environment is viewed egocentrically; 2) the topographic phase of navigation which involves orientation and cognitive map construction also engages allocentric frame of reference, even if the virtual environment is viewed egocentrically; and 3) the procedural phase of navigation can involve both of frames of references, which, in turn, should coincide with Burgess' (2006) affirmation that egocentric and allocentric representations exist in parallel, combining to support behaviour according to the task.

EXPERIMENTAL DESIGN AND METHODOLOGY

The virtual environment paradigm, designed in SL, contains a big water pool with a platform at location B (see Figure 1), around which there a several distant cues like trees and houses. The paradigm is related directly to calculus problem of finding the optimal path from A to B under the condition that available paths must transverse two different mediums, involving different rates of speed (figure 1).This paradigm is adapted from the paradigm in Pennings'(2003).

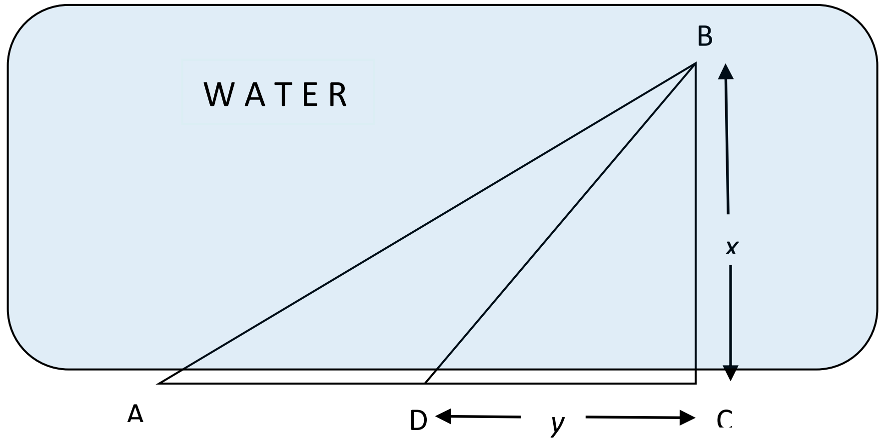

Figure 1. Paths to the platform (adapted from Pennings, 2003)

There are three phases in the experimental paradigm: 1) the exploration phase which allows the participant to learn how to control the avatar, and how to interchange between egocentric and allocentric views, 2) the topographic phase of staying on the platform and learning the environment with egocentric view and memorizing the location of the platform in order to be able to find it when it is invisible in the next phase ; and 3) the procedural phase of reaching the invisible platform from the beach as fast as possible. At this stage the participant can choose whether to use the egocentric or allocentric view. Phases 2) and 3) should be repeated a few times: every time the position of the platform should be changed. In Phase 4) there is problem mathematizing, which involves the reasoning provided below.

To complete the task of time minimization the participant can choose getting into the surf at the point nearest to it and reaching the platform directly (Figure 1, AB). On the other hand, the participant can anticipate intuitively that the speed in water should be slower than the speed on land. So, another option would be to minimize the distance in the pool: the participant can choose sprint down the beach to the point on the shore which he/ she considers being closest to the supposed platform location, C, and then turning a right angle and getting into water. Finally, there is the option of running a portion of the way then plunging into the pool and go diagonally to the platform. This type of problem is in every calculus text and contains the following mathematical background (Pennings, 2003).

Let $T(y)$ represent the time of reaching the platform. Suppose that the participant decides to get into water at D, y metres from C. Let z represent the entire distance from A to C (see Figure 1). Since Time = Distance/ Speed:

$$T(y) = \frac{z-y}{r} + \frac{\sqrt{x^2+y^2}}{s}$$

where r is the running/ walking speed on land, and s is the speed in water. Minimizing $T(y)$ means that $T'(y) = 0$, which gives

$$y = \frac{x}{\sqrt{\frac{r}{s}+1}\sqrt{\frac{r}{s}-1}}$$

The learners can see from the formula that since r and s are fixed, y is proportional to x. They can compare this result with their virtual navigation based on their intuitive path simulation.

The measurements to be analyzed include: the distance between B and C for every changed location of platform B, the distance between A and D, and the voluntary choice of view (allocentric or egocentric) during the procedural phase of navigation. The post- experiment interview includes the following questions:

a) What view did you choose (allocentric or egocentric) and why?

b) What did you have in mind in choosing your particular path to the hidden platform?

c) How does the mathematics describing the process correspond to your mental dynamic model of an optimal path?

CONCLUSIONS

Tall (1991) stressed that students generally have very weak visualization skills in calculus, which, in turn, leads to a lack of meaning in the formalities of mathematical analysis. The proposals in this report of a computer-simulated paradigm provide not only a high fidelity of visualization based on the SL environment, but also an enactive form of unearthing the calculus problem from a real-life situation. It also develops an intuitive understanding of the calculus problem due to revealing and getting aware of the tacit intuitive simulations, and due to opportunities to compare these mental simulations with the formal representation of the problem. Tall (1991), particularly, pointed out that a flexible software, and especially the one containing dynamic movements, may be used to give more powerful intuitions in calculus, and that “intuition naturally leads into the rigor of mathematical proof” (p. 20).

Mathematizing the problem has a certain didactical value as a particular case of RME. Choice of view at the procedural stage of navigation should serve as an indirect indication of what frame of reference is utilized while constructing a cognitive map and simulating mentally the optimal path, which, in turn, would provide better understanding of mental processes during problem solving in this particular calculus situation.

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HOW DO CHILDREN MULTIPLY: COMMUTED PAIRS

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Multiplication is one of the most important abilities gained through school life. Because more advanced topics in the curriculum depend on previously gained arithmetical abilities, teaching of multiplication is crucial. Some recently discussed methods for the teaching of multiplication and multiplication table are claimed to be more efficient, more easily learned and applied faster by students. This study includes interview transcriptions of eight 5th grade students and summarizes different techniques they use for multiplication in terms of efficiency, accuracy and responsiveness.

INTRODUCTION

Notably for the last two decades, mathematics educators started to criticize teaching of multiplication in schools and the meaning of multiplication tables for pupils (Ball et al, 2005; Bogomonly, 2006; Butterworth 2003 & 2006; Thomas & Keung, 1998). Is multiplication taught efficiently? Should multiplication table be memorized? If yes, to what extent, and how? Because the multiplication table includes commuted pairs, it replicates itself for the second half. As an example, should students memorize commuted pairs such as 2x4 and 4x2 distinctly?

The issue also has some cultural aspects. According to the literature, there are some cultural differences for the teaching of multiplication (Thomas & Keung, 1998; Campbell & Hue, 2001; LeFevre & Liu, 1997; Sam, 1997) and some consequences. For example, in China, children are asked to memorize only half of the western version of multiplication tables. The other half is obtained via the commutative law. In most other countries such as Canada, this is not the case, resulting in children having to memorize twice as much as their Chinese counterparts. This is also claimed to be eliminating an opportunity for children to gain an early abstract thinking of commutativity (Bogomolny, 2006). Studies show that Chinese students tend to perform better in the memorization of multiplication tables (Sam, 1997). Research also shows that Chinese adults are more accurate and faster at solving multiplication problems than their Canadian peers (Campbell & Xue, 2001; LeFevre & Liu, 1997). From the linguistic point of view, the Chinese language also seems to have a cultural advantage over most other western languages (Campbell & Xue, 2001; LeFevre & Liu, 1997; Thomas & Keung, 1998).

The multiplications in Chinese multiplication tables begin with smaller numbers in a form of "n x M" (n < M). Do children learn faster and act faster with "n x M" than “M x n”? Butterworth at al. (2003) found that children between 6 and 10 years old reorganize their mental mathematical structures to privilege “M x n” over “n x M”, even if “n x M” was memorized and practiced earlier. This study investigates this hypothesis and the techniques that children use in multiplication in general.

METHODOLOGY AND DATA ACQUISITION

The study was conducted with eight 5th grade primary level students (10-11 years old). They were selected from moderately- and higher-skilled students. Face-to-face interviews were audiotaped and each interview took approximately 4 to 10 minutes. Responses taken within 10 seconds were considered as fast. Students were provided with comfortable physical conditions and were not allowed to use paper, pencil or calculator. Students were encouraged to talk aloud and were asked to multiply a single digit number with another one or two digit number. At the end of each interview, the interviewee was asked to answer the following questions: “Which is easier to multiply? When the first number is larger or when it is smaller?”

RESULTS

STUDENT # 1 (Female)	n x N (n < N)	N x n (N > n)
Correct and Fast answers	3x7, 5x10, 5x8, 2x7, 3x11, 1x4, 3x15	7x3, 4x2
Correct and Slow answers	5x , 6x12	7x4, 9x3, 15x3, 8x6, 14x5
Incorrect and Fast answers		
Incorrect and Slow answers		
Which is easier to do?	Both the same, actually let’s say 6x7 is easier than 7x6, I think. Or 4x12 is easier than 12x4. Also because I know the multiplication table, I can work more comfortably with numbers smaller than 10.	

STUDENT # 2 (Female)	n x N (n<N)	N x n (N>n)
Correct and Fast answers	3x7, 4x6, 7x13, 3x30, 7x40	8x5, 9x3, 11x5, 13x5, 16x8, 18x6, 20x6, 40x8, 52x5, 12x5
Correct and Slow answers	8x12, 9x13, 7x13	
Incorrect and Fast answers		
Incorrect and Slow answers		

Which is easier to do?	Both are very easy, actually maybe 20x7 is easier than 7x20 because it is easier if the first number is larger. For example 6x13 would be harder; if it was 13x6, then I would immediately multiply 6 with 3 which equal to 18, the other with 6 and add 1. So it is definitely harder when the first number is smaller.

STUDENT # 3 (Male)	n x N (n<N)	N x n (N>n)
Correct and Fast answers	5x6, 3x5, 2x10, 2x11	6x2, 9x2, 7x2
Correct and Slow answers	4x7, 3x6	8x3 (2nd try)
Incorrect and Fast answers		
Incorrect and Slow answers	6x9, 3x9	8x4, 8x3, 12x2
Which is easier to do?	Mmm..depends. Let’s say 2x8 id harder than 8x2 but 8x3 is harder than 3x8 because for the first one I count 3 times, 8-16-24 but for the other one is harder because 3-6-9-.. too long!	

STUDENT # 4 (Female)	n x N (n<N)	N x n (N>n)
Correct and Fast answers	2x12, 5x6, 4x11, 5x12, 2x15	12x4, 20x5, 9x3
Correct and Slow answers	3x9, 2x15, 4x13	13x6, 15x3, 19x2, 18x3
Incorrect and Fast answers		
Incorrect and Slow answers		
Which is easier to do?	First of all it is harder if one of the numbers is larger than 10 because I did not memorize those ones. But smaller ones are very easy to do because I know them from the multiplication table. It is harder when you have to add something to the other digit, let’s say 13x3 is easier than 4x16. Also 13x5 is easier than 4x12 because 10x5 is 50 and all you need to do is to add three more 5s: 55-60-65. So multiplying something with 5 is easy. 16x4 is a bit easier than 4x16 because I do not have to switch the order of the numbers; I directly multiply just like I do in my notebook.	

STUDENT # 5 (Female)	n x N (n<N)	N x n (N>n)
Correct and Fast answers	3x7, 2x8, 5x11, 4x11	9x4, 9x5

Correct and Slow answers	3x13	14x5
Incorrect and Fast answers		
Incorrect and Slow answers	6x12	15x6
Which is easier to do?	If the numbers are smaller than 11, doesn't matter, if not, I like if the first one is larger because I am more familiar to calculate that way.	

STUDENT # 6 (Male)	n x N (n<N)	N x n (N>n)
Correct and Fast answers	3x7, 3x8, 5x6, 4x8, 3x6, 4x9, 5x9, 4x9, 6x8	9x4, 12x4, 8x2
Correct and Slow answers	3x11	13x5, 9x5
Incorrect and Fast answers		
Incorrect and Slow answers	3x14, 5x14	
Which is easier to do?	If one of the numbers is 9, then it is easy because I count with my fingers like 9, 18, 27, 36 etc. If one of the numbers is larger than 10, then it is hard, except for 11. 13x5 is harder than 4x12 because you have to add 1 to the second digit but for 4x12, you directly multiply. 2x13 is easier than 13x2 because 2x3=6 and 2x1=2 so 2x13=26 but for 13x2 you have to calculate 13+13 so it is harder. As for 13x3 or 13x4 I again multiply rather than add, but you need to add numbers in case of multiplying with 2.	

STUDENT # 7 (Male)	n x N (n<N)	N x n (N>n)
Correct and Fast answers	3x8, 4x12, 3x9, 2x14, 4x12, 4x13,	9x6, 10x3, 13x5, 9x5, 8x5, 13x2, 15x3, 16x3, 12x5,
Correct and Slow answers	4x14, 8x12	16x4, 17x3, 14x3
Incorrect and Fast answers		
Incorrect and Slow answers	4x17	13x4, 12x9
Which is easier to do?	Let's say 17x4 is harder than 4x17. When calculating 17x4, I first find 4x10=40 than 4x7 and add. To me, 4x17 is easier because finding 4x7 is easier.	

STUDENT # 8 (Male)	n x N (n<N)	N x n (N>n)
Correct and Fast answers	3x7, 4x6, 3x11,	9x5,
Correct and Slow answers	9x12, 5x13	12x4, 13x4
Incorrect and Fast answers		
Incorrect and Slow answers	5x15	
Which is easier to do?	For example 3x9 and 9x3 is the same but let's say 8x4 is easier than others because 8x2=16 and 16+16=32 so 8x4=32. 11x4 is easier than 2x12 because you just calculate 1x4=4 and that applies to both digits so 11x4=44. 13x2 is easier than 12x4 because 13+13=26. So multiplying by 2 is easy in general. I know the multiplication table so the order does not matter.	

DISCUSSION

There is no clearly meaningful difference in performances of the students for nxM or Nxm cases. However as it can be seen from statements such as:

'16x4 is a bit easier than 4x16 because I do not have to switch the order of the numbers; I directly multiply just like I do in my notebook.'

that the order of numbers does matter if one of the numbers has two digits. However this is more like being a learned behaviour rather than being an intuition. Answers in this study reflect that for most of the students, the order of the multiplication does not matter if both numbers are single digits. When the multiplication table is memorized, neither the order nor the numbers matter for the students. Lack of memorization of the table leads some students to face problems as can be seen from the following example:

'2x8 harder than 8x2 but 8x3 is harder than 3x8 because for the first one I count 3 times, 8-16-24 but for the other one is harder because 3-6-9-.. too long!'

Therefore lack of memorization can lead to students having some difficulties when multiplying by higher digit numbers.

Results also show that multiplication by 5, 10 and 11 is calculated faster and more accurately.

CONCLUSION

This study does not verify or falsify the results of Butterworth et al. (2003). Although the results clearly show the importance of the memorization of multiplication table, they do not reflect a clear recommendation for the order of multiplication for commuted pairs. Conversely the results show that after memorizing, the importance of the order loses its priority for the multiplication of single digit numbers. Can the curriculum benefit from the commutativity property of multiplication to eliminate additional work for students? Or should we obligate students to memorize both nxM and Nxm? While students from China and Iceland benefit from the prior view (Sam, 1997; Campbell & Xue, 2001; LeFevre & Liu, 1997), additional research comparing the efficiency of two views conducted in Western schools is needed to enlighten mathematics educators, curriculum developers and teachers to get a clearer idea for the potential advantages and disadvantages of both views.

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OPERATIONS WITH NEGATIVE INTEGERS IN A DYNAMIC GEOMETRY ENVIRONMENT

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We review difficulties elementary students (and teachers) face with the concepts of, and operations with negative numbers and zero (Davidson1992; Streefland 1996; Lincheveski & Williams,1999).We then examine how the dynamic geometry environment (DGE),through use of the Geometer's Sketchpad software, contributes to the understanding of the integer operations in general, and negative numbers and zero in particular. We extend Davidson's (1992) object oriented framework, and Lincheveski & Williams'(1999) semiotic activity framework in a dynamic geometry environment and hypothesize that use of dynamic learning activities changes the way integers, in particular, negative numbers are perceived, and that a dynamic learning environment contributes to an action oriented thinking.

INTRODUCTION

Several studies have reported the learning difficulties elementary students face with negative numbers and zero, and a number of models have been proposed to deal with this issue (Hefendehl-Hebeker, 1991; Davidson 1992; Streefland 1996; Lincheveski & Williams, 1999; Featherstone, 2000), but we believe that there still remains some fundamental issues in this area that require further inquiry. The teaching of number operations at elementary school in general, is done through use of physical objects. All the studies above point to the difficulty of bridging the intuitive gap created when elementary students operate with negative numbers. That is, while natural numbers can be easily linked to physical objects, a similar link between physical objects and negative numbers is hardly straightforward. For instance, for elementary students, having objects, say two pencils, plus three more pencils results in five pencils altogether. Mathematically 2 + 3 = 5. On the other hand -2 + -3 = -5 has no obvious link to physical objects. (Hefendehl-Hebeker, 1991; Lincheveski & Williams, 1999; Featherstone, 2000). The outcome of operating with negative numbers is clearly not an intuitive one to elementary students because they cannot connect with physical objects as they do with positive numbers. Working with negative numbers for many students remains a matter of remembering the formal rules of mathematics. (Lincheveski & Williams, 1999). Our study on negative numbers attempts this problem from a different and essentially dynamic perspective.

As mentioned earlier, different models have been proposed to try and address the intuitive gap created when students operate with negative numbers and zero. Streefland (1996) reports a study in which negative numbers are modelled as cold cubes, and positive numbers are hot cubes, all put together in one container.

If f(C, H, t) denotes the state of cold and hot cubes at a given time t, then an integer, k is represented by a class of (C, H) pairs such that the difference between the cold and the hot cubes is –k. For instance -3 is represented by a class of (C,H) pairs {(0,3),(1,4),(2,5),…}. The model, called "The Witch", is criticized as being mathematically artificial and having a false concretization. However, we think that it attempts to provide a qualitative link between negative and positive numbers, in terms of objects that students can easily identify with. What should be observed with this model though is that the students should be helped to appreciate that negative numbers are not "cold cubes" all the time and, as they [students] upgrade their mathematical knowledge, they should outgrow the metaphor. Another contribution is provided by Lincheveski & Williams (1999) who model the concept of negative and positive numbers based on a process and object semiotic theory. They model people entering a disco hall, and at any time some people will be moving out and others going into the hall. The IN/EXIT numbers are recorded and managed at the gate. Sets of cards marked blue (IN-"positive") or yellow (EXIT-"negative") have numbers on them. For instance if a blue card with marked with an integer "N" and a yellow card marked "K" are drawn, then N people have entered the disco hall and K have exited the hall. The focus of this study is to provide a framework for correct use of mathematical language and signs, and to help children shift meaning when their knowledge is evoked in a different setting. We incorporate elements of Lincheveski & Williams's (1999) action based framework in this study, focusing more on an action oriented mode of working with integers, as compared to an object oriented mode. We model our activities on the dynamic number line adapting Sinclair and Crespo (2006).

Another example of models is given by Davidson (1992). Davidson's study, similar to those described above, is based on children's understanding of integers through links with the physical objects. However, he gives insight to our study in that he de-emphasizes physical objects and emphasizes actions in quantifying negative numbers. He suggests that to quantify an integer, we need its cardinal and ordinal meanings, and that using only one of them to quantify an integer results in an incomplete process. We agree with Davidson's (1992) position and propose that the number concept is thus a synthesis of the cardinal and ordinal meanings of integers, and that understanding these two meanings is important in working with negative, positive integers and zero. Further, we use Davidson's framework to examine the coordination between cardinal and ordinal meanings of integers in a

DGE environment. We argue that the intuitive representation of negative numbers is based on actions, rather than on objects. We hypothesize that the qualitative misconceptions of negative numbers may be informed by uncoordinated, one-sided cardinal or ordinal meanings, and that the precondition for correctly quantifying an integer is the co-existence of both cardinal and ordinal meanings. With these positions, we design our activities on the dynamic number-line adapting Sinclair and Crespo (2006). The specific research question is: “How does the dynamic representation of the number-line affect the way in which operations with integers are conceptualized?”. We propose that because of the dynamic, continuous nature of the number-line (which de-emphasizes the actual value of numbers and requires interaction), attention might shift to actions.

METHOD

Study design

This report is part of an on-going larger study, incorporating mathematical modelling and technology with undergraduate students (non-mathematics majors) and elementary teachers. Four participants were interviewed in a case study but because of space limitation, only one case is reported in this paper. The main factors considered for selection is that the participants are not mathematics majors, and that they have connections with elementary school mathematics. Activities are designed on the Geometer’s Sketchpad (GSP) software, involving use of a dynamic number line. GSP was chosen because of its flexibility in adjusting the scales on the dynamic number line so that participants only needed basic computer skills to use it during the interview. Data collection strategy is based on interview. Each participant is interviewed separately for approximately one hour and the interviews are videotaped. We analyze verbal data and body movement for information relevant to our research question. Since the study is still on-going we only report the work covered so far.

Interview

The interview session is comprised of three main parts: introduction, interaction with dynamic software and debriefing. In the introduction session, participants draw and label a static number line using pencil and paper. Then they state what the number line is used for. They locate integers on the number line and use them to solve some basic arithmetic problems, including multiplication of two negative integers.

In the second part of the interview, participants interact with dynamic software and explore the dynamic number line by observing what happens when they add, subtract or multiply integers on the dynamic number line. The computer session takes more than fifty percent of the total time because it contains more activities.

In the third and final part of the interview, the computer is closed and participants tell their story what they think about addition, subtraction and multiplication of integers. They are also asked about the numbers zero and one. We pay attention to indications of action and object based thinking in the data that we collect. Some of the data we are interested in include gestures, body movement and use of specific words or expressions that signify motion or action.

Validity

Participants were assured of confidentially of their personal information and the interview was carried out individually, not in a groups. We use this to claim that participants answered questions freely without withholding any information they wanted to give. Also the interview protocol had been tested before on some participants and based on the analysis of data, some modifications were made to improve its clarity to the participants. We believe that these precautions provide reasonable credence to the study.

RESULTS

Results are presented following the sequence of data collection as outlined in the research design and interview sections. First, we report on the interview before interaction with dynamic software, followed by a report of the interview during the interaction with software. We end with a report on the debriefing interview.

Before interaction with dynamic software

Participants draw and label a static number line on paper. Then they state what the number line is used for. They locate integers on the number line and use it to solve basic arithmetic problems, including multiplication of two negative integers. In this transcript B is the participant and N is the interviewer.

Addition and subtraction

001 N: what if you wanted to do negative five minus two?
002 B: going backwards… so when you are adding a negative number you go backwards … it’s like subtracting.. It’s just like saying minus a number… am I right?
005 N: yeah , that’s good
006 B: [laughs]

Locating a number on the number line

003 N: ok, on the number line can you locate twenty?
004 B: oh yeah... if it was extended you get two …, just two ten, yeah.
005 N: how about zero point zero one?

006 B: yes, but you need to zoom in, zoom, zoom, zoom in. We would look at the zoom of our number line and we look at the zoom of zero point zero one, wherever we want to put it.

Multiplication on the number line

007 N: explain multiplication, let's say three times five. you don't have to use the number line?

008 B: you could use the number line though…you could use sections of three like taking sections of three five times…

009 B: [another way] I would probably look at it as taking five bushels of three apples [each], you get fifteen apples.

010 N: how about negative three times five?

011 B: …then you will have to explain the rules of math first, like if you are multiplying two negative numbers you get a positive. that's just a rule… minus, minus, plus; plus plus, plus...

Before the introduction of the dynamic software, B shows some elements of action oriented thinking in words like "moving backwards" when adding negative numbers (line 002), and "zooming in …" when locating a rational number on the static number-line, (line 006). But he also shows some object oriented thinking in the use of bushels when multiplying two positive numbers. It is interesting that B cannot extend the bushels metaphor to multiplying two negative numbers, but instead recalls rules of mathematics (Lincheveski & Williams, 1999). Earlier on we had predicted that negative numbers pose a big challenge to students and that the only strategy they have is the use of rules of mathematics.

During interaction with software

Using the dynamic number line, participants drag the points and observe the changes that take place when they add, subtract or multiply two numbers, negative or positive or both. Participants also observe the direction of motion on the number line in each case, and use their experience to state a hypothesis on which direction the movement takes when any two arbitrary integers are added, subtracted or multiplied. Lastly participants use the un-labelled dynamic number line to locate zero and one.

Addition and subtraction

B is interviewed while in action on the dynamic number line.

012 N: is there a situation when you add two numbers the sum is bigger?

013 B: yes for the positive numbers the sum is always going to be bigger, for the negative numbers the sum is always smaller

014 N: ok?

015 B: for adding negative and positive numbers the sum is always going to be in between…

016 N: nice job

B uses the action words “going to”. He actually experiences the motion by dragging the points on the dynamic number line and sees how the results are changing.

Finding zero and one on the dynamic number line

Participants are asked to locate 0 and 1 on the “mystery machine”. The mystery machine is a dynamic number line with a simple mathematical model b + a = 0.The equation is not displayed on the screen, but when two points are dragged along the dynamic number line, they satisfy the model.

017 N: This is another combination…

018 N: can you find zero and one?

019 B: zero and one?

020 B: …um, if this was zero, and this one is two, two plus zero is two […] so…

022 B: b would be zero because you are adding a number to zero which is always a constant...

023 N: you got zero, now could you find one?

024 B: [drags points on number line for some time, and concludes]… we have no idea what the scale is, it could be point zero zero zero zero one…

B argues that from the unmarked dynamic number line, he cannot tell where one is from zero because he does not have enough information. From the underlying mathematical model of b + a = 0, B manages to locate zero but not one.

Role of zero and one

025 N: if you were to say which number is the most important, which number would it be, zero or one?

026 B: in the whole world?

027 N: yeah?

028 B: zero

029 N: why?

030. B: because everything revolves around zero, the number ten is ten because it is ten [units] away from zero. you are a million spaces away from the

number zero. negative number is negative because it is on the other side of zero.

031 N: what would you say to somebody who says that zero is not a number because it is like nothing?

032 B: I don't think the world works without zero… We are 2010 away from zero. you need zero because you need a starting point.

B makes a case about the importance of zero as a number, claiming that everything *revolves* around zero. Besides, the world does not work without zero. We take the active words *revolves* and *works* to indicate some particular way of thinking about zero.

Exploring the product of integers

Participants observe the result of multiplication on the dynamic number line, linking it with previous operations.

033 N: why is that when you multiply two negative numbers you get a positive number?

034 B: the more we move over here is the more we are in debt [pointing to the negative side of the number line]

035 N: just go back to zero. because things get smaller as we keep changing our multiplicand they should just keep going…

036 B: Really, ah!

037 N: now how can you explain negative two times negative three?

038 B: …negative two times negative three. I don't think you can work like that in the negative land. You can't work with six cups of coffee because there are no six cups of coffee [in the negative land]

039 B: I don't know how to explain that but let's go back to continuity on the number line…

040 N: should it follow the path or it should go back

041 B: follow the path [negative side of the number line]

The multiplication of negative numbers takes B to the debt metaphor (line 34), and he does not get much help from it. From there he realises that "there are no objects in the negative land", again confirming our earlier submission. However through moving the points on the dynamic number-line with some prompts from N, B discovers, and is visibly thrilled by, the notion of continuity on the dynamic number line. He refers to this which he refers to as the "path". He is surprised by the change in direction on the dynamic number line (lines 035,036) when two numbers are multiplied, and begins to consider using the continuity strategy in solving the problem of multiplying two negative numbers (lines 039-041).

After interaction with software

This part of the interview was carried out when the computer is closed.

042	N:	how do you imagine the idea of negative three minus five?
043	B:	when you add a number to a negative number, as long as the number is less than zero your sum is always going to get smaller, and as long as the number is greater than zero your sum is going to be larger
045	N:	that is an abstract answer
046	B:	is it?
047	N:	what about multiplying negative three by negative five?
048	B:	because we are multiplying negative number by a negative number, your [product] is getting larger
049	B:	so if you are multiplying negative three by five, your product is going to get smaller. Just follow the path
050	N:	do you see the path?
051	B:	I do yeah, I follow the line there [body swing to the left] and I follow that way [body swing to the right]

It is quite interesting that B has abandoned the object properties which were so strong at the beginning, and is using the action oriented metaphor of following the path, referring to the continuity on the dynamic number line. One hour is not long enough for somebody to completely change their thinking, but we see evidence of dynamic action oriented thinking in B's actions and words, which we attribute to his interaction with the software.

DISCUSSION AND CONCLUSION

Our task in this study was to investigate how the dynamic representation of the number-line affects the way in which operations with integers are conceptualized. We modelled the study on Lincheveski & Williams' (1999) semiotic framework, and Davidson's (1992) object vs. action theory. We incorporated modelling and technology from Sinclair and Crespo's (2006) study. From our results, we do not find it surprising that there is action orientation while interacting with the DGE software. We attribute this to the nature of the dynamic tasks in eliciting action-oriented thinking. The main hypothesis was that because of the dynamic, continuous nature of the number-line (which de-emphasizes the actual value of numbers & which require interaction), attention might shift to actions. We proposed that this hypothesis has been sufficiently justified in this study. Our claim is that action oriented thinking is needed for working with integers and rational numbers. One implication for teaching and learning is that by addition and subtraction might better be grasped by students if they acted it out by walking along authentic paths (Nurnberger-Haag, 2007) which are marked. They can then watch and decide for themselves how the numbers are changing as they walk it off.

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DEALING WITH MATHEMATICAL ABSTRACTION IN TEACHING

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When teachers plan, one of their most important challenges is to deal with abstract mathematical concepts and figure out ways of translating them into understandable ideas. By analyzing mathematics classroom interaction through the lens of reducing abstraction, this paper discusses how teachers deal with mathematical abstraction in teaching.

INTRODUCTION

Abstraction is often seen as the fundamental characteristic of mathematics; it has been a central topic of discussion since the days of Aristotle and Plato. Aristotle himself admitted that "mathematical objects are the result of abstraction" (Lear, 1982, p.161). More recently, "abstraction has been recognized as one of the most important features of mathematics from a cognitive viewpoint as well as one of the main reasons for failure in mathematics learning" (Ferrari, 2003, p. 1225). In this regard, this paper aims to explore the notion of mathematical abstraction in the learning and teaching of mathematics. Here is the brief itinerary for the rest of my paper. First, I will discuss the notion of abstraction in the contemporary educational practices in the context of mathematics education. Second, I will attend to the notion of reducing abstraction as propounded by Hazzan (1999). The reducing abstraction framework was initially developed to examine the mental process of the learner and has been used in different areas of mathematics and computer science (Hazzan, 1999, Hazzan & Zazkis, 2005, Raychaudhuri, 2001). My attempt, however, is to look at the notion of reducing abstraction from a teaching perspective rather than a learning perspective, thus offering a new area of applicability of the framework. In so doing, the notion of reducing abstraction has been redefined and applied to analyze the mathematics classroom interaction. Third, I will explain how the data was collected followed by the results and discussion. Finally, some concluding remarks will follow.

What is Abstraction?

With regard to the nature of mathematical concepts, they are generally classified into two basic referential domains—concrete and abstract. As it is commonly understood, "a concrete concept is one whose referent is demonstrable and observable in a direct way", as Wilensky (1991) puts it, "something tangible solid; you can touch it, smell it, kick it; it is real" (p. 55) as opposed to "[a]n abstract concept is one whose referent cannot be demonstrated or observed directly" (Danesi, 2007, p. 227). From the empiricist perspective, concrete is associated with physical knowledge based on experience whereas the abstract is associated with logical and mental structures such as mathematics (Piaget, 1970; van Oers, 2001).

This view of abstraction in relation to concepts as is *commonly understood* is easy to see in mathematics as mathematical objects or concepts that are not something that can be touched or kicked. In this traditional view of abstraction, decontextualization and disconnectedness are seen to be the essential properties of abstraction. Largely influenced by this view of abstraction, mathematics has gained an image in the public as a decontextualized and disconnected subject from the real world. Recently this view has been challenged by many educators and researcher (see Noss & Hoyles, 1996, van Oers, 2001, Wilensky, 1991).

Among others, Wilenski's (1991) view of abstraction which is based on the relationship between the person and object of thought provides more flexibility and a broader view in regards to the mathematical abstraction. He writes: "concreteness is not a property of an object but rather a property of a person's relationship to an object. Concepts that were hopelessly abstract at one time can become concrete for us if we get into the "right relationship" with them" (Wilensky, 1991, p. 197). From this perspective, mathematical concepts are neither more nor less abstract in their own right; it depends on the internal connection of the learners with the concepts.

LITERATURE AND THEORETICAL FRAMEWORK

Hazzan's (1999) research on how students learn abstract algebra is an important work in understanding the mental process of the students while learning new mathematical concept. Her finding is that learners tend to make unfamiliar concepts more familiar by reducing the level of abstraction, which usually happens unconsciously. It often occurs when they do not have a mental construct to 'hang on to' to cope with the same level of abstraction as introduced by the authorities (teacher, textbook etc.). In other words, when a student sees a mathematical object, he or she will try to make sense of it based on his or her past experiences with other mathematical objects.

Hazzan (1999) categorizes three abstraction levels, each of which interprets students' learning as some way of reducing abstraction level of the concept. These three levels are:

Level 1: Abstraction Level As The Quality Of The Relationships Between The Object Of Thought And The Thinking Person

It refers to the idea that whether the concept is abstract or concrete is not the property of the concept on its own right; it is based on the relationship of the person to the object of thought.

Level 2: Abstraction Level As Reflection Of The Process-Object Duality

It refers to the situation when there is an emphasis on procedures and the techniques (process conception) rather than constructing meaningful mathematical concepts (object conception).

Level 3: Degree of Complexity of Mathematical Concepts

It refers to the tendency of the students to work with an element or subset which usually gives a partial picture of the concept, rather than working with the whole set.

Hazzan (1999) mentions that these three levels of abstraction should not be thought of as hierarchical or disjoint, but as inter-related and one may even emerge from the other. For example, a learner trying to cope up with a concept in a less complex manner (Level 3) or a process (Level 2) can be interpreted as an attempt of the learner to make the concept more familiar (Level 1). So, based on the perspectives one takes, one level of reducing abstraction can be thought of as reducing abstraction in the other level.

What does this tell us about teaching? This clearly points to the idea that mathematics teaching should be directed towards making richer connection between the learners and the new (unfamiliar) mathematical concepts. In other words, the reducing abstraction framework tells us that while introducing new mathematical concepts/ abstraction, it is necessary to maximally use previously acquired knowledge, experience and level of thinking as well as students' familiar contexts so that a richer mental connection between the learner and the concept may be established.

This idea is in line with many other psychologists and educators (see for example, Davydov, 2000; Hershkowitz, R., Schwarz, B. B. and Dreyfus, T. 2001; Piaget, 1970; Vygotsky, 1996). For example, Piaget's idea of developmental psychology and genetic epistemology tells us that children develop abstract thinking slowly, starting as concrete thinkers with little ability to create or understand abstractions. Based on this idea, the genetic approach to teaching mathematics is widespread. Vygotsky (1996) writes: "Direct teaching of concept is impossible and fruitless. A teacher who tries to do this usually accomplishes nothing but empty verbalism, a parrot-like repetition of the words by the child". I agree with Safuanov (2004) when she says:

> "Strict and abstract reasoning should be preceded by intuitive or heuristic considerations; construction of theories and concepts of a high level of abstraction can be properly carried out only after accumulation of sufficient (determined by thorough analysis) supply of examples, facts and statements at a lower level of abstraction" (p. 154).

Safuanov (2004) suggests that teaching involves the process of introducing new abstractions, concretising or semi-concretising them, then repeating at a slightly more advanced level. How can this be done? There is no direct answer to this question. A review of literature shows that, in most cases, providing many examples as well as incorporating tools such as models, technologies, metaphors, metonymies, analogies, gestures, manipulative, etc. has been proved effective (Edwards, 2003; Lakoff & Núñez,1996; Sfard,1991). These tools act as a vehicle to go from the abstract mathematical ideas inherent in the problem to the familiar and concrete domain (source) and then back to the realm of ideals or abstract concepts (target domain). That is, the concepts are concretized and presented to the students in the lower level of abstraction temporarily. The goal is, however, to go to the higher level of abstraction stepping on the lower level. This activity is certainly an attempt to reduce the level of abstraction of the concept on the teachers' part in order to make the concept mentally accessible to the students.

METHODOLOGY

The research questions that guided this work are: 1) How does a teacher deal with the abstraction level in teaching, and 2) Can the reducing abstraction framework suggest a plausible explanation for the sources of teaching activities and its impact on students' understandings or misunderstanding of the concept? To answer these questions, I used two sets of data: a) observation of mathematics classrooms b) informal conversation with students. The strategy for gathering data consisted of an observation of two college preparatory classes (each lasted about an hour and half) taught by two different teachers who are well experienced in teaching this course, and professionally trained mathematics educators. All data were collected by the author, who attended the lecture and took extensive field notes. As much as possible, the phrases, statements or sentences the instructors used to explain the concepts including some observable behaviour such as 'gestures' as well as student responses that the observer found relevant for the study were noted down. An audio or video recording was avoided due to the risk of influencing the natural classroom situation. Due to the space limitation, two examples have been selected, analyzed and presented.

RESULT AND DISCUSSION

Example 1: The instructor posed the following problem to the class:

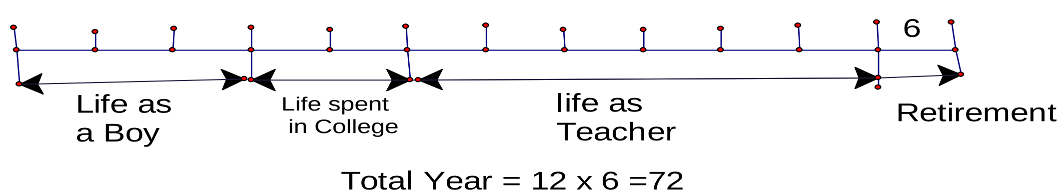
John spent a quarter of his life as a boy growing up, one sixth of his life in college, and one half of his life as a teacher. He spent his last 6 years in retirement. How old was he when he died?

Discussing the problem and considering the total years of his life as one, the teacher writes the different stages of John's ages as fractions as ¼ + 1/6 + ½ = (3+ 2+ 6)/ 12 = 11/12. Then she subtracts the sum of the fractions from 1 in order to get the fraction of John's life spent in retirement (i.e. 1 - 11/12 = 1/12). And finally, she solves the problem by equating the fraction with the given amount of years (6 years in this case). The teacher writes: 1/12 of life = 6 years; 12/12 of life = 12 x 6 = 72 years is the total age.

Some of the students expressed their frustration to this method as they could not make sense of what was going on in the solution process. Students' reactions (pointing to 1 - 11/12 = 1/12) such as "Where does that 1 come from?", "Why did you subtract 11/ 12 from 1?", and "Why did you suppose his total age equals 1…it does not make sense to me!"), clearly show the students' struggle to understand the mathematics behind it.

As an alternative approach, the teacher explained the problem to the students as follows:

First, they were told that since one quarter, one-sixth and one half are the fractions used in the problem, the best number to choose to set up the number line is 12 (for the sake of convenience) because it is the lowest common multiple of those fractions. To do so, they would need to represent the timeline and quantity schemata in a visual way. They were shown how to do this by an appropriate diagramming technique as shown below.



The idea was to allow the students to see the relation between the fractions of the ages in concrete terms, by the use of diagram including 'time as a number in a number line' (source domain) metaphor. They were then able to understand much more concretely that representing John's part of his life as a boy (which is a quarter of his life) suggests moving John's age-point 'to the right by 3'' on the time line. Similarly, life spent in college suggests moving the age-point 2 units to the right and so on. Finally, one partition out of 12 partitions in the time line was left for retirement. Since the retirement is for 6 years, it can be easily seen that each partition in the time-line represents 6 years (time is a quantity metaphor). Hence students could see that John was 6 x 12 = 72 years when he died. It was surprising to find out how easily the students comprehended the abstract concept in the second method. Students' reaction that "this method makes a whole lot more sense than the previous methods" also supports this fact.

One of the reasons that can be attributed for students' difficulties in the first case is that manipulation of the fractions without any concrete referent might be too abstract for some of the students. Here, the teacher started from the abstract idea of fractions inherent in the problem and stayed at a level of abstraction. Noticeable reduction in abstraction did not occur. In the alternative method, the incorporation of the familiar objects/context such as 'the number line' and the metaphor, 'time as a number in a number line' in instruction helped students to establish a close mental relationship with the concept. This can be interpreted as an act of reducing abstraction in the first level (unfamiliar vs. familiar).

Example 2:

The teacher posed the following question:

The three lines are, 2x + 4y = 16; 4x – 3y = 6 and 3x + y = -2. Graph the three lines and label them. Do they form a triangle?

(T = teacher, G = a group of students, S = an individual student):

T: How can you graph these equations? (Pause of about 4 seconds.) Let me show you how. I choose the first equation (pointing to $2x + 4y = 16$) first and show you how to graph it, ok…? Use the cover method. I cover $4y$ (she covers $4y$ with her hand and completely hides it from the scene). Now tell me what is the value of x?

G: 8 (group response)

T: So, We have one point (8, 0).

T: Now if I cover $2x$. (She covers $2x$ with her hand). What is the value of y?

G: 4 (group response).

T: So, the other point is (0, 2). Now we have two points (8, 0) and (0, 2). Let me plot these points on the graph and draw the straight line.

S: Oh, I see. That's easy!

Understanding the relationship between the graphical representation and the algebraic representation of a linear function is one of the most important concepts in this level. Usually, these concepts are introduced in the textbooks by three methods: 1) Given equation is transformed into slope intercept form $y = mx + b$ where m is the slope and b is the y -intercept. Then using the slope and y -intercept the graph of the function is drawn. 2) Making a T-table for x and y and randomly plugging in a few values (usually 3 - 5 numbers) for the independent variable and calculating corresponding values for the dependent variable. Then the coordinates are plotted in the graph to draw the figure. 3) By finding x-intercept and y-intercept in which case student should know that on the x-axis, the y-coordinate is zero and vice versa.

The 'cover up' method as employed by the teacher is not fundamentally different from method 3) above. On the x-axis, the y- coordinate is zero. Hiding $4y$ with her hand (gesture) while finding the value of x, the teacher is using "Zero is the lack of an object" metaphor (Lakoff & Núñez, 2001, p. 372). Her gesture and the use of metaphor significantly reduced the level of abstraction of the concept for the students. This is an attempt from the teacher's part to make the unfamiliar 'intercepts' concept more familiar with the use of gesture and 'zero is the lack of an object metaphor'. From this perspective, this act can be interpreted as the first category of the reducing abstraction.

Viewed from the other perspective, it can be put in the process-object duality because the cover up method emphasizes the process conception (how to do it but not what it means). It should be noted however that the teacher's intention was to use this method to make the process easier while keeping the concept meaningful to the student. However, the students' response in the second question reveals that the students did not understand the concept as the teacher wanted them to do.

T: To draw the line for the second equation (points to the second equation which is $4x - 3y = 6$) we need to find any two points, yeah! Let's find them. (After an instance of mental calculation, the teacher writes (0, -2) and (3, 2) as two points).

S: How did you get (3, 2)? It has to be (1.5, 0). That's what I got.

At this moment, there was confusion and bewilderment among most of the students as to how the teacher arrived at the points (0, -2) and (3, 2). It is evident that the student could find the correct points on the line mechanically but with no meaningful understanding. For them, (3, 2) could not be the point on the line.

T: Oh, I see what you are talking about. Um... cover up the $-3y$ (she covers $-3y$ with her hand). Now tell me what is x?

S: 1.5.

T: I don't like that number. So I choose $x = 1$. Then $y = 0.66666\ldots$, right?

T: umm... still I don't get a 'nice' number. If I choose $x = 2$ then... still I get an 'ugly' number. For $x = 3$, what do I get for y?

G: 2

T: So, (3, 2) is one of the points in the line.

There are no nice or ugly numbers in mathematics on their own right. It all depends on the relationship between the thinking person and the object of thought. For some people, fractions may be ugly number to work with but some people enjoy working with fractions. By evoking the aesthetic dimension of mathematical objects (Sinclair, 2003), the instructor successfully helps student to understand the concept that there are, in fact, many (infinitely many) points in a line but for the sake of simplicity, a 'nice' number (a non-fractional number) was chosen.

One of the interesting questions to ask here would be what is the teacher's intention behind focusing on the process conception at first? Was it because this was how she learned when she was a student? Was it because of the teacher's belief in Sfard's (1991) theory of process-object duality according to which the process conception is less abstract than an object conception and that once the students are familiar with process and after interiorization, they will be better able to learn the concept? Was it because introduction of process conception is easy to introduce and makes life easy for the teacher? Or was it an intentional act to encourage mathematical thinking in the Vygotskian sense? The intention was not known at this time as the teacher was not interviewed; but it could be inferred from the later part of the classroom interaction that the act was intentional and trying to encourage mathematical thinking in Vygoskian sense. Vygotsky (1926/1997) thinks that it is necessary to establish obstacles and difficulties in teaching, at the same time providing students with ways and means for the solution of the tasks posed. Rubinshtein's (1946/1987) idea is not different: "The thinking usually starts from a problem or question, from surprise or bewilderment, from a contradiction" (cited in Safuanov, 2004, p. 4). It is similar to Piaget's phenomenon of equilibration which usually occurs from the violation of balance between assimilation and accommodation. Whatever the reason may be, the teacher's act of reducing abstraction (in the 1st and 2nd level) proved to be a pedagogically effective activity.

CONCLUSION

Reducing abstraction is one of the theoretical frameworks that examine learners' behaviour while coping with levels of abstraction. In this paper however, I attempted to look at the notion of reducing abstraction from a teacher's perspective rather than a learner's perspective thus offering a new area of applicability of the framework. In so doing, some problematic situations have been identified. As Hazzan (1999) mentions, from the learner's perspective, the mental process of making unfamiliar concepts more familiar by reducing levels of abstraction happens unconsciously; it often occurs when learners do not have a mental construct to 'hang on to' to cope with the given level of abstraction. But from the teacher's perspectives, the choice of the words and phrases such as 'unconscious' and 'lack of the mental construct to hang on to' have been identified as problematic. It turned out that while dealing with mathematical abstraction in teaching, teachers reduce the abstraction level of the concept but, in most cases reducing abstraction is intentional and of pedagogical value. Finally, this paper exemplified some instances where 'reducing abstraction' has been proved to be an effective teaching strategy while in some cases it may be misleading. The results emphasize the importance of paying attention to the nature of students' understandings and possible misconceptions that may arise while reducing the abstraction level of the concepts in teaching.

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THE PRAGMATICS OF MATHEMATICAL DIALOGUE THROUGH EMAIL

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This paper investigates the nature of the mathematical dialogue carried out between pairs of high school students using email as a problem solving medium. A set of discourse tools are applied to study the pragmatics of the dialogue and comparisons made to the traditional formats for dialogue. Differences which appear in the use and choice of personal pronouns are examined along with the interactions between the students in their virtual shared space.

INTRODUCTION

The use of electronic means of communication has escalated and can now be considered commonplace among students in Secondary schools. My interest in this investigation stems from a desire to understand and develop the thinking process within the mathematics classroom. Wagner (2007) has reported on the passive resistance and silence he experienced when attempting to engage students in dialogue in a mathematics classroom. Even in a classroom where meaningful dialogue can be generated this is typically dominated by a few students and offers the opportunity for some students to 'hide'. Recording small groups of students working together can offer insight to the process using discourse analysis. There has been a growing body of research on the 'language of mathematics' since the early work of Halliday (1978) and Pimm (1987). This paper examines whether asking students to communicate their ideas about a problem through email can extend this body of research and compares this medium to other forms of mathematical dialogue by analysing its lexico-grammatical structure.

Research into the use of email as a collaborative tool has primarily restricted to subjects other than mathematics. In a business communication setting Kirkley et al. (1998) found that students appreciated being able to negotiate a meaningful response to problem situations through email. While there is a lack of nonverbal cues found in personal face-to-face interactions, which are often a basis for understanding, Kirkley et al. noted that these cues also reinforced hierarchies and dominant personalities and found students often seem to prefer email when communicating *about* problems. The extension in this research is in *doing* problems in a mathematical context.

In examining the dialogue I shall be using a discourse toolkit established in previous studies in this area. I examine the writing with a particular view to hedging (Lakoff, 1972; Rowland, 1995), the use of personal pronouns (Rowland, 2000), modality (Morgan, 1998; Rowland, 2000), and aspects of Politeness theory (Brown & Levinson, 1987, Grice, 1975; Goffman, 1972).

MATHEMATICS EDUCATION LITERATURE

Rowland (1995) writes extensively about the use of hedges in mathematical discourse and their role to blur the edges of a statement in order to make it hard to deny completely. Hedges can be used as *shields* against uncertainty as a *plausibility* shield (*I think. etc.)*, or to implicate a third party as an *attribution* shield (*the text tells us.. etc)*. Alternatively hedges can be *approximators,* in the form of a *rounder (about, etc.)* or an *adapter (somewhat etc.)*. Rowland refers Channell (1985) in identifying goals speakers achieve using vague language, including: saying what you don't know how to say; covering for lack of specific information; expressing politeness; and protecting oneself against mistakes. The presence of hedging is therefore an important part of the communicating process.

The Politeness theory of Brown and Levinson (1987) includes Goffman's (1972) notion of *face*, which is the 'social value a person effectively claims for himself'. Face is categorised in terms of *positive* face, for social approval, and *negative* face, expressing the need for freedom of action. Typical conversation consists of *face threatening acts* and *face-saving acts*. This is another important part of any dialogue and its presence is examined in this research.

Rowland (1992, 1999) brings awareness to the possible function of personal pronouns in mathematics discourse. Of particular interest is the notion of agency being expressed through the shifting use of personal pronouns throughout the users' language. Rowland suggests that users tended to shift from using the *I* term to the *you* term as their understanding of the concept shifted to a more general nature. The *you* term can detach the student from what they are saying and imply that the user of the phrase has confidence that their explanation is generally acceptable, although the students may not be aware of this. Pimm (1987) discussed the use of the *we* term and suggested that the user was associating themselves with an anonymous expert community (pp. 69-70) to lend authority to an utterance. Another usage of the *we* term is to draw complicity between the participants in the discourse (Mühlhäusler & Harré, 1990). The suggestion is, then, that the students' choice of personal pronouns speaks to their sense of agency in relation to their work, and to their confidence in its value. Wagner (2007) considers that students are 'promoting a sense that everyone sees the same things in the same way' (p. 42), which explains the general *you* voice and even the *we* discussed by Rowland (2000) and Pimm (1987).

Another aspect I wished to examine was modality. Modality is used to convey the speaker's propositional attitude (Wagner, 2007), or the attitudes on the part of the speaker towards the factual content (Rowland, 2000). If the email is to represent the students thinking in a meaningful way then it is important that there is a sense of modality coming through the text and that the medium does not render the message sterile so that it is the receiver who associates attitude, justly or not.

Conversation Analysis (CA) is another tool used to look for patterns in discourse. In particular, CA can be used to examine turn-taking and adjacency-pairs such as in agreement and disagreement (Schiffrin, 1994). This seems to naturally fit with the email dialogue where there is none of the usual interruptions of an oral dialogue, The four Maxims of *Co-operative Principle* (CP) (Grice, 1975) are relevant here. *Quality*: Speak only what you believe is true; *Quantity*: Be as informative as is required; *Manner*: Be brief, orderly, and unambiguous; *Relevance*: Be relevant to the matter in hand. As Rowland (2000, p. 125) observes: 'the participants care about the mathematics, but they also care about themselves, their feelings and those of their partners in the conversation.'

METHODOLOGY

To initiate the email dialogue a student from a grade 11mathematics class was partnered at random with a student from a different class and assigned a random geometry problem (n = 15). The problems varied in difficulty and no attempt was made to explain the problem or scaffold the solution. Any requests for help were deflected back to the students. The students were asked to direct any difficulties or questions to their partner. I repeatedly stressed to students that I wanted to see the development of their thinking and, in some cases, the problems may not be easily solvable but a lack of solution might not affect the assessment of their work. I was looking for an honest attempt to address the problem and a dialogue demonstrating their ideas and thinking. Students were instructed to use the 'reply' option so that all correspondence was kept in a single document. When they were satisfied with their result, or felt they could go no further, they were asked to forward the email to me. I examine here the work of one pair, H and J.

Analysis of the Discourse

H and J randomly selected the following problem:

> A semicircle has a radius of 2 cm, and chord *AB* is parallel to the diameter *CD*. If chord *AB* is half as long as the diameter of the semicircle, how far from the diameter is it?

H1 10:10 AM

The first thing we need to do is look at the information that we are given.
We know the semicircle we are looking at has a radius of 2cm and therefore
a diameter of 4 cm. We also know that chord AB is parallel to the diameter

CD, and that chord AB is half as long as the diameter of the semicircle,
therefore 2 cm long. The question is where to go from here any thoughts?

J: 8:49 AM (next day)
HEY H! Okay, time to do some math!!
I agree with everything you just said.. Would you agree that we can solve
this by saying that it is a trapezoid? Just an idea, what do you think?? ;)

H: 8:51 AM
I think that's a brilliant idea! I think it is a trapezoid because the top and
bottom are parallel. This gives us some new information: 1) sides AC and
BD are equal length, and 2) <C and <D, and <A and <B are equal angles.
The question is, how can we use this?

J: 8:56 AM
Now we're cooking ;)
Okay, what if we made three triangles within this trapezoid! Going from
the midpoint of chord CD, to A and B. So, what shall we do now? do i hear
pythagoreas theorem coming on? That's for us to figure out 8-)

H: 9:07 AM
We could use Pythagoras theorem, but not using the three triangles we just
made. What we could do, however, is split the centre triangle down the
middle, making 2 right angle triangles. That gives us sufficient
information: because AB and CD are parallel, perpendicular chord H bisects
chord AB and CD. This makes AF 1cm long. However, since <A is on the
edge of the circle, we know the hypotenuse of this new triangle is 2cm.
And thus, we can use Pythagoras theorem.
a^2 + b^2 = c^2
1^2 + b^2= 2^2
b^2= 5 so b= sqrt 5
And so, the distance between the two chords is sqrt 5. Make sense?

J: 9:20 AM OH MY WORD H it seems as though we've solved it!!
Wow, I am so impressed with us right now.. really.. I am.

Table 1: H & J –email dialogue

The opening exchange was typical of all the groups. The problem was either repeated or summarized without any suggestion of solution and ending, as in line 6, with a request for input. Conversational analysis suggests that this is an offer to ‘turn-take’ while it might also be considered as redressive ‘face-work’ used to avoid a face threatening act. In one pairing where this did not occur the response was abrupt and may have been a result of considering this a face threatening act.

J begins her reply by establishing a *voice* – a light tone - with her casual comment in line 8, *Okay, time to do some math!* This is followed by another redressive face-saving act in line 9 where J makes a point of agreeing with H’s simple reiteration of the problem. The modal ‘would’ can be seen to carry a positive attitude in line 9, while the strong hedge ‘just an idea’ redresses the positive face threatening act from the suggestion. J has made the first move and while she is likely confident that there is a trapezoid in the figure she makes no further suggestion before passing the turn back to H. She is adhering to the Quality Maxim in not pushing her idea even though her second response indicates that she has thought this through further. She is also looking for confirmation.

H immediately replies with a strong face-saving act before using his turn to extend the model. His use of *I think..* (line12) is an epistemic hedged performative (Lakoff, 1973) in validating J’s contribution. Again H adds detail to the suggestion before passing the problem back to H. At this stage H has not offered any fresh insights and still seems to be in a *zone of conjectural neutrality* (Rowland, 2000). He is content to use a form of attribution shield and work with the suggestions J makes.

In her reply, line 18, J’s use of the modal *Okay* marks a shift in her confidence and an indication that she now feels comfortable to make a more thoughtful suggestion without it being perceived as face-threatening. Again she provides only limited insight before using what I would regard as a *complicit* hedge in line 19 by asking ‘*So, what do we do now*?’ Note the use of *so* as an interjection here to add emotion. She suggests the use of Pythagoras equation with an enigmatic ‘*that’s for us to figure out’*. These comments may be perceived as face-maintaining in throwing out ideas without thinking them through. She may be trying to draw complicity from her partner while maintaining a plausibility shield.

H’s response is interesting in that he uses *we* to indicate he is still working with J. However, there seems to be a shift from the use of *we* as a team in line 22 to a general usage in line 23. This shift occurs as the student H gets deeper into his explanation, which takes on a more noticeably formal modality. H finishes with ‘*Make sense*?’ which may be seen as asking for confirmation from his partner that his description is making sense, or his desire to keep J involved.

For her turn, J does not show any signs that she has checked H’s process or reflected further on it. She has not picked up on his error in rearranging the equation or incorrect answer. She is satisfied with the presence of his argument.

It is interesting to observe that there are 15 occurrences of *we* which relate to the collaborative nature of the problem but only 2 of *I* in the form of a personal pronoun related to the procedure. The generalized form of *you* does not occur at all. This pattern is repeated in most of the student's dialogues, only rarely did a student switch to the generalized *you*. This may be a product of this genre in that the students are not formally presenting their work but are relating to each other. This may suggest that the hidden presence of the teacher is not obviously affecting the dialogue.

Other email exchanges illustrated that students were comfortable in using fillers such as *Hm*, *umm* or *er* in their emails in the same way as they do when speaking. Existence of fillers suggests the students were writing as their train of thought developed. Linguistically, fillers are often used when a speaker doesn't want to give up their turn in the conversation while they think about what to say next. Their presence in emails is interesting and worth further study.

SUMMARY AND CONCLUSION

The application of tools for discourse analysis on the students' problem solving through email indicates several comparisons with other forms of dialogue. It can be seen that it is possible to associate modality with utterances in this format by picking up on key words such as interjections. In addition, there still exists the 'half-finished and vague utterances found in spontaneous discourse' (Pimm, 1987). In the majority of cases the emails were short and demonstrated adherence to the basics of Politeness theory. The conversational analysis indicated that students followed a turn-taking process and were careful not to perform face-threatening acts without redressing their statements to make them more acceptable. The exchange included many of the features of a spoken dialogue, such as fillers, expressions of doubt, and requests for their partners' opinion. The dominance of the *we* pronoun in these email exchanges indicated a strong sense of the students working together on a solution and differs from the findings of Rowland (2000) and Wagner (2007) in which the students were describing/presenting their results. In this genre the use of *we* also extends to the generalization of a solution rather than *you* as found in other forms of communication. This suggests further study into implied agency here would be worthwhile. Comparison of several such exchanges to identify markers at which agency shifts from an inclusive *we* to a more personal or general form, such as in lines 22/23 in the above transcript, might hold useful information for a teacher with regard to students' understanding. The response of the student's partner to this change in agency is also a point of interest. Do they respond to it in a passive way (as in the case here) or in a more interactive way?

A disadvantage of this process is the lack of evidence of prosodic hedges – variations in pitch temp or rhythm -and the lack of intimacy in the problem solving process. Gordon Calvert (2001) suggests that individuals need to work in the same intimate space to have a mathematical conversation. Clearly email does not afford this level of intimacy, but it does offer a sense of anonymity allowing students to relax their guard more than in the intimate setting of group work. Hesitations which can unsettle a student in class can be masked in this format as they are afforded time to collect their thoughts without interruption. More direct student feedback regarding this may add fresh insights to improve the process.

While this is not a dialogue in the true sense of the word as students are aware of the final audience as the teacher, perhaps tailoring their responses accordingly, the process opens a rich vein of possibilities. A more in-depth study which compares the approach of the same pair of students using different methods of problem solving might indicate how feasible this kind of communication is and whether it can be used to demonstrate students' thinking in an authentic way. Positive results might open further possibilities to extend the medium into other electronic forms familiar to students, such as texting and blogging.

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MAKING THE FAMILIAR STRANGE: AN ANALYSIS OF LANGUAGE IN POSTSECONDARY CALCULUS TEXTBOOKS THEN AND NOW

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Three calculus textbooks covering a span of about 40 years were examined to determine whether and how the language used has changed given the reform movement and the impetus to make mathematics accessible to all. Placed in a discourse analytic framework using Halliday's (1978) theory of functional components –ideational, interpersonal and textual, and using the exposition of the concept of a function as a unit of comparison, the study showed that language is an integral indicator of the author's view of mathematics and an important factor for textbook adoption in the pursuit of student success.

INTRODUCTION

In the late 1980s, the Calculus Consortium at Harvard (CCH) comprised of eight institutions including universities, a high school, and a community college, was funded by the National Science Foundation to redesign the Calculus curriculum with a view to making Calculus more applied, relevant, and accessible. The intent was to re/think and re/present the content so as to focus on real-world applications, to emphasize concepts and graphical representations, and to take advantage of the increasingly sophisticated technology. This initiative has been extensively embraced leading to the calculus reform movement. As a result, Calculus is now mostly presented in a manner radically different from the traditional approach which relied heavily on abstraction, formal notation and symbolism, and algebraic conventions. Besides the 'regular' Calculus courses, Calculus courses and materials have been developed for specific disciplines (such as Biology and Economics) and for different modes of delivery (such as Calculus with Computer Explorations).

The goal of this research is to see whether and how calculus textbooks designed for the postsecondary level in 'regular' Calculus courses have changed over the years with respect to the language used in the exposition and by inference, the view of mathematics manifested. One concept, that of a function and in particular its definition, is chosen and used to trace the dimensions of the language over the years and the consequent shifts in the view and presentation of mathematics in calculus textbooks. The questions that arise for me are: Has the language of calculus textbooks changed over time and if so, in what ways? From the language, how are the authors' views of mathematics characterized and how have they changed over time? Has the language changed from one that is exclusive (mathematics as an elite subject with an elite community) to one that is inclusive and accessible to all?

The three textbooks I have chosen are *Calculus* by Spivak (1967), *The Calculus of a Single Variable with Analytic Geometry*, 5th edition by Leithold (1986), and *Single Variable Calculus: Early Transcendentals*, 5th edition by Stewart (2003). Spivak and Leithold were both Mathematics Professors from American universities (Brandeis and Pepperdine, respectively) writing for an American audience while Stewart is an Emeritus Professor in Mathematics at a Canadian university (McMaster) writing for a Canadian audience. Each of these textbooks was well-known and well-used in its time. I chose the first two because they were the ones that I still have after the many moves in my life, and the third because it is one that I use in my teaching at this time.

Textbooks may be studied subjectively to describe the interaction between the student and the written material or to describe teachers' use of textbooks and the subsequent effect on the teacher (Remillard et al, 2009). However, following Herbel-Eisenmann (2007), I seek to examine the 'voice' of calculus textbooks over the years as *objectively given structure* (emphasis in the original, p. 396). This examination will be placed in a discourse analytic framework which attends to the aspects of text relating to language, voice, agency and identity. In particular, linguistic markers such as the use of pronouns, imperatives and modality will be traced as a means of addressing the above questions.

ANALYTIC FRAMEWORK

Language has been increasingly seen as an important issue relating to mathematics teaching and learning. Rowland (2000) emphasizes two principles in studying language: the linguistic principle ('language as means of accessing thought') and the deictic principle (language as a means of communication and a 'code to express and point to concepts, meanings and attitudes') (p. 2). In his *Language as a Social Semiotic*, Halliday (1978) identifies three functional components or functions of language– the ideational, the interpersonal, and the textual –from which meaning is apprehended.

The ideational functional component of the text answers the questions: What is the view of mathematics as presented in the text? How is the subject of mathematics envisioned in the mind of the author of the text and in what style is it rendered? The ideational function describes the nature of the subject matter from the ideological and epistemological stance of the author. It is 'the component through which the language encodes the cultural experience and the speaker encodes his own individual experience as member of the culture' (Halliday, 1978, p. 112). It also names the objects, concepts, and processes involved in mathematical activity and indicates agency on the part of the author and reader. The ideational function is composed of the experiential function (dealing with transitivity and agency) and the logical function (relating to continuity and modes of argument).

The interpersonal functional component describes the social and personal roles and relationships among the authors and readers and the ways in which the readers interact with the written text and the textbook itself as a whole. Evidence of this function is discerned by considering the use of personal pronouns (first, I/we/us/our, and second person, you), imperatives, and modality. The interpersonal function is the 'participatory function of language, language as doing something' (Halliday, 1978, p. 112).

The textual functional component describes the content matter or the mathematics presented in the text, the theme and modes of reasoning, the arguments and their forms, and the narratives of mathematical activity. Halliday describes it as 'the component which provides the texture: that which makes the difference between language that is suspended *in vacuo* and language that is operational in a context of the situation' (pp.112-113). Evidence of the textual function is seen in the cohesive devices the text uses to preserve consistency and continuity.

Halliday also introduces three concepts which shed more light on these three functional components, namely, *field*, *tenor*, and *mode*, respectively. The field refers to what is going on in the context of the situation and what the participants are doing, the tenor to the roles of the author and the reader and how they stand in relation to one another, and the mode to the channel or wavelength that the author has chosen to use depending on 'what function the language is being made to serve in the context of the situation' (p. 222).

I will examine each of the textbooks with respect to these three functional components and compare and contrast them as to the "voice" that emerges, the extent of agency, and the construction of the identity of the reader by the text.

METHOD

The data consists of the pages from the three Calculus textbooks that cover the exposition of the concept of a function. Exposition includes the preliminary introductory commentary and the definition (or definitions) of a function. In each textbook, there were many more pages devoted to the important classes of functions such as polynomial, exponential, logarithmic, and trigonometric functions, but I have chosen to limit the analysis to only those pages relating to development of the concept and the definition of a function.

I mined the relevant pages carefully with respect to the markers for the three functions as articulated by Halliday and elaborated by Morgan (1996). I paid close attention to the use of personal pronouns, imperatives, and modal auxiliary verbs. I also considered the use of questions and conditionals, *if*, *if…then*, and *given* and *given that*, as evidence of forms of reasoning and modes of argument.

FINDINGS AND DISCUSSION

The results of the comparison of the textbooks across markers for the functional components of language with respect to the concept of a function are given in Table 1.

Table 1. Comparison across markers for the functional components.

	Spivak (1967)	Leithold (1986)	Stewart (2003)
Pronouns - 1st person	we/us/our 32 instances	we/us 5 instances	we/us 24 instances
Pronouns – 2nd person	you 9 instances	None	you 3 instances
Imperatives-Inclusive	let's 1 instance	call, compare, let, note, observe, recall 6 instances	consider, determine, let, notice, remember 7 instances
Imperatives-Exclusive	None	find, read 4 instances	draw, find, sketch, use 6 instances
Modal verbs	May 2 instances	None	None
Questions	2	None	1
Conditionals	If 6 instances if … then 10 instances	Given 3 instances given that 2 instances	if 3 instances if … then 4 instances

The Interpersonal Functional Component

The most easily-detected functional component of language is the interpersonal which describes the roles and relationships of the author and the readers. This component can be discerned from the incidence of personal pronouns, imperatives, and expressions of modality.

The most striking occurrence is that of 32 instances of first-person pronouns in Spivak as compared with five in Leithold and 23 in Stewart. In Spivak, there were 29 uses of *we*, two of *us* and one of *our*. My reading of this is that Spivak views the reader as someone who is part of the community of people doing or studying mathematics. From the opening paragraph in his liberal use of *we* and *us*, Spivak sets the tone of including the reader in his deliberations. He concludes his opening paragraph with ‘Let us therefore begin with the following:’ (p. 37). An alternative reading of *we* is given by Pimm (1987) who questions the *we* that authors use and wonders how he personally is implicated in the proceedings as to responsibility for what may ensue. Another possible reading is that the use of *we*, *us*, and *our*, suggests a more general form indicative of the register of mathematicians.

In comparison to this substantial use of first-person pronouns, the five occurrences of *we* in Leithold read clinically as in ‘we see that’ or ‘we observe that’. There is an implied *us* in the following sentence from Leithold: ‘In Definition 0.5.1 the restriction that no two distinct ordered pairs can have the same first number assures that y is unique for a specific value of x’ (p. 45), for whom does the restriction assure, if not us, the readers?

In keeping with Spivak’s view of the reader as a thinking and feeling partner in the endeavour, there are nine instances of *you* as he recognizes the presence of the reader, such as ‘It will therefore probably not surprise you to learn that …’ (p. 37) and ‘you may feel that we have also reached the point where…’ (p. 45). However, Spivak does write at one point: ‘You should have little difficulty checking the assertions that…’ (p. 39), an assumption that may leave the reader a little frustrated if some difficulty is encountered. Later, he redeems himself with his regard for the reader in ‘If the expression $f(s(a))$ looks unreasonable to you, then you are forgetting that $s(a)$ is a number like any other number, so that $f(s(a))$ makes sense’ (p. 40). This is an early indication of affect in mathematics learning, in recognizing the role of emotions and feelings. There are no instances of address to the reader in Leithold while Stewart has three: ‘as you can see’, ‘You can see that…’, and ‘when you turn on a hot-water faucet…’.

The use of personal pronouns indicates the presence or absence of humans in the activity and the implied distance and degree of formal relationship between the author and the reader (Morgan, 1996). Spivak and Leithold are at opposite ends of the continuum in this regard. Spivak even employs the construction, ‘Lest you become too apprehensive about …., let us hasten to point out that…’ which is by far the greatest consideration an author can give to a reader. Here again Spivak is demonstrating his recognition of the phenomenon of affect, despite his quaint sentence construction. Leithold deploys his words in a detached ‘scientific’ manner, the very opposite of the kind of writing that Burton and Morgan (2000) exhort mathematicians to adopt.

The frequency of imperatives in a text indicates the degree to which the author wishes to draw the reader’s attention to a point in the text (note that, observe that), to encourage the reader to reflect (consider, compare, recall, remember), or to give a simple command (find, sketch, use). Both Leithold and Stewart use a similar number of imperatives that indicate the usual textbook framing (consider, notice, observe, recall) and that signal the ability of the author (determine, evaluate, find, sketch, use) to tell the reader what to do. The former are examples of inclusive imperatives that characterize the reader as ‘thinker’ while the latter are examples of exclusive imperatives that characterize the reader as ‘scribbler’ (Rotman, 1988). The use of imperatives shapes the relationship between author and reader and serves to construct the reader as a potential member of a community (Morgan, 1996). It is note-worthy that Spivak does not use any of these imperatives but still manages by his use of personal pronouns to convey a sense of introducing the reader to and including the reader in the activity that mathematicians undertake.

The imperative ‘let’ occupies a special place in mathematics (as is commonly found in arguments and proofs, let x be…). Spivak uses it once in ‘Let us therefore begin with…’ (p. 39), which is more of an invitation rather than a call for consideration. Leithold uses the construction, let f be…, three times. There are no instances of *let* in Stewart in the pages under consideration but there is a variety of other imperatives that are roughly equally inclusive and exclusive (Table 1).

Modality, as a feature of language, enables authors and speakers to express their feelings, values, attitudes, and judgments about the propositions in their texts. Halliday (1978) expresses a preference for the term, modulation, rather than modality in that the text is modified or nuanced in some way. Demonstrations of modality include modal auxiliary verbs such as ‘may’ and ‘can’, adverbs relating to the uncertain state of knowledge such as ‘possibly’ and ‘maybe’, the use of moods and tenses, and the use of hedges (Rowland, 2000, p. 65). For these three textbooks there was little or no evidence of modality. There were two instances of ‘may’ in Spivak (‘You may feel that we have also reached…’ and ‘Two consolations may be offered’, p. 45). These have nothing to do with the mathematics involved but indicate concern for and offer solace to the reader. Leithold and Stewart offer no suggestion that that there is any uncertainty related to mathematical activity and by their lack of use of modality, indicate a view of mathematics that strongly holds to an absolute, ideal perspective.

As seen from these markers for the interpersonal functional component, the tenor of the language in the three textbooks is marked differently. Leithold and Spivak are diametrically opposite in the use of the first and second person pronouns and imperatives in engaging and addressing the reader with Stewart striking a moderate note in this regard.

The Textual Functional Component

All three authors use the mode of discourse characterized by exposition (evident of the raison d' être of the textbook) in laying out a clear and concrete treatment of the subject matter. Questions as evidence of a conversational or dialogic style of exposition were barely used; there were two questions in Spivak, none in Leithold and one in Stewart. The one instance in Stewart is a perfect example of the question-and-answer cohesive form:

> 'The graph of a function is a curve in the xy-plane. But the question arises: Which curves in the xy-plane are graphs of functions? This is answered by …' (p. 17).

One instance of a question in Spivak is a particularly engaging example of language that recognizes and cements the role relationships in the text: 'By what criterion, you may feel impelled to ask, can such functions, especially a monstrosity like (12), be considered simple?' This question manages to capture pronoun use, affect, and modality, all in one fell swoop. With respect to forms of reasoning, there are similar numbers of instances of conditionals in all three textbooks as *if*, and *if…then* are widely used in mathematical arguments. Leithold has 5 instances of *Given* and *given that* but these are not used as conditionals in an argument. Instead they are used in examples such as, Given that f and g are defined by $f(x) = \ldots$ and $g(x) = \ldots$. The current usage is more informal: 'Suppose that …' which, as a shortened form of 'Let us suppose that…', is more of an invitation.

The Ideational Functional Component

The ideational functional component in each of the three textbooks is very nearly identical in that the authors' content and meaning are very similar. Each author is interested in communicating the content of the concept of a function. Further, each author conveys the weightiness of the subject matter and the experience of being part of the culture of being mathematicians and doing mathematics. Each is writing of the objects and relations that are under consideration when introducing and discussing the concept of a function. By the degree of use of the linguistic markers analyzed above, each encodes in the text his individual vision of mathematics. The view of mathematics evinced in all three is fixed, absolute, and formal.

Besides the content, the ideational component, in describing which actors carry out which processes, speaks to the concept of agency as it is invited or suppressed. Morgan (1996) elaborates on the use of nominalization in order to suppress or mask agency. Clear examples of suppression of agency occur in Leithold: 'Equation (1) defines a function' and 'This equation gives the rule by which …' (p. 45). There are no similar constructions in Spivak or Stewart.

In summary, the three textbooks are similar in their theme and message but differ considerably in the interpersonal component with Stewart capturing a moderate position between what may be considered the extremes of linguistic markers by Leithold and Spivak.

IMPLICATION

The language of mathematics is often seen as foreign with its own lexicon, grammar, and modes of argument. More than being able to negotiate the language, students of mathematics must become fluent in it. Bakhtin declares that '[e]ach text presupposes a generally understood (that is, conventional within a given collective) system of signs, a language (if only the language of art)' (1953/1986, p. 105). Hence the mathematics textbook has a conventional system of signs which is part of a language that is to be understood if one wishes to be a member of the community involved in mathematical activity.

The differences in language in a textbook account for much of the reader's regard for the textbook. In this paper I have teased out the subconscious linguistic markings in the text and have shown that there is more to the text than meets the eye; that what we have taken as familiar is indeed strange: a nebulous complex of beliefs and ideas about mathematics which we adopt and perpetuate without realizing the implications and consequences. This analysis suggests that it behooves us as teachers to re/examine our practices in making textbook choices for the betterment of ourselves and our students and to be aware of the functions and forms of language that subtly maintain hegemonic practices in the teaching and learning of mathematics.

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MATERIAL AGENCY: QUESTIONING ITS MEDIATIONAL SIGNIFICANCE IN MATHEMATICS LEARNING

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Tools in the mathematics classroom are often not given the credence or the attention they warrant. Considering Vygotsky's view of mediation, tools may play a larger role in mathematics than originally thought. This study presents a framework for looking at tools in student mathematical learning. Using Pickering's analytic framework (1995) distinguishing individual, disciplinary and material agencies, I analyze two students in grade 12 and their interactions with a dynamic geometric software, specifically Geometer's Sketchpad. In the process of solving a problem I will analyze the students' engagement with the tool in terms of the different types of agencies, based on their spoken words and their actions in using the program.

INTRODUCTION

The idea of allocating agency to materials has a sparse academic history but there is growing interest in certain disciplines and the literature is expanding (Pickering, 1995, Malfouris, 2004). Material agency has been given some credence in areas like environment (Oliver, 2009) and information systems (Rose & Jones, 2005). Material agency has gained currency as an alternative to viewing the individual as having the control and recognizing that agency is not a possession of an individual but an expression of the individual in a particular context (Emirbayer, 1997). In this paper I adopt this perspective in looking at tools in a mathematics classroom.

A tension has always existed between those who consider mathematics as being more of a mental discipline and, those who consider that the physical role of objects, materials or machines playing a formative role in the learning of mathematics. While both sides recognize that tools play their role in the practice of mathematics, the "mental" mathematicians may consider that tools or machines play a relatively minor role either to simplify a calculation or to merely serve as a vessel that serves the sole purpose of "getting" to the mathematics. This attitude is not so much explicitly stated as it is practiced. Whether stemming from a Platonist's vision of mathematics as a separate, distinct and pure discipline, that is accessible solely through contemplation (Tarnas, p. 6), production acts often state no reference to materials or tools used in the process. "Scientific laboratories were…places where discoveries are made in a concrete, ad hoc fashion, and only later recast into canonically accepted formalisms; Noble laureates testified that they related to their scientific materials in a tactile and playful manner" (Turkle, 2007, p. 7). While mental discipline advocates argue tools can cloud the very nature of mathematics, advocates for an object-oriented inquiry argue that tools or machines influence how we learn mathematics and are consequently worthy of study. "Piaget believed that mathematical understandings come not from a passive perception of the physical properties of the objects but from children's reflection on the actions they perform on the objects" (in Schliemann, 2002, p. 303).

ANALYTIC FRAMEWORK

The implementation of tools or machines into mathematics classrooms and how they are used is a topic of interest: if mathematics learning is to be fully understood, the tools used in mathematical activity are not to be reduced to an unnecessary step. Wertsch claims that one of Vygotsky's major themes in his theoretical approach was "…that an adequate account of human mental functioning must be grounded in an analysis of the tools and signs that mediate it" (in Daniels, 2008, p. 4). The framework that I would like to propose for analyzing tools in mathematics education is Pickering's distinction of agencies. Pickering (1995) has classified 3 types of agency: individual, disciplinary and material. While one would not usually think of materials or disciplines as having agency, Pickering describes the individual engagement with either of these agencies as a "…dialectic of resistance and accommodation" (p. 52). Pickering has referred to this interplay of resistance and accommodation as a "dance of agency". His view is that mathematics is a product of human activity and therefore individual agency plays a major role in any conceptual and/or material advancement. However, engagement with materials or conceptual systems is not a one-sided activity. In his argument for disciplinary agency Pickering describes how a system can "…carry human conceptual practices along…independently of individual wishes and intents" (p. 115). So although individuals exercise their agency in the form of intentions and actions, they are often met with resistance, an obstacle. Pickering claims this resistance is the agency of the material or conceptual system. The dance of agency is then engaged by having the individual accommodate their actions to appropriate the resistance. This dialectical interaction is the unit of analysis. When a student of mathematics is interacting with an object and an attempt is made by the individual to achieve a goal, resistance to that goal is an example of material agency.

Malafouris (2004) describes agency as not being "properties of things or humans but are properties of engagement" (p. 22). Agency is a result of activity; it is an emerging product resulting from an interaction. It is not a typical perspective to think of material having agency but once one views agency as the interaction between players it seems a bit more reasonable to consider material agency as having, in a subtle and implicit way, intention as well. It has input into the situation with its structures and its restrictions on the subject. The subject has to adapt to the form of the object. Emirbayer describes agency as the "…engagement by actors of different structural environments [which] both reproduces and transforms those structures in interactive response to the problems posed by changing historical situations" (in Oliver, 2008, p. 7). Material agency, then, is an expression to indicate or to point out that the tool or object has something to offer. Wertsch (in Daniels, 2008) maintains that a "focus on mediated action and the cultural tools employed in it makes it possible to "live in the middle" and to address the sociocultural situatedness of action, power, and authority" (p. 58). I suggest one way to think of material agency is that it supports or restricts individual agency.

Boaler uses Pickering's framework to argue that disciplinary agency often dominates the practices in a traditional classroom. Pickering describes disciplinary agency as the negotiated rules and algorithms of mathematics. Thus if student are not given the chance to act independently, the math is given the status to direct and determine the practices of math classroom activity. Boaler argues that good classroom teaching would engage a balance between the two agencies for both are important and essential. Both Pickering and Boaler however do not refer to material agency in mathematics. Pickering offers material agency as only being evident in scientific advancements. So while Pickering is focusing on the emergence of new ideas, theories, and practices I hypothesize that material agency does have significance in the practices of mathematics.

Not a lot of attention on materials in mathematics has been analyzed. More attention, however, of materials/objects is being analyzed in math class. A 2002 journal issue of the Learning Sciences has devoted a whole issue to objects in math class but the focus was on the language used when using artifacts (McDonald et al., p. 116). Boaler and Greeno (2000) argue that practices define the knowledge produced. So students brought up in traditional classrooms of mathematics have a perspective of mathematics as being one of following rules. Consequently their versatility is lacking. Students, however, brought up in an environment of negotiations and problems have a more far reaching capability of practices. Interaction of agencies, draws together, the practices that define the mathematics and the object or tool as a way of mediation. I believe that as educators we must allow students agency to engage with artifacts to interact with disciplinary agency.

Wagner also uses Pickering's framework by acknowledging disciplinary agency but appeals to material agency in mathematics and poses the question: "What is the nature of material agency in mathematics?" (Wagner, p. 43). I borrow from Wagner and ask the question: What is the nature and implication of material agency when students of mathematics are engaged in using a tool?

RESEARCH CONTEXT AND PARTICIPANTS

While there are many tools and/or artifacts that have found themselves in different ways into the mathematical community I am choosing what could be termed a technological artifact. Dynamic geometry software (DGS) can be said to have been made to elicit determined geometrical principles. DGS's options and many features such as built in tools offer many choices for students to engage with. It is the choices they have that allows for them to exercise their own agency. This dialectic engagement is what I choose to focus on.

I collected data in a grade 12 class in a Vancouver high school. Students were working within an enrichment unit and had some problems they were working through. The topics of the problems ranged across a variety of topics. The problem I gave in this one context involved two points, A and B, visible and dragable in GSP. The objective was to determine the mathematical relationship between A and B. That is, if they drag one of the points the other moves in a determined way and they were to figure out what that relation was. They had already had practice with a similar problem where the relationship between A and B was simply a reflection about a hidden line, but, in this particular situation, the relationship was circle inversion. After about five minutes of working on the problem I revealed the hidden circle to the class that was a fairly explicit hint. About a quarter of the class was successful in determining the relation between A and B. While the whole class was involved in the activity I had two students in particular use a computer that had Jing running on it. Jing is a screen capturing software that records both screen activity and audio as well. I was able to analyze these two students' work after the class. The following comments rely on their work.

On the screen are the points A and B and a circle. While dragging different components on the screen Alice says, " A and B don't move when I drag this, but they move when I do this". Alice was dragging the circle but the first time A and B were highlighted. She continues to drag different components of the objects on the screen. At another point she notices that B disappears toward the bottom of the screen and reappears at the top of the screen. She exclaims, "Didn't it just go that way?...wait a minute...B went down and then it came from above". Alice seems to be experiencing resistance to how she expected the points to move. Although she is just moving objects on the screen, she is coming to recognize what the software will allow her to do and what it will not. There were many times she said, "I want to see what happens" which indicates that she has intention and thus agency. But her agency only plays out in the engagement with the software and it is at those times where she recognizes that what she might expect may not happen. Her accommodation is to continue to move components around on the screen. At a later point Joanne starts to take more initiative with the situation.

Joanne: how come it drags with the circle?

Alice: when it's inside the circle it drags the circle, when it's outside the circle it doesn't

Joanne: no it doesn't

Alice: wait a minute!

The girls again are questioning the rules associated with the on-screen objects and are struggling to determine what the mysterious relationship is. These girls did not figure out the relationship of A and B. For the most part, they played with the objects and were determining what they could do and what they could not; all the while trying to find a way to see the mathematical relationship. They, unfortunately, did not get past the resistance offered by the GSP.

DISCUSSION

During the following class I asked all the students what they thought of the activity and of GSP. Since they were involved in problem solving with other meditational means they were in a good position to compare and make comments about their engagement with the program. Since there was a general feeling that Geometer's Sketchpad helped understanding and visualization, a follow up questionnaire was given to all the students. It asked for an elaboration of the specific aspects of Geometer's Sketchpad that helped or did not help. Here are some of the responses:

Josie: GSP allows us to visualize graphs or shapes a lot better. It also allows us to move things around…

Billy: The ability to move the sketches around gave us the freedom…

In both Josie and Billy's account I suggest that the program enhanced or allowed a student agency to emerge. The material agency offered a venue for individual choices and intentions.

Harry: …the information on the computer wasn't very easy to remember.

Linda: …many of the program's functions are in the menu bar and are somewhat difficult to find…

In Harry and Linda's case the material agency was more of a restriction. What the software provided to these students was more of a challenge so that, in fact, the materials got in the way of the mathematics. The resistance of the material became their main point in addressing the program. It would be an interesting study to pursue what effect this kind of experience might have upon their intentions.

CONCLUSION

This report describes the beginnings of a study. There is much to be improved as well as reworked but the idea of material agency, I suggest, is an important and substantive topic in mathematics education. If we are to consider the idea of resistance and accommodation as a process in problem solving, the analysis of this process and of the materials seem to be worthy of study.

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DIFFERENT ROLES IN TEACHERS' COLLABORATIVE DESIGN OF MATHEMATICS TEACHING ARTIFACTS

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The collaboration among teachers and educators in the design of mathematical teaching instruments—such as lessons, activities, or assessment instruments—has been widely used as a means for both mathematical learning improvement and teacher professional development. This study focuses on the interaction among participants in this type of collaborative design, in particular the roles they play during the designing of such artifacts. The data were obtained from three different sources: (1) the video recordings of one collaborative design team over eight months, including group and individual interviews; (2) interviews with participants of another three different cases of collaborative design; and (3) related literature that includes other cases of collaborative design. Domains of variability and similarity were identified resulting in a categorization of the different roles that participants hold during collaborative design.

INTRODUCTION

The collective design of teaching artifacts—including mathematical tasks, lessons and assessment instruments—by teachers and educators has been widely applied as both a means for the improvement of mathematical instruction and teacher professional development (Stigler, & Hiebert, 1999; Marton, & Tsui 2004; Jaworski, 2009; Slavit, Nelson, & Kennedy 2009; Lin, 2010). The type of collaborative design of teaching artifacts—or collaborative design for brevity—in this study includes the following features: (a) the selection/negotiation of the goals for the artifact; (b) its implementation in the classroom; and (c) the debriefing of the results after the implementation. Communities of practice (Wenger, 1999), as well as cultural-historical activity theory (Engeström, 2008) have been used by researchers (Jaworski, 2009; Davis, 2008; Minori, 2009) as frameworks for collaborative work. These frameworks, however, do not fully describe the interactions among participants engaged in the designing of mathematics teaching artifacts.

In an attempt to "construct a collaborative, interactionist model of teacher change," Raimo Kaasalia and Anneli Lauriala (2010) describe one case of "student teachers' collaboration and its relevance to the change of their beliefs and practices" (p. 855). In their attempt the authors, in order to include cultural and situational factors, adopt and interactionist view describing participants' *roles* and *statuses*. In other cases of collaborative design, however, other roles are played. Slavit, Nelson, & Kennedy (2009) have stressed the importance of the *facilitator* in "supported teacher collaborative inquiry" by promoting and enhancing teachers inquiry, particularly asking "interesting questions." The facilitator plays a particular role, in this case, as an expert in some areas of mathematics or mathematics education.

The purpose of this paper is to describe the different roles that people play when engaged in collaborative design. The data, in order to include a variety of cases of collaborative design, were obtained from three different sources: (1) a deep analysis of one team in which the author is also a member; (2) interviews with participants of three other different cases of collaborative design; and (3) selected literature as second hand data. Although this study is not intended to be exhaustive, a comparison of several cases allows for the development of domains of variation and similarity which serve as a conceptual framework for the roles of participants of different 'settings' of collaborative design.

THEORETICAL FRAMEWORK

For this study an interactionist approach has been adopted with a particular focus on the *role* and the *status* in a social group. Kaasalia and Laurilia (2010) consider that "in social situations a person must adopt a social role, which refers to a set of expectations of how a member of a special group or community is expected to act in his/her position" p. 855. Blumer (1969), however, argues that "social interaction is obviously an interaction between people and not between roles; the needs of the participants are to interpret and handle what confronts them … and not to give expression to their roles" (p. 75). In this paper the *role* will be considered as a set of perceptions of how a member is expected to act. A person does not necessarily adopt a social role, rather a social group creates expectations of a person—who also has his or her own self-expectations. Kaasalia and Laurilia (2010) understand the status of a person according to a competence level.

> A person's status characteristic is associated with his/her performance expectations, i.e., with a belief about how a member having a given characteristic is expected to perform. ... Status and role are defined on the basis of competence: The higher the status and role a member of a group has, the bigger contribution other members of the group expect he/she to have in solving the task. So the members who have a higher status are expected to be more active than the members having a lower status (p.855).

The status will be considered in this paper as the perspectives that each member of the group has with respect to the competency of another member in a specific area. A collective activity, such as the collaborative design of teaching artifacts, entails mutual engagement in discussions and actions; the role and the status of a person depend on the performance of this person. Competency may have several dimensions and can be perceived differently from different members in a social group. Therefore, the status of a person entails a level of competency in specific domains. For instance, in a collaborative design team one member can have high status as a mathematician who, however, has lower status regarding his or her knowledge about the curriculum.

Social interactions, however, take part in specific contexts which frame the actions and expectations in a group. The notion of *settings* will be used in this paper referring to the factors of the context in which collaborative design is conducted—such as the norms within a group, the economical support, or the physical arrangement of facilities. Thus, in order to describe the members' role in a collaborative design team, the settings must be described as well.

METHODOLOGY

This study consists of two main stages. Firstly, an analysis of a case of collaborative design, the Lougheed team. Three high school teachers and the author met weekly from September 2008 to April 2009. The meetings were video recorded and group interviews were conducted one after the first three mouths and another at the end of the time we worked together. Individual interviews were conducted as well after we finished the collaborative work. The video recordings were split into small segments and coded along with the transcripts of the interviews. Emerging topics were derived and contrasted with the relevant literature.

Secondly, based on a preliminary analysis of the data collected in the first stage, people conducting collaborative design in three other cases were interviewed—three persons for each case. The first case is based on lesson study. The participants, volunteer teachers and educators from different schools and universities, worked independently. The second case corresponds to the activities within mathematics teacher's professional development programs in the form of workshops or courses as part of a masters program. And the third is a district initiative of collaborative work among teachers. The interviews were transcribed and coded. Emerging themes were compared and contrasted with the data from the first stage refining and generating new themes.

Some related literature was a source of second hand data. Particularly, the works of Fernandez and Yoshida (2004), Elliot and Yu (2008), and Slavit, Nelson and Kennedy (2009) were relevant because they include transcriptions of interviews and excerpts of dialogues within in different cases of collaborative design. The roles described in this paper are the result of a comparative analysis of the data from the two stages described above and in the literature.

FINDINGS

In order to describe the different roles identified in this study, descriptions of the different settings will be discussed including a comparison among the analyzed cases.

The Lougheed team. Three secondary mathematics teachers from the same school and the author formed a lesson study-inspired team. The meetings were hold every week at 7:00 AM in one of the participant teacher's classroom as students arrive after 8:00 AM. Two lessons and one assessment rubric were designed by this team. The implementations were observed by the team in some of the teachers' classrooms. Participation on this case was voluntary with out any economical support for the collaborative work.

The independent group. This group, gathers for four general meetings a year in which teams are formed, the designing of mathematics lessons is initiated, and findings are shared. Place and refreshments are provided by a research institution, being this the only support for this group. Teams meet independently to design, implement and debrief mathematical lessons. The size of the teams and the number of sessions held vary; however, usually teams are small—two to four members—and two sessions are designated to the designing of a lesson.

Professional development programs. All these professional development programs are given by the same instructor, who is a mathematics educator. Participant teachers of these programs design mathematical tasks or assessment rubrics together in the sessions, implement them in their classrooms, and report results back to the cohort for its discussion and refinement.

School district initiative of collaborative work. This has two versions. In one version volunteer teachers meet to design mathematical tasks and then they and other teachers pilot the problem until it is ready to its district distribution for the use of interested teachers. The other version consists of learning communities, not all of them conducting collaborative design. Economical support has been provided for this collaborative activity in the form of grants allowing, for instance, teachers to be taken out of their classrooms for the collaborative work.

We can identify some important differences in the settings of these four cases. Economical support or special budgets would allow teachers and educators to have access to resources as well as encouraging teachers and researcher to participate in these projects. The district initiative case is the only one which has benefited from economical incentives. Other forms of support, however, may be present such as the use of the facilities of an educational institution for sessions and the permission for the artifacts' implementation in the classroom, which is the case of the Lougheed team and the independent group. Attachment to a research institution also represents support for all the four cases in terms or resources and contact with educators and researchers.

In the cases of the professional development programs and the district initiative the implementations of the artifacts were conducted in isolation by the teachers, whereas in the cases of the Lougheed team and the independent group the designed lessons were observed. The differences of these settings entail the distinction of the particular roles of *designers*, *implementers* and *observers* which will be described below.

THE ROLES OF PARTICIPANTS IN COLLABORATIVE DESIGN

Teachers have a special role during collaborative design. They are the ones who use the designed artifacts in their classrooms. *Educators* are usually experts in education, or mathematics education; however, *facilitators* may not have a degree in education or in mathematics. The role of the educator is usually as a counselor who provides advice on mathematics or mathematics education to the teachers. On some occasions the educator also asks questions that enhance critical thinking in the team, making her, or him, a *facilitator* of teachers' learning. A facilitator can participate in collaborative design in a variety of ways: some times fully engaged in the designing of the artifacts, and another times providing resources for the team.

In the case of the facilitator as an expert in education, Slavit and Tamara (2009) argue that facilitators "can directly support the inquiry process by providing resources that build teachers' skills in framing an inquiry focus. ... Facilitators can also play a crucial role by asking questions or rising issues that teachers might otherwise avoid, or fail to see, or be afraid to make explicit" p. 4. Expertise in mathematics, however, may not be necessary in order to enhance teachers' inquiry. In the case of the district initiative, the development coordinator [DC] is in charge of organizing and facilitating the collaborative work among the different teams. He explains his role in the teams:

DC So my job is to coordinate the team. So the date, keep the team on track, I do the word processing, I do all of the background piece, to help them get to where they need to go. So I really facilitate the discussion.

I participate to a certain degree because I don't have a classroom to pilot [the artifact] and because I have not necessarily taught the grades that they have came from. I ask the questions that they may not think about. I can be an outside voice to the group because I can ask them if the language that they are using to the prompts of the question is too difficult, to easy … It is really my job to sort of push the group ahead, facilitate that conversation.

Organizing and scheduling the collaborative work are tasks that facilitators may be engaged in as well. One of the functions of the DC of the district initiative case is the writing of the artifacts under design, the *scribe* role. The role of the scribe was taken by the DC in this case; however, in other cases the scribe is one of the participating teachers. The scribe also participates by graph designing the artifacts. In the professional development case of collaborative design, the instructor identifies people playing this role as the "Graphic Artists – these are the ones who are willing to create graphics for the task."

Being a *liaison* with external experts is another role that the facilitator may play and was mentioned by the DC of the district initiative case. This role also appears as an important component a three years large scale project of Learning study conducted in Hong Kong (Elliot & Yu, 2008). In the case of the independent group some experts in education are already participating within the group; however, external experts are also invited to the general meetings on some special occasions.

Other roles that participants of collaborative design play are the *designers*, the *implementers*, and the *observers*. As already mentioned, in some cases of collaborative design the implementation of the artifacts is observed by other people—for example the independent group and the Lougheed team. Additionally, people who implement an artifact are not necessary a part of the designing team—as it is the case of the district initiative case. This is an example of roles that depend on the settings of the collaborative design.

The status of a member of a collaborative design team is not only related to performance within the group: official 'credentials,' such as an academic degree or a job position, influence participants' perceptions. In the case of the Lougeed team different perceptions of the researcher's status can be identified in the following scripts of teachers' dialogue:

T1 But I think though there is a very special place as a researcher and as you [looking at the researcher] become published that always will set [you] outside of this community.

T2 I don't think publishing gives any more respect or any more trust to what you are saying just because you are published. Just because is written doesn't mean is any more true.

In the previous script T1 perceives the researcher/educator as an outsider of the community of teachers. The status of a researcher is perceived different by T2. I the following script T1 goes on explaining that the the researcher/educator has more authority on what he says, a perception shared by T3:

T1	You [researcher] clearly have to have more authority on what you say. ... I would perhaps give more weight to what you say just because in theory you have more background knowledge. ... You are becoming a professional in this area. So, in theory you should know more.
T3	Like you are the supervisor, you have your own supervisor and you are the supervisor of us, kind of.

During a group interview where participants of the Lougheed team shared their perceptions about everybody else's role, the theme of *expertise* emerged. This expertise is a status that depends on the perceptions of each member of the team and represents a role played by some participants. For instance, T2 was considered as an expert in mathematics by T3 as we can see in the following script.

T3	I saw [T2] as, in this context, as the math expert. .... So I was learning new thing from you. And you are always doing the puzzles. I would be sit and watch you actually figuring out the patterns and coming up with the expressions. So, you were taking a much more active role in the sense that you were trying out and I just sit and watch.

Although expertise in some domain may be perceived by the whole group, it can also be restricted to the perspective of one particular member. In the previous script, the status of T2 as high mathematical competent was only indicated by T3. The role of T2 as an expert in mathematics was perceived mainly by T3. In addition to the *mathematics expert*, other domains of expertise found in the data were: *the data base expert*, and the *technology expert*.

CONCLUSIONS

Different roles of participants in collaborative design from the four cases which make the data for this study have been presented. The particular role of the facilitator has several dimensions which can be taken on by other members of the teams. As expected, the perceptions of the roles and statuses may vary from one member to another. The notion of status in this paper extends the one used by Kaasalia and Laurilia (2010) by allowing for a multidimensional nature in which expertise in different domains replaces the linear version of higher and lower status.

The roles described in this paper, although coming from a small number of cases, serve as a base to conceptualize the 'role' of participants in collaborative design. Comparison with the literature gives generality to some extent. People interested in conducting collaborative design, or in the developing of models or frameworks for the interactions among its participants can be informed by this conceptualization.

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LESSON STUDY IN CIRCLE GEOMETRY: THE EFFECTS OF TEACHER'S PEDAGOGICAL CHOICES IN THE DEVELOPMENT OF STUDENTS' GEOMETRIC REASONING

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Geometric reasoning both in its classical axiomatic approach as well as in its empirical-intuitive approach is one of the hallmarks of mathematical reasoning, and a fundamental aspect of mathematics - in the former, through the development of a deductive proof, and in the latter through experimentation and visualization of the dynamically changing objects of analysis (angles, lengths, shapes, as well as their properties and relationships). As such, it is also one of the major goals of mathematics education. Yet, how to assist students in the development of these skills remains elusive. In this paper we examine the interaction between teacher and student actions in the development of a proof scheme for a theorem in circle geometry. Lesson study as an on-site professional development process acts as a window for our exploration.

BACKGROUND

Lesson study is a professional development process in which teachers systematically examine their practice, with the goal of becoming more effective in their practice of teaching. The benefits of lesson study are many, such as the deepening of mathematics teachers' subject matter knowledge (Watanabe, 2002). Another specific attribute of lesson study, which is of primary focus for this paper relates to teachers' increased knowledge and understanding of how students think about mathematical ideas. The centrepiece of lesson study is the *research lesson*, developed collaboratively, taught by one team member in a real classroom while observed by others, and finally discussed and reflected upon by the whole team. Some research lessons are publicly taught in the form of a lesson study open house, where teachers from other schools, and sometimes also university educators and researchers, come to observe the lesson and subsequently engage in a reflective discussion about its effects on students' learning. This often happens if some instructional innovation is being tested in practice. It should be noted that the term "research" in this context means teacher-led, practice-based inquiry into the teaching and learning of mathematics. This paper focuses on observations from an open house lesson study, and provides an analysis of the teaching and learning situation as it occurred in a class of 9th graders in a junior high school in Nara, Japan. The lesson is situated in a sequence of lessons on circle geometry, where students are expected to actively investigate the properties of angles formed by tangent lines and chords. The team of four mathematics teachers at the school designed a lesson on the Alternate Segment Theorem, which is a theorem typically learned in secondary school mathematics as part of the study of Euclidean geometry. This report stems from an ongoing research on the teaching and learning of geometric reasoning in school mathematics, using lesson study as a context for exploring classroom interactions, didactical situations (Brousseau, 1997), pedagogical choices, and teacher discourse (Sfard, Forman, & Kieran, 2001).

THEORETICAL FRAMEWORK

While there is a consensus that teachers' mathematics-for-teaching (Davis & Simmt, 2006), is a complex, dynamic, and tacit body of knowledge, which is very difficult to assess reliably, there seems to be little agreement on what exactly this knowledge is. Interestingly, while *what* should be known to teach well is elusive, *how* such knowledge should be held has been shown quite explicitly on several specific domains of mathematical knowledge for teaching. For example, Liping Ma's research revealed that mathematical knowledge for teaching rests firmly on what she called "profound understanding of fundamental mathematics", and which she explicated quite extensively for several topics of elementary mathematics, such as multi-digit subtraction with regrouping, multi-digit multiplication, and division of fractions (Ma, 1999). With such understanding, teachers are seen to be able to move in their subject easily, naturally, and in a way that allows them to effectively plan for instruction to avoid the typical student misconceptions, and to respond efficiently to a great variety of possible student errors.

It is less clear how mathematics teachers are to acquire this kind of profound and connected knowledge, how such knowledge is to be held and used in the classroom, how it could be recognized, and what exactly constitutes such knowledge. Lesson study seems to hold some promise as a context in which mathematics teaching could be studied and developed systematically, and in which such knowledge could be deepened both at the level of individual teacher as well as within a community of teachers. It can also act as a window for educational research to examine and explicate teachers' mathematics-for-teaching, which is our aim here.

This paper reports on observations and conversations that occurred during a Japanese research lesson concerning the concepts of circle geometry. In our analysis we use the framework suggested by Harel (1994), which builds on the work of Shulman (1986, 1987), and which purports that three interrelated critical components define teachers' *knowledge base.*

First is the *knowledge of mathematics content*. This affects what the teachers teach and how they teach it; furthermore, it refers to the breadth and depth of the mathematics knowledge as well the "ways of thinking" and "ways of understanding". Harel and Sowder (1998) further elaborated and distinguished the two categories of the content knowledge – *ways of understanding* (WoU) and *ways of thinking* (WoT). A way of understanding in this classification is particular to a situation and refers to things such as interpretation of a term, a justification to validate or refute a proposition, or a solution to a problem. On the other hand, a way of thinking is "what governs one's ways of understanding, and thus expresses reasoning that is not specific to one particular situation but to a multitude of situations" (Harel & Sowder, 1998). This domain includes problem-solving approaches, proof schemes, and beliefs about mathematics.

Second is the *knowledge of student epistemology*. This refers to teacher's understanding of how students learn both from the perspective of didactics of mathematics (how new concepts are constructed) as well as the psychological principles of learning. This type of knowledge may also include teacher's knowledge of how mathematical ideas evolved historically, as this can often relate to epistemological obstacles encountered by the individual learner (for example, irrational number and incommensurability of magnitudes).

The third component of teachers' knowledge base is the *knowledge of pedagogy*. This refers to the fundamental principles of teaching such as motivating students to learn, assessing their current understanding and shaping instruction accordingly, promoting desirable ways of understanding and ways of thinking, providing guidance in learning activity, and helping students solidify the newly learned material.

SUBJECTS AND CONTEXT

A team of four mathematics teachers from Lower Secondary School Attached to Nara University for Women prepared the research lesson on Alternate Segment Theorem. The lesson was implemented in a class of 40 Grade 9 students, and it was observed and subsequently discussed by the team members, a university professor of mathematics education from Nara University, 4 graduate students in mathematics education from the same university, 10 mathematics teachers from various other local schools. There was also a group of 20 teachers and teacher educators from North America who came to Japan to observe and learn from the Japanese practice of lesson study, in a program led by two mathematics educators from United States of America with Japanese educational background who also acted as translators. Lesson plan was distributed to all observers in advance, and both the lesson and post lesson discussion were videotaped. Field notes documenting classroom interactions and the unfolding of the lesson in action (lesson protocol) were taken by the researcher, as well as a recording of the post lesson discussion documenting teachers' and teacher educators' discourse.

The lesson was taught by Ms. Sunomi (pseudonym) who has been teaching for four years. The goal of the lesson was for students to actively investigate the Alternate Segment Theorem, and to try to prove it. As part of the study of circle geometry, students have already been exposed to the Inscribed Quadrilateral Theorem and to the Inscribed Angle Theorem. Both these theorems were proved in prior lessons, and were taken as the basis upon which the new theorem, the Alternate Segment Theorem, was to be built. The opening of the lesson involved a demonstration using the Cabri dynamic software. The team of teachers decided to introduce the new theorem as a "limit case" of the two other theorems which students had already learned before, and thereby connect the three major theorems of circle geometry. The idea was to motivate student learning by connecting the intended new learning (alternate segment theorem) to the two previously learned theorems (inscribed quadrilateral theorem, and the inscribed angle theorem).

RESULTS AND DISCUSSION

First, we present segments of the classroom interaction during the lesson progression, and we offer our discussion and analysis of the situation based on the framework of Harel and Sowder, as described earlier. Second, we present segments of the discourse from the post lesson discussion pertaining to the situations observed in the classroom. These shed light on teacher's intentions, pedagogical choices, and ways of knowing.

Classroom interaction and analysis of the situation

The three theorems were presented in a demonstration by Ms. Sunomi, who used the dynamic geometry software, projecting a single figure on a screen while dynamically moving point X. In the figure, when X moves along the short arc BA, inscribed quadrilateral theorem can be used to see that the external angle at X is equal to the angle at P. On the other hand, when X moves along the long arc AB, angle at X is again equal to the angle at vertex P, this time according to the inscribed angle theorem. But when point X coincides with A, the new theorem, the alternate segment theorem, that students were supposed to learn during this lesson, presents itself. This relationship is what the teacher intended her students to see, so that it could be assimilated and accommodated in the students' minds (Piaget, 1983) as a working knowledge which could be later on used and applied.

The teacher starts with a 10-minute Cabri demonstration, where she makes explicit and repeated requests to students "to pay attention to the exterior angle measure at vertex X, as X moves towards point A".

T: According to the Inscribed Quadrilateral Theorem, which angle measure is equal to the exterior angle measure at vertex X?

None of the students seem to be recalling the previously learned theorem, or they are too unsure to speak. The teacher then produces the following

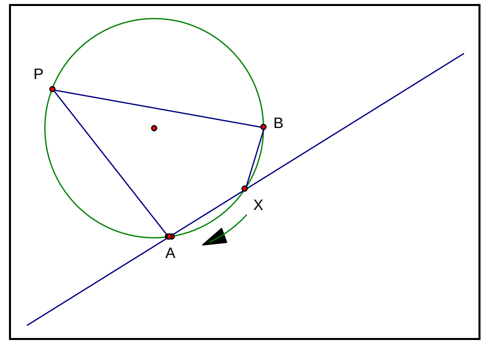

Figure 1: Inscribed quadrilateral theorem

angle measures on the screen: <BXA=150 $^\circ$; <BPA=30 $^\circ$; and points out to students that angle at P has the same measure as the exterior angle at X.

T: When points A and X coincide, we have a point of tangency. This is the theorem we are about to learn today, the Alternate Segment Theorem. It says that the angle between the chord and the tangent at the point of tangency is equal to the angle in the alternate segment.

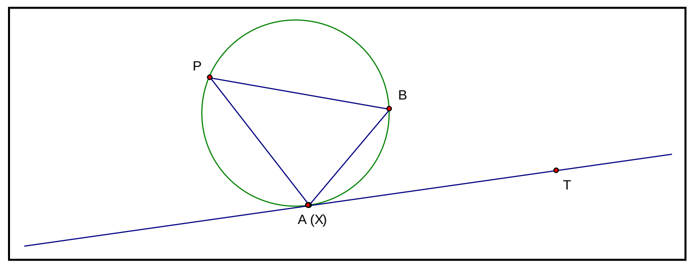

Figure 2: Alternate segment theorem (new theorem)

(pause)

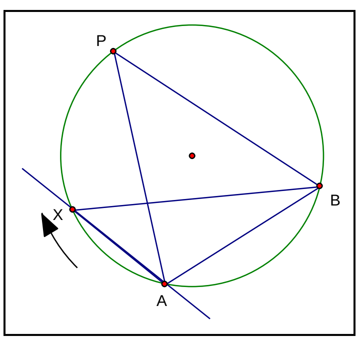

Figure 3: Inscribed angle theorem

T:What happens as point X passes beyond point A? Which theorem appears then?"

Once again the teacher must answer her own question and inform students it is the Inscribed Angle Theorem, according to which <BXA = <BPA. The tangent line in this case transforms into a secant line, but the two angle measures remain equal - in the case of demonstrated example they are 30 ^{o} throughout the process of moving point X. Switching off the computer screen, the teacher then proceeds to the blackboard, and instructs students to write down the theorem statement as she produces it on the board. On the board it now says, "The alternate segment theorem: the angle between a tangent and the chord at the point of tangency is equal to the angle in the alternate segment." She draws a circle using a board compass, with triangle APB inscribed in it, and a tangent line passing through A. She leads students to focus on a chord, either BA or PA, and the angle it forms with the tangent, and asks individual students to name the equal angle pairs. One student, S_1, succeeds in naming <BAT and <BPA equal, and marks the two angles on the board as equal. This happens after some uncertainty about which

chord to use, as there are three (AB, AP, and also BP). The student used the chord AB, probably because it was in the same position as in the Cabri demonstration. He probably realizes that BP does not apply as there is no angle being formed between that chord and the tangent line (at least not at the point of tangency, A), but mentions that there is another chord, AP which he is unsure about.

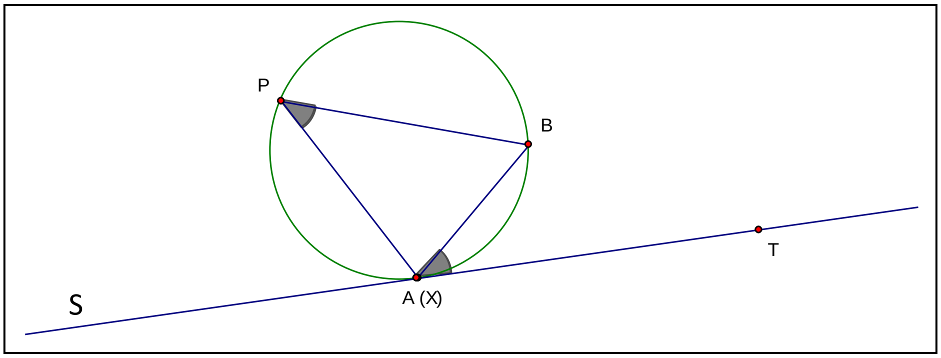

Figure 4: Students making sense of the statement of the new theorem.

T: As S_1 says, PA is also a chord, so where else is a pair of equal angles?

S_2 : Angle PAS is equal to angle PBA.

This phase, which consisted solely of helping students understand what the theorem states took about 20 minutes. Only two students offered responses to the teacher, and came to the board to complete the sketch. It is difficult to say how many of the other 38 students came to grasp the meaning of the statement of the theorem. Even as the teacher tried to invoke the two theorems that students have already studied before, there seemed to be little in the knowledge considered as shared or that has been institutionalised (Brousseau, 2005).

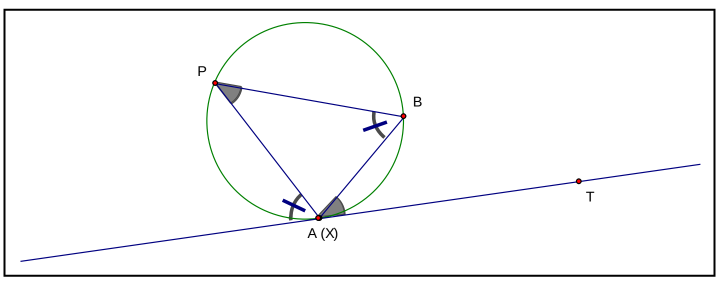

Figure 5: Grappling to grasp the statement of the theorem.

The next phase of the lesson involved motivating students to see the need for a proof.

T: We know now what the statement says – but we need to know if it is true all the time. Think about the proof.

(pause)

T: Let's write the hypothesis and the statement much clearer. What is the hypothesis here? The picture we had on the screen before is really just a specific case. Remember, when I was moving the point in Cabri?

(pause)

The teacher repeatedly refers to something that is not there anymore, something that related to her actions during the demonstration, but which seemed to have escaped the students' minds entirely. Not a single student offered a suggestion what could possibly be meant by “the hypothesis” and by “the proposition”, even after repeated requests.

T: OK, Ms. Moto, any idea? … It’s not easy… OK, keep thinking.

The teacher then wrote the following on the board:

HYPOTHESIS: Line AT is tangent to the circle.

PROPOSITION: < BAT = < APB

T: We have to show that if line AT is tangent to the circle, then the two angles are equal.

In the next segment of the lesson the teacher proceeded with a proof, which consisted of the treatment of three cases: a) when < BAT is a right angle; b) when < BAT is acute; and c) when < BAT is obtuse. The time ran out midway of case two. There was no real discussion or reason provided by the teacher as to why there is a need for the treatment of three cases.

Throughout the episode, students were mainly copying passively from the board, while the teacher developed the proof (actually, part of it). In terms of the teacher’s knowledge of the mathematical content, it was obvious that the teacher was very well prepared, had clear instructional goal, and was proficient in the content there was to teach. However, the teacher’s way of understanding was at great odds from that of the students’. Where she saw the three theorems connected, the students could not even grasp what the new theorem was supposed to be stating. The term “angle in the alternate segment” was utterly unfamiliar to students. The teachers’ appeal for the need of a proof did not seem to convince the students, and certainly not there should be a consideration of three cases. In this sense, her way of thinking, as it concerns proof schemes and the need to distinguish between the hypothesis and the proposition, and then deduce the theorem using a valid reasoning process was also at odds with students’ way of thinking.

Concerning the knowledge of student epistemology, the team of teachers were hoping to gain an insight into the benefits of preceding the formal proving of the theorem with the dynamic geometry demonstration to provide an intuitive background for the development of theorem. However, this did not seem to aid students at all. They could not even formulate what they saw. Mathematically, proving the theorem in the sense of Euclidean approach is indeed much simpler than proving it using a limit approach (as the distance AX approaches 0). Of course there was no intention of proving the theorem using the limits, it was only intended as an empirical learning process to give a sense of what the theorem actually states – but even as such it did not achieve this goal. This could be explained as an instructional approach referred to as *perception-based perspective* (Simon, 1995) on how students learn mathematics. According to this perspective students develop mathematical understandings through their engagement with representations that make the concept under study clearly perceivable. Mathematical relationships exist as an external reality. It could be thought of as the "show and tell" approach to teaching mathematical concepts. Students are supposed to observe how varying the location of a point, relative to another point, affects the relationship between certain angles. What students are learning from such approach is *that* moving a certain point does not alter certain angle measures. They are not in fact learning the logical necessity of the relationship between the two angles (a concept). It is contended that empirical learning process does not result in conceptual learning, because mathematical concepts are the result of *reflective abstraction* and not of empirical learning (Simon & Tzur, 1999).

In terms of the teacher's knowledge of pedagogy, there were some definite shortcomings. Most notably, where the teacher assumed the students to have the background knowledge of the inscribed angle theorem and the inscribed quadrilateral theorem, this was not apparent. In such state they could not make the connections that the teacher intended for them to make. When asked during the post-lesson discussion, what was the reason that the proof was not completed as intended for this lesson, the teacher gave two reasons. One was that there had been a student teacher teaching the class for the past two weeks, and the second was that the Cabri demonstration took away more time than she planned for. She had hoped that the students would discover the relationship through the use of the demonstration, or at least that they would get an access into what the theorem actually states.

In the post lesson discussion the teacher was questioned about why she spent so much time helping students understand what the theorem was about, such as what parts are equal – this turned the lesson into a reading comprehension exercise rather than trying to find out why the theorem is true. There was a general consensus that the demonstration and the proof were not connected too well. One commentator suggested that it would have been necessary to establish the "direction of the proof", that is, for students to be able to say what is hypothesis and what is the conclusion. This would need to start with much simpler theorems, and should have been in place by this time. Other observers asked questions such as, "Do students have any idea of why your were looking at cases?", "Why do you need proof?", "Why we need the right angle case first?", "How do you help students appreciate the merits of cases?". There was a mutual agreement that students need to be more involved in deriving the theorems, that there needs to be equitable access to mathematics and that understanding needs to be fully shared among all students. The different ways in which the lesson can be organized, as well as the form of the lesson, will dictate student activity; for example, was the computer demo and technology here used as a tool to explain or a tool to motivate inquiry.

CONCLUSION

In conclusion, we shared results from a classroom based lessons study of teaching and learning proof in geometry. Teachers' mathematics-for-teaching including the knowledge of student epistemology can be refined over time through the practice of lesson study and teacher collaboration. Pedagogical decisions that teachers make, as manifested in the teacher's actions in the classroom, are a response to both the design of the lesson and to student activity. These choices play a critical role in the type of classroom environment that is established as well is in the opportunity for students to link the new mathematical content to the existing concepts, build chains of reasoning, make conjectures, provide justifications, and engage in proof development activities. Teachers' knowledge of mathematical content and ways in which it can be presented for learning is developed, shared, refined, and transformed in a culture of collaboration, whereby teachers become scholars of the interaction between teaching and learning, and of the subject matter they are to teach.

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