FACULTY OF EDUCATION

SIMON FRASER UNIVERSITY

MEDS-C 2009
PROCEEDINGS

PROCEEDINGS OF THE 4TH ANNUAL MEDS-C:

MATHEMATICS EDUCATION DOCTORAL STUDENTS' CONFERENCE

NOVEMBER 14, 2009

MATHEMATICS EDUCATION DOCTORAL STUDENT CONFERENCE 2009 – PROGRAM

8:45 – 9:15	Welcome and coffee	
9:20 – 9:40	The psychology and physiology of structure in geometry: a study in educational neuroscience	Kerry Handscomb
9:45 – 10:05	Anatomy of an "AHA" moment	Olga Shipulina
10:10 – 10:30	Systemic outreach activities: providing tools of empowerment	Melania Alvarez
10:35 – 10:55	The designing braid: teachers' interactions while designing learning artefacts	Armando Paulino Preciado Babb
11:00 – 11:20	A critique of Ethnomathematics	O. Arda Cimen
11:25 – 11:45	Students reducing abstraction: the case of logarithms	Krishna Subedi
11:50 – 12:10	Sanding the lens: the narrative of a task from the initial planning to the undergrad students' conceptions of inequalities	Elena Halmaghi
12:15 – 12:35	Tensions related to course content in teaching Math for Teachers: the case of Alice	Susan Oesterle
1:00 – 2:00	Lunch: Himalayan Peak Restaurant	
2:15 – 3:00	Plenary: The Accidental Professor	Tom O'Shea
3:00 – 3:30	Plenary Q&A	
3:30 – 3:45	Some useful elements from a survey	Christian J. Bernèche
3:50 – 4:05	Identification of habits of mind inherent in mathematical exploration	Sean Chorney
4:10 – 4:30	Use of DGS to promote kinaesthetic thinking: a case of linear transformation	Shiva Gol Tabaghi
4:35 – 4:55	Technology and modeling as agents of inquiry	George Ekol
5:00 – 5:20	Teachers connecting mathematics through a lesson study on similarity	Natasa Sirotic
5:25 – 5:55	Reflection and vision for the future of MEDS	All

PLENARY SESSION: THE ACCIDENTAL PROFESSOR

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I assume that PhD programs in mathematics education are designed primarily to prepare individuals to take over when the professors in those programs retire, die, or otherwise cease to function. I have observed that doctoral programs, to this end, tend to focus on courses, theses, disputations, and managing political crises, with the guidance of their professors. Students may come to believe that such professors, from their early years, single-mindedly mapped out their future and followed a grand plan to achieving their current academic status. In this session, I will present a case study (myself) to show how fallacious this assumption may be. In fact, I will argue that Brownian motion[1] provides a model that may best explain one's progress (?) through the academic world.

THE PSYCHOLOGY AND PHYSIOLOGY OF STRUCTURE IN GEOMETRY: A STUDY IN EDUCATIONAL NEUROSCIENCE

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[1] *Various definitions are available on the web. Some examples follow. Choose one.*

- *"A zero-mean continuous-time stochastic process with independent increments (also known as a Wiener process)"*

  www.mathworks.com/access/helpdesk/help/toolbox/econ/f6-1001427.html

- *"The random movement of tiny particles, for example of dust or pollen, that results from collisions with the molecules of the gas or liquid in which they are suspended."*

  bscw.cs.ncl.ac.uk/pub/bscw.cgi/S4926fa70/d56519/book%20glossary-final.doc

- *"The movement of microscopic particles caused by Brownies."*

  www.besse.at/sms/glossary.html

This paper is a synopsis of a doctoral dissertation in mathematics educational neuroscience. It presents a psychological model for geometrical thinking and learning and its correlative physiological model. With respect to the latter, geometrical concept formation belongs to the parietal lobe of the cerebral cortex. The cerebellum has a functional role in directing attention to those aspects of a geometrical percept that are essential to the concept under consideration. The theoretical framework is embodied cognition, as informed by Spinoza, which allows coherent integration of psychological and physiological aspects of geometrical reasoning. A conclusion of the research is that decontextualization of geometrical concepts may facilitate student learning of these concepts.

ANATOMY OF AN "AHA" MOMENT

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Specific details regarding students' understanding and learning, and how to identify and observe these details is a most challenging empirical aspect in support of any viable cognitive theory in mathematics education research. This especially pertains to research in mathematical problem solving and, in particular, to capturing and exploring the nature of "AHA" moments. In this paper, ways in which such studies can provide better empirical ground for developing more accurate theories of mental processes during mathematical thinking and learning are introduced and demonstrated using "state-of-the-art" methodologies that go well beyond the traditional dependencies on video-tape recordings – specifically, computer screen capture, eye tracking, and electroencephalography (EEG) for analysis of an "AHA" moment.

SYSTEMIC OUTREACH ACTIVITIES: PROVIDING TOOLS OF EMPOWERMENT

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How we act and relate to the world depends on the knowledge and skills that our parents and community share with us, how effective they are at teaching those skills and knowledge in order to understand and deal with

the world around us, and how we personally interpret and act on that knowledge. This relationship between interpretation and action is what Bourdieu tried to encapsulate in the idea of habitus, which he describes as a system of interchangeable dispositions that bridges the gap between structure and agency. The main goal of the outreach programs we are developing is to empower aboriginal students to be able to overcome the obstacles they encountered as they try to get a high school education. We are working under the hypothesis that when students feel that their mathematical knowledge is adequate they can use this as a tool of empowerment in their school work.

THE DESIGNING BRAID: TEACHERS' INTERACTIONS WHILE DESIGNING LEARNING ARTEFACTS

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In this report I propose a theoretical framework that serves to understand conversations and interactions that teachers and educators undertake when engaged in the collaborative design of mathematics learning artefacts—such as a lesson, a class projects, or an assessment instrument. A constructivist grounded theory approach was used in order to develop such a framework. Three theoretical concepts describe the participants' conversations when designing a lesson in this context: (1) anticipating possible students' approaches and struggles; (2) pursuing coherence within the context of the classroom where the artefact will be implemented; and (3) approaching previously selected goals for the artefact. Comparison with other theories of mathematics teachers' development is made in the concluding section, stressing the focus on teachers and educators' interactions of the proposed theoretical concepts in this paper.

SOME USEFUL ELEMENTS FROM A SURVEY

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This paper presents pre- and post-surveys on attitudes held by Mexican high school teachers, involved in a one-year lesson study project, as a means to understand the *limitations and possible usefulness of this specific survey. Items that are brought to the fore during the analysis will be discussed and may inform research neophytes as they consider various instruments for their own imminent data collection.*

STUDENTS REDUCING ABSTRACTION: THE CASE OF LOGARITHMS

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Reducing abstraction is one of the theoretical frameworks that examine the learners' behaviour while coping with abstraction level. It refers to the tendency of the learners to unconsciously reduce the level of abstraction while learning new concepts to make it mentally accessible for them. Analysing the work of three students through the lens of reducing abstraction, the aim of this paper is to investigate and exemplify some misconceptions and instances of error in students' understanding of logarithms.

SANDING THE LENS: THE NARRATIVE OF A TASK FROM THE INITIAL PLANNING TO THE UNDERGRAD STUDENTS' CONCEPTIONS OF INEQUALITIES

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This report comes from a broader study that investigates undergraduate students' conceptions of inequalities. It comprises the design and refinement of a task with the purpose of making it more engaging for students and of getting results that are more transparent for author's interpretation. The narrative follows one task that has been developed, implemented, interpreted, refined, implemented again and used in the process of deriving university students' conceptions of inequalities. The interpretation of student's work is framed as an emergence of the lens CONCEPTIONS OF INEQUALITIES. The lens is intended to magnify students' work on inequalities for the researcher to better spot the various conceptions and interpret understanding of inequalities.

TENSIONS RELATED TO COURSE CONTENT IN TEACHING MATH FOR TEACHERS: THE CASE OF ALICE

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Instructors of mathematics content courses for prospective elementary teachers are influenced by many (sometimes) competing factors as they strive to meet their goals for their students. Via an analysis of three episodes that occurred during an interview with one such instructor, this report seeks to illustrate some of the tensions that these instructors operate under as they make decisions related to course content.

TECHNOLOGY AND MODELING AS AGENTS OF INQUIRY

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I first state Confrey and Maloney's (2007) four distinct but related approaches to technology use in mathematics instruction, one of which centrally involves modeling. My focus in this paper is on how students use technology in modeling, in light of the Deweyian definition of inquiry. Data from undergraduate students engaged in a modeling task are analyzed and discussed. Based on the results, I concur with Confrey & Maloney that modelling, through the process of inquiry, provides opportunity for the inquirer to progress from an indeterminate to a more determinate situation. However, I refine the definition of mathematical modelling proposed by Confrey & Maloney(2007) and submit that the end product of a modeling process is a description of the determinate situation with respect to the original task.

USE OF DGS TO PROMOTE KINESTHETIC THINKING: A CASE OF LINEAR TRANSFORMATION

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A neuroscience theory suggests that kinesthetic thinking is the very basis of thought. Similarly, Lakoff and Núñez (2000) suggest that dynamism plays an important role in conceptual development. Furthermore, Núñez (2006) argues that mathematical ideas *and concepts are ultimately embodied in the nature of human bodies, language and cognition. In this paper, we examine the role that dynamic interactive representations of mathematical concepts plays in promoting kinesthetic thinking. In particular, we report evidence of students' kinesthetic thinking while they interact with a dynamic interactive sketch of the concept of linear transformation.*

A CRITIQUE OF ETHNOMATHEMATICS

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Starting with its presentation by Ubiritan D'Ambrosio at the International Congress in Mathematics in 1984 and his following paper in the next year entitled "Ethnomathematics and Its Place in the History and Pedagogy of Mathematics", the concept of ethnomathematics, as a new field of study, has been taking a wide place in mathematics educational research. In this paper, I will first focus on how different researchers, who study ethnomathematics, define it. Based on their definitions, firstly I will start with the points in those their positions and descriptions of ethnomathematics differ. Then I will continue with a concluding part to summarize the ideas, philosophies and stances ethnomathematicians share. In addition, I will critique the thesis that ethnomathematicians share, which is taking a position that mathematics is culturally dependent. I will call this thesis as Culturally Relativity Thesis (CRT). And for the following part of the article, I will manifest my own counter epistemological view of mathematics which I will call as Culturally Independence Thesis (CIT) and I will support my thesis in four aspects: etymological, socio-pedagogical, historical-anthropological and with regarding applied mathematics.

IDENTIFICATION OF HABITS OF MIND INHERENT IN MATHEMATICAL EXPLORATION

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In this study students were given an opportunity of posing problems after being presented with a "rich" mathematical object. Students' problems were commented on in the hope of revealing to the researcher a possible framework.

TEACHERS CONNECTING MATHEMATICS THROUGH A LESSON STUDY ON SIMILARITY

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Scaling and similarity, a topic from elementary mathematics, is one where the concepts of number and shape interplay. The concept of similarity was already known the ancient Greeks, and it remains an important topic of study to this day. More importantly, it could be fruitfully employed to the development of the concepts of number and number operations, and for the learning of proportional reasoning. The ideas presented here stem from a lesson study on similarity, but have a wide range of applicability for school mathematics, as they address the connections between magnitudes, quantities, and numbers. In particular, we present the use of geometric representations as a way to uncover the multiplicative relations between quantities and their relative sizes. The report presented here is taken from an ongoing study situated in a school-based community of practicing teachers, who harness the potential of community and workplace to develop their practice of teaching mathematics.

SYSTEMIC OUTREACH ACTIVITIES: PROVIDING TOOLS OF EMPOWERMENT

Melania Alvarez

Simon Fraser University

How we act and relate to the world depends on the knowledge and skills that our parents and community share with us, how effective they are at teaching those skills and knowledge in order to understand and deal with the world around us, and how we personally interpret and act on that knowledge. This relationship between interpretation and action is what Bourdieu tried to encapsulate in the idea of habitus, which he describes as a system of interchangeable dispositions that bridges the gap between structure and agency. The main goal of the outreach programs we are developing is to empower aboriginal students to be able to overcome the obstacles they encountered as they try to get a high school education. We are working under the hypothesis that when students feel that their mathematical knowledge is adequate they can use this as a tool of empowerment in their school work.

"The future isn't something hidden in a corner. The future is something we build in the present"—Paulo Freire.

INTRODUCTION:

My research problem consists in developing, implementing and assessing outreach activities that could help aboriginal students to improve their mathematical knowledge and access to a higher education and a better job. The main goal in the development of these academic outreach activities has been to find mechanisms of empowerment that will allow students to overcome the obstacles they encounter in school in order to continue and graduate from high school. However, as Freire (1998) warns us, real empowerment does not come *from* the educator *to* the educand, this is a paternalistic view which only provides a *benevolent form of oppression,* where the educator assumes to know what the educand needs in order to succeed and sees himself/herself as the one who is able to provide this change for the educand (Freire 1998:6). Real empowerment comes from within, and this change from within does not happen in one day, it is a long process where the educator can only provide the educand with opportunities and tools to deal with those opportunities, but only until the students take them as their own can change happen.

The hypothesis that guides these outreach programs is that if we are able to teach students and provide them with a stronger academic background they will feel more confident in school, and this confidence will empower them to feel better about themselves. They will have more and better choices to access, and they will act upon those choices, given that they will have better academic tools to face the school system and the possibilities that this system provides.

However in order to be able to present opportunities that could be of value we need to understand the habitus of the students. How do they perceive their world? How do they act according to their perceptions? How are they able to overcome or not the obstacles presented to them given the knowledge and resources they have?

The dispositions or tendencies to act in a particular way under certain circumstances can be classified as categories of perception and assessment that subsequently will inform principles of action, what Bourdieu called habitus. The habitus gives the individual a "feel for the game" (Bourdieu 1990: 9), and provides the individual with potentials and possibilities of actions according to his/her knowledge of the world (1990: 9). The habitus very much depends on the individual's personal history, but also the collective history of the community with which the individual interacts throughout his/her life.

THE IMPACT OF STRUCTURE AND AGENCY AND HOW THEY RELATE:

The obstacles Aboriginal children face have a deep impact in their psyche and reality, and in order for change to happen in their lives there needs to be a transformation at both levels which in most cases cannot happen overnight. In *Learning to Labour,* Willis' (1977) claims that a way one can bring change into the school realities and choices of marginalized students is by moving away from just looking at students' discontent and resentfulness especially against authority, and at deterministic views of cultural and social reproduction. We should instead pay more attention to the mutual relations and interactions between structure and agency 'habitus'.

Willis (1977, 1983) states that there is a 'moment' at the end of the line of many moments of reflection and action when individuals make sense of their conditions of existence, and it is at this moment than a pivotal choice and a transition is made.

In *Learning to Labour*, Willis (1977) describes the lives of working class boys living in Birmingham England (they self identify as "the Lads"), and how when the moment comes when they are finally able to enter the world of manual labour, they reject what any further schooling has to offer, and they make the choice of allowing themselves to get working-class jobs. These choices are what make cultural production and in most cases reproduction to occur. Willis defines cultural production as "the process of the collective creative use of discourses, meanings, materials, practices and group processes to explore, understand and creatively occupy particular positions, relations and sets of material possibilities" (Willis 1983:114).

As one can see there are two main elements to this definition: the social structure that the individual has to face, and the understanding and response of individuals to this structure. Willis proposes that in between structure and anticipated/predictable actions made as life choices, there is a crucial moment where creativity is a possibility and change is possible. In the case of Willis' 'Lads', they made a free choice of becoming un-free by electing to be part of the system of working-class exploitation and oppression ‘it is the future in the present which hammers freedom to inequality’ (Willis 1977: 120). Willis sees this moment of choice as the end of the line position for students, a last moment of many moments of a continuum of resistance gathered around a final stance when the student finally makes the choice to drop out of school.

An example were one can see the same point that Willis makes about his ‘Lads’ but applied to an Aboriginal population is the case of the Koori in Greytown Australia. The study by Munns & McFadden (2000) shows that these conditions of rejection in the Koori Population start as early as primary education, and that later on these aboriginal students drop out of high school once they are legally able to.

Munns & McFadden (2000: 62) identified the following conditions as the lead causes for students to take up resistant positions about school: Powerlessness (Much of the suffering is directly caused by endemic institutional and personal racism directed against them); Feeling powerless (They feel that their lives are a constant battle against a system that does not work for them); A sense that school was not working them (As students get older they become aware that most of them spend many years at school for few academic rewards); They reject what feels like an unequal educational experience since teachers usually have low expectations about them and this is translated in their practice; Lack of cultural support (students become aware that the community expects that they will fail and quit school. They sense from the community that as a group they are not able to succeed academically)

Munns & McFadden (2000) point out that if these conditions are not appropriately dealt with in the earlier years, we will see their effect along the continuum of schooling years and beyond.

These conditions are very similar to the ones that many aboriginal students experience in the schooling system in Canada. The aboriginal population has been subjected to this systemic racism for a long time. Many researchers show that being subjected to racism brings feelings of low self-esteem, and it has a deep psychological and emotional effect on individuals (Clarke, 1994). Sellars (1992:85) points out that “When you’ve been programmed to believe you are worth nothing, you unconsciously act out the role and it’s difficult to change that view of yourself”. The First Nations Education Steering Committee Society Report (1997:16) stated, “when a person is told over and over again that they are a ‘lazy Indian’ or a ‘stupid Indian’, eventually they believe it”. In order to change these conditions, given the obstacles presented by the system there needs to be a continuous intervention for change on behalf of the educator. We need to provide the students with the tools that will empower them to make positive choices for their future.

OBTACLES TO EDUCATION FOR ABORIGINAL STUDENTS IN CANADA

Historically Aboriginal peoples have long endured a host of unfair social, economic, and geographical barriers. Like the Kooris in Australia, aboriginal people in Canada usually have to overcome multiple obstacles in order to acquire an education and/or a job. A substantial percentage of them face poverty, unemployment, and poor health (Friesen & Friesen, 2005). Within the reserve and aboriginal communities we find that housing conditions are usually sub-standard (Preston 2008:59), and when they attempt to come to cities looking for better jobs or an education they still find that it is difficult for many of them to find a reasonable place to live (Friesen and Friesen, 2005).

Evidence shows that young aboriginal people are most likely to withdraw from high school between grades 9 and 10 (Government of Canada Background paper, 2004). We find that by this time many at risk aboriginal students have collectively experienced various problems: academic failure, truancy, difficult school/home environment (family breakdown, domestic violence), frequent relocation from one school district to another, problems with police and possible criminal involvement, teen pregnancies, and alcohol and drug abuse. However many aboriginal students who drop out, have not faced these problems but still feel that school is not the place for them.

The following are in my experience and according to research some of the main obstacles that aboriginal students face at school and lead them to abandon it.

- Lack of study skills, course requirements, and academic knowledge, especially in the areas of mathematics and science (Hardes 2006).
- Lack of counselling, lack of mentors and educational role models.
- Low expectations and racism (Richardson and Blanchet-Cohen 2000).
- Schools consistently ignore the Aboriginal perspective, and thus do not prepare students for the world they intend to function in.
- Poverty: there is nothing teachers and staff can do to change the immediate realities of poverty that some children face, however school can help by giving good schooling to children and building connections between parents and community. Neufeld (1990) notes that by seeing school processes as holistic where affective links between school's staff and students are developed, poor children can bring and find some solace from their burdens, which can make a big difference in their overall outlook. Any community programs that the school might develop can be a useful intervention (Nettles 1991).
- Student's mobility and absenteeism. Aboriginal students are among those who are more often absent and more likely to change schools.
- Teenage pregnancy/taking care of siblings or the children of siblings/daycare. I was surprised to learn that many aboriginal students were used as babysitters by their older siblings, which resulted in going late to sleep and many times arriving late and tired to school.
- Low self-esteem
- Alcoholism/Drug Abuse/Sexual Abuse

Of the points above, the third one is the one that according to Richardson, and Blanchet-Cohen (2000), and I concurred, is the greatest obstacle of all. The study by Schissel & Witherspoon (2003), *The Legacy of School for Aboriginal People*, reveals that more than 35% of high school students rated racism as the main barrier to learning. According to Henry, Tator, Mattis, and Rees (2000) racism exists at many levels, from individual to cultural, and the effects are equally harmful regardless of the level.

Systemic racism is usually more difficult to point out given that it includes laws and norms rooted in a social system with unequal distribution of social, political and economic resources among various racial groups (Henry et al 2000:6). When one confronts systemic racism the "inequality is built into institutions in a way which is often invisible both to those who dominate and those who are dominated" is an action "which has the result rather than the intent of disadvantaging persons or privileging them…" (Vickers 2002:197).

Since the teacher is the representative from the school that most directly deals with students, their perception and interaction with students has the greatest impact in them. A study into dropout prevention conducted across four countries --India, Nigeria, United Kingdom and the United States – discovered that reversing the teachers' negative perception about minority children was one of the main factors for maintaining at risk students in the education system (Woolman 2002).

Wotherspoon's (2006) study show that Aboriginal students are highly sensitive to the fact that a teacher's actions and orientation can sometimes make a substantial difference to the specific education and life pathways they follow. In the literature it is widely acknowledged that teachers' perception of students will have an impact on the teaching, learning and assessment outcomes that students receive. Wilson (1991) identified several examples of unfair treatment of Aboriginal students by teachers like rigid and unfair implementation of attendance policies; behaviours that show disrespect towards the students; and inadequate teacher assistance. Other studies show that in general aboriginal students cannot rely on the teachers' support and they feel unwanted at school (Laroque 1991).

There is extensive literature which point out the importance of the need for teachers to acknowledge racism at school and that they should refrain from judgments about their students based on the ethnic background (Lund 2006).The reality is that in schools the existence of racism is usually ignored or denied and the repercussions of low expectation and racism spill on and feed the other obstacles.

OUTREACH ACTIVITIES: SOME PRELIMINARY RESULTS

The Pacific Institute for the Mathematical Sciences (PIMS) has been implementing various outreach activities in several First Nation schools and public schools with significant aboriginal population in British Columbia. Two years ago it started working with Britannia Secondary in Vancouver, looking for ways to improve the high school graduation rate of aboriginal students, as well as to increase the level of math preparation among these students. More than 30% of the students attending Britannia Secondary are aboriginal.

I have been working with PIMS in the development and implementation of various programs to improve academic achievement at Britannia Secondary. These outreach programs focus mainly on acquisition of mathematical knowledge and understanding. It is especially important that students take rigorous math courses in high school, given that this is one of the greatest predictors of successful college completion (Adelman 1999). By leaving behind the philosophy of reduced expectations, introducing new interesting and challenging programs and exciting ways to learn mathematics, PIMS hopes to be able to provide aboriginal students with the tools they need to be able to make a career decision of their choice, including a career in science.

PIMS started offering a Math camp in the summer of 2007 for students attending Britannia Secondary who had failed their grade 10 Math. Five aboriginal students attended this camp and at the end of the summer four of these students were tested by the school and were placed in principles of math 11.

In general researchers recommend that the type of outreach programs, which PIMS is implementing, should begin by eighth grade or earlier and not later than ninth grade (Corwin et al., 2005). Conversations I had while working with aboriginal students at Britannia Secondary seemed to confirm these findings. I realized that many of their "delinquent behaviours" in class or skipping class altogether started in 8th grade due to feelings of not being able to cope with the courses from the beginning, and not being able to foresee any possibilities of going to university and getting an education which could provide them with a better future. In general the transition from seventh to eighth grade is a difficult one for many children, however for aboriginal students it seems to be particularly harsh. For the first time children are streamed and in the case of most aboriginal students, they are placed in courses with the lowest academic expectations.

For this reason I developed and implemented a six-week summer camp for aboriginal children transitioning from elementary school to high school. We were able to organize these camps during the summers of 2008 and 2009 at Britannia Secondary in Vancouver. The goal of these camps was to provide students with a more solid knowledge of mathematics and English when they enter high school. We also invited Aboriginal adults who are successful in a variety of fields to participate in these camps. They talked to the kids about possibilities and how it was possible to deal with the system to be able to realize personal dreams and the importance of education. These role models have been an important part of our program as well.

PIMS has also continued to offer a summer camp for students entering grades 9 to 12 who have been part of our programs.

In addition to the summer camps, mentorship programs are being implemented to help these students with their math courses throughout their high school years and PIMS also provides scholarships throughout the year to students who attend school regularly and have a good work ethic. Not all the students in our programs get these scholarships, they have to earn them.

The result has been that this year for the first time in recent history 2 aboriginal students graduated with principles of math 12 and three with principles of math 11 from Britannia Secondary, and for the next year we expect they will be three more students graduating at this level of math. The two students who graduated with principles of math 12 obtained scholarships to attend college.

When the trustees of the Vancouver School Board asked these students what had made the difference, one of them answered that it was important for them to know that they have people behind them supporting them at school and at PIMS, and that learning and understanding mathematics gave them the confidence and tools to continue their studies. Our long term commitment to their learning and graduating from high school had an impact in their own commitment to continue until the end.

In conclusion, if we want some of these at risk students to succeed we need to provide long term continual assistance with a variety of programs. We need to present opportunities and positive interventions to provide students with more positive outlooks for life, and the tools to deal with the opportunities so that when the moment comes to make a pivotal choice, the students will act in a way that will further their horizons.

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SOME USEFUL ELEMENTS FROM A SURVEY

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This paper presents pre- and post-surveys on attitudes held by Mexican high school teachers, involved in a lesson study project, as a means to understand the limitations and possible usefulness of this specific survey. Items that are brought to the fore during the analysis will be discussed and may inform research neophytes as they consider various instruments for their own imminent data collection.

BACKGROUND

Surveys are commonly used in research work for a variety of purposes; some of which are to gather data to identify trends, to determine opinions on issues and to identify beliefs and attitudes of individuals (Creswell, 2008). The latter is important in Education where one may be interested in accounting for attitudes of research participants before and after interventions or to indicate change over a period of time, etc. Surveys are popular data gathering tools because they are typically administered quickly and results can be rapidly tallied. There remains an important question though as to their relevance and usefulness for a given research project. For this very reason, surveys in education are often employed in conjunction with other instruments such as interviews, narratives, artefacts, etc. This paper, therefore, aims at analysing a specific survey to identify some of its limitations and possible usefulness.

RESEARCH CONTEXT AND PARTICIPANTS

The surveys that are analysed in this report are part of a research project that was conducted in three different high schools in the state of Morelos in central Mexico. Paulino Preciado, an SFU Mathematics Education PhD candidate , used a Lesson Study (Stigler & Hiebert, 1999) format for collaborative work sessions with over twelve teachers in the 2008 fall semester, nine of which are included in the present survey analysis. The Morelos schools received one to two sessions per week over the term, with the exception of one school that had much fewer sessions. Mexico is presently going through educational reforms and *Lesson Study represents an alternative for teachers' professional growth towards progressive teaching practices* (Preciado & Liljedahl, 2008). This research project focused on teacher experiences over the duration of the Lesson Study Project and the survey was but one of the instruments used. For the purpose of this paper, I wish to regard the survey as possible evidence of change. Teachers answered 15 questions on attitudes before and after the project. This information was transferred to a spreadsheet for further analysis which is entirely new for the project.

METHODOLOGY

The fifteen-question survey offered five rating levels, which were given a numerical value as follows:

Strongly Agree:	4 points
Somewhat Agree:	3 points
Somewhat Disagree:	2 points
Strongly Disagree:	1 point
N/A:	0 point

Schools where treated separately for data analysis. The sample size is not sufficient to reach any compelling conclusion but may still provide some feedback on the study. The standard deviations for every question were calculated separately so that the different ratings would be more evident. Again, I stress the fact that changes between individual teachers from the pre- and post surveys are preserved in this manner. Standard deviations for pre- and post-surveys were calculated, cumulated and grouped per school.The total change is calculated as the absolute values of each changed answer from pre- to post-surveys, they are then added and averaged. This is meant to compare the level of change between schools in either direction on the scale to simply indicate *dissatisfaction with existing conceptions* (Posner et al.,1982). The tables below are ordered to reflect the amount of time spent at the different schools; the school that was visited the most is School A at the top of the list and so on. Again the intend of this analysis is to explore the use of a survey to measure change and to identify limitations and usefulness of such a survey.

RESULTS

School A

Standard Deviation Pre-S	Standard Deviation Post-S	Total Change School A	Direction of Change (+)	Direction of Change (-)
8.5	7	8.5	5	-3.5

School B

Standard Deviation Pre-S	Standard Deviation Post-S	Total Change School A	Direction of Change (+)	Direction of Change (-)
8.6	8.1	9	3.3	-5.8

School C

Standard Deviation Pre-S	Standard Deviation Post-S	Total Change School A	Direction of Change (+)	Direction of Change (-)
3.5	2	1.5	0	-1.5

DISCUSSION

Firstly, it is important to recognise that none of the observations have been validated anywhere other than in this set of data and require further investigation with a larger sample size to carry weight. A quick glance at the values in the tables, draws my attention to a correlation between the frequency of the visits and the value of the Total Change. School A, the most visited, has a Total Change of 8.5, School B, 9 and School C, the least visited of the three, has a Total Change of 1.5. It would seem to indicate a direct repercussion of the amount of time spent at a school and the amount of change that occurred within that school.

Again the intent is to consider possible correlations presented here, more analyses of interviews, that are connected to the pre- and post-surveys, would have to be conducted to further support this possibility. It is nonetheless encouraging to see that interventions may have some effects on attitudes held by teachers, even if attributed to a repositioning of the teachers' introspections.

Another interesting observation is that the standard deviations of all schools get smaller in the post-survey. School A goes from 8.5 to 7, School B, from 8.6 to 8.1 and School C, from 3.5 to 2. Of course it's easy to become overly exsited with these reductions which are indicative of the proximity of teacher scores. School A has 19% more compactness after the project, School B has 6% and School C, 43%. The data support a tightening of the group maybe due to a joint enterprise and a shared repertoire as in Wenger (1998). I'm not, however, overly confident that the numbers can justified such a claim. School C, for example had the least amount of contact and seem to have more group unity after the activity but the post- and pre-surveys for this school are almost identical i.e. hardly any change is noticeable. The sample is so small that any conclusion would be premature. The other two schools had three and four teachers each and did spend significantly more time working together. Both school had greatest change and question ratings were tighter post project.

Additionally, positive and negative values are not directly indicative of *desirable* and *undesirable* changes. For more clarity, it is to be noticed that there are 2 orientations to the rating scale, and to *strongly agree* or to *strongly disagree* may be equivalent depending on the question posed. Using Schulman (1986) as a frame of reference, five questions could be viewed as valued attributes for a teacher and would be indicated by a higher score on the survey, 6 questions would have a reversed scale i.e.

a low score would indicate a positive attribute for a teacher, and lastly 4 questions did not fit into these categories.

In conclusion, numbers can easily lead us to brash claims. Analysing a survey can inform us on its validity and limitations. It may also point to areas requiring further investigation and refinement, and further can reaffirm the imperative for several data gathering instruments in research.

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IDENTIFICATION OF HABITS OF MIND INHERENT IN MATHEMATICAL EXPLORATION

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In this study students were given an opportunity of posing problems after being presented with a "rich" mathematical object. Students' problems were commented on in the hope of revealing to the researcher a possible framework.

INTRODUCTION

The activity of exploration, although it has obvious manifestations in the "real" world, is not a common activity in a school environment. The act of exploring might require a search for an unknown feature of an object, or it might entail the pursuit of an answer to a question, or, it may solely involve "looking around". I adopt Hawkins (2001) perspective of exploration. His view is that exploration, in and of itself, is a novel activity whether or not it leads anywhere. He describes exploration as "…a mode of behavior in which the distinction between ends and means collapses; it is its own end and it is its own reinforcement" (p. 116). It is valuable because of what is found along the pathways. This is in stark contrast to current trends of mathematics teaching that Brown and Walter (1983) describe as the "right way" syndrome. Current learning theories can only exist when a goal is defined (Hawkins, 2001). Exploration breaks that tradition. Hawkins continues to indicate what it means to explore, exploration "…makes it plain that goals in problem solving cannot be imposed by external input, but must be evolved, fabricated, set out of the antecedent, ongoing activity of the learner" (p. 116). This directs the focus to problem posing. I suggest that exploration provides a context in which to participate in problem posing. If in fact a student is looking for, searching for, wanting to discover something, then they most likely will be constructing problems.

I believe that in the act of exploration certain mathematical modes of thinking are developed. I have chosen problem posing as a way to identify these mathematical modes. These modes are still in need of articulation but for now I refer to Kobiela et al.'s (2008) use of Goldenburg's habits-of-mind perspective. In one of their studies they identify generalization, relations referred, invariance and scope as actions that reflect mathematical modes of thought. My interest lies in observing what kinds of modes of thought will reveal themselves when students are in the act of exploration, and more specifically, for this study, problem posing. I recognize this is a vague focus; there is still need for refinement as I am still in the process of determining a framework in which to organize the observations or interpret any results. What I have attempted to do in this paper is to outline my thoughts and present a cursory data collection and brief analysis to help me articulate my perspective.

DATA COLLECTION AND ANALYSIS

This presentation is only an initial attempt to notice features and to furnish the occasion of "seeing" something. Students in a grade 9 enriched class were presented with a "rich" mathematical object and were given no other instructions except that of posing problems. The first five rows of Pascal's triangle were posted on the front board and students were requested to pose 5 problems based on the patterns they noticed. Students worked in pairs, were given a period of 10-15 minutes to work together and each pair of students were then asked to post their "best" problem on the front board. The results were interesting. By creating an environment of exploration and by considering a perspective of habits of mind, my interest was to see what might occur with an activity of this sort and to ask myself if there was potential in the posing of problems.
Although there were a fair number of trivial problems posed there were also a good number of non-trivial problems. These non-trivial problems satisfied a variety of dimensions. Currently I am unable to articulate exactly how I will categorize these problems, but for the time being, I will note the ones that appealed, for whatever reason, to me.

Problem posed	**Comments**
1) What pattern do you see in the number of digits in each row?	A non-trivial, non-conventional question. This could end up being a very rich mathematical problem.
2) Why does the third diagonal column form an arithmetic sequence that goes up by 1?	Although mathematically trivial, there is still a recognition of a pattern, a connection to previously learned material and proper use of terminology
3) What is the 2^{nd} number in the 46^{th} row?	A trivial response satisfies this problem but the problem is non trivial. Its extensions are wide reaching.
4) Are there any geometric	An unconventional and interesting problem.

patterns? If so, explain.	
5) Which level will the first three-digit number appear?	Another a-typical, unconventional problem associated with Pascal's triangle. This problem could lead to a generalization of growth

One problem, not stated in the table above, was a more traditional problem associated with Pascal's triangle and referred to the pattern inherent in the sum of each its row. What was interesting to me was that a student who read that question expressed verbally that it was a good question because it pointed out to him, in reading it, that there was a pattern that he had not originally noticed. It is not surprising to me that one needs to understand underlying features of the math environment to formulate these types of problems. It is quite exciting to see the potential in both the problems and the enthusiasm of the kids.

CONCLUSION

Although this study has no properly defined framework or methodology I believe that it is of interest to reflect on the results. I found the overall activity to be extremely engaging for students, and I feel there is potential to identify habits of mind. Although trivial problems were posed, there was also evidence of problems that were thoughtful and sophisticated. After the activity, I asked the students what they thought of exploration. One comment stood out, inspiring me to continue to pursue the idea of problem formulation in an exploratory situation. I will conclude with the quotation: "Since we don't know everything about anything (e.g. Pascal's Triangle), the lack of knowledge would most likely arouse more curiosity/questions".

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A CRITIQUE OF ETHNOMATHEMATICS

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Starting with its presentation by Ubiritan D'Ambrosio, the concept of ethnomathematics, as a new field of study, has been taking a wide place in mathematics educational research. In this paper, I will first focus on how different researchers, who study ethnomathematics, define it. Based on their definitions, firstly I will start with the points in those their positions and descriptions of ethnomathematics differ. Then I will continue with a concluding part to summarize the ideas, philosophies and stances ethnomathematicians share. In addition, I will critique the thesis that ethnomathematicians share, which is taking a position that mathematics is culturally dependent. My critique will be detailed in four aspects: etymological, socio-pedagogical, historical-anthropological and metaphor of applied mathematics.

INTRODUCTION

Before detailing my paper, I would like to shade light on some points to provide the background to the reader to better understand following sections. Firstly I will direct my thoughts towards all scholars who deal with the field of ethnomathematics, because apart from their slight differences in understanding of the implications of the field, as I will detail in the following part of the article, they generally share the same philosophical, historical, anthropological, educational and political stances. That's the reason why in his article about ethnomathematics, which is known as the first article in which the term of ethnomathematics was introduced for the first time; D'Ambrosio (1985) uses pronoun 'we' instead of 'I' to address the scholars who are studying ethnomathematics and I will use the term *ethnomathematician* which is also used by Gerdes (1994) to address these scholars. I also would like the stress that, because linguistically the plural and singular forms of mathematics are the same word, the idea of having many mathematics*es* or argument of having *the* mathematics as a single is not available to be expressed and to be used to emphasize the difference between viewpoints of CRT and CIT in English grammar. Therefore readers should pay attention to that the word 'mathematics' is used sometimes in plural by CRT standpoint while it is always used in singular by CIT standpoint.

According to its first definition by D'Ambrosio (1985), ethnomathematics is the mathematics which is practiced among identifiable cultural groups such as national-tribe societies, labor groups, children of certain age brackets and professional classes. So this practiced mathematics of cultural groups can be different from its well-known and recognized form, which is defined by ethnomathematicians as *Eurocentric mathematics* (Powell & Frankenstein, 1997). While these cultural groups can be thought of based on their ethnicity, also in his definition, D'Ambrosio (1985) uses the term of 'cultural group' in an expanded form that also covers different social groups within a society (such as carpenters, street sellers etc.) those use mathematics in its uniquely developed forms. So ethnomathematics can be summarized as the mathematics which is practiced with members of a cultural group who share similar experiences and practices with mathematics which can be in a unique form. All these different cultural groups have their own language and specific ways of obtaining their practical mathematics and ethnomathematicians study their techniques (Gilmer, 1995). In their chapter in the book which is entitled 'Ethnomathematics: Challenging Eurocentrism in Mathematics Education', Powell and Frankenstein (1997) emphasize the existence of different definitions of ethnomathematics associated with different perspectives. For example while Gerdes's (1988) definition of ethnomathematics is quite compact, it has no specific emphasis on culture: "The mathematics implicit in each practice". Ascher's (1986) definition of ethnomathematics is less inclusive and more focused on non-literate cultures: "The study of mathematical ideas of a non-literate culture". In his second attempt, D'Ambrosio (1987), includes the term of *codification* as a difference and he expands 'mathematics' and instead uses 'reality'. This definition has an emphasis on the systematization of ethnomathematical ideas of cultural groups and manifesting reality through their own system of codification: "The codification which allows a cultural group to describe, manage and understand reality". In his definition of ethnomathematics, rather than focusing on the term ethnomathematics for some specific cultural groups, Bishop (1988) is defining mathematics itself as a cultural product: "Mathematics…is conceived as a cultural product which has developed as a result of various activities". In her later definition, Asher (1991) introduces two additional components in expressing her description of ethnomathematics. Firstly she adds the word of 'presentation' to (possibly) emphasize her new position, which evokes that ethnomathematics is not only at the implicit level or just a composition of ideas, but also it was explicitly practiced in reality, presented and still being presented by different cultural groups. Secondly she switches her use of the term 'non-literate culture' to 'traditional people'. The reason for her change of definition is possibly because of her wish to include other cultural groups that have presented or being presenting their mathematics literally: "The study and presentation of mathematical ideas of traditional peoples". In his definition, Pompeu (1994) points out the requisite of recognition of ethnomathematics by Western anthropologists for its manifestation, therefore his definition evokes its continuing Western dependency: "Any form of cultural knowledge or social activity characteristic of a social group and/or cultural group that can be recognized by other groups such as Western anthropologists, but not necessarily by the group of origin, as mathematical knowledge or mathematical activity". And lastly Knijnik's (1998) definition of ethnomathematics has more socio-cultural emphasis and she does not use either the word 'culture' or any other word may address ethnicity of social groups. However by the use of the term 'subordinated', her definition of ethnomathematics can be thought of as having a more political emphasis than the other definitions above: "The investigation of the traditions, practices and mathematical concepts of a subordinated social group".

In spite of their slight differences of definitions for ethnomathematics, ethnomathematicians share four common assumptions. The first of these assumptions is regarding with their epistemological view of mathematics. They share the argument that mathematics is the creation of human, regardless of their theoretical positions of cognition and learning, they expose their common position by expressing that mathematics is not universal as traditionally believed (D'Ambrosio, 1985) and expressing that, it is human creation (Bishop, 1988). This assumption not only gives ethnomathematicians the philosophical ground on which they can rely and establish their theory, but also flexibility and comfort to be able to assert the cultural relativity of mathematics. Based on an assumption of universality of mathematics, mathematics would not be a creation of human beings, therefore there would not have any possibility to claim the relativity of mathematics among cultures. Therefore it can be inferred that under only this epistemological assumption, it can be argued that every culture has its own mathematics, because the members of this specific culture create their own specific mathematics. The second assumption of ethnomathematicians is their attributions towards the anthropological and historical findings of mathematics (Ascher & Ascher, 1981; Ascher & D'Ambrosio 1994; Kats, 1994; Zazlavsky, 1994; Gerdes; 1994). They use these findings as empirical evidence to support cultural relativity of mathematics, that is, along the history of human being, different cultural groups created their own special mathematics. However there are some exceptional views, like Bishop's, that all humans engage in the same basic activities, which lead to mathematics. Thirdly, almost all ethnomathematicians emphasize that Western mathematics is imposed to the other cultural groups by colonization, and share the emergent need of searching the derivations of the mathematics of Third World countries. In other words, they have a common political attitude and reaction against imperialism and Westernization and a supporting position for Third World countries (D'Ambrosio, 1985; Bishop, 1988, 1994; Ascher & D'Ambrosio ; 1994, Gerdes, 1994, Vithal & Skovsmose, 1997). And fourthly, because almost all ethnomathematicians are also mathematics educators, the last common assumption that ethnomathematicians share is the implacability of their findings to mathematical education research. They connect their positions with mathematics education, addressing some contemporary issues in mathematics educational research and offer solutions and implications relying on their ethnomathematical point of view (D'Ambrossio, 1985; Bishop, 1988, 1994; Bassanezi, 1994).

For sure all of these components are interconnected; therefore it may not be possible to not touch other assumptions while attempting to discuss only one. Nevertheless, in this article, my focus specifically will be on cultural relativity of mathematics which ethnomathematicians hold. Here I also want to clarify that epistemology of mathematics is quite connected with the view of its cultural dependency. However my discussion will not be on this, in other words, my intention is not to argue if mathematics is a creation of God or of human beings. My argument is whether it is embedded in universe. Therefore I want to clarify that arguing if mathematics is a universal fact or not, does not necessarily mean the same thing as, although it is connected to, arguing its foundation, which is not my focus in this paper. Also I will not argue that whether teaching techniques developed or can be developed based on ethnomathematical point of view or are effective in cognition of mathematics.

To systematize the rest of my article, based on the view I critique and the view I hold, I feel the need of naming and clearly describing what these two views are. The thesis supported by ethnomathematicians is that mathematics can be relative among cultural perspectives and social groups, so it can be developed as a result of various activities based on practices and experiences of these cultural groups, therefore it is a cultural product rather than being cultural-free and universal (Bishop, 1988; Gerdes, 1994). I will name this thesis as the *Cultural Relativity Thesis (CRT)* about mathematics. My counter thesis to CRT is basically based on the idea of cultural independency and universality of mathematics. I will name this thesis as *Culturally Independence Thesis (CIT)*. According to this thesis, independent from its symbolizations, understandings, processes of development and the ways of practices, applications or implications used by different cultural groups, mathematics as it is today is a universal value of all humanity, in other words, it is not Eurocentric. For the following section of my paper, I will discuss these two theses in four grounds: etymological, socio-pedagogical, historical and regarding with applied mathematics.

1. Etymological:

When deriving a term for a new interdisciplinary field, it is required to pay attention to each sub terms, with respect to their etymological consistency. The word "mathematics" comes from the Greek term μάθημα (máthēma), which means learning, study, science (Wikipedia) and τικός (tikos) means art. All these three components of its original meaning; learning, study and science are shared values of all humanity, in other words, they are universal, because all humans learn, study to learn and develop techniques or to discipline their knowledge to improve its applicability to their lives. This etymologically manifests that *mathema*tics itself is universal. On the other hand the Greek prefix *ethnos* stands for a group of people living together. This can be interpreted as something belongs to a specific social or ethnical group. Therefore the ethno-mathematics etymologically refers to ethnical (or ‘socially different from others’) mathematics. In my opinion the word ethnomathematics can be an oxymoron because etymologically it is contradictory to use ethno (having a meaning of relativity) combining with mathematics (having a meaning of universality). Some researchers sharing the need to distinguish mathematics from the ways of doing it feel the need of deriving new terms to critique cultural dependency of mathematics. One of them, Robert Thomas (1996) defines the term ‘real mathematics’ to be able to distinguish what ethnomathematicians mean by mathematics from what he thinks mathematics is. However I do not feel a need to derive or refer to new terms to describe what mathematics is. I think that it is not necessary to derive new terms to address this phenomenon here, because it is the mathematics itself. Therefore because the mathematics is universal and real itself, using adjectives like 'universal' or 'real' for mathematics is pleonasm.

2. Socio-Pedagogical:

Beyond its etymological construction, the word ethnomathematics sometimes can be problematic in its use in society and education. Even if from the literature on ethnomathematics it is clear that ethnomathematics is based on a broad interpretation of the notion ethno including different cultural groups, not necessarily ethnically; the prefix still evokes race or ethnicity (Vithal & Skovsmose, 1997). Consequently, although researchers who study ethnomathematics emphasize that *ethno* should not be thought of as ethnically (D'Ambrosio, 1985, 1987; Gerdes, 1994), some studies show that it can be taken in this account, contradicting with the intention of derivation of ethnomathematics and political position of ethnomathematicians. A study addressing this issue is conducted in South Africa by Vithal and Skovsmose (1997). In some societies with sensitive historical relations among different races within, mainly due to colonization, ethnomathematics can be associated with meanings to relate to the racism (ibid). As a result, due to the prefix *ethno* can carry strong divisive and negative connotations in these kinds of societies, with regard to its educational implications; its practical use in schools can be problematic as taken anthropologically (ibid). As a socio-political aspect, I also would like to critique Eurocentric mathematics (Powell & Frankenstein, 1997) or Western mathematics (Bishop, 1988). Even if we assume that CRT is true and mathematics is a cultural product, we still would not argue that the currently used mathematics is purely Western. Some historical findings show that the mathematics we are using was not only derived by Greeks. Before Greeks, in China, India, Anatolia, Mesopotamia and Africa math was used in similar ways of Greek's and affected Greek mathematics. Even if their ways, symbols and language they were using to communicate mathematically were different, the phenomenon they were dealing with was the same mathematics as we do. Along the dark age of Western world, Islamic civilizations took it over, for instance the word 'Algebra' is derived from the name of book 'Al-Jabr' of a Muslim mathematician named Al-Kharezmi. Therefore, even under the assumption of cultural dependency of current mathematics, currently used mathematics is still not Eurocentric, rather could be said it is multicultural mathematics and therefore is still a shared value of all cultures.

3. Historically-Anthropological:

Along history, all cultures from all over the world shared the same concerns to deal with the same problems they are faced with in their practices. Their ways or interpretations to express and practice these problems can be different. Their levels of depth on exposition of these problems can be relative based on the appearance of these problems in the environment or sociality they were situated. However this does not mean that these problems or realities *themselves* were different. In other words, different cultures developed similar solutions to similar problems, just in different representations. We can give numbers and counting as the example. Along their practices, different cultures used different symbolizations and ways to express counting, numbers and arithmetic. For instance in ancient times Quipus were used by Incas, knotted cords were used by Chinese, many other ancient societies like Babylonians, Indians found their own ways to express numbers and to solve their problems require counting and arithmetic. This is also an argument which is supported by CRT however it attributes relativity to the problems that are shared by all cultures, instead of attributing it to the ways, expressions or symbolizations used by these cultures. This point of view held by CRT misses that while many cultures were dealing with mathematics in different shapes; the phenomenon these cultures were dealing with was the same thing. That's actually why we can clearly understand these anthropological findings even today, by connecting our ways of doing mathematics with their ways. After translating their language to English and their own expressions to currently used mathematical symbolization, we clearly understand this fact. For example, Babylonians were dealing with algebraic formulas and theorems such as Pythagorean Theorem, and we are currently using the exactly same mathematics currently. Likewise after translating their texts and understanding the tools they were using to speak mathematically, we are able to conclude that not only Babylonians, but also Greeks, Egyptians, Indians, Chinese and maybe many others, for those we could not find anthropological and archaeological evidences yet, were expressing *the same thing* in their own languages. This shared *thing* they were all dealing with was the *mathematics itself* that we are dealing with.

I would like to express some analogies by telling a story to clarify what I'm trying to distinguish from what. Let's think about two different societies living in ancient times. One is living just near a coast to an ocean; another is near lake. Both cultures are trying to find a way to go over along the surface of the water (for nourishment or exploration). Both cultures are trying to use swimming as a way to achieve their goals, but they both realize that swimming is an individual practice and it is tiring to swim each time for fishing or exploration. Seeing the activity of 'swimming' as a bridge leading a better idea, they realize that they do not sink into water immediately. So they both get conscious of the universal law of Hydrodynamics. This manifestation is directing them to construct a tool which can swim, can transport more than one person and requires less or no power to be moved. After some brainstorming within each context, both come with a vehicle which makes them able to deal with the problem. Because of the geographical conditions they are in, after some unsuccessful trials, the one close to ocean may need to construct a wider and stronger ship which is more resistant to storms and waves of the ocean. On the other hand for the other society it can be enough to construct a smaller boat which works on the lake. Maybe while designing their ships, they will be affected by the environment they are living in, such as the shapes of animals they are used to see. Maybe the materials they will be using to construct their ships can be different, such as if one is living near a forest; this society may try to use woods. If the other society is more familiar with reed beds, they may try to use reeds to build their boats. In this story, my analogies were between hydrodynamics as a physical rule (universal and culturally independent) and the counting or arithmetic, between trading and water, between wooden ships or reed boats and quipus or tablets. Maybe different societies used different ways, materials to deal with the problems they faced in practice but, what their problems were about was the same thing, which is mathematics.

4. Applications and Applied Mathematics (AP):

The final aspect I want to discuss is about applied mathematics and applicability of mathematics to the non-human universe. Bishop (1998) mentions six universal activities of mathematics as counting, locating, measuring, designing, playing, and explaining, He argues that even if these activities are universal, the rest of mathematics is culturally dependent. However, according to CIT, these components are just first applications of mathematics, in other words, we can call as foundations of AP. Nevertheless these activities are not all applied mathematics, based on these six roots; AP has been developing. These developments of AP found their places in other disciplines like physics, cosmology, chemistry, medicine, technology, biology, and so on. As a result, all cultural groups from all over the world using these benefits of AP. So if the rest of AP was Western, it wouldn't be applicable to many other disciplines and also its products would not be useful for all humans. For example through the use of mathematics, we can successfully send satellites to the space and they are working in a good harmony; can derive statistics which is used also in many disciplines, such as medicine, so helpful to save many people's lives, all having 'human body'; can build constructions comfortable, compact and resistant to earthquakes. The golden ratio is another example which is can be introduced as an evidence for how mathematics is embedded in the universe. It is not only embedded in human body but also in plants, solar system and so on. Many other examples as the evidences for CIT can be given to explain based on its applicability to the universe, how mathematics in its current shape is in a harmony with universal facts and how it is useful and important for all humans independently from their cultural origin.

CONCLUSION

Ethnomathematics is relatively a new field of study which is supported by many researchers in the field of math education. However researchers should evaluate this new theory multi dimensionally, not only basing on how ethnomathematicians describe it or what they hope what ethnomathematics will implicate in education. Although I believe that paying attention to students' shared experiences can be helpful in education, some of the concerns I addressed in this paper should be clarified. Apart from its theoretical and anthropological aspects, for the future, more empirical studies are needed addressing 'What the groups of people under the focus of ethnomathematics think about ethnomathematics?', rather than assuming that this theory, which is created, developed and argued by Western mind, is reflecting the facts of these cultures and applicable to their educational system.

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TECHNOLOGY AND MODELING AS AGENTS OF INQUIRY

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I first describe Confrey and Maloney‘s (2007) four distinct but related approaches to technology use in mathematics instruction, one of which centrally involves modelling. My focus is on how students use technology in modelling, in light of the Deweyian definition of inquiry. Data from undergraduate students' modelling task are analysed. The results concur with Confrey & Maloney's (2007) that modelling through the process of inquiry, provides opportunity for the inquirer to progress from an indeterminate to a more determinate situation. However, I suggest a slight change in the definition of mathematical modelling proposed by Confrey & Maloney (2007) and submit that the modelling process is ultimately a communication of a solution derived from the model world to the real world.

INTRODUCTION

Though mathematical modelling has been used by scientists and engineers throughout the ages, its importance as a discipline to be studied has been realized only during the last three or four decades. Modelling in mathematics education has historical developments and connections with the demands from the work place for graduates with problem solving skills. (Mclone 1973; Davis, 1994; Challis, Gretton, Houston & Neill, 2002).

Over the years, other perspectives to the classical approach to modeling have developed. The opponents of the classical model argue that modeling involves many cycles and multiple stages, which the linear model does not adequately address (Lesh & Doer,2000).I take the view that mathematical modeling is an attempt to understand an already complex problem and reduction in the number of variables is necessary.On the other hand, a mixed method may be considered to cater for any extra variables that may be of concern in the model.

This work is motivated by Confrey &Maloney (2007) who provide four distinct approaches to technology use in instruction. These are: i) teaching skills without technology and then providing the technological tools as resources after mastery; ii)introducing technology to make patterns visible more readily, and to support mathematical concepts iii) teaching new concepts with the necessary technologically enhanced environment; and iv) focusing on applications, problem solving and modelling, and using technology as a tool for solving such problems. Their results suggest that technology mediated modelling may be a good strategy for instruction in a single classroom with different levels of experience and skills. From the above, approaches, my hypothesis is that technology mediated modelling improves the clarity of modelling task and the communication of its results. The specific questions investigated are 1) how does technology relate to mathematical modelling in an undergraduate classroom? 2) How does mathematical modelling relate to Dewey's definition of inquiry?

The Deweyian definition of inquiry is stated as
"the controlled or directed transformation of an indeterminate situation into one that is so determinate in its constituent distinctions and relations as to convert the elements of the original situation into a unified whole". (Dewey, 1938, p. 226.)

CONCEPTUAL FRAMEWORK

A mathematical model consists of an extra mathematical domain of interest E, and a mathematical domain X, and a mapping from E to X, i.e. $E \rightarrow X$ .Mathematical inferences are made within X and the outcome is translated back to E, $E \leftarrow X$, and interpreted as solutions concerning that domain.

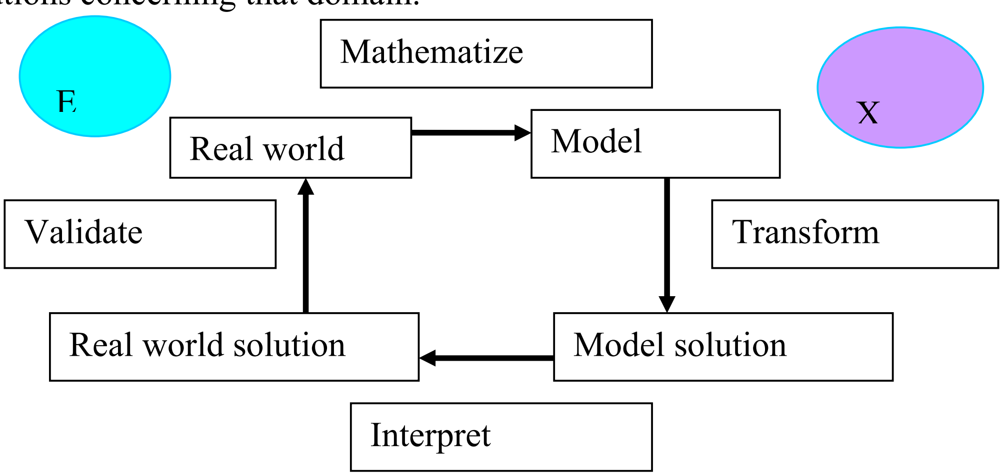


Fig.1: The modelling cycle Doerr & Pratt (2008)

The modelling cycle may be iterated several times on the basis of validation until the conclusions concerning E are satisfactory according to the initial purpose of the model. (Niss, Blum, & Galbraith, 2007).
Using the above framework to analyse my data, I pay attention to students' mathematical representations using technology; reasoning/arguments; how they arrive at their solutions and communicate the final results. Attention is also paid to whether or not solutions are realistic and relevant to the extra mathematical domain.

METHODOLOGY

Participants

Participants are drawn from 37 undergraduate non mathematics majors. The group attended a regular semester geometry class three hours a week for ten weeks. Each person worked on a computer during instruction and had a weekly take home assignment. Although a large group received instruction as part of their semester credit, data from three participants are presented. Since the group was taught together, it is assumed that their solutions are likely to be similar in approach. The three cases were picked for analysis because their solutions look different.

Materials

All the work was done on the computer. Geometer's Sketchpad was used as the main software for the delivery of the course. The computers in the lab are connected to the internet and students got their study materials delivered to them electronically. They in turn submitted their work to the Professor electronically.

Caveat

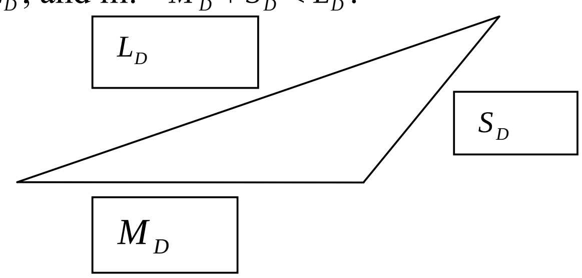
This work is based on the analysis of only one modelling assignment. It does not include any follow up interviews. These issues will be taken up in the on-going study.

Modelling task

You have a choice at the local pizza place: for the same price you can get either one large pizza or both small and medium. Determine which way you get more if you know the diameters of the different sizes of pizza. Describe the procedure to figure out the best deal to choose. Caveat: You can't use the area formula for the circle.

The mathematical model

Let the diameters of the pizza be: L_D for the large size; M_D for the medium size, and S_D: for the small size. Assume that the three diameters meet at the three vertices of a triangle. Then there are three possible scenarios, i. $M_D^2 + S_D^2 = L_D^2$, ii. $M_D^2 + S_D^2 > L_D^2$, and iii. $M_D^2 + S_D^2 < L_D^2$.

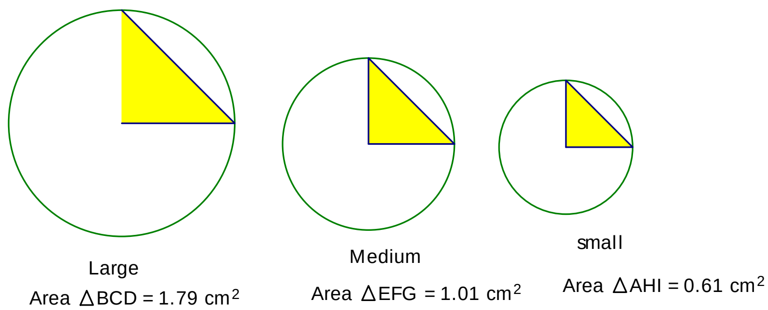


Scenario I: Taking one large pizza, or the medium and small together, makes no difference (no deal); *Scenario II*: Taking the medium and small pizza together makes the best deal; *Scenario III:* Taking the large pizza, makes the best deal.

RESULTS

Three solutions from students, S1, S2, and S3 follow. A sketch accompanied by a description from each student are presented and briefly discussed.

[1] S1:

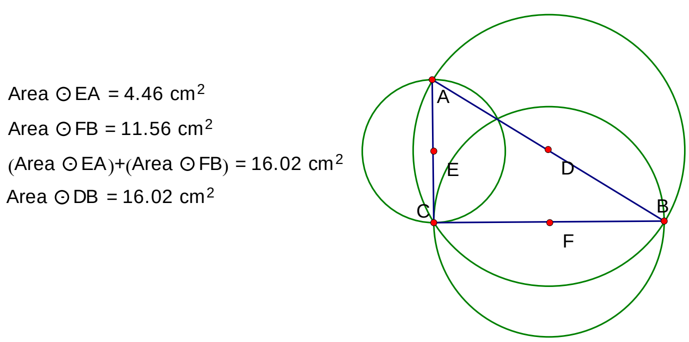


[2] S1:

If the diameter is known for each size of pizza a right triangle can be made from each pizza. Using the diameters, the area of each right triangle can be found and then compared to see which is bigger. In this example that I have drawn the areas of the medium and small pizzas can be added together and compared to the large pizza to see what would be the better deal. In this case that I have drawn, the medium and small pizza do not equal more than the large pizza and therefore, the better deal would be to go with the Large pizza. However, this may change depending on the exact sizes of each pizza in reality.

S1 models the problem in terms of similar areas of the isosceles right triangles. He argues that to get a better deal, one should add the small and the medium areas, and compare it with the large triangle. If the area of large triangle is bigger, then the deal is to take the large pizza. If not, then take the other two pizzas. S1 follows through the modelling cycle up to the validation stage, when he implies a revisit of his solution. From the statement, "*this may change depending on the exact sizes of the pizza in reality*", he is aware that his solution is not final, but may require some iterations for it to be realistic. He makes an attempt to represent the problem as clearly as possible and to communicate his solution quite realistically.

[3] S2:

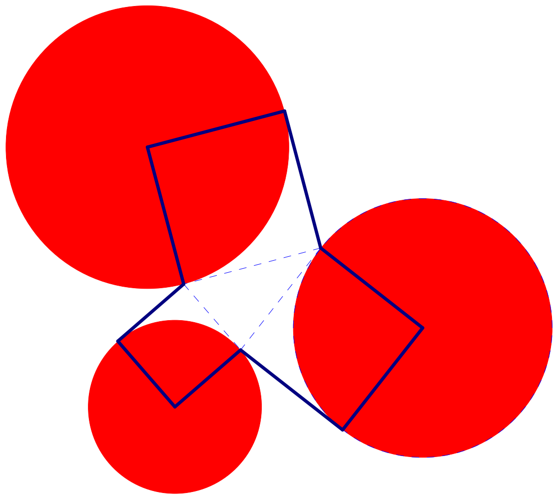


[4] S2:

Using the diameters of large,medium and small pizzas as 3 sides of a triangle. If the triangle is a right triangle, it don't matter which deal you choose because the area of one large pizza is the same of one medium pizza plus one small pizza.
However, if the triangle is acute, choose the small and medium pizzas because the area on one side of an acute triangle will always be smaller than the sum of the areas on the other two sides.
On the other hand, it it is an obtuse-angle triangle, choose the large pizza.

S2 uses the radii of the pizza to form a triangle. By changing the shape of the triangle he determines the respective areas of the three pizzas. The dynamic property of the Geometer's Sketchpad makes it possible to vary the diameters and hence change the shape of the inside triangle. Overall there is clarity in the communication and presentation of S2's results. In fact he mentions all the three possible solutions presented earlier in the mathematical model, even if his sketch shows only the first scenario.

[5] S3:



[6] S3:

In terms of Area,
given that the area of a circle whose diameter is equal to the hypotenuse of a right triangle will equal the sum of the area two circles formed from the base and height of the same triangle,
and
given that in an acute triangle, the area of a circle whose diameter is fomed from the longest side will always be less than the the sum of the area of circles whose diameters are fomed from the remaining two sides,
and
given that in an obtuse triangle, the area of a circle fomed from the longest side will always be greater than the sum of the area of the circles whose diameters are formed from teh remaining two sides,
Therefore
It stands to reason that should the diameter of the three sizes of pizza form an acute triangle, you would get more pizza by ordering a medium and a small;

should the diameter of the three sizes of pizza form a right triangle, it makes no difference as to whether you should order a single large, or a medium and a small;

should the triangle formed from the three sizes form an obtuse triangle, then you would get more pizza by ordering a single large.

S3's solution includes all the three scenarios and the results are presented in great detail, linking the solution with the context of the task. She does not show the mathematical expression in her drawing but from the written work, it is evident that the three scenarios in mathematical model are addressed.

DISCUSSION AND CONCLUSION

Even if the class received the same instruction, we see that students use different approaches to deal with the modelling task. However the role played by technology seems to be one unifying factor in the solutions. Technology substantially enhanced the clarity of modelling task and the communication of its results by the students.

Depending on their experience and skills, they all make reasonable attempts to solve the mathematical model and interpret the solution back to the real world. To that extent the, transformation of the original unknown situation to a more realistic one is my claim on the Deweyain definition of inquiry. This position agrees with Confrey and Maloney's (2007).

However, with respect to the definition of modelling proposed by Confrey and Maloney (2007),

Mathematical modelling is the process of encountering an indeterminate situation problematizing it, and bringing inquiry, reasoning, and mathematical structures to bear to transform the situation. The modelling produces an outcome a model- which is a description or a representation of the situation, drawn from the mathematical disciplines, in relation to the person's experience, which itself; has changed through the modelling experience. (2007, p.60)

I wish to propose the following refinement:

Mathematical modelling is the process of encountering an indeterminate situation problematizing it, and bringing inquiry, reasoning, and mathematical structures to bear, and then communicating the solution derived from the model world back to the real world.

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USE OF DGS TO PROMOTE KINESTHETIC THINKING: A CASE OF LINEAR TRANSFORMATION

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A neuroscience theory suggests that kinesthetic thinking is the very basis of thought. Similarly, Lakoff and Núñez (2000) suggest that dynamism plays an important role in conceptual development. Furthermore, Núñez (2006) argues that mathematical ideas and concepts are ultimately embodied in the nature of human bodies, language and cognition. In this paper, we examine the role that dynamic interactive representations of mathematical concepts plays in promoting kinesthetic thinking. In particular, we report evidence of students' kinesthetic thinking while they interact with a dynamic interactive sketch of the concept of linear transformation.

INTRODUCTION

The impact of technology in teaching and learning mathematics has become a focus since the late 1980s. Technologies such as Dynamic Geometry Software (DGS) enable the designing of models to represent the relationships between symbolic and geometric representations of concepts. The dynamic and interactive features of DGS provide grounds for students to explore these relationships, perform multiple actions, and generate a large number of examples effortlessly (Hollebrands, 2007; Mariotti, 2000). The majority of research in this area has focused on the design of tasks and milieu, learners' use of technological tools and its relationship to construction of knowledge , and the role of representations (symbolic and geometric) of concepts (Hollebrands, Laborde & Straber, 2008). Given that representations are fundamentally dynamic, representing relationships and behaviours over time, and the importance of the role of time and motion in mathematical thinking (Thurston, 1994) we have become interested in investigating the effect of dynamic mathematical representations on students' modes of thinking. We assume that the dynamic representations will change students' ways of thinking about concepts and hypothesise that this change can facilitate conceptualization.

THEORITICAL BACKGROUND

More recent research, drawing on neuroscience theories, suggests that kinesthetic thinking is the very basis of thought. According to Seitz, "we think kinesthetically" (p.24) and there exist three central cognitive abilities for the bodily basis of thought (Seitz, 2000). The first is motor logic and organization that deals with the articulation and ordering of movements. The second is kinesthetic memory that enables human to think in terms of movement by mentally reconstructing objects and imposing motion on and positioning them in space. The last, kinesthetic awareness is having information about our body and objects coming in contact with our body (Seitz, 2000). These theories challenge the inattention to (and sometimes ignorance of) the role of time and motion in mathematical thinking: not only does mathematics tend to detemporalise mathematical processes (Balacheff, 1988), but several mathematicians have expressed discomfort at the idea of moving objects (see Frege, 1970).

Similarly, Lakoff and Núñez (2000) suggest that dynamism plays an important role in conceptual development (Lakoff and Núñez, 2000). Furthermore, Núñez (2006) argues that mathematical ideas and concepts are ultimately embodied in the nature of human bodies, language and cognition. He has also shown that static objects can be unconsciously conceived in dynamic terms through imposing real or fictive motion; as he illustrates cases of the concepts of limits, curves and continuity. However, ways of understanding concepts and objectification that come through a range of multimodal sensations and experiences vary from individual to individual and may not necessarily include motion modality, specifically school mathematical concepts. But, research on university students' and mathematicians' ways of thinking reveal that their thinking involve a dynamic nature (Zazkis et al., 1996; Burton, 2004). Furthermore, a recent study shows the evidence of kinesthetic memory in mathematicians' thinking about mathematical concepts (Sinclair and Gol Tabaghi, 2009). These studies motivated us to further probe university students' modes of thinking in the presence of dynamic representations of concepts. More precisely, we want to investigate whether dynamic representations of concepts affect students' ways of thinking and promote kinesthetic thinking.

In order to probe their ways of thinking, we study students' linguistic expressions and their use of affordances of technology. We also refer to McNeill's gesture classification to analyse their hand and arm movements. Recent research, however, has shown that speech and gesture are two facets of the same cognitive linguistic reality. In particular, research claims that gestures provide complementary content to speech content (Kendon, 2000) and that gestures are co-produced with abstract metaphorical thinking (McNeill, 1992). From a mathematics education perspective, gestures play an important role in cognition and can contribute to creating mathematical ideas (Arzarello et al., 2005, 2007; ESM special issue 2009). Therefore, gestures might be another way of seeing evidence of kinesthetic thinking (Núñez, 2006). In particular, given the motion aspect of gesturing, we hypothesis that analysing gestures will provide more insight into the kinesthetic thinking processes.

RESEARCH CONTEXT AND PARTICIPANTS

In our larger study, we asked students to interact with sketches designed on a variety of linear algebra concepts such as vectors, spans, linear transformations and eigenvectors. We designed our interviews using a set of tasks aimed at eliciting students' ways of thinking while they were interacting with sketches. We interviewed five students who enrolled in a linear algebra course at the time of interviews. Each interview lasted between 30 and 40 minutes. Interviews were videotaped and transcribed. In this paper, we only present the analysis of two interviews focusing on the concept of linear transformation. Our participants, Julia and Mary, were second year university students who volunteered their time to participate in our study. Excerpts from their speeches and snapshots of their gestures are presented to reveal the evidence of kinesthetic thinking.

ANALYSIS OF STUDY

We refer to Seitz's theory of kinaesthetic thinking and the three central cognitive abilities for bodily basis of thought to analyse the participants' linguistic and non-linguistic expressions. We also use McNeill's gesture classification and transcription to analyse participants' hand and/or arm movements as they interact with sketches.

INTERVIEW SKETCH AND TASKS

First, participants were asked to describe linear transformation. Second, they interacted with the linear transformation sketch (see figure 1) that presents a linear transformation of a non-symmetrical object (blue-coloured F) under the matrix of transformation T to describe their observations.

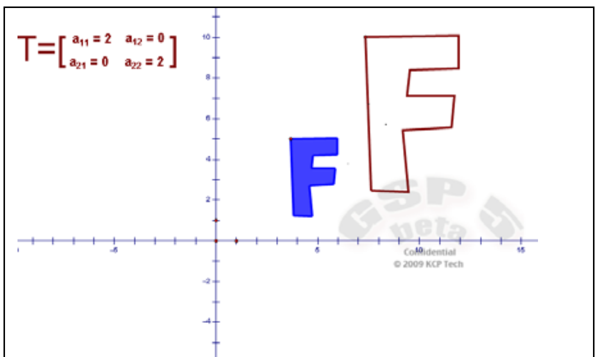

Figure1. linear transformation sketch

They were also asked to predict the image of F under different transformations such as $T = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, $T = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, and $T = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$. After they described their thoughts, they were able to change the components of transformation matrix and visualize the image of F, so that they could reflect on their initial thoughts.

ANALYSIS OF SPEECH AND GESTURES: LINEAR TRANSFORMATION

Julia describes linear transformation as "changing vectors into something else so like mapping vectors to something else so could be a matrix or anything else". It seems that she recalls linear transformation in terms of matrix multiplication. She doesn't have the idea that any object can be transformed, expect perhaps a vector or a matrix. It's interesting that she starts with "changing of vectors", instead of the change of the object to which the vectors are applied. This shows that she has developed a procedural understanding of the concept of linear transformation as reported in the literature (see Stewart, 2008).

After her description, she is given the sketch (see figure 1) that represents the transformation of an object under the transformation matrix $T=\begin{bmatrix}2&0\\0&2\end{bmatrix}$. By looking at the sketch, she starts describing her observations and points out that the pre-image F is mapped into another F that its area is four times more than the area of the pre-image F. It is interesting that she compares the area of image and pre-image. She is then asked to predict transformation of pre-image F under the transformation matrix $T=\begin{bmatrix}3&0\\0&3\end{bmatrix}$. She predicts that $T=\begin{bmatrix}3&0\\0&3\end{bmatrix}$ maps the pre-image F into another F that would have three times expansion in its dimensions and nine times enlargement in its area. She is asked to interact with the sketch and change the values of T into $\begin{bmatrix}3&0\\0&3\end{bmatrix}$ to experiment her predication. To find out the image of F under $T=\begin{bmatrix}2&1\\0&2\end{bmatrix}$, she whispers as she does mental calculations. She further uses her right index figure (see figure 2) to write up the calculation.

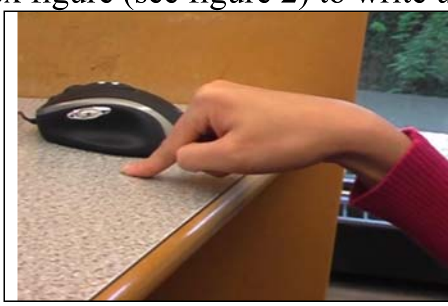

Figure 2. Julia's use of her right index figure to write up while she does mental calculation

When we asked her to explain her thought processes, she said "I am just thinking of as, um so that would be the matrix and then say that I have my x_1 and x_2 that is my vectors so then if I have that vector times the matrix". She considers an arbitrary vector $\begin{bmatrix}x_1\\x_2\end{bmatrix}$ and does matrix multiplication mentally. Her right index finger accompanies her thought processes and her speech which could be an evidence of kinesthetic thinking. She further says "so if you multiply them would be two x_1 and then plus one x_2 [$2x_1+1x_2$] and the bottom would be zero x_1 plus two x_2 [$0x_1+2x_2$]". She completes the matrix multiplication and says "then I do not know how that changes the figure but in what way." Her discourse shows that she has developed operational knowledge of linear transformation concepts and become fluent in computational skills, but she could not articulate on the image of F under the given transformation matrix, $T=\begin{bmatrix}2&1\\0&2\end{bmatrix}$, and so the importance of linear transformation concept. After predicting, she was very keen to use the sketch to find out about the image of F. Her gesture, using her index finger to write up the calculation, was a dynamic gesture and could be classified as an

iconic or a metaphoric gesture using McNeill's classification.
We further asked her to predict the image of F when a_{12} become -1, i.e. $T=\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$. She used the representation that the sketch presented for $T=\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ to reason about the image of F when $T=\begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$. As she says "So now it is slanted that way, -1 would be the slanted other way" as she gestures; moving her right hand from right to left (see figure 3). This shows the benefit of using interactive sketches in promoting kinesthetic thinking. Furthermore, her gesture could be classified as an iconic gesture.

	
Figure 3. shows Julia's gesture as she describes a shear linear transformation	Figure 4. shows Julia's gesture as she refers to the previous image of F on the sketch

Our next prompt was asking about how the image of F changes under $T=\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.
She again refers to the previous representations on the sketch and says "so two one zero two was like that before". Her gesture (see figure 4) shows the direction of the image, that is an evidence of kinesthetic thinking.
She hesitates and further says "it would be more fat, um, I do not know". We asked her to interact with the sketch and change the components of the transformation matrix into $T=\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ to see the image of F. She is at first surprised when she sees that the image is a line, however, she immediately says "oh yeah because they are linearly dependent". The sketch enables her to recall the concept of linear dependency so she justifies the outcome of the transformation under $T=\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$.
Mary, our second participant, describes linear transformation as "like taking one vector and something, and transforming it using a set of like rules so then becomes something else." She recalls linear transformation in terms of transformation of a vector into something else. Although the use of term "something else" gives a sense that she is presented with the abstract idea of being able to transform anything, but she does not have any example of what could actually be transformed. While she interacts with the sketch, she points out that the

transformation matrix, $T = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$, maps the pre-image F into another F that is bigger and wider. In response to our next prompt (how would the image of F change under transformation $T = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$), she changes the transformation matrix into $T = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ and visualizes the image of F. She immediately recalls the name of this kind of transformation saying "oh, it is a shear transformation". She recalls shear transformation, but she doesnot gesture at that time. While changing the components of transformation matrix, she realizes that $T = \begin{bmatrix} 0 & -2 \\ 0 & 3 \end{bmatrix}$ transforms F into a line and attempts to justify this transformation. She goes on and says "oh, okay because a_{11} and a_{21} are zero" and gestures a line as she moves her right hand horizontally from right to left (see figure 5). To predict the image of F under $T = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$, she says "twice as large as now and sheared" and she gestures the direction of the image of F (see figure 6).

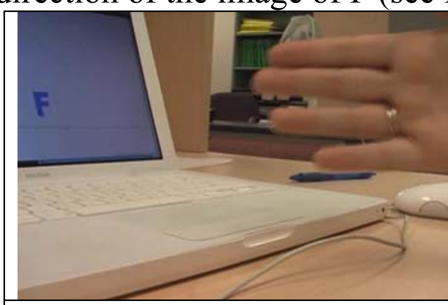	
Figure 5. shows Mary's hand as she gestures a line	Figure 6. shows Mary's gesture to represent the slanted image of F

This again is an evidence of kinesthetic thinking. It seems that the use of interactive sketch promotes kinesthetic thinking. Although, her arm and hand depict a right-slanted image of F, the image would be left-slanted. After her prediction, we asked her to interact with the sketch and reflect on her thinking to evaluate her initial prediction.

DISCUSSION AND REFLECTIONS

The results of our study indicate that: first, dynamic interactive representations of concepts promote kinesthetic thinking. Second, students use a variety of gestures to express their thinking about concepts. Third, the sketch enables them to reflect on their operational understanding so that may possibly support structural understanding. It could also act as a scaffold to help them re-construct their understanding. In this case, the instant feedback from the sketch played a significant role in enabling them to reflect on their thought processes. We also noticed that their use of gestures is slightly different from thoes of mathematicians. Students'gestures are co-produced with their mathematical thinking as they are engaged in prediction tasks. In contrast, mathematicians, in some cases, described concepts using metaphors, as they gestured same metaphors (see Sinclair and Gol Tabaghi, 2009). In both cases, we had difficulties distinguishing iconic from metaphoric gestures, but this is beyond the scope of this paper. We also identified several instances of deictic gestures in students' non-linguistic expressions trigered by the use of the sketch; they pointed to the sketch as they voiced their thinking. We suggest that the use of sketches to enhance gesturing and to promote kinesthetic thinking warrants further study.

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SANDING THE LENS: THE NARRATIVE OF A TASK FROM THE INITIAL PLANNING TO THE UNDERGRAD STUDENTS' CONCEPTIONS OF INEQUALITIES

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This report comes from a broader study that investigates undergraduate students' conceptions of inequalities. It comprises the design and refinement of a task with the purpose of making it more engaging for students and of getting results that are more transparent for author's interpretation. The narrative follows one task that has been developed, implemented, interpreted, refined, implemented again and used in the process of deriving university students' conceptions of inequalities. The interpretation of student's work is framed as an emergence of the lens CONCEPTIONS OF INEQUALITIES. The lens is intended to magnify students' work on inequalities for the researcher to better spot the various conceptions and interpret understanding of inequalities.

The focus of this study is shaped around the conceptions of inequalities held by university students as evidenced in their work on one refined task: learner-generated worked examples of inequalities. The concept image-concept definition (Vinner and Tall, 1981) and the theory of variation as introduced by Marton (1981) and refined and adapted to mathematics by Watson and Mason (2005) postulates a framework for the initial design of the task and the interpretation of preliminary data. Skemp's framework for understanding mathematics (Skemp, 1976) employed in the initial task and the unit of description in phenomenography (Marton & Pong, 2005) applied in the revised task provide the language for communicating the various conceptions of inequalities grounded in the data.

THE *WORKED-EXAMPLE* TASK AND THE PREPARATION FOR LOOKING AT DATA

The *worked-example* of an inequality task was initially given to a class of FAN students. After carefully looking and interpreting preliminary data, the task was refined and implemented again in another class of FAN and a Math 100 class. In this section I will introduce the initial task, the normative solution and the initial work on sorting, coding and interpreting data. The following sections comprise the preliminary results and the emergence of a framework that will potentially become the lens for a major study.

The task

a) Create a worked example that will show someone how to solve linear inequalities.

b) Is the one example provided in part a) sufficient for someone to learn how to solve inequalities by following your work? Do you think you need to create more examples to demonstrate the full breadth of linear inequalities? If so, how many more examples you think you need?

The normative solution to item a) – constructing a worked example of an inequality – could be an example that incorporates, if possible, all axioms that will help transform a given inequality into an equivalent inequality which are the following:

- Suppose that *a* and *b* are (real) numbers such that $a < b$, and *c* is another (real) number different than zero. Then the inequality $a < b$ is equivalent to:
  1. $a + c < b + c$ (adding/subtracting the same amount from both sides of the inequality)
  2. $ac < bc$ for $c > 0$ (multiplying by a positive number)
  3. $ac > bc$ for $c < 0$ (multiplication/division by a negative number)

as well as the conventions related to writing the solution in interval form and graphing the solution on a number line. For example $-2 + 3x < 6x + 7$ could be a possible response. The worked example follows:

$-2 + 3x < 6x + 7$	- Separate terms containing the variable by adding 2 and subtracting 6*x* on both sides
$3x - 6x < 7 + 2$	
$-3x < 9$	- Divide by -3 and reverse the inequality symbol
$x > -3$	
Solution: $(-3, +\infty)$	- Solution in interval form
	- Graphical representation of the solution

For b) the normative answer could be "I think that a few more examples where *less than* as well as *less than or equal to* are used would benefit exemplifying the different types of intervals necessary for writing the solution." Also, a few examples when inequalities produce no solution or all real numbers as solution will be expected to be found in some papers.

The first sorting of data followed a rubric of anticipated work, similar to a rubric for marking assignments. The rubric comprises five distinct categories of responses (examples of linear inequalities) which were labelled with numbers from 0 to 4. The numbering is inspired by the potential marks for the work, given that the task would have been collected with the purpose of testing students' knowledge of inequalities. As the rubric was being created, the focus was mostly on the following questions: (1) Did the respondent attend to the given task? (2) If yes, what types of examples are presented and how is work accomplished? **Table 1** comprises the anticipated categories of examples. I will call the table 'the initial rubric'.

Category	Description of anticipated work
0	No inequality is exemplified
1	An inequality is given. No attempt is made to solve it.
2	A simple inequality is given. Solving follows the pattern of equations. Some good steps. The solution is incorrect.
3	A simple inequality is given. Solving correctly follows the axioms of inequalities. The solution is correct.
4	A pilot example is given. Solving incorporates maximum variation and aspects related to inequalities, including multiplication or division by a negative number.

Table 1: The initial rubric

BACKGROUND

Examples play a key role in both the evolution of mathematics as a discipline and in the teaching and learning of mathematics. There is an abundance of research that acknowledges the pedagogical importance of examples in learning mathematics (e.g.: Atkinson, Derry, and Renkl, 2000, Watson and Mason, 2005; Zhu and Simon, 1987; Kirschner, Sweller and Clark, 2006, Bills, Dreyfus, Mason, Tsamir, Watson, & Zaslavsky, 2006). Zazkis and Leikin (2007) used examples as a research tool that enabled the researcher to gain insight about students' understanding of mathematics. My study extends on that type of research that involves transferring the responsibility of generating examples to students and the responsibility of learning about students' understanding of mathematics to researchers.

SETTING AND METHODOLOGY

The setting for the study on inequalities is Simon Fraser University and the participants are three classes of students: two FAN X99classes and a Math 100 class. The initial task was given to a class of FAN students. The other two classes – a FAN class and a Math 100 class – worked on the revised task.

FAN X99 (Foundation of Analytical and Quantitative Reasoning) is a non credit mathematics course, designed for students who need to upgrade their quantitative background in preparation for quantitative courses. Number sense strengthening, mathematical reasoning, problem solving and math study skills are the main components of the course. This course is recommended to students who wish to refresh their math skills after several years away from mathematics. The group of students varies greatly in skills level. Math 100 (Precalculus) is a course designed to prepare students for first year Calculus. The course is very condensed and includes language and notation of mathematics; problem solving; algebraic, exponential, logarithmic and trigonometric functions and their graphs. The group of students taking Math 100 is very heterogeneous as well.

The Syllabus of both FAN x99 and Math 100 contain inequalities. In FAN x99, linear inequalities are studied as one independent topic in the second part of the course. In Math 100 inequalities are everywhere: Starting form recalling linear inequalities learned in grade 11, continuing with solving quadratic, polynomial, and fractional inequalities, and crowning with using inequalities to find the domain of logarithmic, irrational, or trigonometric functions.

In the first iteration of the task, data was collected 8 weeks into a FAN class where the subjects had been exposed to problem solving, discovery and making connections, rather than lecturing. Generating examples, mostly in class, was a daily routine. A document camera, which allowed for a student to project for the entire class the notebook with the work, was part of the class' equipment. Samples of different normative examples or various solutions to the word problems were presented, projected to the class, discuss and interpreted on a daily basis. The extreme or peculiar examples, the counter examples as well as the examples that were not attending to the task were not ignored: These examples, usually, created some of the most rewarding teachable moments. The survey was given as an open book, individual, half an hour, class work. In all there were 43 participants.

When teaching-learning mathematics follows the transmission-assimilation metaphor, students "strive to make sense of the examples they are offered, use the terms their teachers use to describe generalities, and ultimately are expected to construct new objects and understandings that match those of their teachers" (Watson and Mason, 2005). When the focus is on 'using learner's experience' in revealing the general in particular rather than the definition of a class of objects, the students were invited to think and make connections, which will benefit understanding. With this in mind, the work done with my students prior and during the collection of data was twofold: one was to collect data that is transparent of students' understanding and the other one was to help students construct their concept of inequalities.

The second iteration of the task was given to a FAN class, which had been exposed to a similar treatment as the other FAN class as well as to a Math 100 class. Math 100 is not a seminar as FAN x99; it is a 3-hour per week lecture format. The survey was implemented at the end of a lecture and the students were given 15 to 25 minutes to work on it. For Math 100 the preparation for the task was minimal – prior to writing the survey, the students had a review on inequalities and they had an assignment which comprised mostly of linear and rational inequalities. The students were not exposed to generating examples prior to the task.

To the best of my knowledge, no published study on mathematical inequalities used learner generated examples as a source of data. Usually, a solve-an-inequality task does not provide much variation in students' solutions. Students' work very often reproduces the procedure learned from the teacher or from the examples offered by the textbook. Research on inequality with data coming from solving inequalities tasks show some variation in students' errors, but not much variation in the example space that students have to access to solve the task. For example, when given a task which reads “Can x=3 be a solution to an inequality?” (Tsamir and Bazzini, 2001, p.1) the typical answers could be (1) No, the inequality results in inequality not in equality or (2) Yes, inequalities of the form ≤ can result in equality (Tsamir and Bazzini, 2001). Asking them to generate an inequality of some sort and then to work it out such that someone who follows this work to be able to learn how to solve that type of inequality, students are invited to be creative, to search their personal example space of inequalities, to access different registers of presenting inequalities, to connect the different snapshots that create the concept image for that concept, therefore to think. Showing a correctly solved solution is not guaranty of students’ understanding of inequalities; they could have memorized procedures or followed step by step algorithms. I argue that this novel approach of collecting data will allow seeing more of the students understanding of inequalities solving inequalities tasks. Starting from nowhere often involves undoing, which is harder than doing since it does not start with a memorized step 1 and then a step 2, and so on. This very fact is reflected on students’ responses as well as on their reflection on the task. Here is excerpt from one participant’s notes which acknowledged the difficulty of a construct-an-example task:

> For me question number 1 [the “solve the following inequalities” items] was more clear in the sense that [an inequality] is there & we have to [solve it]. I got kind of lost with questions 2 & 3 because we’re not used to coming up with questions [examples] so question 3’s instruction wasn’t something we’ve seen in the past. I was more successful in answering question 1 because I was used to seeing that kind of question…

The student acknowledged the fact that generating examples require more than solving a problem by analogy with a previous work. Seeing something in the past, retrieval from studied examples makes a task more approachable than constructing one’s own example. This could be evidence that the task under discussion here is complex and the results will give enough variation to capture respondents’ thinking.

RESULTS AND DISCUSSION

As mentioned earlier, the first sorting of data followed ‘the initial rubric’. The following two questions guided the rubric creation: Did the respondent attend to the given task? If yes, what type of example is presented and how the work on it is carried? **Table 2** comprises the anticipated categories of examples as well as the percentage of respondents falling in each category:

Category		Description of anticipated work
0	38%	No inequality is exemplified
1	0%	A simple inequality is given. No attempt to solve it.
2	16%	A simple inequality is given. Solving follows the pattern of equations. Some good steps. The solution in wrong.
3	36%	A simple inequality is given. Solving follows the axioms of inequalities. The solution in good.

4 Table 2	10%	A pilot example is given. Solving incorporates maximum variation and aspects related to inequalities.

Table 2: The initial results

As visible in the percentages that accompany the categories of data, even though the task was open book and the class was prepared for the type of work required to accomplish it, 38% of the respondents did not give any example of an inequality. Some of them exemplified a linear equation in one variable; others exemplified linear equations in two variables.

The second sorting of the data was done with other two questions in mind: (1) What are the recurrent themes present in the data? and (2) Is the individual's understanding of inequalities visible in the data?

At the second scanning of data, five different categories emerged again, the same number of categories as in the anticipated answers. Pondering further each category, I was able to identify the recurring idea that helped the classification and thereby decided that 5 is the number of visible variations in the data. My next step was to look if there is a one to one correspondence between the categories of the first scanning and the classifications that emerged at the second round. For the categories with scores 3 and 4, I could make a one to one correspondence between the first and second stage of interpretation, but some papers from category 1 moved to a different pile labelled 'contextual understanding of inequalities'.

Skemp's (1976) classification of mathematical understanding provides the language to describe the five levels of understanding emerging from data. **Table 3** comprises the emergent themes and qualifies the level of understanding, from misunderstanding to the higher level of relational understanding.

Category	Level of understanding	What the example was telling about inequality
0 32%	Misunderstanding - A different concept, such as a linear equation with two variables exemplified	*Inequalities* have foreign representations - images of other concepts, such as linear equations in two variables incorporated in the concept image of *inequalities*
1 16%	Traces of procedural understanding of *equations*	*Inequality* is perceived as some sort of equation, thus the $<$ sign is replaced by the = when solving the example
2 6%	Contextual understanding of *inequalities*	*Inequalities* describe real life situations, thus their examples focus more on the context rather than the concept of inequality

3 36%	Procedural understanding of *inequalities* Traces of relational understanding of *inequalities*	*Inequalities* have special behaviour when multiplied or divided by a negative quantity - focus on different representations of *inequalities* as well as on the particular aspects that separates equations from inequalities. The axioms of transforming inequalities into equivalent one are correctly used.
4 10% Table 3	Relational understanding of *inequalities*	*Inequality* is a mathematical concept that must be learned in connection with recognizing the symbols, understanding intervals and some axiomatic preparation - focus on pilot examples that will incorporate maximum variation and aspects related to *inequalities.*

Table 3: Levels of understanding

Phenomenography sees learning "as a change in learners' capability of experiencing a phenomenon", and understanding as the capability of spotting and following a pattern of variation (Åkerlind, 2005). 'Conception' is the unit of description in phenomenography (Marton & Pong, 2005). A third sorting is completed in which the focus of the lens shifted from students' understanding of inequalities to conceptions of inequalities. The new scanning of data followed the questions: (1) What are students' conceptions of inequalities? and (2) What can the conception of inequality tell us about an individual's understanding of inequalities?

The third scanning of data validated the five different categories of understanding that emerged previously and attempted to see the portrayed concept images of inequalities. The framework CONCEPTIONS OF INEQUALITIES was emerging. The descriptions of concepts were firm. However, the conceptions 'names were not yet selected. Possible options for the names were listed.

The second iteration of the task

Liljedahl et al (2007) observed that very often our own approach to the task obscures important aspects of solutions. My focus on learner generated worked example influenced the creation of the task. A worked example on inequalities was seen as a step-by-step demonstration of how to solve inequalities. The normative solutions – aka my own approach to the task – contained not only all the steps for solving it but also some peculiar examples that show that inequalities cannot be easily put in patterns for solving them. However, no respondent addressed this in the initial task. Adjustments to the task seemed to be necessary. Learning from a worked example seems more meaningful that creating a worked example just like that. As such, in the first part of the second iteration of the task, I introduced Jamie, who is taking Principles of Math 11, who needs help with inequalities, and who is going to learn from the example. What do Jamie knows about inequalities at this moment? Scaffolding for Jamie could inform about the concept image of the respondent. Part two of the initial task invites the participants to think if their example covered the whole complexity of linear inequalities and if not to say how many examples would serve that purpose. As mentioned previously, nobody attended to the idea that an inequality can produce an empty set as a solution, for example. All the provided examples ended in intervals. Therefore, I also decided to create a cognitive conflict in part two of the task, to force the respondents to rethink their example to incorporate that aspect of inequalities in their response. The limitations imposed by this paper will not allow a discussion about the understanding informed by part b) of the task. Also, the new task comprised three items, from whom only the third one is the one which is referred to as the second iteration of the task. The other two parts of the survey will be included in the major study.

The refined task:

> 3) You know that the best way to learn something is to teach somebody; therefore you have agreed to tutor your cousin Jamie who is taking Principles of Math 11 this year. You are available for him any time and through any means.
>
> a) You've got a text message from Jamie that reads: "*Missed the class on linear inequalities. I have to do my homework. Don't know how to start. Help me with the steps of solving a linear inequality*." E-mail him back the steps for solving linear inequalities. On the space below show the message as well as your preparation for sending it.
>
> b) Half an hour later an e-mail from Jamie arrives: *"I followed your steps and solved a whole bunch of inequalities. Thanks. Then I attempted this one:* $1-2x>2(6-x)$. *I worked out the algebra and got this* $1-2x>12-2x$ *and then ended up with:* $0>11$. *Here I got stuck. Please help.*" E-mail him back. On the space below show the message you will send to your cousin Jamie. The message should contain your feedback on Jamie's work as well as your input to Jamie's further understanding of inequalities.

Having a lens to magnify the responses, reading and sorting the data from the second iteration were a bit less laborious than the first wave of coding. In general it was easy to fit data into the five categories produced by the first iteration of the task. However, for part a) of the task, in the data coming from Math 100, somewhat 15% of the data contained an aspect completely unanticipated – e-mailing the steps for solving the inequality without being accompanied by an example. After rethinking over this new aspect, the issue was easily addressed by correlating respondents' work with another portion of the individuals' surveys. Thus, those papers went either in category 1 or 3, depending on the clarity of the given steps for solving linear inequalities.

CONCEPTIONS OF INEQUALITIES

To use a metaphor, I can say that *the students painted different images of inequality*. Their images were analysed in detail. As a result, five conceptions of inequality were indentified:

- **Conception 0**: *Inequality as an amalgam of images or symbols encountered in a mathematics setting*
- **Conception 1**: *Inequality as a strange relative of an equation*
- **Conception 2**: *Inequality as a tool used in optimization contexts*
- **Conception 3**: *Inequality as a dynamic scale metaphor*
- **Conception4**: *Inequality as seen by mathematicians – a complex mathematical concept that could be expressed in different registers – symbolic, interval, or graphic; and could perform different functions – compare quantities, express and resolve constrains or deduce equality.*

Research on inequalities tried to answer many different questions such as: What is typical correct and incorrect reasoning? What are common errors? What are possible sources of students' incorrect solutions? What theoretical frameworks could be used for analysing students' reasoning about algebraic inequalities? What is the role of the teacher, the context, different modes of representation, and technology in promoting students' understanding? What are promising ways to teach inequalities (Bazzini and Tsamir, 2004)? Studies reported mostly on students' misconceptions on inequalities or on obstacles in understanding inequalities (Linchevski & Sfard, 1991; Bazzini & Tsamir, 2001, 2003, 2004; Tsamir, Tirosh & Tiano, 2004; Boero & Bazzini, 2004; Sackur, 2004). One of the main questions proposed by the Discussion Group meeting at the 1998 PME 22 – What are students' conceptions of inequalities? – is still waiting for an answer. A framework that permits the decomposition of the inequality concept into the structural features that the research participants discern and focus on could help a study that aims to inform about what makes some students better at manipulating inequalities than others. This paper is a snapshot of a process that opens the door to further investigation into learners' understanding of inequalities via conceptions of inequalities.

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THE PSYCHOLOGY AND PHYSIOLOGY OF STRUCTURE IN GEOMETRY: A STUDY IN EDUCATIONAL NEUROSCIENCE

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This paper is a synopsis of a doctoral dissertation in mathematics educational neuroscience. It presents a psychological model for geometrical thinking and learning and its correlative physiological model. With respect to the latter, geometrical concept formation belongs to the parietal lobe of the cerebral cortex. The cerebellum has a functional role in directing attention to those aspects of a geometrical percept that are essential to the concept under consideration. The theoretical framework is embodied cognition, as informed by Spinoza, which allows coherent integration of psychological and physiological aspects of geometrical reasoning. A conclusion of the research is that decontextualization of geometrical concepts may facilitate student learning of these concepts.

INTRODUCTION

The aim of this paper is to present a summary of some of the main points of my doctoral dissertation (Handscomb, 2009). Much of the content herein is adapted directly from that document. According to Schoenfeld (2000), research in mathematics education can be pure or applied. The purpose of pure research is to "understand the nature of mathematical thinking, teaching, and learning" (p. 641), whereas applied research addresses the efficacy of particular pedagogical techniques. According to Schoenfeld, without a deep understanding of the pure aspect of mathematics education,"no sustained progress on the 'applied front' is possible" (p. 641). My primary motivation has been to "understand the nature of mathematical thinking, teaching, and learning," in Schoenfeld's words, specifically with regard to geometry. My current research builds on Handscomb (2005), which presents a theoretical model for the process of geometrical reasoning.

Initially, my objective was to utilize the methods and insights of cognitive neuroscience to substantiate the model in Handscomb (2005), in accord with the emerging research paradigm of educational neuroscience (e.g., Campbell, 2006). The psychological model should have implications with respect to neural activity, and if these implications are borne out, then the psychological model would be substantiated.

It became clear, however, that the data and methods of cognitive neuroscience cannot be utilized without a neurophysiological model for geometrical reasoning that corresponds to the psychological model. If such a model can be developed, then the psychological and neurophysiological models can be mutually informing and constraining. This mutuality between the subjective, psychological approach and the objective, physiological approach is the research method of educational neuroscience (cf., Varela, 1996, 1999).

A large part of Handscomb (2009) is an attempt to develop such a neurophysiological model for geometrical reasoning. Although I do present one major implication for the teaching of mathematics, the major thrust of my research, which is grounded in cognitive neuroscience, may appear distant from classroom practice. Nevertheless, I claim that my research is foundational with respect to understanding the nature of geometrical thinking and learning. It belongs, according to Schoenfeld's (2000) distinction, to the realm of *pure* mathematics education research.

THE EMBODIED MIND

A necessary metaphysical justification for the research framework of educational neuroscience is that there should be an intelligible relationship between body and mind—i.e., between the psychological and neurophysiological models for geometrical reasoning.

The relationship between body and mind that I have adopted is the theory of embodied cognition, as developed in Varela, Thompson, and Rosch (1991), extended by Campbell (2001, 2003), and informed by Spinoza (1677/1996). The material in this section has been adapted from Handscomb (2007) as well as Handscomb (2009).

A principal idea behind embodied cognition is *double embodiment*: we are beings who exist in the world; but also we are beings who perceive the world. According to Merleau-Ponty (1945/2005), whose phenomenology was a major inspiration for embodied cognition, "The world is inseparable from the subject, but from a subject which is nothing but a project of the world, and the subject is inseparable from the world, but from a world which the subject itself projects" (pp. 499-500). Campbell (2001) adds, "We are both embodied within the world and the world is embodied within us: *we are the world within itself*" (p. 6, author's italics).

Note that the "world that the subject projects" and the "world that projects the subject" should both be regarded as epistemic categories, mind and body, respectively. Idealism and realism are matters of perspective, corresponding to Husserl's phenomenological and natural attitudes, respectively (Campbell, 1998; Campbell & Handscomb, 2007). The *subjective* mental world of cognitive function flows from the idealist stance, and the *objective* world of physical activity that can be observed and measured flows from the realist stance.

The embodied point of view takes the body as the locus of experience. The very fact of being embodied and thereby embedded in the world, taking the natural attitude, means that the organism receives external stimuli that change the internal milieu. A change in the internal milieu in turn changes the way the organism acts, altering the subsequent stimulus it receives. Varela et al. (1991) describe this in poetic terms as "organism and environment enfold into each other and unfold from one another in the fundamental circularity that is life itself" (p. 217). Organism and world in which it is embodied are a single, interactive structure.

It would be easy to generate confusion at this point. The "fundamental circularity" and "exchange of stimuli and responses" should be regarded as entirely within the objective domain. Physical activity does not *cause* subjective cognition, or vice versa. Fuster (2003) writes, "A cognitive order, no matter how it is construed, cannot be causally related to a brain order," (p. viii). The naïve belief is that the physical event of stubbing one's toe "causes" the psychological event of pain. It does not—stubbing one's toe *does* cause objective nociceptive activity in the nervous system, but that activity is not in itself pain. Pain is a subjective experience belonging to the epistemic category of mind rather than body.

Cognition, in the natural attitude, is defined by Varela et al. (1991) as "Enaction: A history of structural coupling that brings forth a world" (p. 206). The *history of structural coupling* is the dance between organism and the world. They move together in perfect synchrony, neither taking the lead, but both moving to the same melody. The boundaries of the organism do not stop at the physical shell of the body, but include organs, blood, and nerves—the body is itself part of the world that it enacts—and therefore cognition arises also in the *body's interaction with itself*. In this regard, the most characteristically human aspects of cognition, it may be assumed, are manifested in the neural activity of the brain.

The view of embodied cognition, in which the subjective and objective are epistemic categories, is not, according to Campbell (1993), the original understanding of Varela et al. (1991). They appear, in fact, to endorse, albeit implicitly, a Cartesian dualism. Campbell (2001) suggests modifying embodied cognition with a monist ontology.

Embodied cognition, in this modified sense, a monist ontology and dual epistemology, is a *neutral monism*. Handscomb (2007) reviews the remarkable similarities between Campbell's (2001, 2003) formulation of embodied cognition and the classical neutral monism expounded by Spinoza (1667/1996). According to Spinoza, the world consists of a single substance, which can be known in two ways, thought and extension. These two attributes correspond, respectively, to the subjective and objective epistemic categories.

According to Spinoza (1667/1996) there is a precise correlation between the domains of thought and extension: the "*order and connection of ideas is the same as the order and connection of things*" (E2 P7, original italics). Spinoza, in other words, argues that the structures of the two epistemic categories are identical.

Spinoza's metaphysics and theory of knowledge, I believe, is an unacknowledged precursor of embodied cognition. Spinoza was hundreds of years ahead of his time in this respect. Cognitive neuroscientists are only now beginning to acknowledge Spinoza's insights (e.g., Damasio, 2003).

In order to help situate my research within the academic discipline of mathematics education, it will be informative to compare my approach with the cognitive metaphor understanding of embodied cognition, as represented in Lakoff and Núñez (2000), which has attracted considerable attention in mathematics education.

According to Lakoff and Núñez (2000), "Mathematical objects are embodied concepts—that is, ideas that are ultimately grounded in human experience and put together via normal human conceptual mechanisms, . . . [such as] conceptual metaphors" (p. 366). For example, a collection of physical objects, metaphorically, may be regarded as a number. Mathematics is created by human agency on the basis of these conceptual metaphors; mathematics is *embodied* in the world in the form of conceptual metaphors. Embodiment is understood, therefore, in a gross, behavioural sense instead of embodiment in neural activity.

Metaphysical assumptions, such as those that I make with respect to the relationship between mind and body, appear not to be present in Lakoff and Núñez (2000). Reading their work, one has the sense of a Cartesian dualism, in which mind observes action in the external world and uses these observations, by means of metaphor, to compose mathematical ideas.

GEOMETRICAL REASONING

Mathematics is concerned with certain *structural* aspects of perception. Geometry, in particular, is concerned with structural aspects of *visual* perception. There may be other structural aspects of visual perception, of more relevance to the artist, for example, than the mathematician, so that geometry does not exhaust the structure of visual perception. Moreover, geometry is not delimited by the structural aspects of perception, but entails also procedures, such as logical reasoning or arithmetical reasoning, which engage cognitive functions other than perception. However, geometry is the mathematical science of visual perception par excellence, and I am primarily concerned with the intersection of geometry and visual perception.

With which structural aspects of visual perception is geometry primarily concerned? Euclidean objects such as points, straight lines, and circles still form the backbone of the high-school geometry curriculum (Handscomb, 2005). These objects and their combination and interaction are the focus of my discussion.

Now, a triangle in perception will have various *incidental* properties, such as colour, size, orientation, and shape. These properties are irrelevant to its mathematical triangularity. On the other hand, the property of having three sides is *essential* As far as the triangle is concerned, incidental properties have nothing to do with the geometry of the triangle in itself. If incidental properties of the triangle remain at the forefront of perception, then mathematical reasoning risks confusion and inaccuracy. A goal of geometry education should be to enable students to attend to those properties of the percept that are essential.

I refer to perception of structure, in the sense above, as *schematic perception*. I argue at length in Handscomb (2009) that the famous phrase “seeing the general in the particular,” from the seminal paper in mathematics education by Mason and Pimm (1984), may be reinterpreted as schematic perception.

Schematic perception is a psychological idea. According to the framework of embodied cognition outlined in the previous section, it should have a neurological correlate in the activity of the brain.

OUTLINE OF THE THEORY

Concepts, as understood herein, are associations of percepts, such that each particular percept instantiates the concept. The concept is nothing more. Geometrical reasoning is the main focus of the dissertation, and therefore *visual* concepts and percepts are pertinent to the discussion. Visual percepts are specific and particular in spatial terms, whereas visual concepts are non-specific and general in spatial terms.

In Handscomb (2009) I investigate the neurological interpretation of visual concepts, visual percepts, and visual properties, according to current research in cognitive neuroscience. Visual properties correspond to cognitive networks of the occipital cortex; visual percepts to cognitive networks centred in the temporal lobe, and visual concepts to widely distributed cognitive networks centred in the parietal cortex.

At any moment in cognition, concepts are resolving to percepts. Equivalently, percepts are being associated to concepts. There is a kind of “standing wave” of cognition that is simultaneously top-down and bottom-up, leading from the general to the particular in one direction and from the particular to the general in the other direction. My argument presupposes the cognitive network structure of the cerebral cortex (Fuster, 2003) and the philosophy of duration (Bergson, 1896/2005).

Concepts, as associations of percepts, are fuzzy, indistinct, and inadequate for mathematical purposes. A further level of processing is required in order to produce the crisp, pure concepts that are the raw material for mathematical reasoning. Indeed, those aspects of the concepts, and the percepts to which they resolve, that are *essential* for the mathematical purposes must be attended to, and those aspects of concepts and percepts that are *incidental* to the mathematics must be suppressed from attention. I argue that this process must happen simultaneously in the concept and the percept to which it resolves. The resulting, pure concepts and percepts are *schematic*.

Handscomb (2009) contains several interlocking arguments with respect to the cognitive neuroscience of the cerebellum. The conclusion is that a functional role of the cerebellum is to schematize those visual concepts of the parietal cortex. The cerebellum is a neural structure that, traditionally, is not noted for its contribution to higher cognitive function. In fact, textbooks on the brain emphasize the idea that the cerebellum is responsible for facilitating smooth, efficient motor behaviour. However, in the latter part of the twentieth century research evidence, and the accompanying explanatory theories, began to accumulate that the cerebellum was indeed involved in areas of cognition other than motor behaviour. The cerebellar schematization hypothesis belongs to this trend.

Concept formation in the cerebral cortex may be regarded as *generalization*, whereas the action of the cerebellum may be regarded as *abstraction*. The theory sheds light on the relationship between these two terms.

IMPLICATIONS FOR MATHEMATICS EDUCATION

Although my work is largely a theoretical *pure* mathematics education research, there are implications for actual classroom practice. One of these is that the decontextualization of geometrical concepts will facilitate student learning of these concepts, where *decontextualization* refers to representing the mathematics as starkly and purely as possible, without incidental distractors.

After all, an important component of geometrical reasoning is schematic perception. It seems to be intuitively obvious that if a mathematical concept has already been "schematized" to an extent in the very manner of its presentation, then schematic perception is facilitated. In Handscomb (2009) I use an argument based on the cognitive neuroscience of the cerebellum to demonstrate that schematization by the cerebellum is indeed facilitated if mathematical concepts are decontextualized.

There is substantial research in mathematics education both for and against the notion of decontextualization. Two big ideas in mathematics education theory are situated learning and constructivist learning. I argue in Handscomb (2009) that there are concerns with respect to the universal applicability of these ideas.

On the other hand, some mathematics education research appears to favour decontextualization as a pedagogical tool. I will mention one such study here. More are cited in Handscomb (2009). Kaminski et al. (2008) experimented in teaching the concepts of group theory to students by means of concrete (i.e., contextualized) representations and by means of abstract (i.e., decontextualized) representations. The learning of their subjects was enhanced with the abstract representations. According to the authors, "concrete information may compete for attention with deep to-be-learned structure (*6-8*). Specifically, transfer of conceptual knowledge is more likely to occur after learning a generic instantiation than after learning a concrete one (*7*)" (p. 454).

CONCLUSION

As far as possible, I have developed a rigorous, foundational approach to understanding geometrical thinking and learning. An application of this pure mathematics education research is the apparent efficacy of decontextualization. In a nutshell, given the metaphysical assumptions, and aspects of the cognitive neuroscience of the cerebral cortex and the cerebellum, decontextualization may in certain circumstances facilitate geometrical learning.

On the other hand, my research does not take into account social, cultural, and linguistic factors. In any given situation, one or more of these factors may outweigh the benefits of decontextualization. According to Schoenfeld (2000), questions that ask whether one pedagogical technique is better than another "tend to be unanswerable in principle" (p. 642).

My work so far has been theoretical in the sense that I have not conducted empirical research. However, I do make extensive reference to empirical studies in cognitive neuroscience. Moreover, the main motivation behind this study was to develop a neurophysiological model for geometrical reasoning that could form the basis for empirical research. This has now been done, and the next step should be to substantiate (or falsify) the conclusions empirically.

The theoretical framework and methods of educational neuroscience can enrich mathematics education research and provide additional validation for theories of mathematics education.

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TENSIONS RELATED TO COURSE CONTENT IN TEACHING MATH FOR TEACHERS: THE CASE OF ALICE

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Instructors of mathematics content courses for prospective elementary teachers are influenced by many (sometimes) competing factors as they strive to meet their goals for their students. Via an analysis of three episodes that occurred during an interview with one such instructor, this report seeks to illustrate some of the tensions that these instructors operate under as they make decisions related to course content.

INTRODUCTION

Prospective elementary school teachers are often required to take a mathematics content course before or as part of their teacher preparation programmes. The recent report of the National Mathematics Advisory Panel (2008) has recommended that these courses be specifically tailored to the needs of prospective teachers, and that they be taught by mathematics instructors in mathematics departments. Such courses are already offered at many institutions, and though their course titles vary, will be referred to here as "Math for Teachers" (MFT) courses.

In her 2002 plenary address to the PME-NA, Ball (2002) commented that "we have not put in the foreground the "who" of teacher learning as often as we might". This paper reports on a small portion of a larger project whose aim is to understand who teaches these Math for Teachers courses; specifically how they perceive and respond to their role(s). The focus here will be on one of these instructors and the tensions that she operates under as she tries to make decisions on content for her course.

BACKGROUND AND RELATED RESEARCH

There is a vast amount of literature which makes recommendations for the knowledge, beliefs and attitudes that teachers of elementary school mathematics should have (e.g. Ball, Lubienski, & Mewborn, 2004; Philipp, 2007). Although policy documents (e.g. Greenburg & Walsh, 2008) provide the clearest mandate for MFT courses, there is little advice or agreement on how to set priorities within the long lists of qualities they present. Furthermore, the extent to which MFT instructors have contact with this literature, either directly or indirectly, is unclear.

In fact, very little research has been done on post-secondary instructors in general, let alone instructors of MFT courses (Artigue, 2001). However, understanding these instructors is a crucial component in understanding the role they play in teacher preparation. As post-secondary instructors they enjoy considerable autonomy in their classrooms, and as a result MFT courses-as-delivered can vary significantly (Oesterle & Liljedahl, 2009).

This report will offer a glimpse via one case study into some of the tensions encountered by MFT instructors when making course content decisions. The notion of tensions has been used and recommended by Berry (2007), who engaged in a self-study project related to her own experiences in becoming a (biology) teacher educator. She observed that:

> it captured well the feelings of internal turmoil experienced by teacher educators as they found themselves pulled in different directions by competing pedagogical demands in their work and the difficulties they experienced as they learnt to recognize and manage these demands (Berry, 2007, p. 119).

METHODOLOGY

Data for the larger project was gathered through semi-formal interviews conducted with instructors of MFT courses at post-secondary institutions in British Columbia. Theoretical sampling was used to select instructors who represent a range of years of experience teaching the MFT course, and come from a wide variety of post-secondary institutions. Most of the participants were previously known to me on a professional basis, as I am also an MFT instructor. Member checking was employed to mitigate bias due to my own prior experiences.

In the interviews, participants were asked about their educational backgrounds, their preparation for teaching the MFT course, their aspirations for their students, and their approaches. They were also asked to reflect on how successful they were in achieving their goals. Audio-taped interviews were transcribed and coded using constant comparative analysis (Corbin & Strauss, 2008). Forty-eight focussed codes emerged from the preliminary analysis, which could be tentatively arranged under 8 (not disjoint) themes: instructor identity, tensions, and resources, student knowledge, affect (beliefs and attitudes), orientation to mathematics, orientation to teaching, and classroom environment. Instances of tensions were identified through instructors' expressions of ambivalence or even guilt, and through apparent contradictions between reported intentions and described realities.

This report will focus on the data from the interview with Alice (a pseudonym).

ALICE'S INTERVIEW

During Alice's interview she expresses feelings of ambivalence or regret several times, often while she is talking about issues involving course content. In order to set a context for the particular tensions that arise, I will begin with a summary of Alice's background and her goals for her students. This will be followed by a description of three episodes that occurred during her interview which shed light on some of the tensions she encounters.

Like most of the participants in the study, Alice has an advanced degree in mathematics, teaches in a mathematics department at a post-secondary institution, and has not participated in any formal education courses as part of her educational background. She was selected for this study because she is new to teaching the MFT course, having taught it only once, though she has been teaching mathematics for over 10 years.

Early in the interview, Alice was asked about the topics she teaches in her MFT course. In her list of topics she includes: problem-solving; arithmetic operations on whole numbers, integers, rationals and reals (including varieties of models); some geometry; and some probability and statistics. (The topic list is very similar across different post-secondary institutions. In the context of the BC system, this is largely due to the need, for community colleges in particular, to offer MFT courses that will be transferable to the large universities that offer teacher training programmes.)

Further questioning revealed that Alice aspires to teach her students much more than simply how to do the mathematics subsumed under these topics. Her expressed goals in teaching the MFT course reflect a strong concern for addressing students' attitudes and beliefs. She explains:

> For me the main thing would be that they would, kind of, not be afraid of math, and like math, like think it's actually interesting, think it's fun... So I try to make it fun, and, playful, because many of them, I believe, are a bit hesitant about the whole math thing. And they postpone it as late as they can. So the goal was to make them feel a little bit improved on their confidence, for some of them...and for the others, to bring in some interesting, funny, questions. Like, little fun problems to solve, that they won't just have to teach the kids later on to add numbers over and over and over.

Her main concerns are combating her students' mathematics anxiety, building confidence and showing them that there is more to math than repetitive drill exercises. From her responses it is unclear what emphasis she places on improving her students' mathematical knowledge. For many of the other participants, goals for improving students' mathematical skills were mentioned in their goal descriptions. It is notable that this did not occur in Alice's interview.

Episode 1: Probability or Symmetry

This first episode begins early in the interview, when Alice is concluding her description of the course topics. She comments:

> Alice: Probability is there too. Statistics and probability. Statistics is basic. Probability branches...one could branch off. I guess I should not have.
>
> Interviewer: You shouldn't? You mean you did?
>
> Alice: I did. I did a little because I like it, and I shouldn't have. Next time maybe I'll play more with the symmetries and the tessellations...but as I look at elementary school curriculum, the probability is not there at all!

Alice couples her regret over having (possibly) done too much probability with the thought that she may not have spent enough time on symmetry. This competition between different topics for the limited time available in the course is a common tension experienced by many teachers in many courses. More interesting tensions become evident upon a closer examination of her reasons for both her regret, and for choosing to teach more probability and less symmetry in the first place.

Alice uses the elementary school curriculum as a reference to inform what her priorities should be. Indeed she goes on to explain that she has children in elementary school who have done work on symmetry, and in consequence:

> I figured maybe that [symmetry] was more important for me to cover than the actual probability which I enjoy much more than symmetries.

On the other hand, her rationale for teaching probability is that she enjoys it. There may be an assumption here on her part that if she enjoys it, then her students will as well. Teaching probabilities could facilitate her goal of helping her students see mathematics as fun, but she is dismayed that it is not part of the elementary curriculum. There appears to be a tension here between what she wants to teach, and what she feels she should be teaching as indicated by the elementary curriculum.

One other factor emerges as she elaborates on not spending time on symmetry. She laments that "she cannot draw [the symmetries] clean and pretty" and as a result she feels uncomfortable demonstrating them. Instructor expertise (or in this case lack of it) contributes to the tensions in this instance. Her aversion to teaching symmetry is in conflict with her belief that she should be spending more time on it.

She resolves:

> ...probability for me was a much more fun topic, but next time I'll practice a little with my symmetries, and with my drawings... or use a computer.

This pledge to practice more with symmetries before she teaches again, especially since she is still linking the topic of probability with symmetry, might be seen to be a move within the tension towards making a greater effort to spend more time on a topic that she sees as a significant component of the elementary curriculum, despite her inclinations or her aversion.

However, at a much later point in the interview, when asked whether probability and statistics would still be part of the course if she taught it again, Alice responds:

> Probability...I [have] more mixed feelings [compared to Statistics], even though I like probability, but I guess...the quantity and quality balance in there is hard to make, so maybe...[long thoughtful pause]…. I would still do it!

The tensions remain. Although she has determined to teach more about symmetries, there is no indication that she will give up on the topic she loves to teach, despite her acknowledgement of the "quantity and quality balance" dilemma.

Episode 2: Fractions or Tessellations

The second episode arises when Alice is asked whether she does anything with her students specifically because they will be teachers of mathematics one day. In response she describes having a discussion with her students about the elementary school curriculum and its neglect of the important topic of fractions:

> In the curriculum these days, from what I observe with my kids, is that fractions are covered at the end of a school year. For some weird reason, fractions are in June or May. And then…they don't always make it…. Fractions are VERY important! It almost signals to me why we have such a horrible situation with fractions. Everybody's so scared of the fractions, they push them towards the end. So I gave them a big speech that they promise to me, once they get into the workforce, they will fight to move fractions earlier.

The "horrible situation" she refers to is the lack of skill in working with fractions that she encounters regularly as a teacher of post-secondary mathematics. It is unclear who the "they" are who are so afraid of the fractions that "they push them towards the end", but it could be either the teachers themselves or the elementary curriculum designers. From Alice's perspective, not only are fractions neglected, but time is spent on less worthwhile topics:

> … it's frustrating, you know, for there are no fractions.… There are tessellations, they're done for a month, but fractions are done for a few weeks in June. And so I tried to relate … what I think is more important…

She goes on to muse:

> I don't know why I keep talking about tessellations. I guess because it was new to me—tessellations were completely new to me, and, so, they're lovely, but then again, how much of curriculum should be devoted to tessellations....Could [they] be maybe combined with art classes?

Alice experiences no difficulties in deciding the relative merits of fractions vs. tessellations. Her past experience as a mathematics teacher informs her knowledge that most students exhibit poor understanding of fractions, and that this creates difficulties for them. At the same time the fact that tessellations are new to Alice may support her view that they are less important: if she has managed well without them then perhaps the students will not have a great need of them.

The tension arising here is between what she believes students need to know based on her own experience and what she perceives to be emphasised in the elementary curriculum. This is in contrast to the first episode in which she appears to feel she should use the elementary curriculum to inform her priorities in the MFT course.

Not surprisingly, much later in the interview, when asked what she would do differently if she teaches the course again, Alice replies: "I would probably do less tessellations." However, this quickly gives way to an expression of guilt:

> I feel bad about it, because geometry is being so abandoned, but then again, it's a cycle, a vicious cycle. If I teach less, then they will not want to teach it...

Her regret in this case does not seem to be about tessellations in particular, but rather about her perception that geometry is not being given its due. At the same time it reveals the import she attaches to her content choices. What she chooses to teach or not teach sends a message to her students about the relative importance of mathematical topics, and will in turn have an impact on their future practice as teachers. This concern contributes to the tensions around her content decisions.

Episode 3: Knowledge or Attitudes

In a more philosophical discussion near the end of the interview Alice replies to a question about whether her students have sufficient mathematics knowledge to be teachers. Her reply reveals some uncertainties:

> That's a very good question. That, that's a very deep question. Because we don't teach so much math in that class, you know. We don't drill them on whether they can do those fractions. Mmm, so we kind of believe they have the elementary math [...] but how much above it they should be...You see they always say that you should be significantly above what you want to teach, because then you have the big picture, you see the troubles and all that. I don't know that much about that. [...] Many of them are [ready to teach] and many at least will not be afraid to go for it. But I still think there are people who will be afraid....I still think I let people go in there being afraid.

She goes on to comment that those who are still afraid will likely avoid the mathematics as much as possible in their future classrooms, though they may be "wonderful at some other subjects". She laments the fact that there are not specialist mathematics teachers at the elementary school level.

A careful parsing of this passage reveals some of the different forces contributing to the tensions that Alice operates under. As she thinks out loud, her pronoun use changes from "we" to "they" to "I". "We" likely represents her institution as she describes what doesn't happen in the course: there is not much math and no skill drill. As well she seems to explain why mathematics skills are not emphasised: "we kind of believe they have the elementary math". The hedging with "kind of believe" may indicate that she is in fact aware that many of her students do not have those presumed skills. (This is confirmed by other comments in her interview). In the phrase "they always say..." , the "they" seems to point to education experts, or at least to those who have an informed opinion. She understands why having a higher level of mathematics knowledge might be advantageous for a teacher, but she switches to the pronoun "I", and quickly disassociates herself from the "they", asserting that she doesn't know that much about these matters. We see in this episode references to her institution, expectations regarding students' prior knowledge, education lore, and her own feelings of inadequacy with respect to educational issues, all of which inform and influence her content decisions.

Furthermore, although she appreciates the importance of the question, in the end Alice does not decide whether her students are prepared mathematically. She instead shifts to consider whether her students "at least" will not be afraid of the mathematics, even if their content knowledge is not strong. Throughout Alice's interview and in her stated goals, affective concerns are paramount and they emerge here again. She expresses some satisfaction that many will have overcome their fears, although after a pause, she observes that this goal isn't always successful either.

DISCUSSION

Although tensions are often described in terms of two opposing forces, those experienced by instructors in MFT courses cannot be described so simply. In this context there is a plurality of influences which sometimes compete and sometimes combine in various ways to affect instructors' choices.

Episodes 1 and 2 show Alice making choices about particular topics. The elementary school curriculum as she experiences it via her own school-aged children is an important consideration, but it is sometimes in conflict with her personal inclinations and/or aversions (episode 1), or with her knowledge and experience as a mathematics instructor (episode 2). The tension created by these sometime competing influences is amplified by the necessity to make either/or decisions given the limited time she has with students in the course.

Episode 3 alludes to tensions that operate at a more theoretical level, revolving around fundamental questions about what role the MFT course should play in the development of elementary teachers. How important are goals for improving mathematical proficiency relative to goals for building positive student affect? For some participants in the study, affective aims are sabotaged by an emphasis on building mathematical proficiency, while for others the mathematics proficiency is a necessary step towards improving students' attitudes to mathematics. Alice chooses to emphasise affective goals, while at the same time acknowledging that others (education experts?) may not concur. In her particular case, she deals with this tension by deferring authority for deciding these priorities to others at her institution, and suggesting that deficiencies may need to be addressed at the systemic level (i.e. with specialist teachers.)

We see here tensions that operate among goals (e.g. mathematics proficiency, attitudes) and points of authority (e.g. the instructor herself, the elementary curriculum, her institution). Alice operates within the tensions without coming to any definitive resolution, rather she engages in an on-going search for balance.

Conclusion

The tensions described here with reference to Alice's transcript represent only a glimpse into a few of the many influences that affect decisions of MFT instructors. Students' prior knowledge and expectations, along with other peer and institutional factors are among those significant influences that could not be addressed within the space restrictions of this report. Moreover, these tensions do not only arise in the context of content decisions, but also in choices around methodology.

Although the tensions experienced by Alice will not be identical to those experienced by others, a consideration of her situation can improve understanding of the possible tensions that instructors of MFT courses may encounter. With this understanding, these instructors may become more aware of what influences their decisions, laying the groundwork for future research into how they might best be supported in their endeavours.

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THE DESIGNING BRAID: TEACHERS' INTERACTIONS WHILE DESIGNING LEARNING ARTEFACTS

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In this report I propose a theoretical framework that serves to understand conversations and interactions that teachers and educators undertake when engaged in the collaborative design of mathematics learning artefacts—such as a lesson, a class projects, or an assessment instrument. A constructivist grounded theory approach was used in order to develop such a framework. Three theoretical concepts describe the participants' conversations when designing a lesson in this context: (1) anticipating possible students' approaches and struggles; (2) pursuing coherence within the context of the classroom where the artefact will be implemented; and (3) approaching previously selected goals for the artefact. Comparison with other theories of mathematics teachers' development is made in the concluding section, stressing the focus on teachers and educators' interactions of the proposed theoretical concepts in this paper.

TEACHER COLLABORATIVE DESIGN

The generation, improvement, and implementation of learning strategies for mathematics teachers' professional growth has been an increasing issue in the research of mathematics education (Linares, & Krainer 2006). Among a variety of programs and interventions designed to improve teachers' professional practices, the collective design of teaching/learning artefacts, usually lessons, by teachers and educators has been widely applied in different contexts and countries—e.g. lesson study (Stigler, & Hiebert, 1999), learning study (Marton, & Tsui 2004), and communities of inquiry (Jaworsky, 2009, p. 12). This study is a part of a wider research project named “Teacher's Professional Growth by Collaborative Lesson Design,” and concerns itself with understanding the process that teachers and educators undertake when participating in the collaborative design, implementation and results' debriefing of a mathematics lesson—or any other learning artefact such as a class project or an assessment instrument. I call this collaborative activity *Teachers Collaborative Design* (TCD), which includes the following features: (1) there is a specific goal for the artefacts such as teachers' interest, curricular requirements, or school district initiative; (2) there is a description or plan for the delivery of the artefact, including the roles of teachers and students; and (3) there is a mathematical content that includes themes, selected examples, and exercises or problems.

While the benefits of TCD in mathematics education have been documented (Lewis, Peery, & Hurd, 2009; Minori, 2009; Goldsmith, Doerr, & Lewis, 2009; Ling, & Runesson, 2007), a focus on the teachers' interactions in this context barely appears in the literature. Communities of practice (Wenger, 1999), as well as the "Cultural-Historical Activity Theory" (Engeströn, 2008) have been used by researchers (Jaworsky, 2009; Davis, 2008; Minori, 2009) in order to study interactions among teachers and educators when engaged in TCD. Those theoretical frameworks are adapted from social theories to mathematics teachers' development, which on one hand, provide a general socio-cultural perspective, but on the other hand, may not address the particularities of mathematics teachers and educators in the context of TCD. In this report I present a theory aimed to understand the conversations and interactions within the designing process of TCD. The theory is grounded from the designing process of two lessons developed by a team consisting of three mathematics secondary teachers and me as a researcher and facilitator.

We went through two different projects of lesson design: one in the Fall of 2008, where five meetings were used to design a lesson in a grade 9 classroom; and another during Springer of 2009, with six meetings for the design of a lesson for a grade 8 class. Additional meetings were held with other purposes such as setting goals and debriefing the lessons; however, the focus in this part of the research project is on the designing process.

METHODOLOGY AND THEORETICAL CONSIDERATIONS

In order to focus on the involved interactions and processes in a team of TCD, I followed a *constructivist grounded theory* (Charmaz, 2006) approach as underlying methodology. Accordingly, the theoretical concepts I am presenting offer an interpretation of a single case. However, my intention is to generate theory that might be used, modified, or adapted in order to interpret other cases of TCD.

Constructivist grounded theory is based in *symbolic interactionism,* "a theoretical perspective which assumes that people construct selves, society and reality through interactions" (Charmaz 2006, p.189). According to this perspective, social life is made of processes; meanings arise from, and influence, actions. I can have neither an objective perspective as an observer in this research nor do I believe that such perspective is possible; I constructed and shaped, jointly with participant teachers, the settings and trends of actions in the project.

This study differs from common grounded theories in two aspects: (1) the use of a single case to generate theory, and (2) my double role as a member of the team and as a researcher. However, on one hand Glaser and Strauss (1967, p.153) explain that it is possible to generate grounded theory from a single case using the constant comparative analysis method. On the other hand, research related to mathematics teachers' professional development programs and interventions is often conducted by the same facilitators (Llinares, & Krainer, 2006, p. 451).

The data used for this study was collected by recording the meetings held in order to design the two lessons, as well as conducting semi-structured interviews—both group and individual. After each meeting I conducted an *open coding* consisted on splitting the recordings into small segments, labelling each one and writing and explanation of what we were doing in such segment—sometimes a transcription of the conversation was added, as well. The time of the segments was indicated, so I had easy access to them required. At the end of the Fall of 2008, I conducted a group interview in order to capture participants' perceptions of the process, as well as corroborate or modify my interpretations at that moment. Refinement of codes and categories was done after coding the data from the meetings in Spring 2009. More interviews, group and individual, were conducted in order to *saturate* the theoretical concepts and *verify* my conclusions with participant's meaning of the process. Writing *memos* was an active part of the methodology during all the process of data collection and analysis.

THE DESIGNING BRAID

The *genesis of an artefact in TCD* is the process of designing the teaching/learning artefact where teachers act and interact, not only when they meet together, but also outside the working sessions. The *Designing Braid* is made up of three theoretical concepts for the description of conversation and activities during the genesis of the artefact: (1) *anticipating*, (2) *approaching goals*, and (3) *pursuing coherence*. Additionally, there were other activities such as discussing and negotiating the schedule for the lessons, and distributing labours for each participant in the team; these interactions comprise another category which I call *team organisation*.

Anticipating includes both predictions of students' performance before, during, and after the implementation of the artefact, as well as proposals of teachers' actions in order to approach what they initially predicted. I called the former *forecasting*, and the later *commitment*. Participants in a TCD are involved in forecasting, for example, when they use either a learning theory or their own experience to predict students' struggles or success in the implementation of the artefact. Forecasting also includes the piloting of the mathematical tasks as part of the designing process.

Examples of commitment are: the use of tables in students' handouts as a means of guiding their process; the planning of teacher's responses to some forecasted student's struggle during the implementation of the artefact; and exposing students to similar problems or activities in previous lessons in order to get them used to working in some specific setting.

In the process of designing a teaching/learning artefact in TCD, participants set the mathematical content and students' learning in a larger context—e.g. course unit, grade level, or post-secondary studies—in order to make decisions about the tasks and format of the artefact as a part of the course. Such decisions go beyond topics in mathematics, including for instance the developing of a micro learning culture in the classroom. This is what I called *pursuing coherence*.

Teachers and educators are involved in TCD are concerned with the previously established goals for the artefact. *Achieving goals* refers to the participants' interactions towards fulfilling such goals. Proposing mathematical tasks or activities for students is a part of the achieving goals category. Interesting cases are when teachers discuss, and sometimes dismiss, proposed activities for the artefact—an instance of this will be showed in the next section. Achieving goals entails critical reflection of the means and goals of the artefact under design.

While anticipating, pursuing coherence, and achieving goals are interwoven actions, the team organization guides the direction of the project and comprise the activities that TCD participants do while focus on the designing an artefact. In Figure 1 the three interwoven concepts are presented as forming parts of a braid that is directed by team organization.

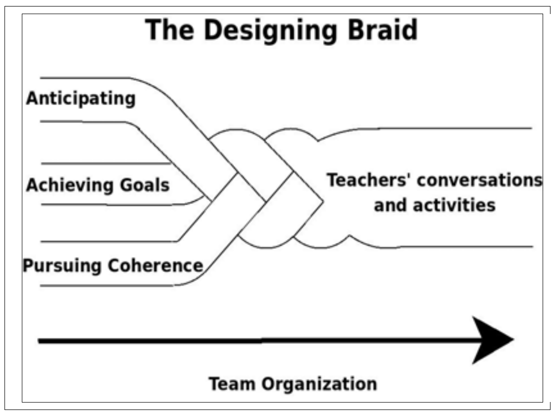

Figure 1. The designing braid.

A CASE OF TEACHERS COLLABORATIVE DESIGN

In order to elucidate the dynamic of the genesis of the artefact process, I present some excerpts from the first lesson—Fall 2008—that we designed in the TCD project. The participating teachers' pseudonymous are Arnold, Brad, and Sofia. The goal of the lesson was to get students translating word problems into algebraic expressions, and we decided to use patterns in order to achieve this goal.

After reviewing some problems in a textbook and discussing which ones could be used in the lesson we were designing, Brad wondered about the students' possible approaches and Sofia forecasted a students' way of starting those problems:

Brad: What kind of ideas can we expect kids to come up with in place of 5n or 3n+2n? What might they say is the pattern?

Sofia: Well, I think that they will start with the recursive thing.

Additionally, Arnold had been concerned with scaffolding students in the lesson and forecasted that some students wouldn't use a table to represent their data. In response, Brad showed a page in the textbook containing some tables and shapes for patterns problems and suggested:

Brad: If you haven't worked on this page on functions, there are some patterns, tables, and shapes that might help to get started, even for the weaker kids.

Brad's suggestion is a commitment to approach the possible difficulty of students using a table or a chart in order to represent their findings that Arnold had forecast. We also discussed that the use of patterns and problem solving should be general strategies for learning mathematics instead of isolated parts of the course:

Sofia: In a larger picture what I eventually want to have is a good lesson, but also good sequences ... the patterns: it should be something that they are familiar with already. ... So at the end of the year, looking at all, they have seen patterns in a lot of different ways.

Such a perspective of students being familiar with the use of patterns and problems solving is an attempt to make coherence in terms of learning mathematics through the whole course.

After analysing different problems, we decided to use the following one for the lesson, the 'cube' problem:

> If you paint a solid cube formed by small cubes of the same size, how many of those small cubes have three faces painted? How many cubes have two faces painted? And how many cubes have one face painted?

We discussed students' knowledge and abilities required for that problem. Teachers commented that students should be exposed to similar tasks in advance: students must be familiar with the use of patterns at classroom. Thus, a sequence of activities for prior lessons was proposed:

Sofia: Yesterday we looked at the problem of the cubes, ... how we'll introduce it. So, in the previous lesson, we thought, we can look at a few patterns without necessarily coming up with the formula, but looking at the sequence of triangular numbers ... square numbers, cube numbers. But, just so that they may be ... better trained to recognise the square numbers when they would come up within the next lesson.

At least three issues can be reed from previous transcription: (1) teachers are having discussions outside of the regular meetings for designing the lesson; (2) they are concerned with students' struggles in recognizing square numbers in the lesson under design; and thus, (3) they have the commitment of introducing a set of mathematical sequences in prior lessons in order to get students being able to approach the 'cube' problem. In this case anticipating joins with pursuing coherence, under the context of the team organization that included additional work outside the weekly meetings.

By reviewing the chosen goal for the lesson we changed our mind about the task for the lesson. Initially, we considered the 'cube' problem interesting because it was rich in mathematical content, had easy entry, and was hard enough so that advanced students could keep working on it while others were still approaching the problem.

However, after comparing the context of the course and the goal of the lesson, some doubts arose, as we read in the following transcription:

Sofia: I do think it is a great problem [the cube problem], but I'm not sure that is what it is necessary; what we want for this lesson.

Teachers' beliefs and perspectives about learning mathematics can become visible when discussing whether the lesson under design actually helps to reach the intended goals. For instance, Brad was questioning whether the use of patterns in the way we were discussing would be effective in making students translate words into algebraic expressions.

Brad: After doing all those puzzle-solving [problems] and getting their own solutions and writing them down and talking about it, how is that help with specifically this task of translating [words into algebraic expressions]?

Designing a lesson in order to achieve our pursued goal was one part of the discussion; however, making such a lesson fit coherently into Arnold's class was problematic. We came up with the design of a prior lesson, consisting of a set of patterns, in order to create the coherence needed to have students working on the 'cube' problem. Finally, that prior lesson became the lesson we designed, implemented, and debriefed, leaving the 'cube' problem for another lesson that teachers might use afterwards.

Note: There were many interesting off-task moments were we deviated the conversation from the designing of the artefact. Such moments included: (1) general discussion on mathematics education; (2) participants' consults and sharing of teaching strategies, and (3) clarification of mathematical concepts that are part of the curriculum. Those off-task moments represented also an opportunity for teachers' learning.

CONCLUSION

The designing braid is a perspective to understand the interactions participants undertake in the process of designing a teaching/learning artefact in the context of TCD. Anticipating, pursuing coherence, and achieving goals are—in addition to team organisation—the conversation and actions that participants of TCD engage in while designing an artefact. Although the designing braid is a theoretical concept grounded in a single case, it resonates with other theories describing teachers' interactions in their practise. Teachers working in the context of TCD are exposed to: sharing ideas and knowledge, which potentially wide their repertoire of strategies; reviewing literature on research and other resources related to education on mathematics; sharing and discussing beliefs about mathematics learning; interacting with peers, promoting in this way the building of a community with its own knowledge; and reflecting about their practise as well as the means and goals of their teaching. The designing process offers an occasion for teachers' professional growth while engaged in TCD, and resonate with the theoretical model of lesson study proposed by Catherine C. Lewis, Rebecca R. Perry, and Jacqueline Hurd (2009), who identify “three pathways through which lesson study improves instruction: changes in teachers’ knowledge and beliefs; changes in professional community; and changes in teaching–learning resources” (p. 285).

Anticipating, as part of the designing braid, entails considering students' actions, and thinking, as well as their corresponding teachers' responses. By examining data from three empirical studies, Goldsmith, Doerr, & Lewis (2009) conclude that “attention to and analysis of student work is an important process within the 'black box' of teacher improvement that deserves principled attention in future research” (p. 103). The designing braid contributes to the understanding of how teachers put attention to students' work in the designing process of a learning artefact.

The three concepts of the braid are not exclusive to TCD, Rowland (2008) identifies four categories of use of examples in teaching mathematics by novice teachers: (1) taking account of variables, (2) taking account of sequencing, (3) taking account of representation, and (4) taking account of learning objectives. While taking account of variables and taking account of representation are part of anticipating, taking account of sequencing is part of pursuing coherence, and taking account of learning objectives is part of achieving goals. This resonance suggests not only validity for the proposed theory of this paper, but also a broader scope considering cases outside TCD. For instance, in the “knowledge Quartet,” a theoretical framework developed by Rowland, Huckster, and Thwaites (2005), “connection” is a dimension of teachers' knowledge which describes the decision teachers make based on the sequence and connections of the content. In pursuing coherence teachers make decisions not only related to the “connection” dimension of the quartet knowledge, but also to the sequencing of activities that make students get used to working in a specific learning environment—as required for the lesson under design. The later aspect relates not only to the teacher's pedagogical knowledge, but also to the teaching and learning *culture* of the classroom.

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"ANATOMY" OF AN "AHA" MOMENT

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Specific details regarding students' understanding and learning, and how to identify and observe these details is the most challenging empirical aspect in support of any viable cognitive theory in mathematics education research. This especially pertains to research in mathematical problem solving and, in particular, to capturing and exploring the nature of "AHA" moments. In this paper, ways in which such studies can provide better empirical ground for developing more accurate theories of mental processes during mathematical thinking and learning are introduced and demonstrated using "state-of-the-art" methodologies that go well beyond the traditional dependencies on video-tape recordings – specifically, computer screen capture, eye tracking, and electroencephalography (EEG) for analysis of an "AHA" moment.

INTRODUCTION

Learning to think mathematically by means of mathematical problem solving is an effective pedagogical approach in contemporary instructional design in mathematics education. Detailed analysis of cognitive processes can provide a deeper insight into what is going in the minds of learners and eventually can serve as a scientific background for designing pedagogies that can lead to improvements in helping learners to learn more effectively. According to Mason, Burton, and Stacey (1982), three kinds of involvement are required when thinking on mathematical problems: physical, emotional and intellectual. Among the different states of this cognitive process the authors distinguished an 'insight' state, referred to as an 'AHA! moment'. Liljedahl (2005) has described an 'AHA! moment' as "the moment of illumination, the AHA! experiences, that instance when the connection is made is a part of the culture of mathematics" (p. 231).

Traditionally, educational researchers engaged in empirical studies of cognition and learning have relied upon behavioral data gathered from interviews, field notes, self-reports, and audiovisual recordings. The most challenging aspect of any viable learning theory concerns identifying specific details regarding the way in which learners' understanding are, or can be made manifest, in behaviour, and how to promote, observe, identify, and objectively assess such manifestations (Campbell, 2003, p.72).

One novel approach to capturing an "AHA" moment was provided by Campbell (2003). His dynamic tracking methodology involved simultaneous and contiguous real – time audiovisual recording of the learner and the learner's computer screen.

Campbell demonstrated simultaneous recording of body behaviour and her procedural actions on the screen in the course of solving a geometrical problem using Geometer's Sketchpad. These data allowed identifying some behavioral manifestations of learner's cognitive shift connected with an "AHA" moment. According to Campbell, this research was a starting point for founding a new Educational Neuroscience Laboratory at the Faculty of Education at SFU, on the basis of which, and under the supervision of Prof. Campbell, this research was conducted.

EYE MOVEMENTS AND EYE TRACKING METHODOLOGIES

Eye - movements have been studied as an indicator of attention and memory (Kramer & McCarley, 2003). Eye movement and eye fixation analyses have been efficiently used as a method of research into strategies of successful and unsuccessful arithmetic word problem solvers (Hegarty, Mayer, & Monk, 1995). Pupil diameter change can also provide information about a cognitive state (Just, Carpenter, & Miyake, 2003). For example, the pupil size has been found increased in response to the complexity of the mathematical cognitive processing (Granholm & Steinhauer, 2004). The pupils change their sizes due to many factors, including such as change in affective states (Barreto et al., 2007; Partala & Surakka, 2003). In Partala and Surakka's (2003) study it was shown that pupil size is significantly larger after negative and positive stimulation in comparison with the neutral state. According to Liljedahl, the "AHA! moment is accompanied by a strong positive emotion (2005), and this should be reflected in the pupil size increase (Partala & Surakka, 2003).

So, eye movement data should provide useful information about various cognitive states during mathematical problem solving and, particularly, in capturing and analysis of an "AHA" moment.

Eye movement measuring techniques

One of the world leaders in eye tracking systems today is Swedish-based Tobii Technology (Tobii Eye Tracker and ClearView analysis software, 2006). Tobii Eye Trackers use the method of tracking light reflected from the cornea and the lens external and internal surfaces. Monitoring the reflections provides measurements of eye movements and pupil dilation with an accuracy of eye movement measurements of 1^0 of visual angle.

A conceptually different approach for determining eye movements is used in *electroöculographic* devices. In the middle of nineteenth century Emil du Bois-Reymond observed that the cornea of the eye is electrically positive relative to the retina, situated in the back of the eye (Malmivuo & Plonsey, 1995). This source behaves, as if it were a single electromagnetic dipole oriented from the retina to the cornea, thereby forming the so-called corneoretinal potential. Signals from this source are measured using electroöculography (EOG). The data acquisition is provided by a number of electrodes placed around the eyes. The accuracy of measurement is approximately 2° of visual angle (Ding, Tong, & Li, 2005).

Both methods provide high accuracy measurements for educational research experiments and for mathematics education research experiments, in particular.

Application of eye tracking methodologies for capturing an "AHA" moment

The role of EOG for exploring the experience of an "AHA" moment has been described in more detail elsewhere (Shipulina, Campbell, & Cimen, 2009). The experiment in the study was based on a paradigm of Dehaene, Izard, Pica, and Spelke (2006). Six diagrams were presented to a participant on the computer screen. Five of the six diagrams were connected by a common mathematical concept, and the task of the participant was to identify the one diagram that does not conform. The participant's uttering of "ahh" can be related to "AHA!" moment, but further analysis of the data set is required to explore this hypothesis. Figure 1 illustrates eye-movements on the screen stimulus recorded by Tobii Eye Tracker during the 10 second experiment.

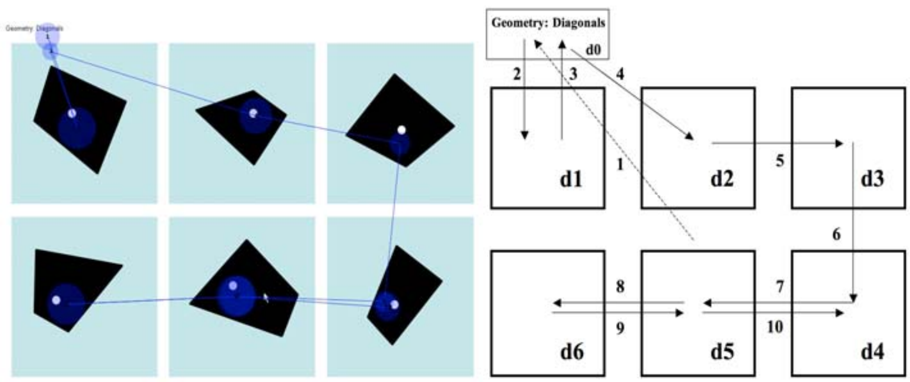

Figure 1. Actual eye-movements on the screen stimulus recorded by Tobii Eye Tracker. Blue lines correspond to eye movements. Blue circles correspond to eye fixations: The bigger circle, the longer fixation (after Shipulina, et. al., 2009).

Using this eye- tracker screen capture a schematization of sequential eye movements during the experimental visual-cognitive activity was designed. EOG was used as an integrated part of the eye - tracking methodology.

Figure 2 illustrates data recorded from two EOG electrodes, attached just off the corners of the participant's left and right eyes, highlighting "gaze" regions, and "move" values (1, 2, 3, ..., 10).

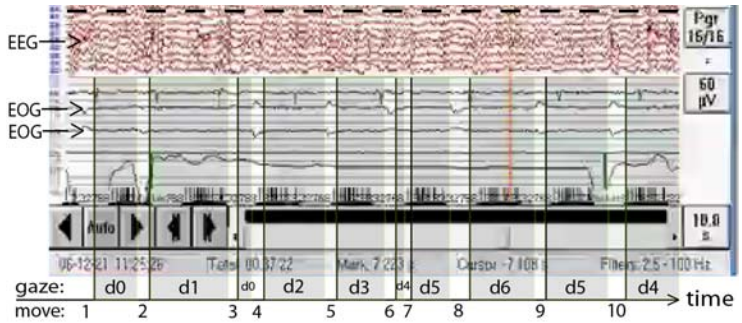

Figure 2. Data that were recorded from two EOG electrodes, attached just off the corners of the participant's left and right eyes, highlighting "gaze" regions (d0, d1, ..., d6), and "move" values (1, 2, 3, ..., 10) (after Shipulina et. al., 2009).

Thus, eye tracking include two main approaches to measuring and monitoring these indexes of the ocularmotor activity: One approach is based on tracking the light reflected from the eye surfaces; another approach is based on the existence of the potential difference between the cornea and the back side of the eye. Integration of both provides a ground for analysis of the overt part of visual - cognitive activity during solving the described above geometrical problem, which includes capturing an “AHA” moment.

The main limitation of eye tracking methodologies is that they provide measurements, recordings, and analysis only overt eye-related part of the visual system. That is, the analysis of complex visual – cognitive tasks is conducted only on the basis of one overt part of the whole complex visual system. To conduct detailed and accurate analysis of complex visual – cognitive tasks another *cortical* part should also be taken into consideration.

BRAIN ACTIVITY AND EEG METHODOLOGIES

A major reason for growing interest in educational neuroscience in mathematics education research is a need for better empirical grounds for developing theories of mental processes (Campbell, 2006 a, b).

Electroencephalography (EEG)

Electroencephalography (EEG) refers to a technique for measuring electrical activity produced by the brain, as recorded from electrodes placed on the scalp. EEG offers high temporal resolution, is non-invasive, and is the least uncomfortable for educational research purposes (Campbell, 2010).

One of the main problems of EEG is that every electrode records a composite signal from many different electrical sources located in the brain. For extracting individual signals from the mixtures recorded on the scalp a signal processing technique known as independent component analysis (ICA) was used. After ICA the EEG data set is transformed into independent (unmixed) signal components related to unknown independent electrical sources in the brain. The problem of localising these electrical sources was resolved by one of the non-linear optimization methods (DIP FIT) available in a software package called EEGLAB

(Delorme, Makeig, 2004). There is also another very informative way of analysing brain signals available in EEGLAB, namely, instantaneous frequency decomposition (Figure 3d).

Having approximate locations and time amplitude courses (signal components) of a number of brain activators (dipoles), the next step of EEG data interpretation is extraction of components which relate to cognitive brain activity from other components identified as artifacts.

INTEGRATIVE APPROACH

Simultaneous recording EEG, Tobii eye-tracking (ET), EOG and audiovisual (AV) data sets constitutes an integrative approach for detailed analysis of the problem solving process described above. The integrative approach includes also the detailed analysis of every individual component obtained after EEG data have been decomposed by ICA.

Figure 3 illustrates the integrative analysis of the component 1. As it was shown above (see Figure 2) eye-tracking data enabled a segmenting of the EEG data into time windows related to "gaze" and "move" zones . This segmentation was used to create a 'mask' for time frequency analyses of signal components for the purposes of identifying frequencies related to "gaze," "move," and other, more subtle, saccadic events.

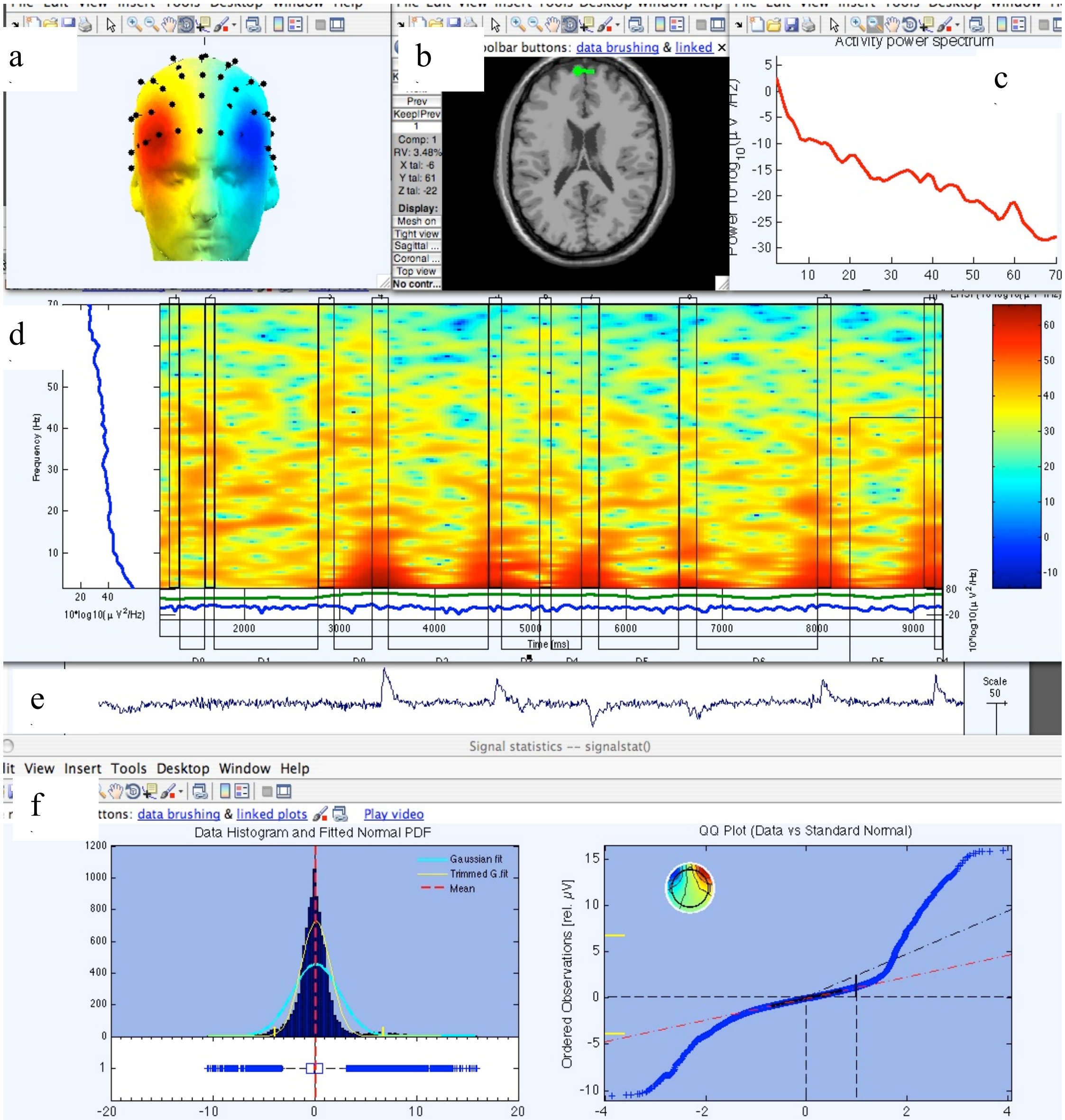

Figure 3. An integrative analysis of component 1: a) scalp topography produced by the component brain source; b) location of the source; c) spectral analysis; d) time- frequency transformation with a 'mask' separating "gazing" from eye movement periods; e) time – amplitude course of the component; f) statistical analysis for identification whether the component relates to muscle artifact or to brain source activator.

From Figure 3d it is seen that bursts of low frequencies occurs exactly during the eye movement periods. This is strong evidence that this component relates to an eye movement artifact, since such burst of frequencies are caused by the firing of motoneurons enervating the eye muscles. Statistical characteristics illustrated in Figure 3f provide more evidence that component 1 is indeed an artifact: the trimmed distribution curve is located too far from Gaussian distribution curve. The source location (Figure 3b) shows that the source is located near the eye muscles.

These integrative aspects of the analysis indicate that Component 1 should be considered as an artifact.

CONCLUSIONS

Integration of contemporary eye tracking and EEG methodologies provides detailed measurements of overt eye related behavior and covert brain related behavior of learner during mathematical problem solving process which includes an “AHA” moment. We called such analysis an ‘anatomy’ of the process. In such integration, eye tracking methodologies play twofold role: One is informative (recording actual eye movements and pupil diameter change by ET); another is bridging and calibrating with EEG data sets, thereby connecting brain and behavior. Data obtained from eye tracking recordings allow creation of a segmentation ‘mask’ for detailed analysis the EEG data set, in particular, for analysis time frequency transformations of individual signal components after transforming EEG data with ICA.

As it appears from this study, the most detailed and accurate analysis of visual – cognitive activity during mathematical problem solving and capturing an “AHA” moment can be conducted when both eye-tracking and EEG methodologies are used in tandem.

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TEACHERS CONNECTING MATHEMATICS THROUGH A LESSON STUDY ON SIMILARITY

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Scaling and similarity, a topic from elementary mathematics, is one where the concepts of number and shape interplay. The concept of similarity was already known the ancient Greeks, and it remains an important topic of study to this day. More importantly, it could be fruitfully employed to the development of the concepts of number and number operations, and for the learning of proportional reasoning. The ideas presented here stem from a lesson study on similarity, but have a wide range of applicability for school mathematics, as they address the connections between magnitudes, quantities, and numbers. In particular, we present the use of geometric representations as a way to uncover the multiplicative relations between quantities and their relative sizes. The report presented here is taken from an ongoing study situated in a school-based community of practicing teachers, who harness the potential of community and workplace to develop their practice of teaching mathematics.

BACKGROUND

Lesson study is a professional development process in which teachers systematically examine their practice, with the goal of becoming more effective in their practice of teaching. The strength of the lesson study is manifold, but one that is of primary focus here relates to the deepening of mathematics teachers' subject matter knowledge (Watanabe, 2002). The centrepiece of lesson study is the *research lesson*, developed collaboratively, taught by one team member while observed by others, and finally discussed and reflected upon by the whole team. It should be noted that the term "research" in this context means teacher-led, practice-based inquiry into the teaching and learning of mathematics.

While there is a consensus that teachers' mathematics-for-teaching (Davis & Simmt, 206), is a complex, dynamic, and tacit body of knowledge, which is very difficult to assess reliably, there seems to be little agreement on what exactly this knowledge is. Interestingly, while *what* should be known to teach well is elusive, *how* such knowledge should be held has been shown quite explicitly on several specific domains of mathematical knowledge for teaching (Ma, 1999). From Ma's research we learn that mathematical knowledge for teaching rests firmly on what has become known as the "profound understanding of fundamental mathematics". With such understanding, teachers are seen to be able to move in their subject easily, naturally, and in a way that allows them to effectively plan for instruction to avoid the typical student misconceptions, and to respond efficiently to a great variety of possible student errors.

One of the tenets of profound subject matter knowledge is its connectedness to the neighbouring and more remote mathematical ideas. Historically, when two seemingly remote mathematical landscapes became recast as one, say by finding that one landscape is isomorphic to another, then this alone offered a great amount of insight into the less known area of mathematics. Connections are also emphasized as one of the process standards of school mathematics. Although the study of mathematics is commonly partitioned into separate units of study, there is a widespread consensus that instruction should be such as to allow students to experience mathematics as an integrated field of study and to see the interplay among mathematical ideas. The importance of connections is also underscored in one of the most influential documents ever written with a specific purpose of guiding teachers in their role of assisting students in the development of mathematical thinking. There it is stated that, “An emphasis on mathematical connections helps students recognize how ideas in different areas are related. Students should come both to expect and to exploit connections, using insights gained in one context to verify conjectures in another.” (NCTM, 2000).

It is less clear how mathematics teachers are to acquire this kind of profound and connected knowledge, how such knowledge is to be held and used in the classroom, how it could be recognized, and what exactly constitutes such knowledge. Lesson study seems to hold some promise as a context in which mathematics teaching could be developed systematically, and in which such knowledge could be deepened both at the level of individual teacher as well as a community of teachers. It can also act as a window for educational research to examine and explicate teachers’ mathematics-for-teaching, which is our aim here. In this paper we focus on the mathematical connections afforded by the lesson study on the topic of similarity. We report here on the connections uncovered by the lesson study team. The team was composed of several mathematics teachers, a mathematician, and a mathematics educator. The latter two are experts in the fields of mathematics and mathematics education respectively, and we shall refer to them here as the “knowledgeable others”. The role of the mathematician was to ensure a sound mathematical basis, point out interesting connections, and fill in the details where needed. The role of the mathematics educator was to interpret the subject from the perspective of teaching and learning, and point out what educational research has to say about the common difficulties encountered in the process of learning the particular topic.

SUBJECTS AND CONTEXT

"Lesson Studies", in various formats, have become popular in the teaching community as a means for professional development. Three teachers from Southpointe Academy, where the research lesson was implemented in an open house format, and two teachers from another district comprised the "lesson study team" for this lesson study cycle. About 20 teachers from other schools and districts came to observe the lesson. All observers participated in critiquing and reflecting on the lesson during the post-lesson discussion. The research lesson was implemented in the authentic environment of a real classroom with real students. There were 24 students in the class of mixed mathematical ability. Students were used to working in groups as well as individually. In addition, whole class discussions were always part of the learning process, where students shared their thinking and evaluated one another's approaches.

The previous unit of study involved an introduction to geometry. Students learned a number of basic geometric constructions (using compass and straightedge), such as how to draw perpendicular and parallel lines, how to replicate an angle, how to bisect an angle and also a line segment, and how to construct certain angles. They studied and derived properties of angles on intersecting lines (angles on a line add up to 180°, vertically opposite angles are equal, and angles at a point add up to 360°). Using the fact about equality of corresponding angles on parallel lines crossed by a transversal, they deduced simple facts about relationships between angles on parallel lines. In addition, students deduced a number of facts such as, "In a triangle, angles add up to 180°", "In a parallelogram, opposite angles are equal, and adjacent angles are supplementary". The unit of study in which the research lesson was situated was a unit dealing with similarity of geometric figures. In addition to building on what was learned about angles on parallel lines, students used their knowledge of ratio and proportion, and also of the solving of equations in one unknown (solving proportions), studied earlier in the school year. The research lesson was the second lesson of the unit. The team decided to build the idea of similar figures using the notion of scaling factors, applied to a line segment, then to a triangle, then a quadrilateral, and finally to any shape, such as an outline of a the boarders of Canada on a map.

Figure 1: Three cities on two maps of Canada form two similar triangles. "What about these triangles is the same and what is different?" must be formulated beyond,

LINE SEGMENT AS A FUNDAMENTAL GEOMETRIC SHAPE UPON WHICH NUMBER CONCEPTS CAN BE BUILT

In this lesson, the goal was to describe similarity more precisely, beyond *sameness of shapes*. Students were expected to observe and formulate the general properties of similar figures: (a) that the measures of corresponding angles are equal, and (b) that the ratios of the lengths of corresponding sides are equal. The intention of the instructional sequence was for students to develop a way of thinking about proportion geometrically (for example, breaking a given line segment into *n* equal parts and then taking *m* parts to create the scaled image of the original line segment). Students were expected to develop a sense for what is entailed, mathematically, with statements such as "scaling up" and "scaling down" by a given rational number factor, $k = m/n$ (i.e., *m* greater than *n*, or less than *n*, respectively).

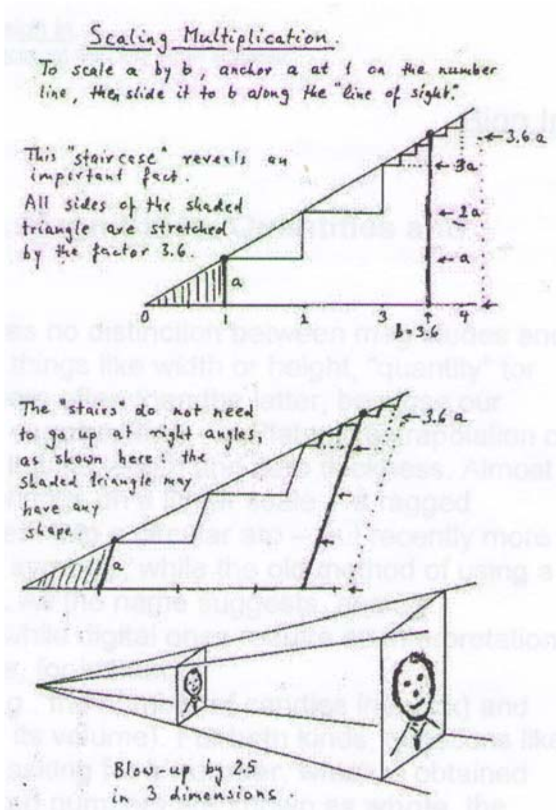

Figure 1: Scaling by

As mentioned, the concept of similarity was introduced through the idea of a scaling factor, connected to multiplication as a scaling of a line segment. Students had previous experience with tasks such as, "Given a line segment AB, construct point B' such that AB' is 3 times as long as AB". This was extended to encompass rational number multipliers, or scale factors, for example, "Given a line segment AB, construct point B' such that AB' is 5/7 of the length of AB". Using geometric construction of equally spaced parallel lines, students were shown that a given line segment AB can be cut into *n* equal parts. If one assigns a measure of 1 unit to AB, then point A corresponds to 0 and B to 1. This way we get a custom unit of *1/n* from which all fractions with the denominator of *n* get their positions on the number line as distances from 0 to the respective point, which is at some multiple of the custom distance *1/n* away from 0. For example, the *m*-th point on the number line would be distance *m* times as far as *1/n* is from 0, so it is a constructed point corresponding to the fraction *m/n*. This was intended to prepare students for understanding the ideas of relative size, and what we refer to as "scaling multiplication".

It is not obvious that a real number multiplier would also uphold the scaling multiplication. Essentially, this was the monumental work of Eudoxus around 400 BC, who reworked the theory of proportions to include irrational magnitudes, by which irrationality received its first proper treatment, as this appears in Euclid's Elements (Eves, 1990). In a prior study on preservice mathematics teachers' understanding of irrational numbers, we explored participants' conceptions about ways to find the exact location of a constructible irrational number $\sqrt{5}$, and we found that the geometric representation of irrational numbers was strangely absent from the concept images of many participants (Sirotic & Zazkis, 2007). The common conception of real number line appeared to be limited to rational number line, or even more strictly, to decimal rational number line where only finite decimals receive their representations as 'points on the number line'. This is in agreement with the practical experience that finite decimal approximations are both convenient and sufficient, which could be the source of these conflicts.

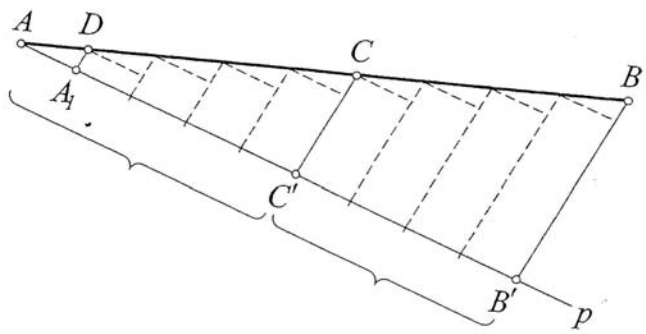

Figure 3: Towards the idea of similarity via scaling of line

Being aware of this, the team of teachers designed the instructional sequence so as to ensure that students were exposed to the construction of various fractional lengths, as Figure 3 suggests.

Nonetheless, for the purpose of the learning task in the research lesson, teachers decided to use "friendly" numbers, such that would allow for whole number scale factors to be employed in the process of solving the problem. Figure 4 shows the task that was decided upon as the main learning ground for the concept of similarity to be applied to. It is based on the story of how Thales found a way to compute the distance to an enemy's ship far in the waters of the Aegean Sea using indirect measurement

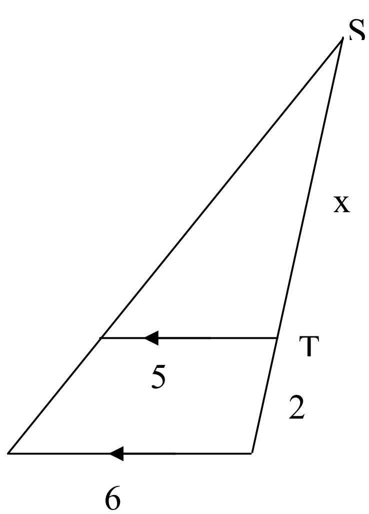

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Figure 4: The task, "How far is the Ship?"

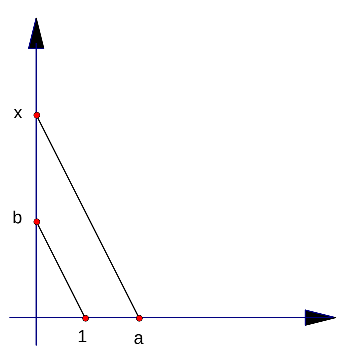
and the relations between corresponding sides in similar triangles.

Then students looked at this more specifically for the case of similar triangles, and they applied their knowledge to solve a practical problem of "How far is the Ship?" The scope of this paper does not permit for the discussion of the various approaches that students took in solving the problem, and what their main challenges were. Upon examination of students' solutions the lesson study team found that a variety of approaches were employed by the students, and that 20 out of the 32 students solved the problem correctly, 5 of which used more than one solution process, and one student used as many as 6 different solutions processes. Of the 12 that did not complete the task while working independently, 3 made algebraic errors in their computations, 3 set their proportion statements not attending to the proper ordering of sides, or correspondence, 1 showed no sign of even starting, and the rest made other types of non standard errors.

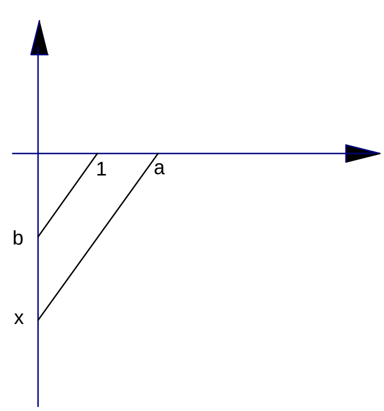
MULTIPLICATION IN THE COORDINATE PLANE

The idea of scalar multiplication, which was presented in the first part of this paper can be extended to include multiplication by negative numbers. The dilemma of why the multiplication of two negative numbers results in a positive product could be resolved using such representation. In a proportion statement $a:b=c:d$ we can call the pair of a, c and b, d analogues. That is, when these ratios are written as fractions, the numerators are analogues and the denominators are analogues. The product ab can then be interpreted geometrically as a length of a line segment, which comes from the construction of similar triangles. When we have a proportion, $1:a=b:x$, then the rectangle with sides 1, x is equal in area as the rectangle with the sides a, b (that is, $1\cdot x = a\cdot b$, or $x=ab$). This means that the line segment of length x is constructed by first drawing the line segment between the two known analogue lengths of the proportion we are considering, and then constructing a parallel line segment through point at a.

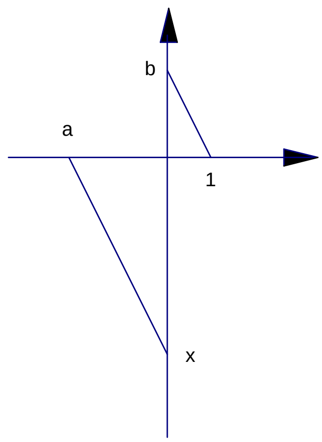
(i) $a > 0$ and $b > 0$; $ab > 0$



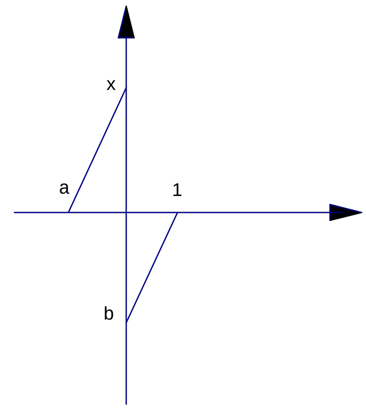
(ii) $a > 0$ and $b < 0$; $ab < 0$



(iii) $a < 0$ and $b > 0$; $ab < 0$



(iv) $a < 0$ and $b < 0$; $ab > 0$



DISCUSSION

As teachers consider and attend to the connections within mathematics, they are more prepared to uncover the same for their students, under the condition they had first uncovered these connections for themselves. We see the process of teachers' uncovering of the mathematical connections and deepening of their subject matter knowledge as a result of three streams of influence on this professional learning community.

Firstly, these connections are being uncovered through teachers' independent study of the subject matter and their subsequent sharing of their findings and ideas during the stages of planning, preparation, and designing of instruction. During this phase, teachers study the historical background of the concepts they are choosing to teach in their research lesson. They examine other curricula and instructional materials, and consult the reports of other lesson studies on related topics. They consider possible approaches to presenting specific mathematical concepts, and a variety of ways in which these concepts can be represented and transformed for learning purposes. Jaworski speaks of "inquiry communities", where inquiry is used as a fundamental theoretical principle and position for engaging critically with key questions of the mathematical content that is to be taught, as well as issues of practice, such as the use of mathematical tasks in classrooms (teachers' practice and perspective), anticipation of student thinking and reactions, assessment of student emerging understandings, and so on. She proposes the use of "inquiry as a tool" as a strategy that can lead to developing "inquiry as a way of being" (Jaworski, 2006). An inquiry community, as we understand it, is similar to "professional learning community" (DuFour, 2004), which is a means for teachers to generate knowledge about their practice, in this case knowledge of mathematics teaching and learning. As such, lesson study could be seen as a special form of inquiry community, where the inquiry is from a perspective of developing teaching. It is not about how teaching is now, or what it should look like if it were to be effective. Rather, it is a close-up look inside of the process as it is changing, developing, and growing in participants' awareness and action in the classroom.

Secondly, they are uncovered by the close observation of learning as it takes place in real time, during the research lesson implementation. When these individual observations are combined, a complete picture of students' experience of the lesson emerges. Students make surprising connections, often quite non standard and unanticipated, and sometimes entirely mathematically sound but never considered by the teacher, all of which plays an important role in teachers' learning from their own practice. It is during this phase that teachers' instructional designs get tested in practice. It is critical that observers take accurate notes of student learning, as this becomes the empirical evidence for the study of how the research lesson impacted student thinking, learning, and understanding. The quality of post lesson discussion very much depends on the quality of the observation data collected by the teachers, as well as on the focus questions that they bring to their observation.

The third stream of influence on the uncovering of mathematical connections comes from the interactions with the "knowledgeable others". This happens most notably during the post lesson discussion, when all participants engage in collective reflection on the research lesson and the ways it impacted student learning. It is during this phase that the connection between multiplication of real numbers in the Cartesians coordinate plane and similarity was made explicit by the mathematician.

For some teachers this was an entirely new connection and a perspective they had not considered before, and for others it was familiar but not really useful for teaching. We disagree with this view, and suggest that further research is needed to establish how number concepts could be more effectively built upon a geometric foundation. It is widely accepted that mere manipulation of numbers is pointless. As we have shown here, and as we know from experience, most operations get their sense in geometric surroundings, and can be readily visualized. The role of visualization and visual thinking has also been emphasized as one of the important processes for the development of mathematical thinking and in problem solving.

Geometric representations are widely employed in mathematics education, both as *presentational* models (used by adults in instruction) and as *representational* models (produced by students in learning). It is recognized that these models play significant roles in instruction and its outcomes (Lamon, 2000). A longitudinal study on five classes of children, each using a different interpretation of the symbol *a/b* that differed from the standard "part of the whole" conducted by Lamon, showed that the geometric model, which is referred to as the measure model, had the highest transfer rate compared to other interpretations.

In conclusion, we once again observed how through this process teachers' knowledge of mathematical content and ways in which it can be presented for learning is developed, shared, refined, and transformed. Teachers become scholars of the interaction between teaching and learning, and of the subject matter they are to teach.

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REDUCING ABSTRACTION: THE CASE OF LOGARITHMS

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Reducing abstraction is one of the theoretical frameworks that examine the learners' behaviour while coping with abstraction level. It refers to the tendency of the learners to unconsciously reduce the level of abstraction while learning new concepts to make it mentally accessible for them. Analysing the work of three students through the lens of reducing abstraction, the aim of this paper is to investigate and exemplify some misconceptions and instances of error in students' understanding of logarithms.

1. INTRODUCTION

Logarithmic function is an important concept that plays vital roles in advance mathematics. Unfortunately, students often find the concept difficult to understand which causes hindrance to their conceptual understanding of the logarithmic functions. Students' understanding of the concept of function (in general) has achieved a lot of attention in the mathematics education research community over the past decades (e.g. Dubinsky & Harrell, 1992; Sfard, 1992); there has been however very little research done that has looked specifically at the students understanding of logarithms (Berezovski, 2004). Hence my purpose in this paper is to understand students' understandings and their mental process of working with logarithms. I aim to investigate and exemplify some misconceptions and instances of errors in students' conceptual understanding of logarithmic notation and rules for working with logarithmic equations.

2. THEORETICAL FRAMEWORK

The theoretical framework used for this study is Reducing Abstraction. The framework of reducing abstraction was first introduced by Hazzan (1999) to examine the mental process of undergraduate students' learning of abstract algebra. According to Hazzan & Zazkis (2005):

> "Reducing abstraction is a theoretical framework that examines learners' behaviour in terms of coping with abstraction level. It refers to situations in which learners are unable to manipulate concepts presented in a given problem; therefore, they unconsciously reduce the level of abstraction of the concepts involved to make these concepts mentally accessible" (p.101)

Building on the work of Wilensky (1991) and Sfard (1991), Hazzan (1999) categorizes three abstraction levels, all of which interpret students' learning as some way of reducing abstraction level of the concept. These three levels are:

a) Abstraction level as the quality of the relationships between the object of thought and the thinking person.

On the basis of this perspective, the level of abstraction is measured by the relationship between the learners and the concept (mathematical object). It is based on Wilensky (1991) assertion that abstraction is not an inherent property of the entity, "but rather a property of a person's relationship to an object" (p. 198). It refers to the tendency of the students to make unfamiliar mathematical concept to more familiar by reducing the level of abstraction of the concept. In other words, when a student sees a mathematical object, he or she will try to make sense of it based on his or her past experiences with mathematical objects. This idea shares much with Hershkowitz et al's. (2001) model of abstraction which views abstraction process from socio-cultural perspectives. They maintain that abstraction is a process of "…vertically reorganizing previously constructed mathematics into a new mathematical structure" (p.2). This vertical reorganization activity "…indicates that abstraction is a process with a history; it may capitalize on tools and other artefacts, and it occurs in a particular social setting" (p.2). This is in fact in line with constructivist theory which claims that new knowledge is constructed based on the existing knowledge. For example, Hazzan & Zazkis (2005) mention that when asked to add numerals such as 12 & 14 in base 5, Sue (one of their students) avoids base 5 additions by converting back to base 10, performing the operation in base 10 and then calculating the result in base 5, thus reducing level of abstraction from unfamiliar base 5 addition to familiar base 10 addition.

2) Abstraction level as reflection of the process-object duality

It refers to the tendency of the students to work with the problem by following step-by-step procedure (process conception) rather than meaningful mathematical concept (object conception). This is based on Anna Sfard (1991) theory of process-object duality according to which the process conception is less abstract than an object conception. For example, Hazzan & Zazkis (2005) observes that when asked whether $33\times 52\times 7$ is divisible by 7, Mia (one of their students) calculates the products (= 1575) and then divide by 7 (i.e. 1575/ 7) to get the answer rather than analysing the object of divisibility thus by reducing the level of abstraction.

3) Degree of complexity of mathematical concepts

This refers to the idea that "the more compound mathematical entity is, the more abstract it is" (Hazzan, 1999, p. 82).

Example:

> " Int: Do you think there is a number between 12358 and 12368 that is divisible by 7?
>
> Nicole: I'll have to try them all, to divide them all, to make sure. Can I use my calculator?" (Hazzan & Zazkis, 2005, p. 112)

Nicole is expected to consider the interval of ten numbers and see the divisibility of 7 but she prefers to work each number separately to check the divisibility. That is she is reducing level of abstraction by working with a subset (a particular number), rather than working with the larger set (interval of numbers) itself.

The theoretical framework of Reducing Abstraction has been used to examine students mental process on learning mathematical concepts in different areas of advance mathematics and computer science such as abstract algebra (Hazzan, 1999), differential equations (Raychaudhuri, 2001), data structure (Aharoni, 1999), computing science (Hazzan, 2003a & 2003b) as well as school level mathematics (Hazzan & Zazkis, 2005). This paper emerged in my recent attempt to understand student's interpretation of logarithmic notation and how these interpretations inform students' understandings of rules for working with logarithms.

2. METHODOLOGY

Based on the analysis of their written work (quiz), three students; Ali, Beth and Cayce (pseudo name) were selected to participate in this study. They were taking foundation course on Mathematics at a private college in BC, Canada. This course is offered to those who either do not meet the required credit hour for enrolment into calculus I (Differential Calculus with Applications to Commerce and Social Sciences) course or who were identified as lacking the required level of mathematics skills to enrol into Calculus I. The data comes from two sources: a) students' written work (quiz) and b) informal interview with the students. The quiz was administered a week after the completion of the unit on Logarithms. The prescribed textbook for this course was "Math power 12 (Western Edition)" and the total instructional time spent on this unit was 4.5 hours. In order to better understand their interpretation of logarithmic notation and their understandings of rules for working with logarithms, I decided to conduct an informal interview with the students. The interview was conducted after a short break of the quiz (during which we briefly analysed their written work) and each interview was lasted about 15 minutes. The interview was audio-taped and analysed.

3. RESULTS

3.1 Relationships between the object of thought and the thinking person

Ali's Case:

One of the questions in the quiz required students to simplify the logarithmic expressions. Ali's written work provided some interesting insights into his understanding of logarithms. It illustrates his tendency to change unfamiliar logarithmic expressions to more familiar simple algebraic expressions. Ali saw no meaningful relationship in the log symbols that supported mathematical activity. For him, $\log_3 4$ and $4\log_3$ are equivalent expressions and he seemed to use this rule consistently throughout the quiz.

Evaluate:

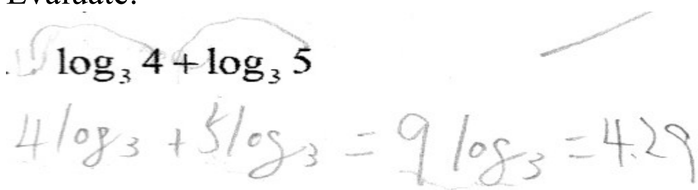

Fig. 1: Ali's written work

Interviewer: (pointing to his answer) how did you get that?

Ali: log three four and four log three (writes $\log_3 4$ and $4 \log_3$) are basically the same, the order really doesn't matter here. It is just like 'a' times 'b' equals 'b' times 'a'. But there is a word for this rule… (Tries to remember the rule) Oh, no … I forgot.

Interviewer: Oh, I see. So, you are using commutative law here?

Ali: Yes, yes … commutative law. We have done many problems on commutative law in our previous math class.

Interview: OK. Then …

Ali: Then um…do the same thing to other (pointing to $\log_3 5$) and add them up because they are like terms. Then use your calculator, its easy!

[Notice how he reads the logarithms notation]

Interviewer: Do you remember how we solved this kind of problems in class?

Ali: Yes, but we did differently in class.

Interviewer: Can you tell me how we did it?

Ali: (writes in a paper) you can change $\log_3 4$ to log 4/ log 3 and do the same thing for $\log_3 5$ and plug in those in your calculator to add. But this method is hard. I don't know why we did that way in class.

It is evident that Ali can recall the properties of logarithms and remember the procedures of doing it in class but was hesitant to follow as he finds his method so much easier. He solves many problems without any acknowledgement of his method violating any laws of mathematics. This supports Gray and Tall's (1994) idea that due to the misunderstanding related to symbols and syntax in mathematics, some students develop their own technique based on their personal interpretation of the symbols. It can be argued that perhaps students' misconceptions may be attributed to the insufficient explicit teaching of the concept, but as above excerpt illustrates, it could also be that the student's tendency to make unfamiliar concept more familiar somehow caused major problem for students preventing them from forming an appropriate mathematical concept. This, according to Hazzan, is an act of reducing abstraction.

Beth's case:

Surprisingly, Beth has developed yet another method for the same question. For Beth, $\log_3 4 + \log_3 5 = \log_3 9$. As we can see in her written work which is supported by her interview, Beth seemed to use the following rule consistently: $\log_a x + \log_a y = \log_a(x+y)$.

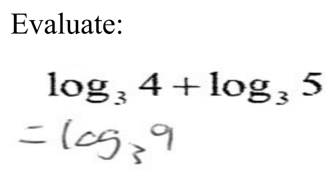

Fig. 2: Beth's written work

Interviewer: So, log 9 to the base 3 is your answer. (Pointing to the question)

Beth: Yes, I got log base three of nine.

Interviewer: How did you get that?

Beth: Um... It's easy. ..like simplifying. If you had (pointing to the question) the log base three of four plus the log base three of five it would be the log base three of nine, because you just take the common thing which is log base 3 (writes $\log_3$ (4+5)) and then add these left over numbers (4 and 5). So, you have log base 3 of 9. It is just like distributive property…

Beth's written work as well as her argument during interview provides an insight on her understanding about the logarithmic notation and rules for working with logarithmic expressions and equations. Her tendency to connect this unfamiliar logarithmic representation to her familiar knowledge of the distributive property can be interpreted as an act of reducing level of abstraction.

Cayce's Case:

Another problem on the quiz required students to solve for x given the equation $\log_5(x+1) + \log_5(x-3) = 1$. It is interesting that Cayce used the properties of logarithm correctly to simplify expression such as $\log_3 4 + \log_3 5$, but she could not use the same concept to solve the equation. This may be partly because of the complexity involved in the problem itself. She mentions in her interview that the equation is not easy to solve. It involves binomials and she remembers from her previous class doing problems on binomials using distributive property or FOIL method. So, she takes log 5 as a common factor and writes the remaining factors in the parenthesis as $\log_5(x+1 +x-3)$. She consistently uses this rule whenever she encounters such scenario.

Solve for x:

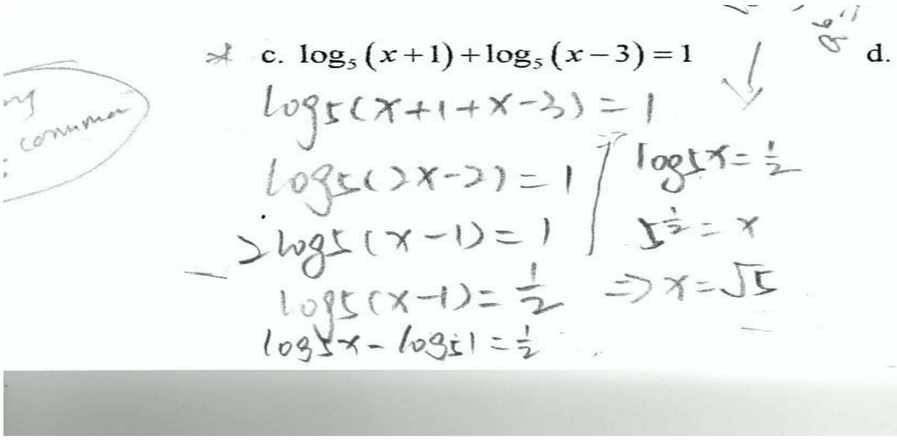

Figure 3: Cayce's written work

Her tendency to relate unfamiliar logarithmic equation to more familiar binomials and treat them with distributive property is another example of how students tend to reduce level of abstraction in learning mathematics.

3.2 Process-object duality

The notion of process –object duality of reducing abstraction is illustrated by Beth's work below.

One of the questions required students to evaluate $\log_3 9$ and find the answer as a single numerical value. As mentioned above, Beth's written work as well as her interview demonstrated that she has some misconceptions about logarithms. But surprisingly, she evaluates $\log_3 9$ (and other problems of this kind) correctly. Therefore, we were interested to know her understanding of the concept involved in the task.

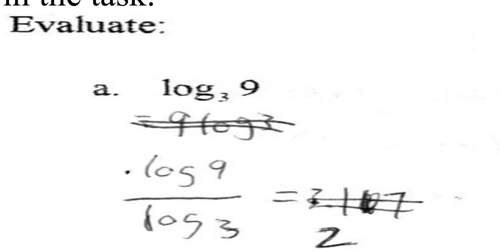

Figure 4: Beth's written work

Interviewer: (pointing to Beth's solution) what did you do here?

Beth: Um ……Don't we have this formula (writes in the paper) $\log_a b = \log b / \log a$? I just converted $\log_3 9$ to $\log 9 / \log 3$ using the formula.

Interviewer: OK. But it seems that you did something different at the beginning?

Beth: Um… Yes, I thought we could write $\log_3 9$ as $9 \log_3$ but then I remember the formula.

Interviewer: What base we are in now?

Beth: if you convert using that formula, the base just disappears. I mean…. I mean you don't have to write the base… but why so? I don't know… but the base 3 is there on the bottom any way, right?

Interviewer: OK. Why would you do that?

Beth: Um … Otherwise you can't plug in to your calculator. How can you plug in the base in calculator? So…

Beth arguments for transferring the logarithm expression (using the formula) in to a form so that she can use her calculator and get the answer shows that her conception of the logarithmic function is based on rules and memorized facts, but not a meaningful knowledge. She knew how to do it (process), but did not understand what it means (object). This behaviour can be interpreted as an act of reducing abstraction through Hazzan's (1999) perspective.

4. Discussion of the Results and Conclusion:

Reducing abstraction as a theoretical framework has proved helpful in my attempt to understand the thought process of the learners while coping with unfamiliar (and complex) mathematical concept. As a way of coping with the complexity of the unfamiliar logarithmic function, Ali finds the rules given by the authorities (book or teacher) difficult and avoids using them in the quiz. Based on his previous experience, he develops his own faulty rules to solve the problems. This supports Gray and Tall's (1994) idea that less-able students are not learning correct techniques more slowly, but are instead developing their own techniques. Beth and Cacey's tendency to reduce the level of abstraction seems to be mathematically inappropriate. Similar to the case of one of the classical examples of overgeneralization that claims that $\sin (a + b) = \sin (a) + \sin (b)$, Beth and Cacey both seemed to over-generalize the problem as $\log_a x + \log_a y = \log_a (x+y)$. This is what Matz (1982) calls "misapplication of linearity" or an "overgeneralization of distributivity". Furthermore, it is important to note that Beth seemed to correctly evaluate $\log_3 9$ as evidenced in her written work, but my conversation with her reveals that she does not have meaningful understanding of the concept but relies on memorization of the rules and facts.

In mathematics education research some researchers such as Berezovski (2006), Kastberg (2001) have done some work on students' understanding of logarithmic functions with in the context of different theoretical frameworks. In this paper, my contributions are two folds: first, by examining students' mental process on learning concept of logarithms through the lens of Reducing Abstraction, I tried to expand the applicability scope of this theoretical framework to logarithms. Second, I exemplified some misconceptions and instances of errors in students' interpretation of logarithmic notation and understandings of rules for working with logarithms with in the context of the theoretical framework. Finally, the result suggests that we, as educators/ teachers, should be aware of the nature of our students' understandings and possible misconceptions in order to develop more effective teaching strategies that will enhance meaningful understanding on students' part.

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