Electricity & Magnetism
Lecture 9

Today's Concept:

Electric Potential - calculations

Electric Potential and Field

The electric potential difference between two points is the energy required to move a unit positive charge between the two points:

SI Units: Volts

$$\Delta V_{a\to b} \equiv \frac{\Delta U_{a\to b}}{q} = -\int_a^b \vec{E}\cdot d\vec{l}$$

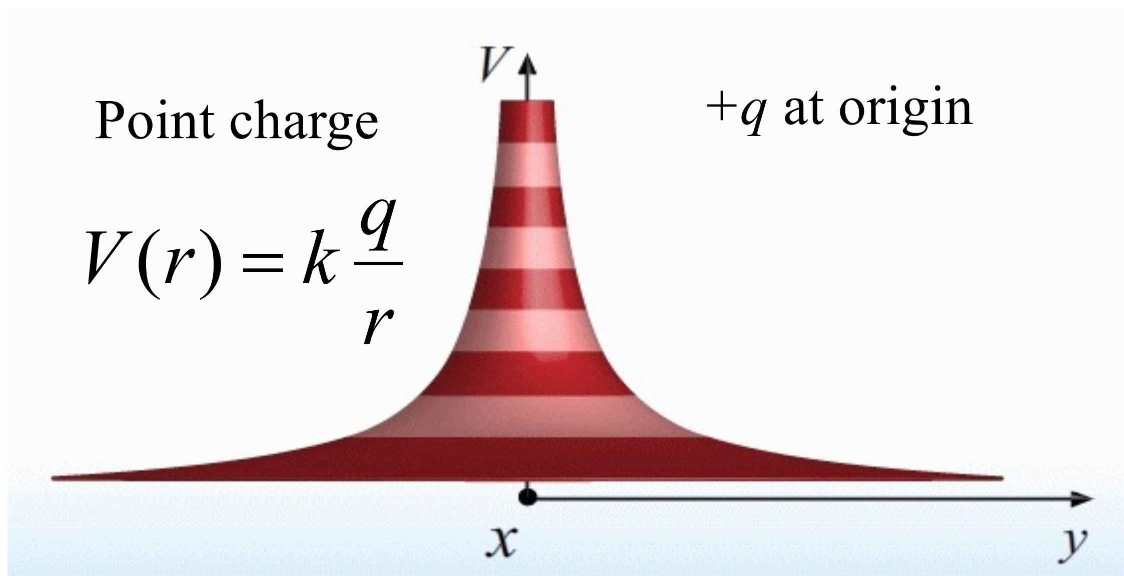
$$[V] = \frac{[J]}{[C]}$$

The electric potential function, $V(\vec{r})$, with respect to a reference point $\vec{r}_0$ is given by:

$$V(\vec{r}) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E}\cdot d\vec{l}$$

For simplicity, we pick $V(\vec{r}_0) = 0$

Usually it is $r_0 = \infty$ where $V \equiv 0$



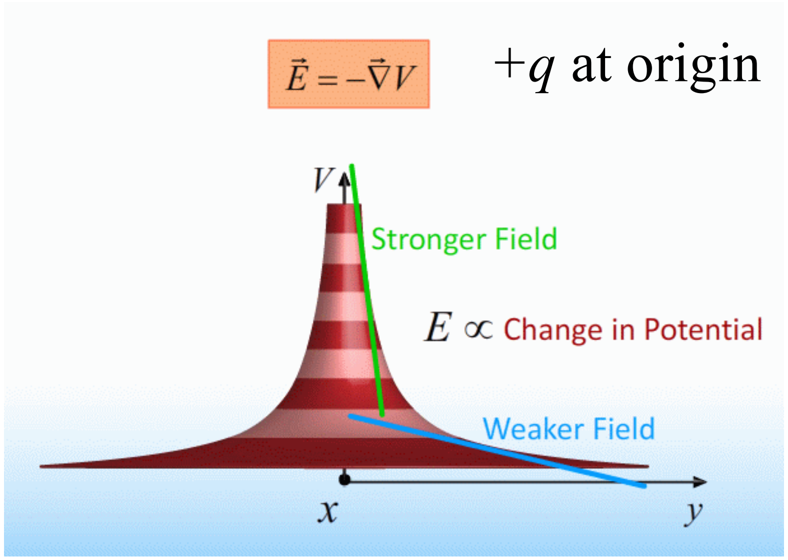
Electric Potential and Field

$$V(\vec{r}) - V(\vec{r}_0) = -\int_{\vec{r}_0}^{\vec{r}} \vec{E} \cdot d\vec{l}$$

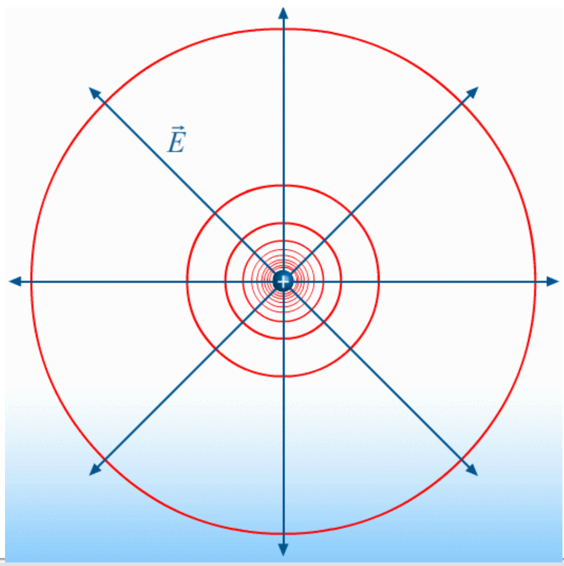
The electric field is the negative gradient of the potential:

$$\vec{E} = -\vec{\nabla} V$$

$$\nabla f(x,y,z) \equiv \hat{x}\frac{\partial}{\partial x} f + \hat{y}\frac{\partial}{\partial y} f + \hat{z}\frac{\partial}{\partial z} f$$



Equipotentials and field for a point charge *+q*



Equipotentials are ALWAYS perpendicular to the electric field lines.
The electric field points in the direction of fastest decrease in the potential.

Finding Electric Field from Potential

Example: Point charge at origin

$$V(\vec{r}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{\sqrt{x^2+y^2+z^2}} \qquad \vec{E} = -\hat{x}\frac{\partial}{\partial x}V - \hat{y}\frac{\partial}{\partial y}V - \hat{z}\frac{\partial}{\partial z}V$$

$$\frac{\partial}{\partial x}\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{-x}{(x^2+y^2+z^2)^{3/2}} = \frac{-x}{r^3}$$

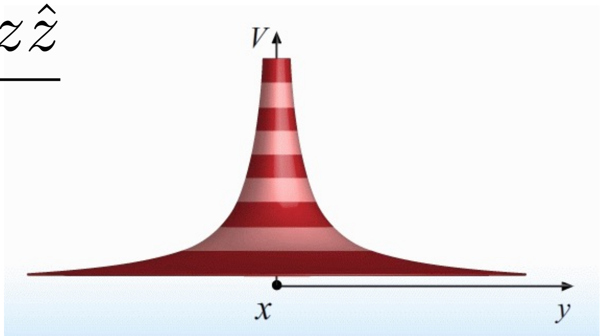
Treat y and z as constants, and take derivatives with respect to x

$$\frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{-y}{r^3} \qquad \frac{\partial}{\partial z}\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{-z}{r^3}$$

$$\vec{E} = \frac{q}{4\pi\varepsilon_0}\frac{x\hat{x}+y\hat{y}+z\hat{z}}{r^3} \qquad \hat{r} = \frac{x\hat{x}+y\hat{y}+z\hat{z}}{r}$$

$$= \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}$$

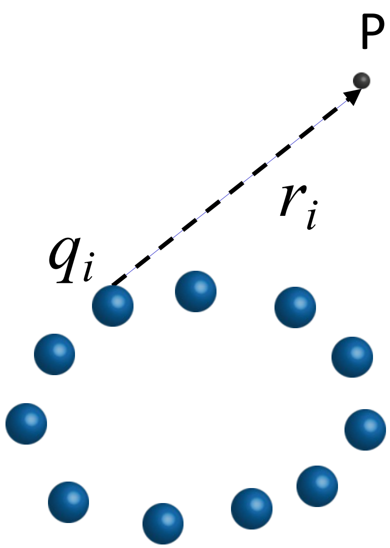
Coulomb's Law!

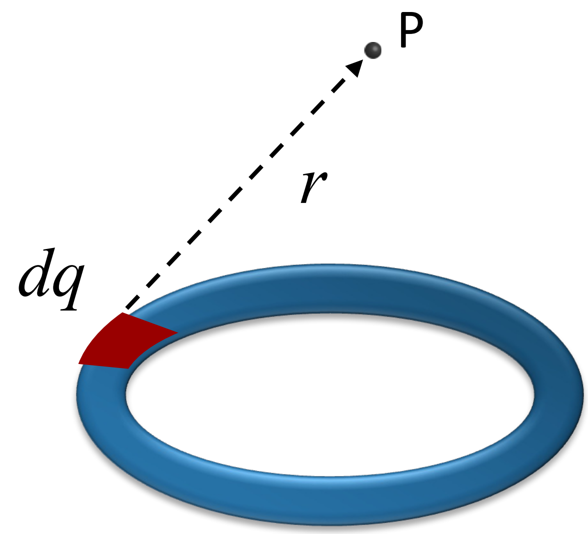


V for Continuous charge distributions

assuming $V = 0$ at infinity

$$V = \sum_i k\frac{q_i}{r_i} \quad \rightarrow \quad V = \int k\frac{dq}{r}$$





$$\lambda = Q/L$$

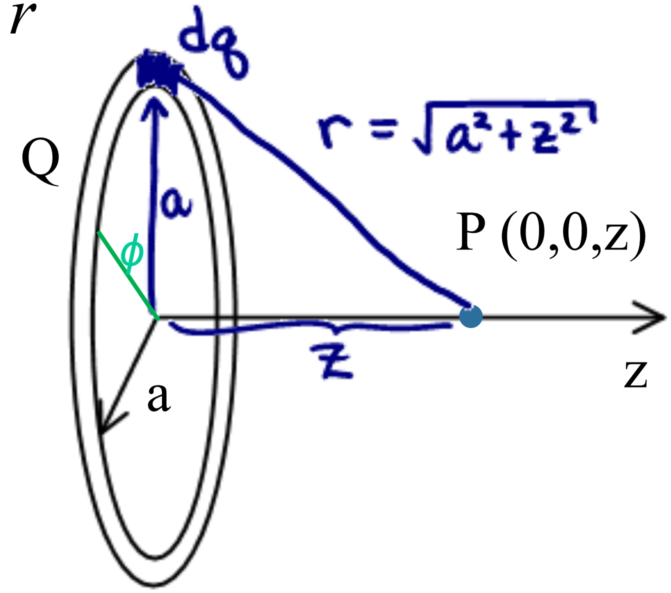
Uniform ring of charge

Find the electric potential at the point *P(0,0,z)* on the axis of the ring.

$$\lambda = \frac{Q}{2\pi a}$$

$$V = \int dV = \int \frac{1}{4\pi\varepsilon_0}\frac{dq}{r}$$

$$dq = \lambda a\, d\phi = \frac{Q}{2\pi} d\phi$$



$$dV = \frac{1}{4\pi\epsilon_0}\frac{dq}{r} = \frac{1}{4\pi\epsilon_0}\frac{Q}{2\pi}\frac{d\phi}{\sqrt{z^2+a^2}}$$

$$V = \int dV = \frac{1}{4\pi\epsilon_0}\frac{Q}{\cancel{2\pi}}\frac{1}{\sqrt{z^2+a^2}}\cancel{\int_0^{2\pi} d\phi} = 2\pi$$

$$\therefore\ V(0,0,z) = \boxed{\frac{1}{4\pi\epsilon_0}\frac{Q}{\sqrt{z^2+a^2}}}$$

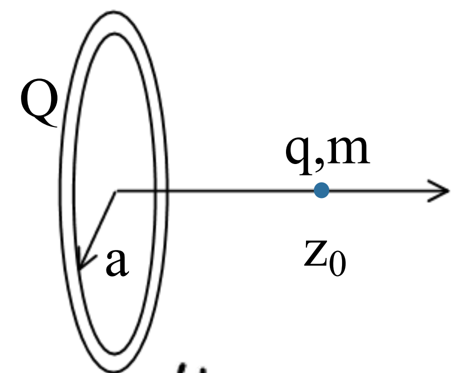
$$z \gg a: \quad V \approx \frac{1}{4\pi\epsilon_0}\frac{Q}{z}.$$

Note $E(z=0) = 0$

but $V(z=0) = \dfrac{Q}{4\pi\epsilon_0 a} \neq 0$

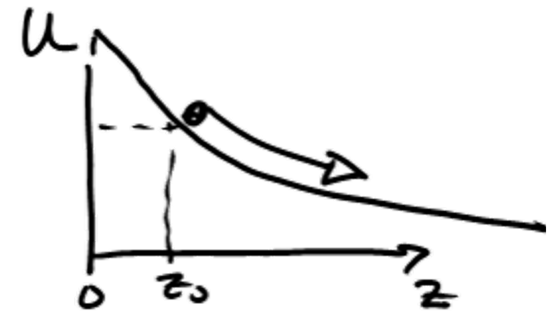
Example: a charged ring

Suppose the ring has charge $Q>0$, and a particle of mass m and charge $q>0$ is released from rest at point $z=z_0$ on axis. What is the speed of the particle when it is very far away from the ring? Assume effects of gravity are negligible.



Use conservation of energy.

$$U(z_0) = qV = \frac{KqQ}{\sqrt{z_0^2+a^2}}$$

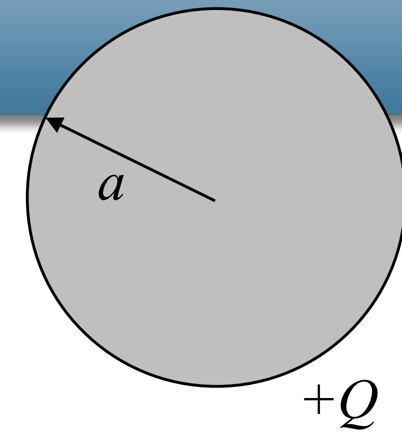


$$\cancel{K_i} + U_i = K_\infty + \cancel{U_\infty}$$

$$K_\infty = \frac{1}{2}mv^2 = \frac{KqQ}{\sqrt{z_0^2+a^2}} \qquad \therefore\ v = \sqrt{\frac{2KqQ}{m(z_0^2+a^2)^{1/2}}}$$

Calculation

Find the electric potential everywhere for a solid conducting <u>sphere</u> with radius a and total charge $+Q$.



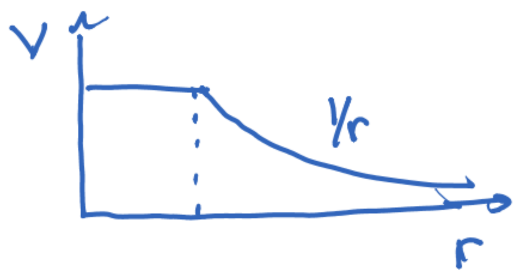
$E_{in} = 0$, $E_{out} = \frac{Q}{4\pi\epsilon_0 r^2}$ (Gauss' Law)

$r > a$: $V(r) - \cancel{V(\infty)}^{0} = -\int_\infty^r E_{out}\, dr$

$= \frac{Q}{4\pi\epsilon_0 r}$

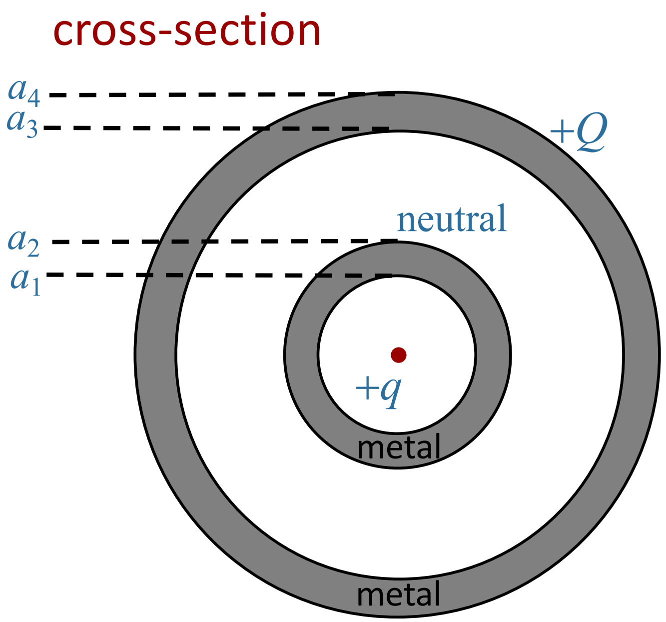
$r < a$: $V(r) = \Delta V(\infty \to r) = -\int_\infty^a E_{out}\, dr - \int_a^r \cancel{E_{in}}^{0}\, dr$

$= \frac{Q}{4\pi\epsilon_0 a} + 0 = \underline{\text{constant}}$ throughout conductor



- conductors are equipotentials

Another Calculation of Potential



Point charge q at center of concentric conducting spherical shells of radii a_1, a_2, a_3, and a_4. The inner shell is uncharged, but the outer shell carries charge Q.

What is V as a function of r?

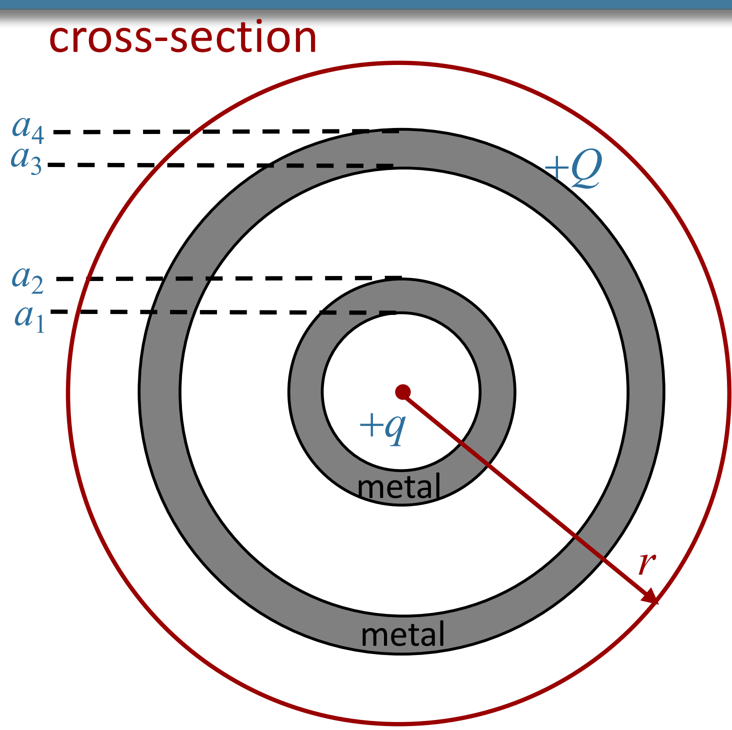
Conceptual Analysis:

- Charges q and Q will create an E field throughout space
- $$V(r) = -\int_{r_0}^{r} \vec{E} \cdot d\vec{\ell}$$

Strategic Analysis:

- Spherical symmetry: Use Gauss' Law to calculate E everywhere
- Integrate E to get V

Calculation: Quantitative Analysis



$r > a_4$: What is $E(r)$?

A) 0

B) $\frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$

C) $\frac{1}{2\pi\varepsilon_0}\frac{Q+q}{r}$

D) $\boxed{\frac{1}{4\pi\varepsilon_0}\frac{Q+q}{r^2}}$

E) $\frac{1}{4\pi\varepsilon_0}\frac{Q-q}{r^2}$

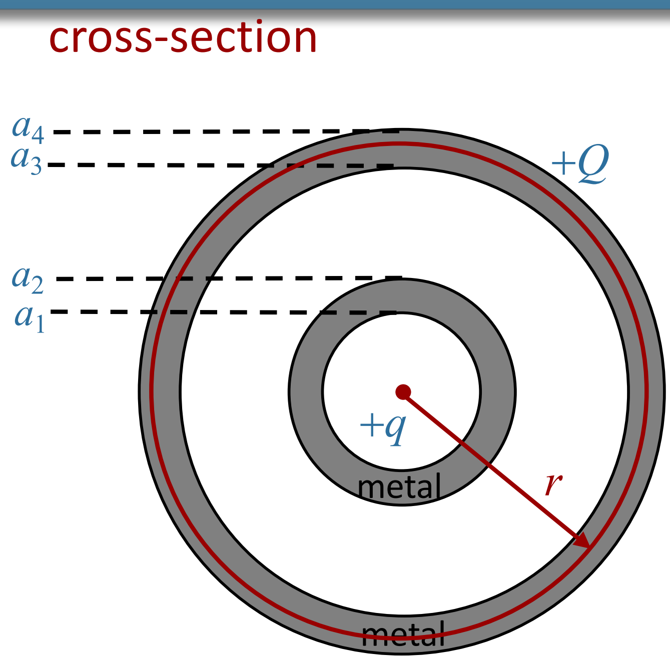
Why?

Gauss' law: $\int \vec{E}\cdot d\vec{A} = \frac{Q_{enclosed}}{\varepsilon_0}$

$$E4\pi r^2 = \frac{Q+q}{\varepsilon_0}$$

$$\rightarrow \quad E = \frac{1}{4\pi\varepsilon_0}\frac{Q+q}{r^2}$$

Calculation: Quantitative Analysis



$a_3 < r < a_4$: What is $E(r)$?

$\boxed{\text{A)}\ 0}$ B) $\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}$ C) $\frac{1}{2\pi\varepsilon_0}\frac{q}{r}$

D) $\frac{1}{4\pi\varepsilon_0}\frac{-q}{r^2}$ E) $\frac{1}{4\pi\varepsilon_0}\frac{Q-q}{r^2}$

Applying Gauss' law, what is $Q_{enclosed}$ for red sphere shown?

A) q B) $-q$ $\boxed{\text{C)}\ 0}$

How is this possible?

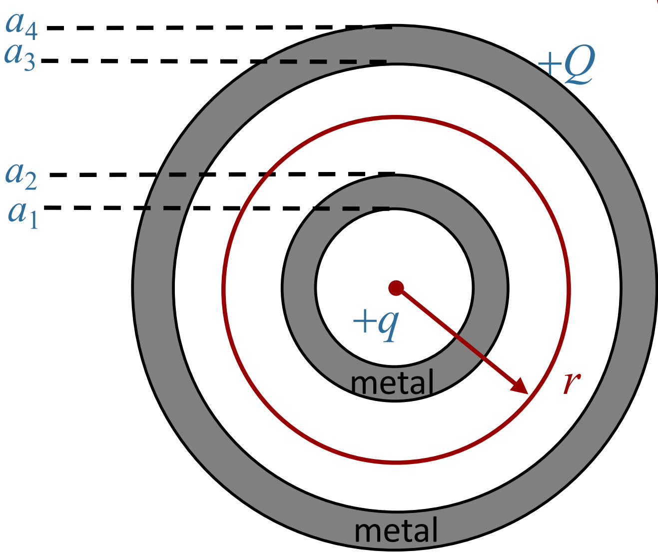
$-q$ must be induced at $r = a_3$ surface → charge at $r = a_4$ surface $= Q + q$

$$\sigma_3 = \frac{-q}{4\pi a_3^2}$$

$$\sigma_4 = \frac{Q+q}{4\pi a_4^2}$$

Calculation: Quantitative Analysis

cross-section



Continue on in…

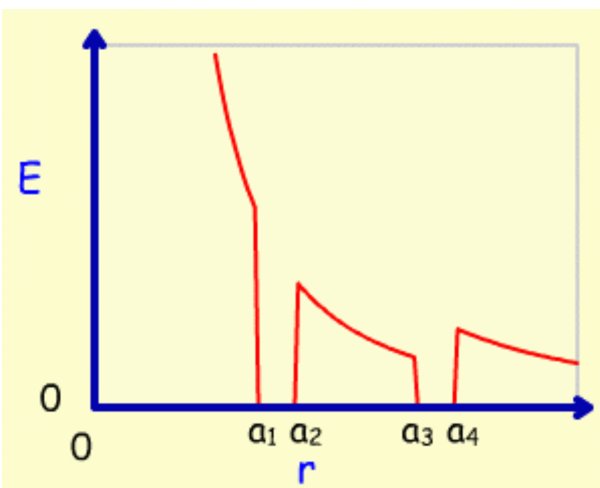
$r > a_4$: $\boxed{E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q+q}{r^2}}$

$a_3 < r < a_4$: $\boxed{E = 0}$

$a_2 < r < a_3$: $\boxed{E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}}$

$a_1 < r < a_2$: $\boxed{E = 0}$

$r < a_1$: $\boxed{E = \dfrac{1}{4\pi\varepsilon_0}\dfrac{q}{r^2}}$



To find V:

1) Choose r_0 such that $V(r_0) = 0$ (usual: $r_0 =$ infinity)
2) Integrate!

$r > a_4$: $V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q+q}{r}$

$a_3 < r < a_4$:

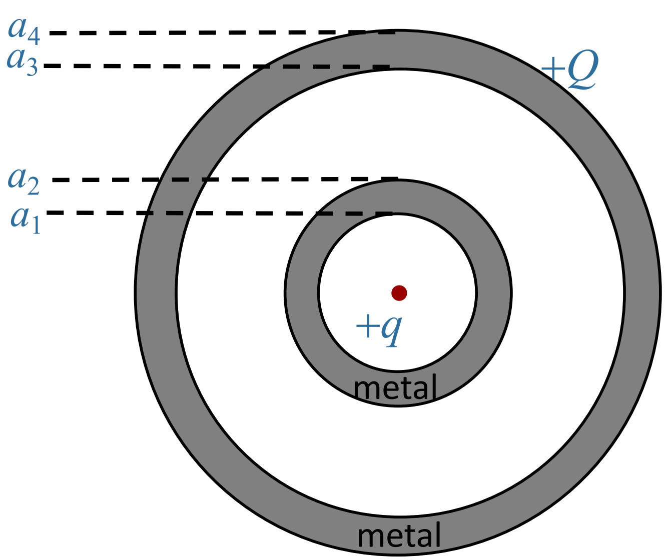
A) $V = 0$

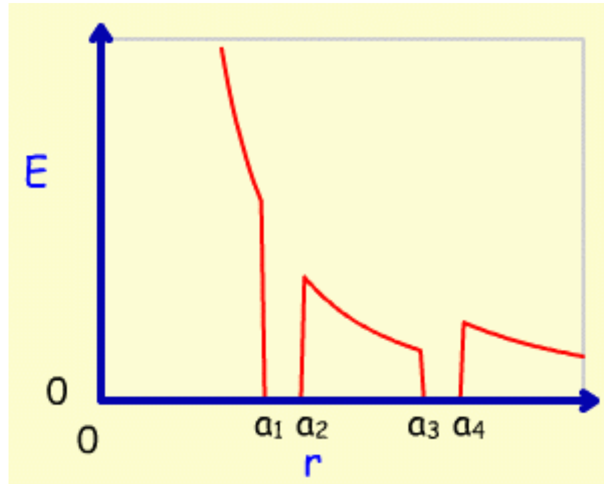
B) $\boxed{V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q+q}{a_4} = \Delta V(\infty \to a_4) + 0}$

C) $V = \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q+q}{a_3}$

Calculation: Quantitative Analysis

cross-section





$$r > a_4: \quad V = \frac{1}{4\pi\varepsilon_0}\frac{Q+q}{r}$$

$$a_3 < r < a_4: \quad V = \frac{1}{4\pi\varepsilon_0}\frac{Q+q}{a_4}$$

$$= -\int_{a_3}^{r} E\,dr = -\int_{a_3}^{r} \frac{q}{4\pi\epsilon_0}\frac{dr}{r^2}$$

$$= \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r} - \frac{1}{a_3}\right]$$

$$a_2 < r < a_3: \quad V(r) = \Delta V(\infty \to a_4) + 0 + \Delta V(a_3 \to r)$$

$$V(r) = \frac{Q+q}{4\pi\varepsilon_0 a_4} + 0 + \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r} - \frac{1}{a_3}\right)$$

$$\boxed{V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q+q}{a_4} + \frac{q}{r} - \frac{q}{a_3}\right)}$$

$$a_1 < r < a_2: \quad V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q+q}{a_4} + \frac{q}{a_2} - \frac{q}{a_3}\right)$$

$$0 < r < a_1: \quad V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q+q}{a_4} + \frac{q}{a_2} - \frac{q}{a_3} + \frac{q}{r} - \frac{q}{a_1}\right)$$

Final Thoughts

1. Written homework 1 is already assigned
2. Online homework 3 due next Monday (May 25) 11:59pm
3. Open Class 2 today at 230pm
4. See you at 1130am