UNIT 13:
ANGULAR MOMENTUM AND TORQUE AS VECTORS

Pure logical thinking cannot yield us any knowledge of the empirical world; all knowledge of reality starts from experience and ends in it.

-- A. Einstein

OBJECTIVES

1. To understand the definitions of torque and angular momentum as vector quantities.

2. To understand the mathematical properties and some applications of the vector cross product.

3. To understand the relationship between torque and angular momentum.

4. To understand the Law of Conservation of Angular Momentum.
This unit presents us with a consolidation and extension of the concepts in rotational motion that you studied in the last unit. In the last unit you studied the analogy and relationships between rotational and linear quantities (i.e. position and angle, linear velocity and angular velocity, linear acceleration and angular acceleration, and force and torque) without taking into account, in any formal way, the fact that these quantities actually behave like the mathematical entities we call vectors. We will discuss the vector nature of rotational quantities and, in addition, define a new vector quantity called \textit{angular momentum} which is the rotational analogue of linear momentum.

Angular momentum and torque are special vectors because they are the product of two other vectors – a position vector and a force or linear momentum vector. In order to describe them we need to introduce a new type of vector product known as the \textit{vector cross product}. We will explore the definition and unique nature of the vector cross product used to define torque and angular momentum.

We will study the relationship between torque and angular momentum as well as the theoretical basis of the Law of Conservation of Angular Momentum. At the end of this unit you will experience the effects of angular momentum conservation by holding masses in your hands and pulling in your arms while rotating on a platform. You will be asked to calculate your rotational inertia with your arms in and with your arms out by making some simplifying assumptions about the shape of your body.
SESSION ONE: TORQUE, VECTORS, AND ANGULAR MOMENTUM

Observation of Torque when \( \vec{F} \) and \( \vec{r} \) are not perpendicular.
In the last unit, you "discovered" that if we define torque as the product of a lever arm and perpendicular force, an object does not rotate when the sum of the torques acting on it add up to zero. However, we didn't consider cases where \( \vec{F} \) and \( \vec{r} \) are not perpendicular, and we didn't figure out a way to tell the direction of the rotation resulting from a torque. Let's consider these complications by generating torques with spring balances and a lever arm once more. For this activity you'll need:

- A horizontal pivot
- A clamp stand to hold the pivot
- Two identical spring scales
- A ruler
- A protractor

Activity 13-1: Torque as a Function of Angle
(a) Suppose you were to hold one of the scales at an angle of 90° with respect to the lever arm, \( r_h \), and pull on it with a steady force. Meanwhile you can pull on the other scale at several angles other than 90° from its lever arm as shown below. Would the magnitude of the second force be less than, greater than, or equal to the force needed at 90°? What do you predict? Explain.

(b) You should determine exactly how the forces compare to that needed at a 90° angle. Determine this force for at least four different angles and figure out a mathematical relationship between \( \vec{F} \), \( r \), and \( \theta \). Set up a spreadsheet to do the calculations shown in the table below. **Hint:** Should
you multiply the product of the measured values of \( r \) and \( F \) by \( \sin \theta \) or by \( \cos \theta \) to get a torque that is equal in magnitude to the holding torque?

### Holding Torque

<table>
<thead>
<tr>
<th>( r_h ) (m)</th>
<th>( F_h ) (N)</th>
<th>( \tau_h ) (N ( \cdot ) m)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Applied Torque

<table>
<thead>
<tr>
<th>( r_{app} ) (m)</th>
<th>( F_{app} ) (N)</th>
<th>( \theta ) (deg)</th>
<th>( \theta ) (rad)</th>
<th>( \cos \theta )</th>
<th>( \sin \theta )</th>
<th>( r_{app} F_{app} \cos \theta ) (Nm)</th>
<th>( r_{app} F_{app} \sin \theta ) (Nm)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(c) Within the limits of uncertainty, what is the most plausible mathematical relationship between \( \tau \) and \( r \), \( F \), and \( \theta \)?

The activity you just completed should give you a sense of what happens to the magnitude of the torque when the pulling force, \( \vec{F} \), is not perpendicular to the vector, \( \vec{r} \), from the axis of rotation. But how do we define the direction of the rotation that results when the torque is applied to an object that is initially at rest and not balanced by another torque? Let’s consider the directions we might associate with angular velocity and torque in this situation.
Activity 13-2: Angular Rotation, Torque, and Direction

(a) Suppose a particle is moving around in a circle with an angular velocity that has a *magnitude* of $\omega$ associated with it. According to observer #1, does the particle appear to be moving clockwise or counter clockwise? How about the direction of the particle's motion according to observer #2?

(b) Is the clockwise vs. counter-clockwise designation a good way to determine the direction associated with $\omega$ in an unambiguous way? Why or why not?

(c) Can you devise a better way to assign a minus or plus sign to an angular velocity?

(d) Similar consideration needs to be given to torque as a vector. Can you devise a rule to assign a minus or plus sign to a torque? Describe the rule.

Discussion of the Vector Cross Product

An alternative to describing positive and negative changes in angle is to associate a positive or negative vector with the axis of rotation using an arbitrary but well accepted rule called the right hand rule. By using vectors we can describe separate rotations of many body systems all rotating in different planes about different axes.

By using this vector assignment for direction, angular velocity and torque can be described mathematically as "vector cross products". The vector cross product is a very strange type of vector multiplication worked out many years ago by mathematicians who had never even heard of angular velocity or torque.

The peculiar properties of the vector cross product and its relationship to angular velocity and torque is explained in most introductory physics textbooks. The key properties of the vector which is the cross product of two vectors $\vec{r}$ and $\vec{F}$ are that:

1. the magnitude of the cross product is given by $rF\sin \theta$ where $\theta$ is the angle between the two vectors; $|\vec{r} \times \vec{F}| = |\vec{r}| |\vec{F}| \sin \theta$. Note that the term $F \sin \theta$ represents the component of $\vec{F}$ along a line perpendicular to the vector $\vec{r}$.

2. the cross product of two vectors $\vec{r}$ and $\vec{F}$ is a vector that lies in a direction $\perp$ to both $\vec{r}$ and $\vec{F}$ and whose direction is given by the right hand rule. Extend the fingers of your right hand in the direction of the first vector $\vec{r}$ and then rotate your fingers towards the second vector $\vec{F}$ and your thumb will then point in the direction of the resultant cross product $\vec{\tau}$.

These properties of the cross product are pictured below.

![Diagram of the Vector Cross Product](image)

**Figure 13-1**: Diagram of the Vector Cross Product

The spatial relationships between $\vec{r}$, $\vec{F}$ and $\vec{\tau}$ are very difficult to visualize. In the next activity you can connect some thin rods of various sizes to each other at angles of...
your own choosing and make some "vector cross products".
For this activity you will need the following items:

- Rods and connectors – Bamboo skewers and modeling clay
- A protractor

Activity 13-3: Making Models of Vector Cross Products

(a) Pick out rods of two different lengths and connect them at some angle you choose. Consider one of the rods to be the \( \vec{r} \) vector and the other to be the \( \vec{F} \) vector. Measure the angle \( \theta \) and the lengths of \( \vec{r} \) and \( \vec{F} \) in metres. Then compute the magnitude of the cross product as \( rF \sin \theta \) in newton-metres (N\( \cdot \)m). Show your units! **Note:** You should assume that the magnitude of the force in newtons is represented by the length of the rod in metres.

\[
|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}| |\vec{F}| \sin \theta =
\]

(b) Attach a "cross product" rod perpendicular to the plane determined by \( \vec{r} \) and \( \vec{F} \) with a length of \( rF \sin \theta \) Sketch the location of \( \vec{r} \) relative to \( \vec{F} \) in the space below. Show the direction and magnitude of the resultant torque \( \vec{\tau} \). **Finally, show your cross product model to an instructor or teaching assistant for confirmation of its validity.**
(c) In the diagrams below the vectors \( \vec{r} \) and \( \vec{F} \) lie in the plane of the paper. Calculate the torques for the following two sets of \( \vec{r} \) and \( \vec{F} \) vectors. In each case measure the length of the \( \vec{r} \) vector in metres and assume that the length of the \( \vec{F} \) vector in cm represents the force in newtons. Use a protractor to measure the angle, \( \theta \), between the extension of the \( \vec{r} \)-vector and the \( \vec{F} \)-vector. Calculate the magnitude of the torques. Place the appropriate symbol to indicate the direction of the torque in the circle as follows:

\[ \bigcirc = \text{arrow into the page} \quad \bigotimes = \text{arrow out of the page} \]

\[
\begin{array}{cc}
\vec{r} & \vec{F} \\
\bigcirc & \bigotimes \\
& \\
\end{array}
\]

\[
\begin{array}{cc}
\vec{r} & \vec{F} \\
\bigotimes & \bigcirc \\
& \\
\end{array}
\]

\[
\begin{array}{cc}
r = \_\_\_\_\_ m & r = \_\_\_\_\_ m \\
F = \_\_\_\_\_ N & F = \_\_\_\_\_ N \\
\theta = \_\_\_\_\_ \quad \theta = \_\_\_\_\_ \\
\tau = \_\_\_\_\_ Nm & \tau = \_\_\_\_\_ Nm \\
\end{array}
\]
Momentum and its Rotational Analogue
Once we have defined the properties of the vector cross product, another important rotational vector is easily obtained, that of angular momentum relative to an axis of rotation.

Activity 13-4: Angular and Linear Momentum
(a) Write the rotational analogues of the linear entities shown. **Note:** Include the formal *definition* (which is different than the analogue) in spaces marked with an asterisk (*). For example the rotational analogue for velocity is angular velocity $\omega \equiv |d\theta/dt|$ rather than $v/r$.

<table>
<thead>
<tr>
<th>Linear Entity</th>
<th>Rotational Analogue</th>
<th>Analogue Definition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$ (position)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$v$ (velocity)</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>$a$ (acceleration)</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>$F$ (Force)</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>$m$ (mass)</td>
<td>*</td>
<td></td>
</tr>
<tr>
<td>$\vec{F} = m\vec{a}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(b) What do you think will be the rotational *definition* of angular momentum in terms of the vectors $\vec{r}$ and $\vec{p}$? **Hint:** This is similar mathematically to the definition of torque and also involves a vector cross product. Note that torque is to angular momentum as force is to momentum.
(c) What is the rotational analogue in terms of the quantities \( I \) and \( \omega \)? Do you expect the angular momentum to be a vector? Explain.

(d) Summarize your guesses in the table below.

<table>
<thead>
<tr>
<th>Linear Equation</th>
<th>Rotational Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ \vec{p} = m\vec{v} ] (definition in terms of ( r ) and ( p ))</td>
<td>[ \vec{L} = ]</td>
</tr>
<tr>
<td>[ \vec{p} = m\vec{v} ] (analogue using ( I ) and ( \omega ))</td>
<td>[ \vec{L} = ]</td>
</tr>
</tbody>
</table>

---

**Observing a Spinning Bicycle Wheel**

If a bicycle wheel is spinning fairly rapidly, can it be turned easily so that its axis of rotation points in a different direction? If its axis it perfectly vertical while it is spinning will the wheel fall over? Alternatively, does it fall over when the wheel is not spinning? To make these observations we will use:

- An old bicycle wheel mounted on an axle
- A piece of string to wrap around the rim of the wheel to start it spinning

---

**Activity 13-5: Is Spinning More Stable?**

(a) Do you expect it to take more torque to change the axis of rotation of a wheel that is spinning rapidly or one that is spinning slowly? Or do you expect the amount of torque to be the same in both cases? Explain.
(b) Hold the wheel axis along a vertical line while the wheel is not spinning and change the axis from a vertical to a horizontal direction. Describe the "torque" it takes qualitatively.

(c) Have someone help you get the wheel spinning rapidly while you hold the axle vertical. While the wheel is spinning change the axis to the horizontal direction. Describe the "torque" it takes qualitatively. How does the torque compare to that needed to change the direction of the axis of rotation of the wheel when it is not spinning? Did you observe what you expected to observe?

(d) Does the magnitude of the angular velocity vector change as you change the axis of rotation of the wheel? Does its direction change? Does the angular velocity vector change or remain the same? Explain.

(e) Does the angular momentum vector change as you change the axis of rotation of the spinning wheel? Why or why not?

(f) If possible use your answer to part (e) above to "explain" what you observed in part (c) above.
Torque and Change of Angular Momentum

Earlier in this course you applied a very brief force along a line through the centre of mass of a rolling cart. Do you remember how it moved? What happened when you applied a gentle but steady force along a line through the centre of mass of the cart? Let's do analogous things to a disk which is free to rotate on a relatively frictionless bearing, with the idea of formulating laws for rotational motion that are analogous to Newton's laws for linear motion. For this observation, you will need:

- A rotational motion apparatus
- A clamp stand to mount the system on
- String
- Mass hanger and masses

Figure 13-2: Rotational Motion Apparatus.

Figure out how to use a system like that shown in Figure 13-2 to observe the motion of the disk under the influence of a brief torque and a steady torque. In describing the Laws of Rotational Motion be sure to consider vector properties and take both the magnitudes and directions of the relevant quantities into account in your wordings.

Activity 13-6: Applied Torques and Resultant Motion

a) What happens to the angular velocity and hence the angular momentum of the disk before, during, and after the application of a brief torque? State a First Law of Rotational Motion (named after yourself, of course) in
terms of torques and angular momenta. **Hint:** Newton's first law states that the centre-of-mass of a system of particles or a rigid object that experiences no net external force will continue to move at constant velocity.

The Rotational First Law in words:

The Rotational First Law as a mathematical expression:

(b) What happens to the magnitude and direction of the angular velocity (and hence the angular momentum) of the disk during the application of a steady torque? How do they change relative to the magnitude and direction of the torque? If possible, give a precise statement of a Second Law of Rotational Motion relating the net torque on an object to its change in angular momentum. **Note:** Take both magnitudes and directions of the relevant vectors into account in your statement. **Hint:** Newton's second law of motion states that the centre-of-mass of a system of particles or rigid object that experiences a net external force will undergo an acceleration inversely proportional to its mass.

The Rotational Second Law in Words:

The Rotational Second Law as a Vector Equation:
**SESSION TWO: ANGULAR MOMENTUM CONSERVATION**

**Fast vs. Slow \( dL/dt \) Action**

Recall that \( \tau = dL/dt \). Suppose that you start a wheel spinning so that its \( \vec{L} \) vector is pointing up, and that you then flip the wheel so that its \( \vec{L} \) vector points down. Which requires more torque during the "flipping time" – a fast flip or a slow one? To find out, you will need the following:

- An old bicycle wheel mounted on an axle
- A piece of string to wrap around the rim of the wheel to start it spinning

*Activity 13-7: Fast Flips and Slow Flips*

(a) Which action do you predict will require more applied torque on a spinning wheel – a fast flip or a slow flip? Explain the reasons for your prediction.

(b) Start a wheel spinning fairly rapidly. Try flipping it slowly and then as rapidly as possible. What do you observe about the required torques?

(c) Did your prediction match your observations? If not, how can you explain what you observed?
Angular Momentum Conservation
Now you can use the vector expression for Newton's second law of Rotational Motion to show that, in theory, we expect angular momentum on a system to be conserved if the net torque on that system is zero.

Only three things in this world are certain—death, taxes and conservation of momentum.

Activity 13-8: Angular Momentum Conservation
Using mathematical arguments show that, in theory, whenever there is no net torque on an object or system of particles, angular momentum is conserved.

Flipping a Rotating Wheel – What Changes?
In Activities 13-5 and 13-7 you should have discovered that it takes a healthy torque to change the direction of the angular momentum associated with a spinning bicycle wheel. Let's observe a more complicated situation involving a similar change of angular momentum. Consider a person sitting on a platform that is free to move while holding a spinning bicycle wheel. What happens if the person applies a torque to the bicycle wheel and flips the axis of the wheel by 180°? This state of affairs is shown in the diagram below.
Figure 13-3: Spinning wheel being flipped on rotating platform

For this observation you will need the following equipment:

- A rotating platform
- A person
- A bicycle wheel mounted on an axle
- A piece of string to wrap around the rim of the wheel to start it spinning

Activity 13-9: What Happens When the Wheel is Flipped?
(a) What do you predict will happen if a stationary person, sitting on a platform that is free to rotate, flips a spinning bicycle wheel over? Why?

(b) What actually happens? Does the result agree with your prediction?
(c) Use the Law of Conservation of Angular Momentum to explain your observation in words. **Hints:** Remember that angular momentum is a vector quantity. Does the angular momentum of the wheel change as it is flipped? If so, how does the angular momentum of the person and stool have to change to compensate for this?

---

**Changing Your Rotational Inertia**

In this activity you will verify the Law of Conservation of Angular Momentum qualitatively by rotating on a reasonably frictionless platform with your arms extended. You can then reduce your rotational inertia by pulling in your arms. This should cause you to rotate at a different rate. This phenomenon is popularly known as the *ice skater effect.* Since people can reconfigure themselves they are not really rigid bodies. However, in this observation we will assume that you can behave temporarily like two rigid bodies – one with your arms extended with masses and the other with your arms pulled in with the masses.
You can observe this effect qualitatively by using the following apparatus:

- A rotating platform
- Your body
- Two 2-kg masses

**Figure 13-4:** Simplified model of a human body as a combination of cylindrical shapes.

**Activity 13-10: The Effect of Reducing Rotational Inertia**

(a) According to the Law of Conservation of Angular Momentum, what will happen to the angular speed of a person on a platform if his or her rotational inertia is decreased? Back up your prediction with equations.
(b) Try spinning on the rotating platform. What happens to your angular speed as you pull your arms in?

If you were asked to verify the Law of Conservation of Angular Momentum quantitatively, you would need to calculate your approximate rotational inertia for two configurations. This process is a real tour de force, but it does serve as an excellent review of techniques for calculating the rotational inertia of an extended set of objects.

Assume that the platform has a rotational inertia given by $I_p = 1.0 \text{ kg m}^2$. Assume that each of your arms with the attached hand has a mass that is equal to a fixed % of your total mass as shown in the table below. Idealize yourself as a cylinder (rather than a square) with long thin rods as arms. You may have to look up some data in the text book to do the rotational inertia calculations.

<table>
<thead>
<tr>
<th>Arm/Hand</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td>Women</td>
<td>4.8%</td>
</tr>
<tr>
<td>Men</td>
<td>5.8%</td>
</tr>
</tbody>
</table>

**Figure 13-5:** Percentage of the mass of a typical person’s arm and hand relative to that person’s total body mass. Ref: Plagenhoef, Stanley. Patterns of Human Motion (Englewood Cliffs, NJ: Prentice-Hall, 1971), Ch. 3.

**Activity 13-11: Your Rotational Inertia**

(a) Find the total rotational inertia of the rotating system consisting of you, a pair of masses, and a rotating platform. Assume that you can hold the 2.0 kg masses at a distance of 5.0 cm from the axis of rotation when your elbows are in. **Hint:** Don’t forget to account for the mass and rotational inertia of the platform. Show all your work carefully.
(b) Find the total rotational inertia of the rotating system if you are holding a 2.0 kg mass in each hand at arm's length from your axis of rotation.

(c) Which part of system has the largest rotational inertia when your arms are extended (i.e. the trunk, arms, 2.0 kg masses, or platform)? Is the result surprising? Explain.
UNIT 13 HOME WORK AFTER SESSION ONE  (WEDNESDAY)

Before Friday November 25th:

• Read Chapter 12 in the Textbook *Understanding Physics*

• Work Supplemental Problem 13-1 listed below

• Work Textbook Exercises 12-16, 12-21 & 12-28

SP13-1) A particle is located at the vector position \( \vec{r} = (4\hat{i} + 6\hat{j}) \) m and the force acting on it is given by \( \vec{F} = (3\hat{i} + 2\hat{j}) \) N. What is the torque acting on the particle about the origin?

**Hint:** The vector cross product between two vectors \( \vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \) and \( \vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k} \) can be calculated as follows:

\[
\vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_x)\hat{k}
\]

UNIT 13 HOME WORK AFTER SESSION TWO  (FRIDAY)

Before Monday November 28th:

• Work Textbook Exercises 12-33,12-40 & 12-54

• Complete Unit 13 entries in the Activity Guide
INSTRUCTOR NOTES

11/25/92 Version 3.5 printed for use with students in Physics 131 at Dickinson College. The Unit number was changed from Unit 11 to Unit 14 to accommodate changes in the Newtonian dynamics portion of the Activity Guide. This Unit was reduced from 3 sessions to 2 sessions. Several activities were eliminated and the quantitative experiment in U11 S3 on angular momentum conservation was omitted for two reasons: (1) no time for it, and (2) it depends on the use of photogates which the students have no experience with this semester. Photogate functions have been replaced with video analysis. There was an abortive attempt to analyse the angular momentum of a merry-go-round on a playground with students moving on it. The video frames were taken from the Apple Visual Almanac video disk. The analysis involved too many approximations and the results were poor. Something else needs to be developed to demonstrate L conservation quantitatively.

11/30/92 Version 3.6 was created. The table in Activity 13-1 was improved for clarity (pp. 13-3 and 13-4).

11/93 Version 3.6 renumbered as version 2.0

11/20/93 Version 2.1 problem Assignments updated.