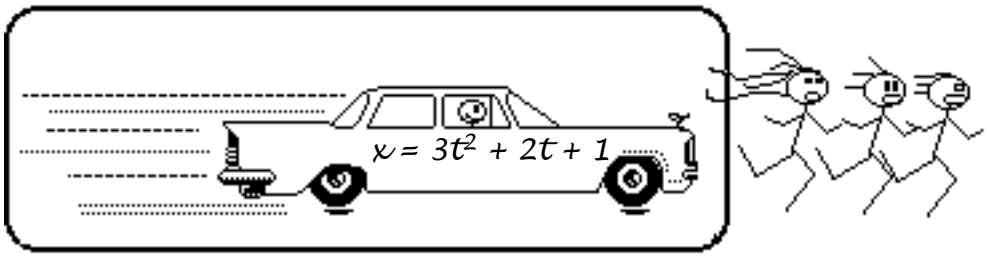
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UNIT 4: ONE-DIMENSIONAL MOTION II
A Mathematical Description



At the point where a speeding driver is caught by a cop, the cop comes up to the speeder and says, "You were going 60 miles an hour!" The driver says, "That's impossible, I was travelling for only 7 minutes. It is ridiculous – how can I go 60 miles an hour when I wasn't going an hour?

Richard Feynman, adapted from a joke in
The Feynman Lectures on Physics, V. 1

OBJECTIVES

To learn how physicists use mathematical equations to describe simple one-dimensional motions by:

1. Understanding the mathematical definitions of both *average* and *instantaneous* velocity and acceleration as well as the meaning of the slope of a position vs. time graph and of a velocity vs. time graph.

2. Learning to use different techniques for measuring length and time and to use mathematical definitions of average velocity and acceleration in one dimension to determine these quantities from fundamental measurements.

3. Checking the validity of some of the *kinematic equations*, which describe the motion of uniformly accelerating objects.

4. Using *mathematical modelling* techniques to determine equations describing one-dimensional motions you observe for uniformly accelerating objects and to compare these equations with the *kinematic equations*.

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OVERVIEW

By now you should have a good intuitive feeling for how to describe motions in terms of the changes in position, velocity, and acceleration that an object might undergo. You can tell people about these motions, graph them, or draw a picture of an object at various times with velocity vectors showing its relative speed and direction. However, the most concise quantitative description of simple motions is obtained by using mathematical equations. This unit is primarily concerned with the mathematical description of the motion of objects moving at constant velocity or constant acceleration.

Equations can describe motion more quantitatively than words or graphs. The equations used to describe simple motions can be graphed and conversely one can guess an equation and then see if it fits a graph of motion based on experimental data.

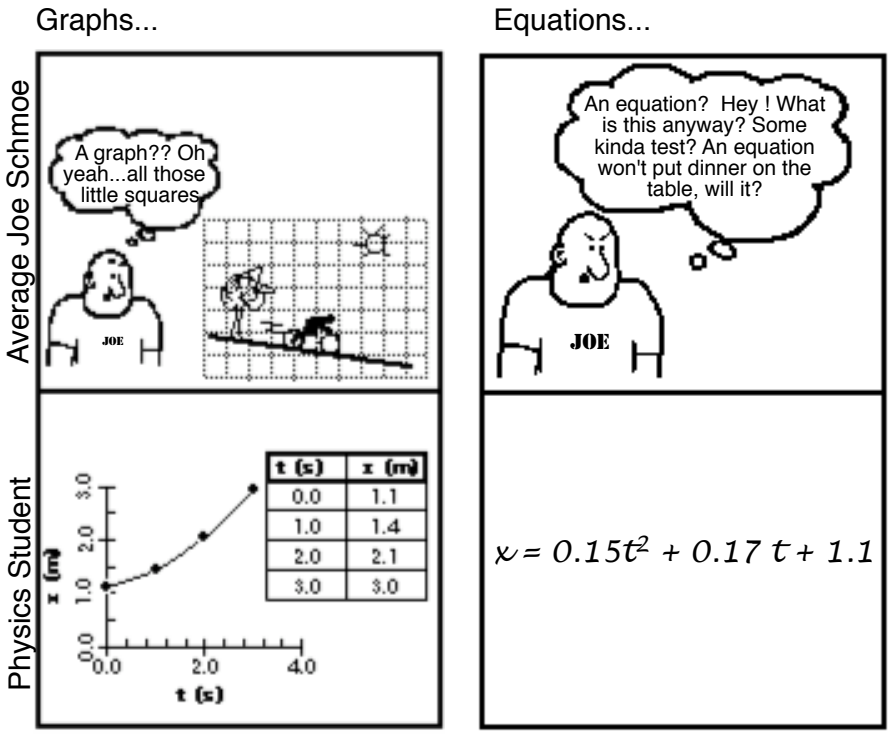


Figure 4-1: The key elements of this figure are a graph and equation representing a uniformly accelerated motion.

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There are two tasks in describing motion with mathematical equations. The first is to be able to use equations that define average velocity and acceleration to calculate these quantities from experimental data. The second is to use mathematical modelling to find idealized equations that describe how the location of objects change over time.

This unit begins with an exploration of the more formal mathematical definitions of average and instantaneous velocity and acceleration to represent "rate of motion" and the change in the "rate of motion" respectively.

In the last session you will explore the motion of a cart which is uniformly accelerated under various conditions and learn to find the equations which describe various motions. The equations that describe uniformly accelerated motion are known as *kinematic equations*. The kinematic equations can be deduced logically based on definitions of instantaneous velocity and acceleration and accepted rules of mathematics. These kinematic equations will then be compared to those you obtain by fitting equations to data. This allows you to verify the validity of the kinematic equations. These modelling activities will complete your quest to learn to represent one-dimensional motion in *words, graphs, pictures, and equations.*

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SESSION ONE: EQUATIONS TO DEFINE VELOCITY & ACCELERATION

Discussion of the Changing Motion Homework
You should share your observations on slowing down, speeding up and turning around and any conclusions you drew from them with your classmates.

Measuring Position as a Function of Time
When you used the motion detector, the computer took care of all the length and time measurements needed to track motion automatically. In order to understand more about how the motion software actually translates measurements into one dimensional velocities and accelerations it is helpful to make your own length and time measurements for a cart system.

Consider the type of uniformly accelerating cart motion that you studied in Unit 3. Suppose that instead of a motion detector you have a video camera off to one side so you can film the location of the cart 30 times each second. (This is the rate at which a standard video camera records frames.) By displaying frames at regular time intervals it is possible to view the position of the cart on each frame as shown in Figure 4-2 below.

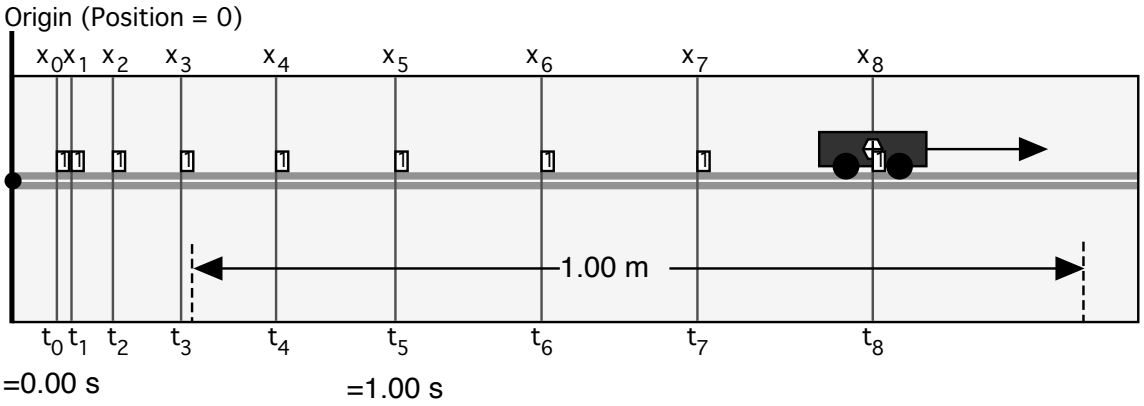


Figure 4-2: A scale diagram of the position of an accelerating cart at 8 equally spaced time intervals. The cart actually moved a distance of just less than 1 metre. Every 6th frame was displayed in the cart movie, so that 5 frames were recorded each second. *At each time the centre of the cart is located in the upper left corner of the rectangle with a number 1 in it.*

For the next activity you will need:

- a ruler with a centimetre scale on it.

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✍ Activity 4-1: Position vs. Time from a Cart Video

(a) Let's start the measuring process by recording the key scaling factors; this will allow you to calibrate. How much time, Δt, has elapsed between frame 0 and frame 1, between frame 1 and frame 2, etc. What is the calibration factor (i.e., how many real metres are represented by each centimetre in the diagram)?

Δt = _____ s m/cm = _____

(b) Use a ruler to measure the cart's distance from the origin (i.e., its position) in cm at each of the times 0.00 s, 0.20 s, etc. and fill in columns 1 and 2 in Table 4-1 below. Alternatively, if a computer-based video of this cart motion is available, determine the cart's distance from the origin in pixels (1 pixel = 1 picture element).

TABLE 4-1

		cm (or pixels) from origin in diagram	Elapsed Time (s)	Actual distance from origin (m)
		1	**2**	**3**
Frame #	Position	***d*( )**	***t*(s)**	***x*(m)**
0	x_0		0.000	
			0.100	
1	x_1		0.200	
			0.300	
2	x_2		0.400	
			0.500	
3	x_3		0.600	
			0.700	
4	x_4		0.800	
			0.900	
5	x_5		1.000	
			1.100	
6	x_6		1.200	
			1.300	
7	x_7		1.400	
			1.500	
8	x_8		1.600	

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(c) Use the scaling factor between "diagram centimetres" and real metres to calculate the position in metres of the cart. You'll want to use a spreadsheet for this. Place the results in column 3.

How Do You Define Average Velocity Mathematically?

By considering the work you did with the motion detector and with the measurements you just performed in Activity 4-1, you should be able to define average velocity along a line in words or even mathematically. *Remember that velocity is a rate of change of position divided by the time interval over which the change occurred.*

Note: Mathematically, change is defined as the difference between the final value of something minus the initial value of something.

Change = (Final Value) – (Initial Value)

✍ **Activity 4-2: Defining Velocity in One Dimension**

(a) Describe in *words* as accurately as possible what the word "velocity" means by drawing on your experience with studying velocity graphs of motion. **Hint**: How can you tell from the graph the direction an object moves? How can you tell how fast it is moving?

(b) Suppose that you have a long tape measure and a timer to keep track of a cart or your partner who is moving irregularly along a line. For the purposes of this analysis, assume that the object of interest is a mere point mass. Describe what you would need to measure and how you would use these measurements to *calculate* velocity *at a given moment in time.*

(c) Can you put this description in mathematical terms? Denote the average velocity with the symbol $<v>$. Suppose the distance from the origin (where the motion detector was when it was being used) to your partner is x_1 at a time t_1 just before the moment of interest and that the distance changes to x_2 at a later

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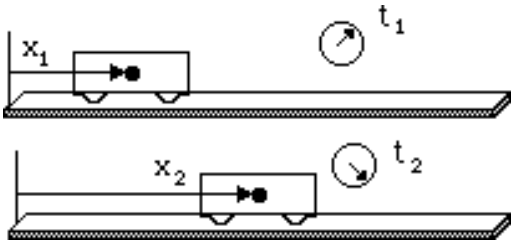


Figure 4-3: Rolling on a track!

time t_2 which is just after the moment of interest. Write the equation you would use to calculate the average velocity, $\langle v \rangle$, as a function of x_1, x_2, t_1, and t_2. What happens to the sign of $\langle v \rangle$ when x_1 is greater than x_2?

(d) Use the mathematical definition in part (c) to calculate each of the average velocities of the cart motion described in Activity 4–1 and fill in column 4 in Table 4-2 below. You will want to use a spreadsheet for this calculation. Show at least one sample calculation in the space below. **Important note:** $t_2 - t_1$ represents a time interval, Δt, between two measurements of position and is not necessarily the total time that has elapsed since a clock started.

TABLE 4-2

Elapsed Time (s) **2**	Actual distance from origin (m) **3**	Average velocity (m/s) **4**	Average acceleration (m/s) **5**
***t*(s)**	***x*(m)**	**$\langle v \rangle$(m/s)**	**$\langle a \rangle$(m/s/s)**
0.000			
0.100			
0.200			
0.300			
0.400			
0.500			
0.600			
0.700			
0.800			
0.900			
1.000			
1.100			
1.200			
1.300			
1.400			
1.500			
1.600			

Defining Average Acceleration Mathematically

By considering the work you did with the motion detector, you should be able to define average acceleration in one

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dimension mathematically. It is similar to the mathematical definition of average velocity which you developed in Activity 4-2. All of the circumstances in which accelerations are positive and negative are described by the equation that defines them.

A pirate is about to throw you overboard. As a consequence you start to walk more and more slowly along a plank. You are accelerating! Why?

✍ Activity 4-3: Defining Average Acceleration

(a) Describe in *words* as accurately as possible what the word "acceleration" means by drawing on your experience with studying velocity *and* acceleration graphs of motion.

(b) Suppose your *average* velocity is $\langle v_1 \rangle$ at a time t_1 and that the *average* velocity changes to $\langle v_2 \rangle$ at a later time t_2. Write an equation for the average acceleration in the space below.

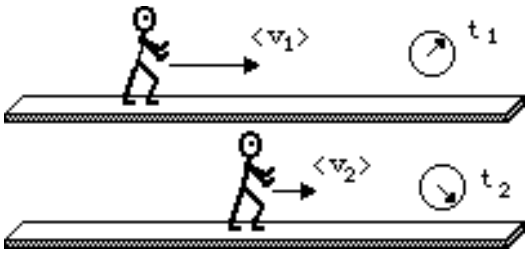


Figure 4-4: Walking the plank slower and slower!

(c) Use the mathematical definition in part (b) to calculate the average acceleration of the cart motion depicted in Activity 4-1 and fill in column 5 in Table 4-2 above for each time interval (i.e., 0.00 to 0.20 s, 0.20 s to 0.40 s, etc.). You will want to use a spreadsheet for this calculation. Show at least one sample calculation in the space below.

(d) Suppose you are walking away from a motion detector as shown above. How does your rate of your walking change if $\langle v_1 \rangle$ is greater than $\langle v_2 \rangle$? Is your acceleration positive or negative? Use the mathematical equation in part (b) to explain your answer.

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(e) Suppose you are walking toward a motion detector. How is your speed (i.e., magnitude of velocity) changing for $\langle v_1 \rangle$ greater than $\langle v_2 \rangle$? Is your acceleration positive or negative? *Be very careful with your mathematics on this one. It's tricky!*

(f) Is the sign of $\langle a \rangle$ always the same as the sign of the velocities? Why or why not? Hint: What happens when you walk the plank?

Instantaneous Velocity and the Slope of an *x* vs. *t* Graph

What happens if we want to know the velocity of an object at a single instant? That is, we'd like to estimate the velocity of an object during a time interval which is too small to measure directly. Since velocity is a measure of the change in position over time, it is possible to use techniques developed in calculus to estimate how a continuously varying function of position vs. time changes during a very short time interval. Let's start by considering how we might determine the slope of a continuous function and proceed from there.

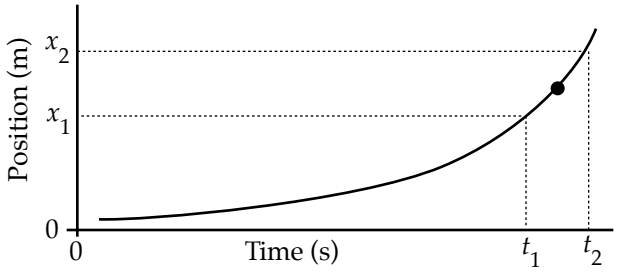


Figure 4-5: A General Graph of Position vs. Time

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✍ Activity 4-4: Defining the Slope or Tangent

(a) In Figure 4-5 above what is the equation for the *average slope* of the curve at the highlighted point in terms of x_1, x_2, t_1, and t_2?

(b) How is the value of the slope related to the average velocity in the time interval $[t_1, t_2]$?

(c) Since the rate of change of position is increasing as time goes on (so that the position "curve" is not linear), how can you calculate a more accurate value of the slope? **Hint:** Feel free to use different x and t values in your "calculation" to correspond to a different time interval.

(d) How would you find the "exact" value of the slope at the point in time of interest?

(e) Look up the formal definition of *derivative* in your calculus book and list it below. Usually it has to do with $f(x)$ and how it changes with x.

(f) Notice that in our position vs. time graph we are interested in how x changes with t. Thus, we would use the notation $x(t)$ to indicate that x is a function of t. By letting x play the role of f and t play the role of x, rewrite the definition of the derivative.

(g) How might the instantaneous value of velocity at the highlighted point be related to the derivative of x with respect to t at that same point?

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(h) Suppose that $x(t) = t^2 + 1$, where x is in centimetres and t is in seconds. What is the derivative of this function with respect to time? What is its instantaneous velocity?

(i) What is the instantaneous velocity, v, in cm/s at $t = 1$ s? At $t = 2$ s?

Acceleration as the Slope of a Velocity Graph

Just as velocity is the rate of change of position, acceleration is the rate of change of velocity. How do we find the acceleration of an object at a single instant (i.e., during a time interval which is too small to measure directly). Since acceleration is the rate of change of velocity, the acceleration of an object is given by the slope of a smooth curve drawn through its velocity vs. time graph. (See Appendix D for tips on fitting curves to data with uncertainties.)

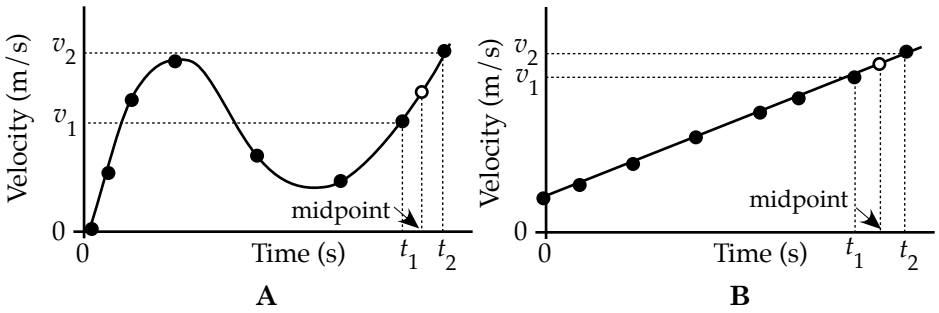


Figure 4-6: Graphs of velocity vs. time based on data with little uncertainty. A smooth curve can be sketched near the data points. The instantaneous acceleration can be approximated as the slope of the curve that fits the velocity data. Graph A shows non-uniform acceleration; graph B shows uniform acceleration.

Let's apply this graphical analysis approach to the task of finding the instantaneous acceleration for the cart motion described in Activity 4-1.

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✍ Activity 4-5: Accelerations from the Cart Data

(a) Refer to the data that you analysed and recorded in Table 4-2 above. Create a graph of the average velocity, $<v>$, as a function of time and affix it in the space below.
Note: If it is available, you'll want to use the Workshop Physics Excel Modelling Template to enter data and plot this graph.

(b) Find the slope of a line that seems to fit smoothly through most of the data points. The slope represents the acceleration of the cart. Report its value in the space below.

Note: If you have an Excel *Modelling Worksheet* available, you should use mathematical modelling to find the best line that fits the data. (See Appendix E and the Excel Modelling Tutorial Worksheet for hints on how to do mathematical modelling.)

(c) How does the slope you found in part (b) compare with the averages reported in column 5 of Table 4-2? Find the average and standard deviation of the elements in column 5 of Table 1.

$<<a>>$ (m/s^2) =

standard deviation, σ (m/s^2) =

(d) Does the acceleration determined by the slope lie within one standard deviation of the average of the average accelerations?

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Important Note: You should have just verified that finding the average velocities and then average accelerations for a uniformly accelerated cart leads to the same average acceleration as finding the slope of the velocity vs. time graph. This method of determining acceleration by finding the slope (or, as they say in calculus, the derivative) of the velocity vs. time function is a method you will use many times in physics. It is one of the easiest and most accurate techniques for describing accelerations.

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SESSION TWO: FINDING THE DISPLACEMENT FROM THE VELOCITY

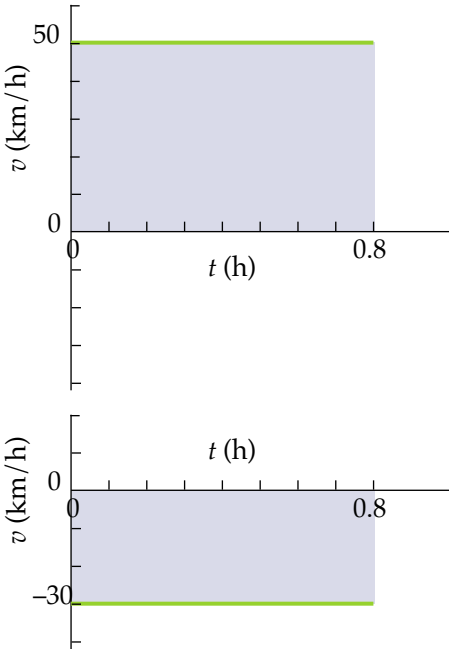


Figure 4-7: The area under a velocity vs. time graph represents the displacement. Top: the shaded area is above the axis representing a positive displacement. Bottom; In this case the velocity is negative and the displacement is also negative

Displacement at constant velocity

If a car is travelling at a constant velocity, v=50 km/h for a time of Δt=0.8 h then the displacement is given by

$\Delta x = v\Delta t$
Δx = (50 km/h)(0.8 h) = 40 km

Plotting a velocity vs. time graph shows this in a graphical representation of the displacement.

The displacement is represented graphically by the *area* under the line representing the velocity. Actually it is the area bounded by the velocity line, the t axis, and the vertical lines at t = 0 h and t = 0.8 h. The graphical area uses the units of each axis in its calculation. So instead of the area being calcuated in cm^2 the units would be (km/h)•h = km

If the car is going in the negative direction then both the velocity and the displacement are negative. For example a speed of 30 km/h in the negative direction would be v = –30 km/h. Travelling for 0.8 h would give a displacement of –24 km.

$\Delta x = v\Delta t$
Δx = (–30 km/h)(0.8 h) = –24 km

In this case the area representing the displacement is below the t axis which means that the displacement is negative.

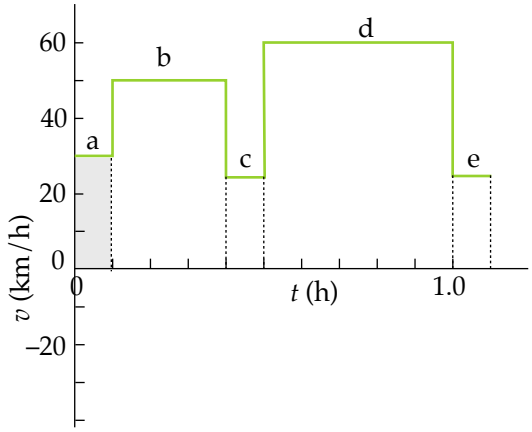


Figure 4-8: The displacements during intervals of different velocites accumulate. The total displacement is the sum of displacements during the constant-velocity intervals.

Let us apply the graphical analysis to complicated motion. Suppose that the motion of a car along a straight road is represented by the data in the following table:

Time Interval Number	Duration of Interval	Velocity During Interval	Displacement in km
a	0.10 h	30 km/h	3.0
b	0.30 h	50 km/h	15
c	0.10 h	25 km/h	2.5
d	0.50 h	60 km/h	30
e	0.10 h	25 km/h	2.5

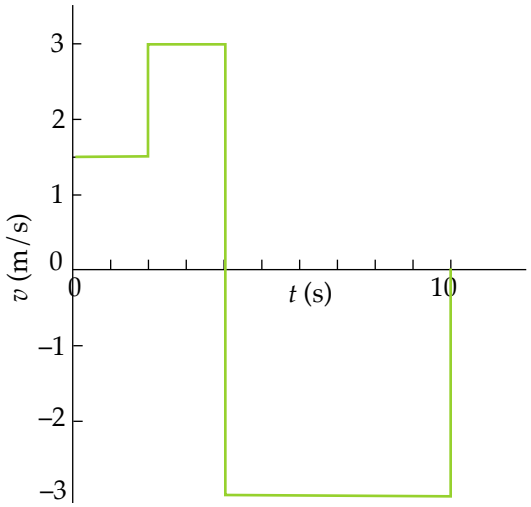


Figure 4-9: Find the displacement during each constant-velocity interval and the total displacement represented by the graph.

In each interval the car travels at a constant velocity, but the velocities are different. To find the total displacement one starts by computing the displacement in the first interval using the formula $\Delta x = v\Delta t$. Then do the same computation for each succeeding interval. Adding up the results for all intervals one finds that the total displacement is 53 km.

✍ Activity 4-6: Determining displacement from a *v* vs. *t* graph

The following graph represents the motion of a cart along a one-dimensional track. Make a table of the displacement in each interval of constant velocity and compute the total displacement in the 10-second interval.

Time Inteval Number	Duration of Interval	Velocity During Interval	Displacement in m

Total displacement = _____

Displacement with continuously changing velocity

When there is constant velocity during an interval of time it is obvious that the area under the velocity vs. time curve represents displacement during that time. Of course no real object can change from one velocity to another instantaneously. Often, as in the case of a car travelling from one speed zone to another, it may not be a bad approximation: the regions of changing velocity would be scarcely noticeable on the velocity vs. time graph.

There are many cases where the velocity is changing all the time and the v vs. t graph is not horizontal. Does the area under the velocity vs. time curve represent the displacement in this case as it does in the constant velocity case? This may seem plausible but a rigorous justification is needed before we use this recipe in such cases.

A simple example will illustrate how to justify using the area under a changing velocity curve to find displacement. Consider a cart, "cart O", that is changing velocity at a constant rate from 0 m/s to 8 m/s during an 8 s time inter-

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val. In this case we would say that the cart's acceleration is constant at 1 m/s/s or 1 m/s^2. Below is a graph of the cart's velocity and the shaded area beneath the v vs. t curve can be computed using the rule for the area of a triangle: Area = ½(Base×Height) = ½(8 s) × (8 m/s) = 32 m.

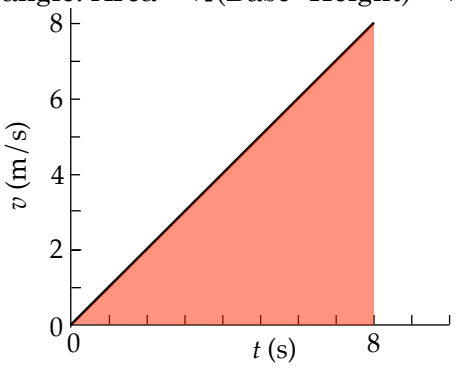


Figure 4-10: In this case the velocity is constantly changing. There is no interval of time with a constant velocity.

Is this 32 m the displacement of cart O? To justify that this is actually the displacement one imagines two other carts, A and B which both travel at constant velocities during subintervals of the 8 s. We divide the 8-s interval into several constant time slices. Carts A and B always travel at constant velocities during each subinterval, and instantly jump to different velocities between subintervals. Cart A starts at 0 m/s and increases its velocity in jumps so that at the start of each constant velocity interval it is travelling at the same velocity as cart O. Cart B starts at a constant velocity so that at the end of each interval it is going the same velocity as cart O. Therefore, Cart A is always going as slow or slower than O and Cart B is always going as fast or faster than O. It should be obvious that cart A will travel less distance than O and cart B will travel more distance.

There are three cases we will consider.
1. Two four-second time intervals,
2. Four two-second time intervals,
3. Eight one-second time intervals.

In each case cart A travels less far than O and cart B travels farther, but the difference should become less and less as the number of subdivisions increase from case 1 to case 3. If we would continue to subdivide the interval into 16, 32, 64, ... or more intervals the distances travelled by Cart A and Cart B should converge towards the distance actually travelled by Cart O which is constantly accelerating cart. If the three distances seem to converge to the same value, then we conclude that the area represented by the

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triangle is, in fact, the displacement of the accelerating cart.

✍ Activity 4-7: Determining displacement of an object with constantly changing velocity

Below are the graphs of the velocities of cart A and cart B for the three ways of subdividing the 8-s time interval during which cart O is accelerating.

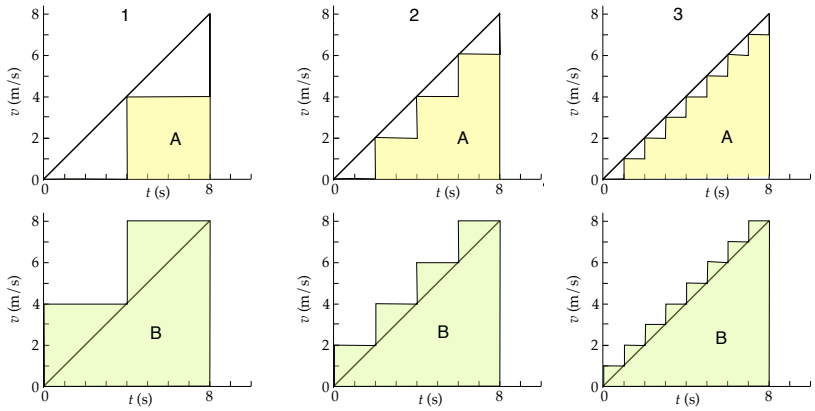


Figure 4-11: One can approximate the continuously changing velocity by finer and finer intervals of constant velocity. In case A the displacement in the approximation will be larger, and in case B smaller, than the actual displacement.

Calculate the displacements of carts A and B for the three cases and enter the values in the table. You may use a spreadsheet for this if you wish.

subdivision	cart A's displacement	cart B's displacement
2 four-s intervals		
4 two-s intervals		
8 one-s intervals		
limit		

You could continue subdividing into finer and finer intervals. Predict what displacements you would get for both cart A and cart B in the limit of infinitely many, very tiny, intervals and fill in the last row of the table.

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Finally compare these limits to the area under the triangle formed by cart O's velocity:

_____

Thus you should notice that it is reasonable to say that the area under a varying velocity curve gives the displacement of the cart. This same method can be used for any changing velocity, not only constant acceleration. As you study integral calculus in a mathematics course you will see a more rigourous proof.

Symbolically the constantly changing velocity represented by the graph is given by

$$v(t) = at$$

And the area under the curve is then the "integral" from the beginning time t_1 to the final time t_2.

$$\Delta x = \int_{t_1}^{t_2} at\,dt = a\int_{t_1}^{t_2} t\,dt = a\left[\frac{1}{2}t_2^2 \; - \frac{1}{2}t_1^2\right]$$

a can be moved in front of the integral because it is constant. In the illustrated case $t_1 = 0$ s and $t_2 = 8$ s and $a = 1$ m/s^{2}.

✍ Activity 4-8: Determining displacement of a real object by integrating the *v* vs *t* graph

Equipment needed
- Motion detector, computer and interface
- Logger Pro software
- Object that you can move in front of the detector (cart, book, small box, etc)
- Tape to mark positions of the object

Set up the motion detector to measure the velocity of an object that you move in front of it. You can choose to move ar cart on the track just any object that is easily detected.

Start the Logger Pro program and initialize the detector. Make sure it can reliably detect the object as you slide it back and forth. Set up to show only a velocity vs time graph of the object.

After it is all set up and ready to go, start a real data run by pressing "Start" and moving the object, starting fairly close to the detector, but more than 0.2 m. slide it away about 1 m and then back, stopping some distance from the starting point. One

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of your team should place a piece of tape to mark the first, last and turn-around position of the object's motion that was recorded by the computer.

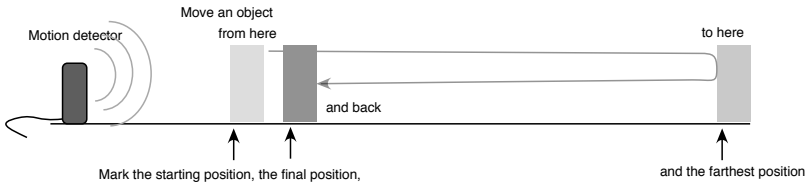
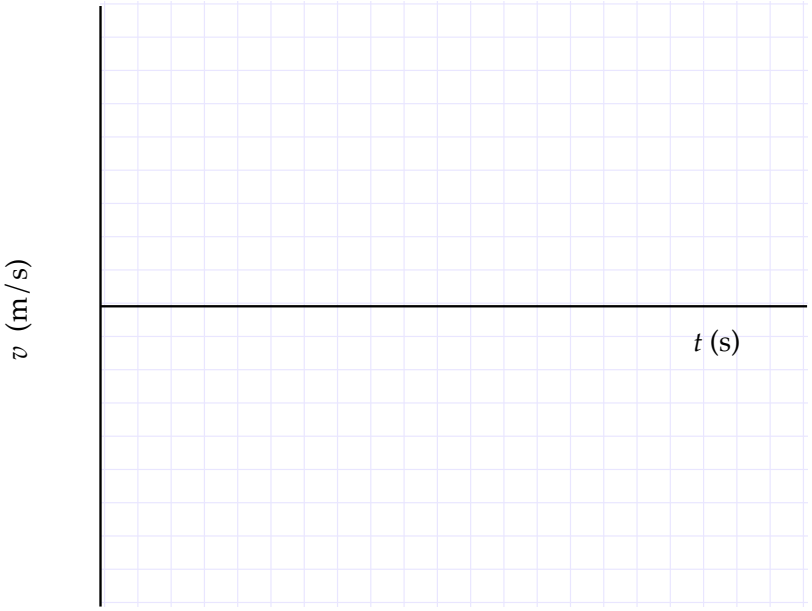


Figure 4-12: Move an object away and back while measuring its v vs. t graph with the motion detector. Mark the initial, final and farthest positions. The final position should not be coincident with the initial position but can be closer or farther away from the motion detector.

Sketch, as accurately as possible, the v vs. t graph that the computer produced. Be sure to fill in the scale on the axes.



Estimate the areas under the positive-velocity section of the curve and under the negative-velocity section. Write these estimates on the graph with proper units.
On the computer highlight the region of the v vs. t curve corresponding to the movement from the initial position to the turn-around point. Then choose "Integral" under the "Analyze menu. The area under the curve, otherwise known as the integral, is displayed. This should correspond to the distance travelled away from the detector.

Integral of $v(t)$ from initial position to farthest position _____
Distance travelled as measured with ruler _____

Now do the same for the region of the curve representing the return trip.

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Integral of $v(t)$ from farthest position to final position _____
Distance travelled as measured with ruler _____

Now integrate the entire $v(t)$ graph from initial to final position, including the turn-around point. This should be compared to the net displacement of the object as measured from initial position to final position

Integral of $v(t)$ from initial position to final position _____
Distance travelled as measured with ruler _____

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SESSION THREE: THE KINEMATIC EQUATIONS

Homework Discussion

Determining Instantaneous Velocities and Accelerations from Simple Equations

Instantaneous velocity is defined as the time derivative of the function which describes how position changes in time. Similarly, instantaneous acceleration is defined as the time derivative of the function which describes how the instantaneous velocity varies in time. Thus, for an object moving in one dimension

$$v \equiv \frac{dx}{dt} \quad \text{and} \quad a \equiv \frac{dv}{dt}$$

Note: The triple bar symbol (≡) is stronger than an equality. It means "defined as".

Beware:
The kinematic equations you are about to derive in the space below ONLY APPLY WHEN AN OBJECT UNDERGOES CONSTANT ACCELERATION.

In the special case in which the variation of a position or velocity with time can be represented by a power of time (bt^n), the derivative can be taken quite easily. The instantaneous velocity is given by

$$v \equiv \frac{dx}{dt} = nbt^{(n-1)}$$

For example, if $x = 5t^3$ then

$$v \equiv \frac{dx}{dt} = 5(3)t^2 = 15t^2$$

The acceleration can be found by taking the time derivative of the velocity function, v_x.

✍ Activity 4-9: Determining v and a by Differentiation

(a) Suppose $x = 4t^2$ m. Find a general expression for v_x as a function of time. What is v at t =0 s? At t = 2 s? (Don't forget units!)

$$v \equiv \frac{dx}{dt} =$$

(b) Use the general expression for v_x you found in part (a) to find a general expression for the x-component of acceleration, a_x, as a function of time. What is the value of a at t =0 s? At t = 2 s? Is the acceleration constant or does it change in time?

$$a \equiv \frac{dv}{dt} =$$

(c) Suppose $x = 3t^4$. Find a general expression for v_x as a function of time. What is the value of v_x at t =0 s? At t = 2 s?

(d) Use the general expression for v_x you found in part (c) to find a general expression for the x-component of acceleration, a_x, as a function of time. What is the value of a_x at t =0 s? At t = 1 s? Is the acceleration constant or does it change in time?

(e) If the derivative of the sum of two functions is equal to the sum of the derivatives of the two functions, what is the instantaneous velocity of an object whose position is given by $x = 4t^2+3t^4$. **Hint:** Note that this is the sum of the functions in parts (a) and (c)

1D Kinematic Equations for Constant Acceleration

So far, you should have concluded that all of the falling motions in the lab have resulted in accelerations that are more or less constant. *There is a standard set of equations (which can be derived using the principles of calculus) that describe the motion of an object that undergoes constant acceleration.* These equations are called the *kinematic equations* and they are derived in your text book. By re-examining the graphs you have sketched which describe objects moving with constant acceleration, and by using the definitions of instantaneous velocity and acceleration, you can verify that the kinematic equations describe uniformly accelerated motion. We will use the following symbols for this exercise:

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x = position along the x axis (which can vary with time)

v = instantaneous velocity along the x-axis (which can also vary with time)

a = constant acceleration along the x-axis (It does not vary in time because we have chosen to consider only those cases for which a is constant.)

x_o = initial position at t=0

v_o = initial velocity component along the x-axis at t=0

There are four constant-acceleration kinematic equations commonly found in physics textbooks. The most fundamental kinematic equation derived in any physics textbook is the equation describing x as a function of t when the initial position is x_0 and the initial velocity is v_0:

The Fundamental Kinematic Equation: $$x = \frac{1}{2}at^2 + v_0t + x_0 \qquad [4.1]$$

This equation indicates that a graph showing the position as a function of time of any motion with constant acceleration is a parabola of some sort.

Note: *All other kinematic equations can be obtained from this one and the definitions of instantaneous velocity and acceleration.*

✍ Activity 4-10: Verification of the Kinematic Equations

(a) Use the definition of instantaneous velocity, $v \equiv \frac{dx}{dt}$, and take the derivative of x with respect to time to show that if kinematic equation #1 is valid then the second kinematic equation would be

Kinematic Equation #2: $$v = at + v_0 \qquad [4.2]$$

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(b) Use the definition of instantaneous acceleration, $a \equiv \frac{dv}{dt}$, to show that the second kinematic equation is valid (i.e., that a = constant) by taking the derivative of equation #2 with respect to time.

The other two kinematic equations given in the text can be derived by combining equations [4.1] and [4.2]. However, since all constant acceleration problems can be solved using equation [4.1] for x(t) and the rules for taking a derivative to get equation [4.2], we do not recommend trying to memorize the other equations listed in the text.

Parabolas and the Fundamental Kinematic Equation
In this session we will explore the parabolic nature of the fundamental kinematic equation, which describes the position of a uniformly accelerated object as a function of time. Then you will do some mathematical modelling on some of your old data. In particular, you will practice finding the appropriate equations to describe the position vs. time data you obtained using the motion detector.

In most basic mathematics courses the parabola is studied extensively as a simple example of a polynomial equation. To be fancy, we'd say that a parabola is a second-order polynomial. Mathematicians usually give the equation for the parabola as follows:

$$f(x) = c_1x^2 + c_2x + c_3$$

where c_1, c_2, and c_3 are constants. Although all parabolas have the same basic shape, some are skinny, some are fat, some start one place on a co-ordinate axis and some start another place. Some parabolas are even upside down! The magnitudes of the constants and whether they are positive or negative determine how a given parabola looks.

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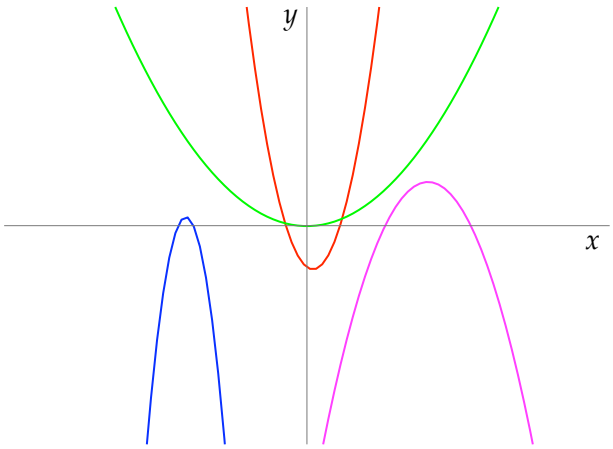


Figure 4-13: Some skinny, fat, and upside down parabolas

In the study of kinematics a different notation is used for the parabolas and each coefficient represents a physical quantity. Thus, using kinematic equation #1 (Eq. 4-1) we get:

$$x = \frac{1}{2}at^2 + v_0t + x_0$$

Thus, we say that position, x, is a function of the elapsed time, t. What kind of parabolas do we get for various values of acceleration (a), initial velocity (v_0), and initial position (x_0) for a uniformly accelerated object? Plotting some parabolas using computer software should provide you with some quick answers to this question. Most of the parabolas you will plot will only be half parabolas. Can you figure out why?

✍ Activity 4-11: Parabolas for Different x_o Values

(a) Explore the effect of the value of x_o on the location of the parabola represented by the equation

$$x = \frac{1}{2}at^2 + v_0t + x_0$$

By examining the equation, predict what you think the effect of the initial position will be on the graph of x vs. t. Explain the reasons for your prediction.

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(c) Test your prediction by drawing some graphs. Let a = 1.00 m/s^2 and the initial velocity v_0 of the object be 0.00 m/s. Let the values of time, t, range from 0.000 s to 1.600 s and do a series of theoretical graphs for different values of x_0. For instance, you might try x_0 = 0.00 m, +0.70 m, -0.50 m, etc. Sketch the shapes of the resulting graphs in the space below.

(d) What is the effect of x_0 on the location of the parabola? Is it what you predicted? Explain.

✍ Activity 4-12: Parabola Shapes for Different a Values

(a) Explore the effect of the value of a on the shape of the parabolic function describing how x changes over time. Assume the object starts from rest at the origin at t=0 so that v_0 = 0 m/s and x_0 = 0 m.

How do you think increasing the value of a will effect the shape of the parabola? What do you think will happen if a is negative? Explain!

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(b) Test your predictions by plotting the equation

$$x = \frac{1}{2}at^2$$

for a=+1.00, +3.00, +5.00 m/s^2, etc. and for a=-1.00, -3.00, -5.00 m/s^2., etc. Let t range between 0.000s and +1.600 s again. Sketch the results below and explain how changing a in various ways affects the shape of the parabolas you get

(c) Does the nature of the change agree with your observations in Lab 2 when you used a motion detector to record the positions of a cart speeding up and when you recorded the positions of a cart speeding up even faster? Explain. Re-sketch the two graphs you observed.

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Modelling: Fitting an Observed Motion with an Equation

You should use the Excel modelling worksheet and transfer time and distance data from one of your past experiment files containing motion detector data for an accelerated cart.

✍ Activity 4-13: Modelling Data

(a) Where does the data you plan to use for distance and time come from? Summarize it in the space below.

(b) Examine your data. What are the values of v_0 and x_0 corresponding to the motion you tracked?

(c) Enter theoretical values for position and time into the theory column in the modelling worksheet. Adjust the value of a until the theoretical curve passes very near or through most of the experimental data points. Upload your spreadsheet with the curve on it to your WebCT assignment submission for this activity.

(d) Write down the equation that describes the motion you studied.

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