Astrophyical Applications of the Tolman-VII Solution

A Physically Realizable Solution to the Einstein Equations

Ambrish Raghoonundun, David Hobill

Department of Physics and Astronomy

University of Calgary

Exact Solutions to Einstein Equations

- Mathematical Problem: Find or Construct Solutions
- Physical Viability Problem: Apply Physically Realistic Conditions
- Physical Realization Problem: Compare with Observation

Example: Spherically Symmetric Static Solutions

- Over 130 known exact solution (with perfect fluid sources)
- Less than 10% obey conditions for being physically realistic
- What systems do physically viable solutions describe? If any?

Spherically Symmetric Newtonian Star

(1) Hydrostatic equilibrium (Pressure gradient = gravitational force density)

$$\frac{dP}{dr} = -G\frac{M(r)\rho}{r^2}$$

Pressure is a decreasing function of radius ($P(r_b) = 0$)

(2) Mass conservation: (mass = volume $\times$ density)

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho$$

Mass is an increasing function of radius

Need an equation of state (EOS) $P = P(\rho, T, \Pi)$

Pressure depends on particle interactions

$$P_{\mathrm{idealgas}} = \frac{\rho k T}{\bar{m}} \qquad \mathrm{or} \qquad P_{\mathrm{radiation}} = \frac{1}{3}aT^4$$

Spherically Symmetric Exact Solutions In General Relativity

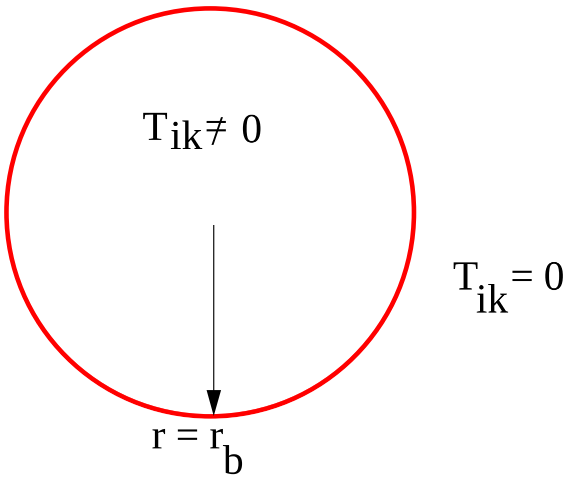
A Representation of a Massive Object in General Relativity

$$R_{ik} - \frac{1}{2} R g_{ik} = \frac{8\pi G}{c^4} T_{ik}$$

Energy - momentum Tensor T_{ik} must be realistic (perfect fluid)

$$T_{ik} = \begin{cases} (P+\rho) u_i u_k - g_{ik} P, & r \leq r_b \\ 0, & r > r_b \end{cases}$$

Simplest solutions are spherically symmetric - match to vacuum Schwarzschild solution at fluid boundary



STATIC SPHERICALLY SYMMETRIC FIELD EQUATIONS

Line element with an areal (Schwarzschild) radial coordinate, r:

$$ds^2 = e^{\nu(r)}dt^2 - e^{\lambda(r)}dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\phi^2.$$

Field Equations:
Einstein equations become ($G = c = 1$, $\kappa = 8\pi$):

$$\frac{\lambda_r}{r}e^{-\lambda} + \frac{1}{r^2}(1 - e^{-\lambda}) = \kappa\rho \qquad (1)$$

$$\frac{\nu_r}{r}e^{-\lambda} - \frac{1}{r^2}(1 - e^{-\lambda}) = \kappa P \qquad (2)$$

$$e^{-\lambda}\left[\frac{\nu_{rr}}{2} - \frac{\lambda_r\nu_r}{4} + \frac{(\nu_r)^2}{4} + \frac{\nu_r - \lambda_r}{2r}\right] = \kappa P \qquad (3)$$

Three equations for 4 unknowns $[\nu(r), \lambda(r), \rho(r), P(r)]$

Add an equation of state: $P = P(\rho)$ to close system

Alternative method - Tolman, Oppenheimer-Volkoff

Bianchi identity $T^i_{k;i} = 0$ leads to:

$$\frac{dP}{dr} = -\frac{1}{2}(P + \rho c^2)\left(\frac{d\nu}{dr}\right)$$

define Mass aspect function:

$$M(r) = 4\pi \int_0^r \rho r^2 dr$$

Elimination of λ and ν leads to

$$\frac{dP}{dr} = -\frac{G(\rho c^2 + P)(4\pi P r^3/c^2 + M(r))}{r(c^2 r - 2GM(r))} \qquad \text{(TOV)}$$

Relativistic equivalent to Newtonian hydrostatic equilibrium solution

Steeper pressure gradient: (extra P terms in numerator and $r^2 \rightarrow r(rc^2 - 2GM)$)

Numerical integration most often required to integrate outward in r

First Exact Analytic Solutions for a Fluid Sphere

EXAMPLE: Schwarzschild Solutions

(a) Exterior Solution (1916) ($\rho = P = 0$)

$$ds^2 = \left(1 - \frac{2GM}{r}\right) - \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2$$

(b) Interior Solution (1919) ($\rho = \mathrm{constant}$)

$$ds^2 = \frac{1}{4}\left[3\sqrt{1 - Ar_b^2} - \sqrt{1 - Ar^2}\right]^2 dt^2 - \frac{1}{1 - Ar^2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta d\varphi^2$$

$$A = \frac{8\pi}{3} G\rho \qquad \frac{8\pi G}{c^2} P = 3A \frac{\sqrt{1 - Ar^2} - \sqrt{1 - Ar_b^2}}{3\sqrt{1 - Ar_b^2} - \sqrt{1 - Ar^2}}$$

Conditions for Physically Realistic Interior Solutions

1. The metric functions match to the exterior (Schwarzschild) metric functions at the fluid-vacuum interface.
2. The mass density is positive and finite every where inside fluid.
3. The integrated mass increases outward. Equals Schwarzschild mass at the boundary.
4. The pressure P is positive and finite everywhere inside the fluid.
5. The pressure vanishes at the fluid boundary with the vacuum.
6. Both the pressure and mass density are decreasing functions of r: $dP/dr < 0$ and $d\rho/dr < 0$.
7. The speed of sound $v_{\rm s} = (dP/d\rho)^{1/2}$ is causal ($0 \leq v \leq c$).
8. The speed of sound decreases monotonically from centre to outer surface.

Two sub-classes

Natural case: ($\rho(r_b) = 0$), Self-bound case: ($\rho(r_b) \neq 0$)

Surveys of Known Solutions obeying Physical Conditions

Massive fluid sphere solutions (uncharged) with isotropic pressure

1 M.S.R. Delgaty and K. Lake, Comput. Phys. Commun. **115**, 395, (1998) [arXiv:gr-qc/9809013].

2 M.R. Finch and J.E.F. Skea, unpublished preprint, www.dft.if.euerj.br/users/Jim_Skea/papers/pfrev.ps

Studies of over 130 Known Explicit Solutions using Computer Algebra

1. Using MAPLE
2. Using SHEEP

Conclusion: Only eight (8) solutions satisfy ALL physical properties.

Physically realistic solutions are RARE

"Physically Interesting Solutions " allow an explicit EOS $P = P(\rho)$

Finch and Skea

TOLMAN VII Solution

Give an ansatz for g_{rr} solve for ρ, then g_{tt} and finally P

$$-e^{-\lambda(r)} = 1 - \frac{r^2}{R^2} + \frac{4r^4}{A^4}$$

$$\kappa\rho = \frac{3}{R^2} - \frac{20r^2}{A^4}$$

$$e^{\nu(r)} = B^2 \sin^2 \left[\ln \left(\frac{e^{-\lambda/2} + 2r^2/A^2 - A^2/4R^2}{C} \right)^{\frac{1}{2}} \right].$$

where A, B, C and R are constants
Match to Schwarzschild Exterior at Boundary $r = r_b$

$$-e^{-\lambda} = e^{\nu} = 1 - \frac{2m}{r}$$

“ The dependence of P on r [$\cdots$] is so complicated that the solution is not a convenient one for physical considerations.”

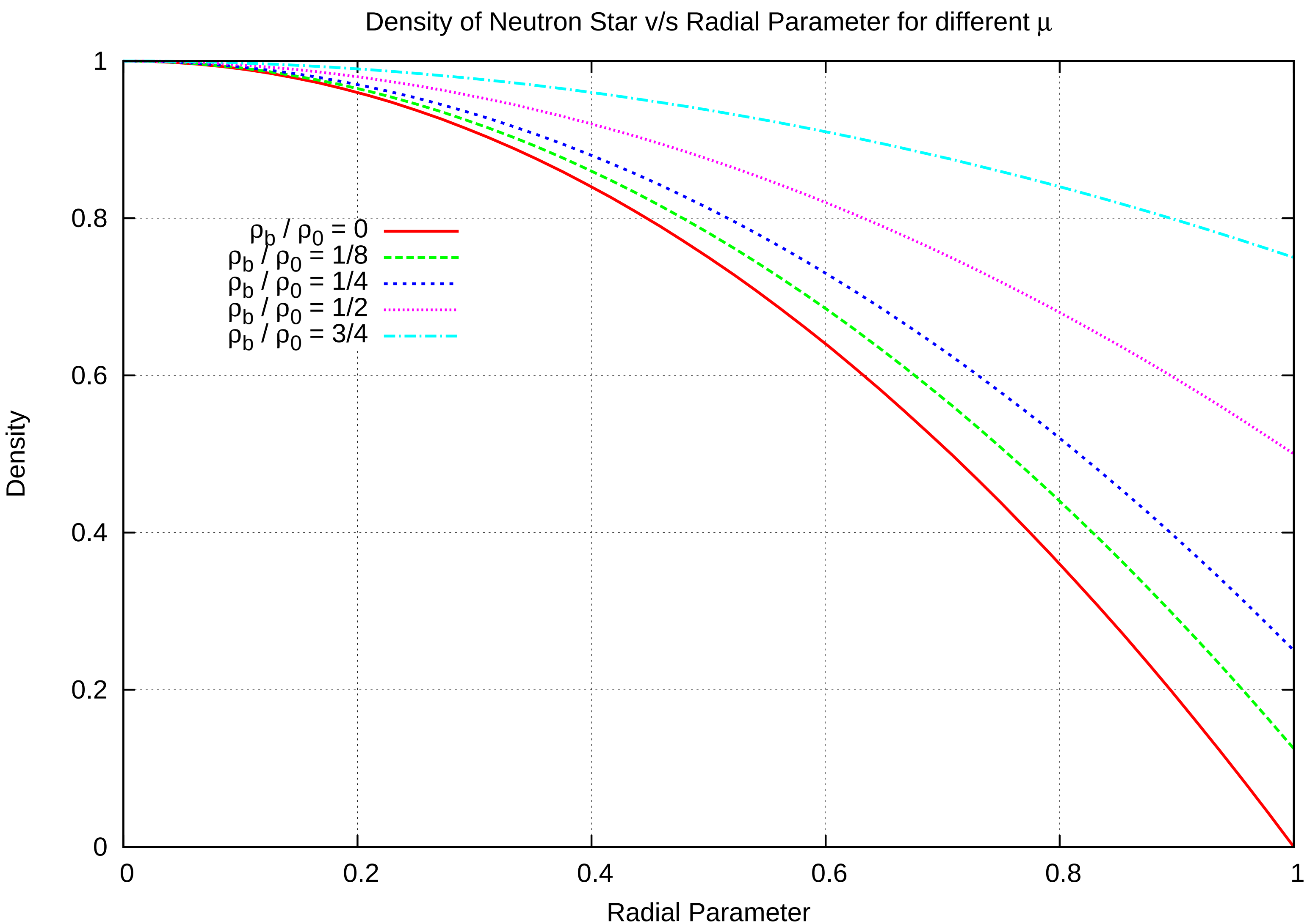
Tolman (1939)

What kind of Object does Tolman VII describe??

Tolman VII Density function

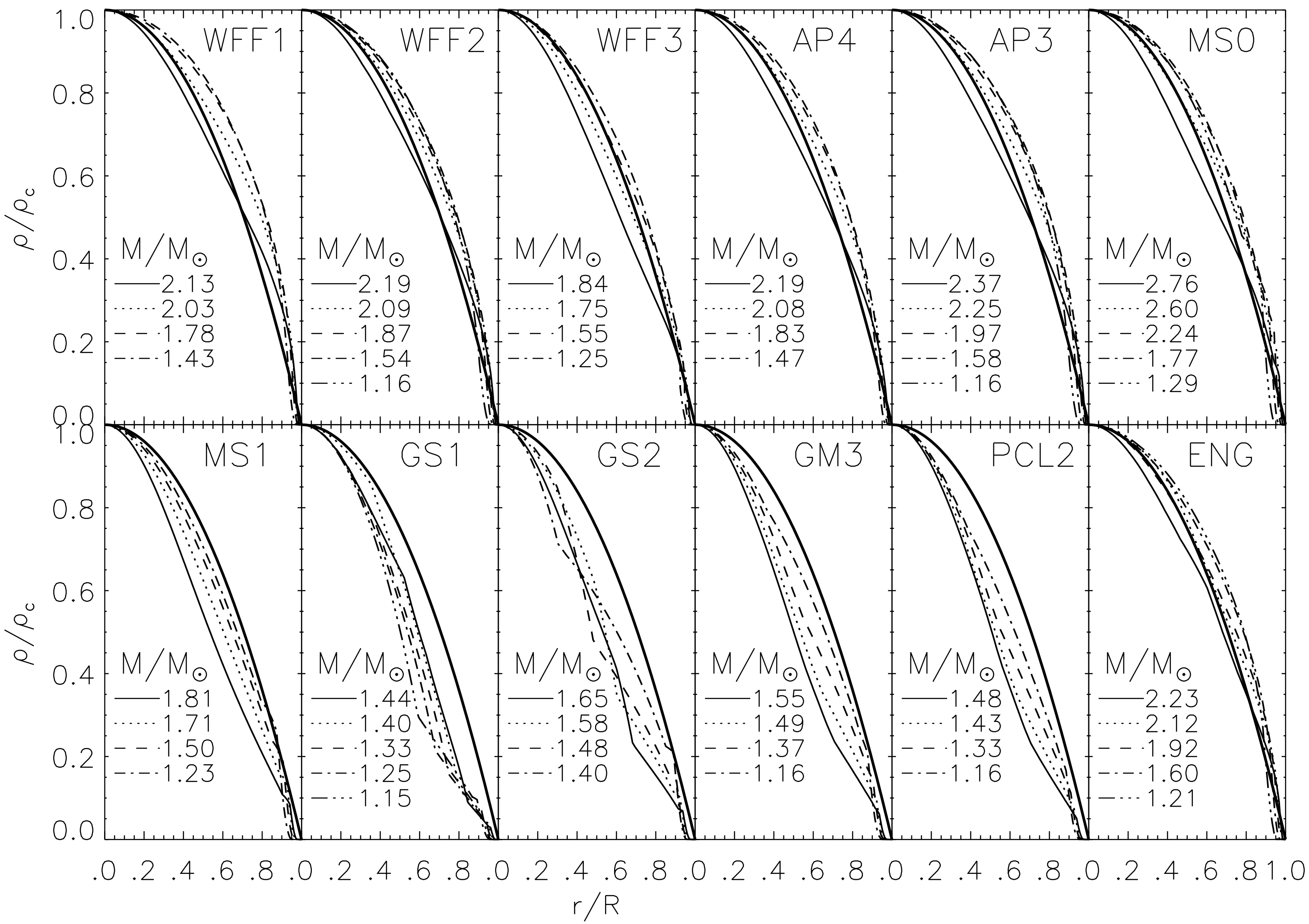
Write density as $\rho(r) = \rho_0(1 - \mu r^2/r_b^2)$

$\rho_0 =$ central density, $\mu = 1 \Rightarrow$ natural model $\mu = 0 \Rightarrow$ Schwarzschild)
Densities do not need to vanish at boundary if $r_c > r_b$ (where $\rho(r_c) = 0$)



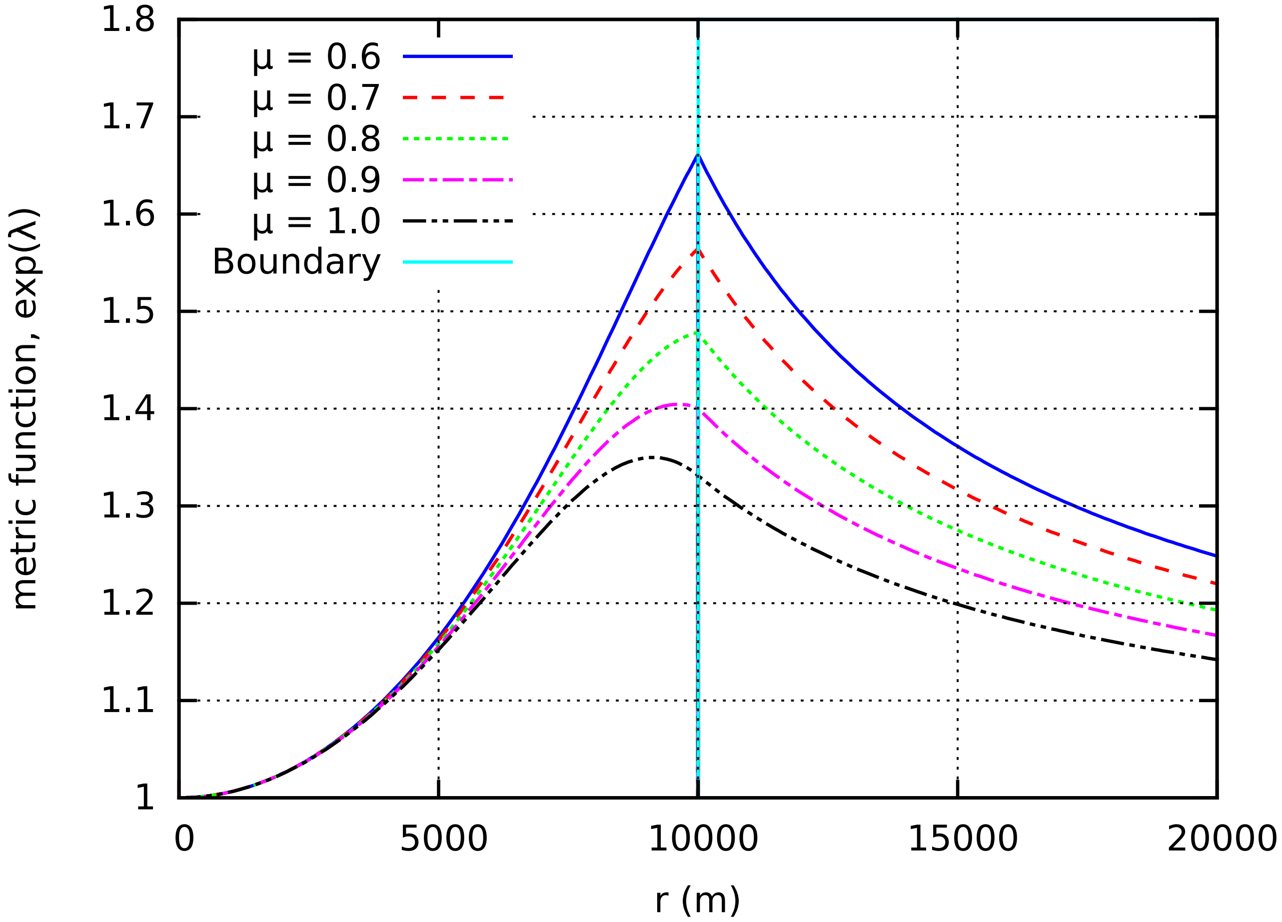
Tolman VII density compared to many neutron star models

Lattimer and Prakash, Ap.J., 550, 426-442 (2001).



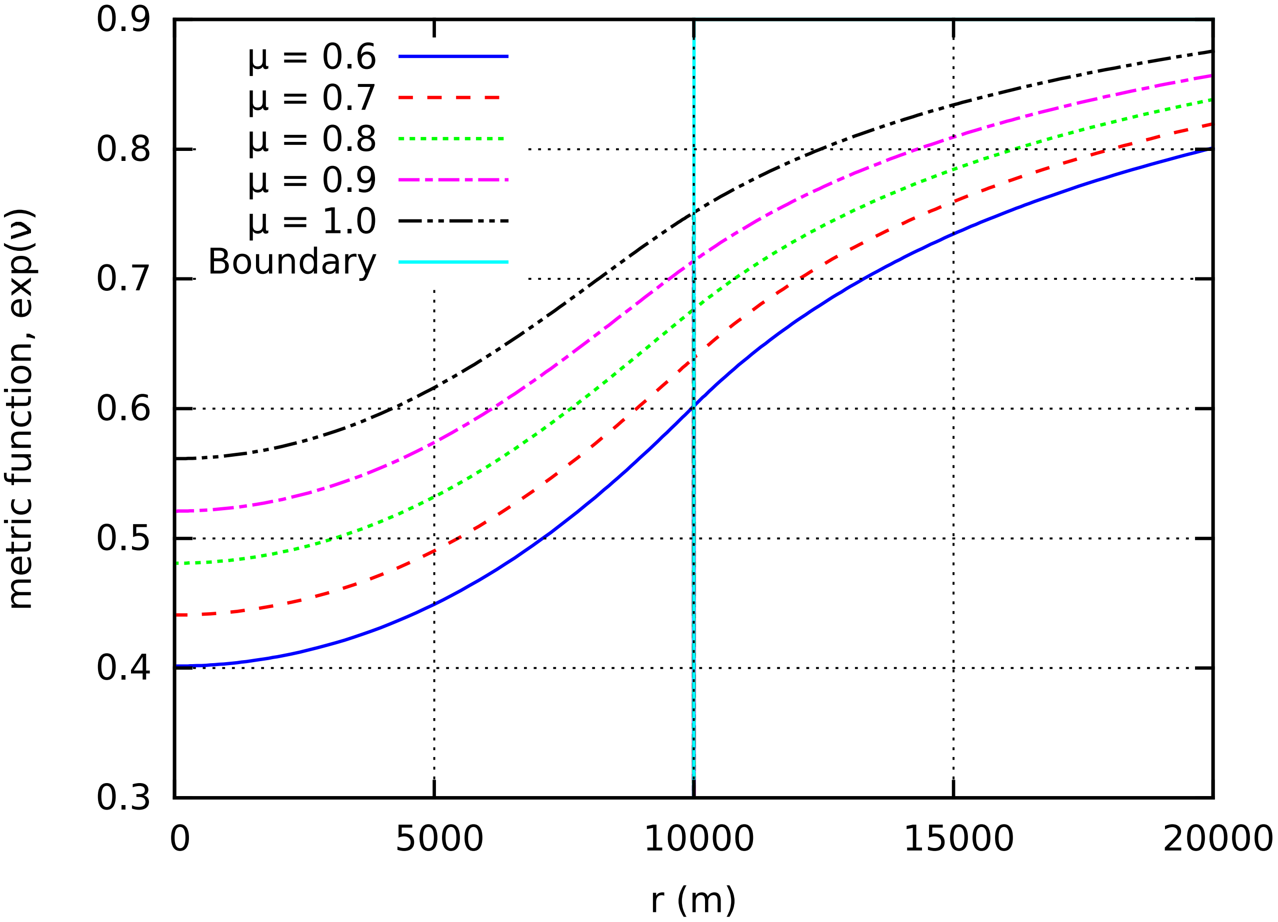
Tolman VII metric function: $-g_{tt} = \exp(\lambda)$

Computed for different values of density profile parameter μ



Tolman VII metric function: $g_{rr} = \exp(\nu)$

Computed for different values of density profile parameter μ



Tolman VII Pressure function

$$\kappa P(r) = -\frac{1}{R^2} + \frac{4r^2}{A^4} + \frac{4}{A^2}\sqrt{1 - \frac{r^2}{R^2} + \frac{4r^4}{A^4}} \times$$

$$\left\{ \sin^{-2}\left[\ln\left(\frac{\sqrt{1 - r^2/R^2 + 4r^4/A^4} + 2r^2/A^2 - A^2/4R^2}{C}\right)^{\frac{1}{2}}\right] - 1 \right\}^{\frac{1}{2}} .$$

here $\sin^{-2} x = 1/(\sin^2 x)$ (not $\arcsin^2 x$).

Can an explicit EOS be obtained from this function to obtain an “interesting solution” ?

setting $\rho(r) = \rho_0(1 - \mu r^2/r_b^2)$

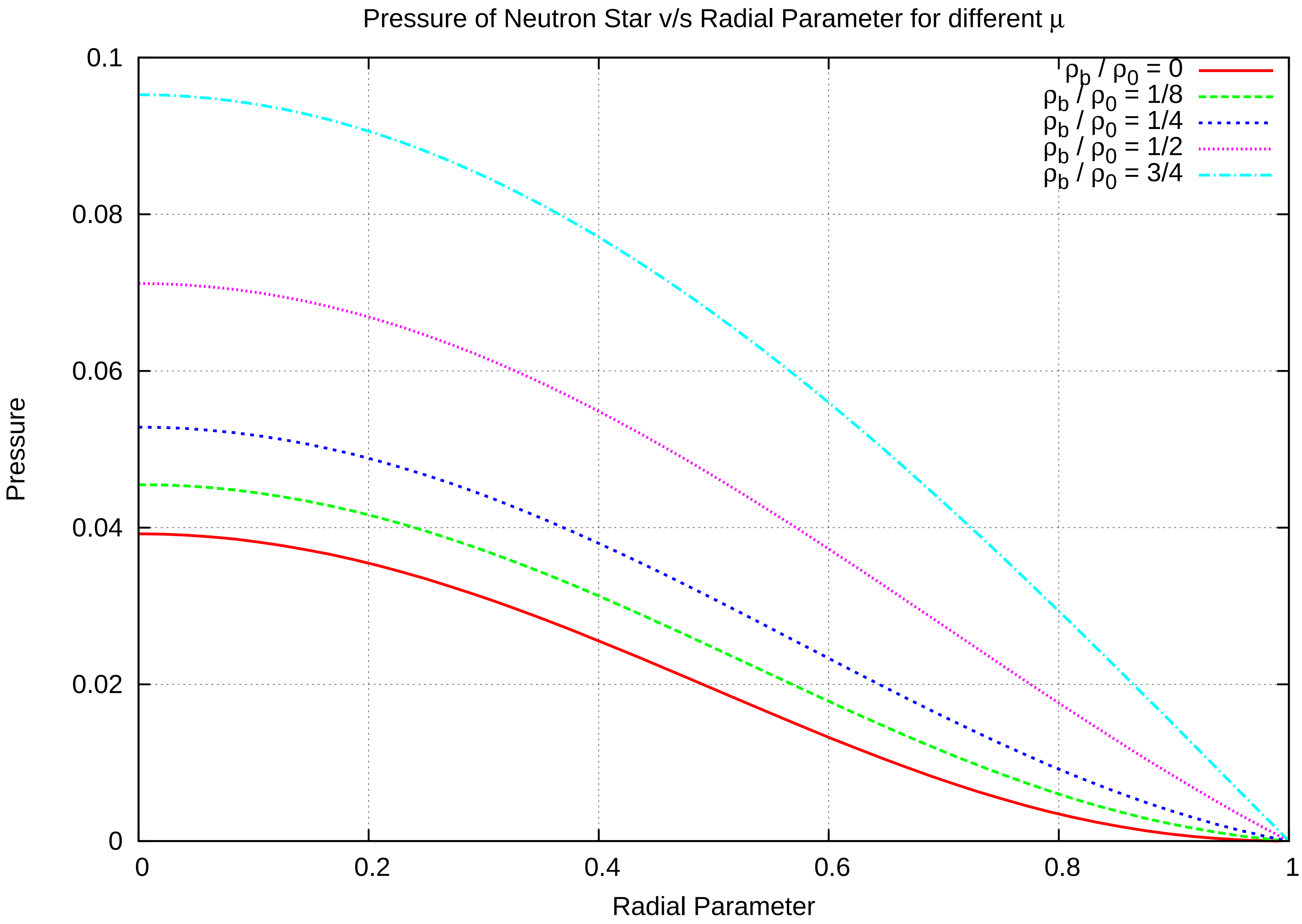
$$\text{one has } r = \sqrt{\frac{\rho_0 - \rho}{\mu\rho_0/r_b^2}}$$

$$R = \sqrt{\frac{3}{\kappa\rho_0}} \qquad A = \sqrt[4]{\frac{20}{\kappa\mu\rho_0/r_b^2}}$$

$C = C(\mu, \rho_0, r_b)$ depends on matching conditions at boundary r_b.

Tolman VII Pressure function

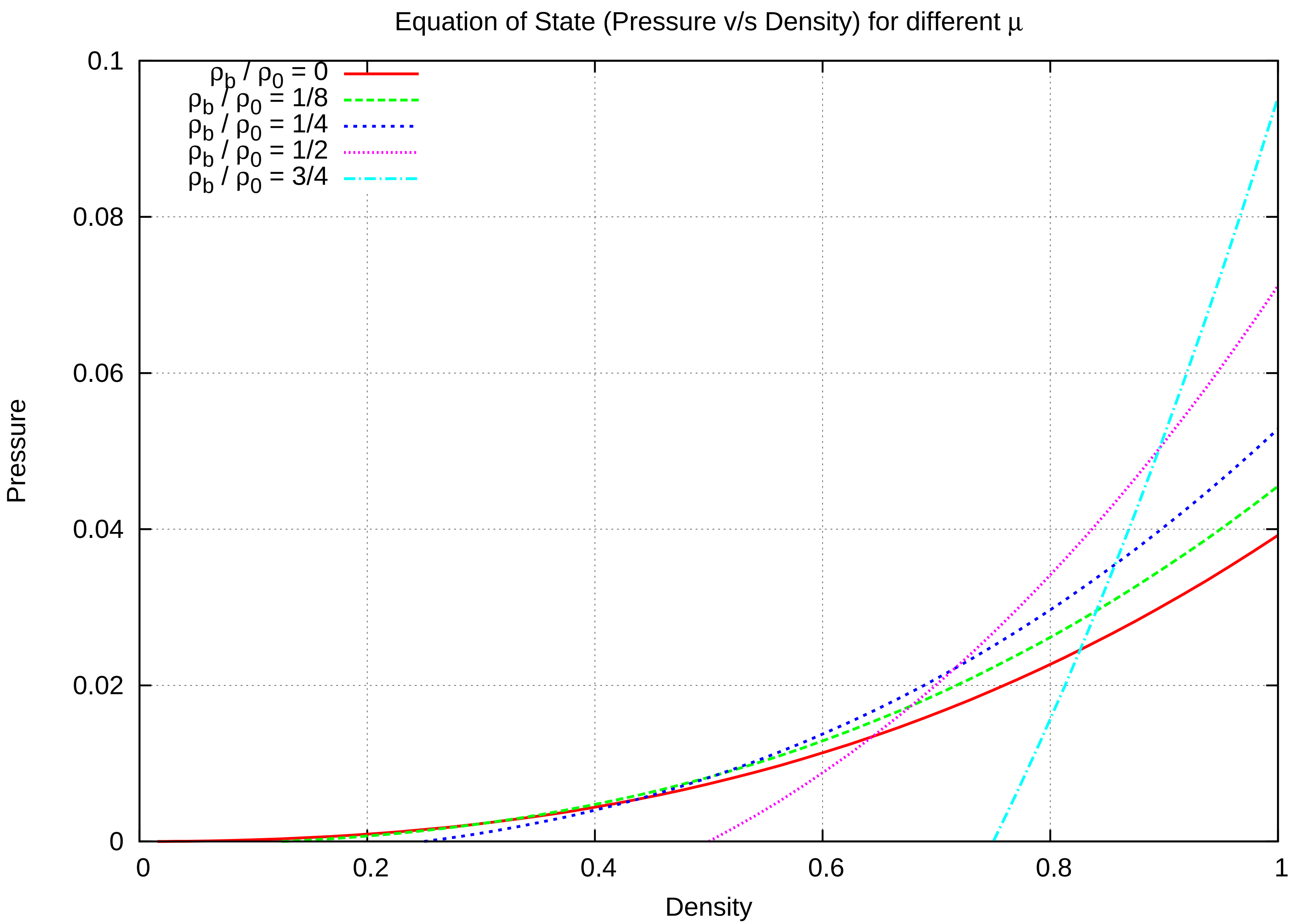
Pressure must vanish at the boundary (this determines B and C)



Pressure as a function of radial coordinate

Tolman VII EOS function

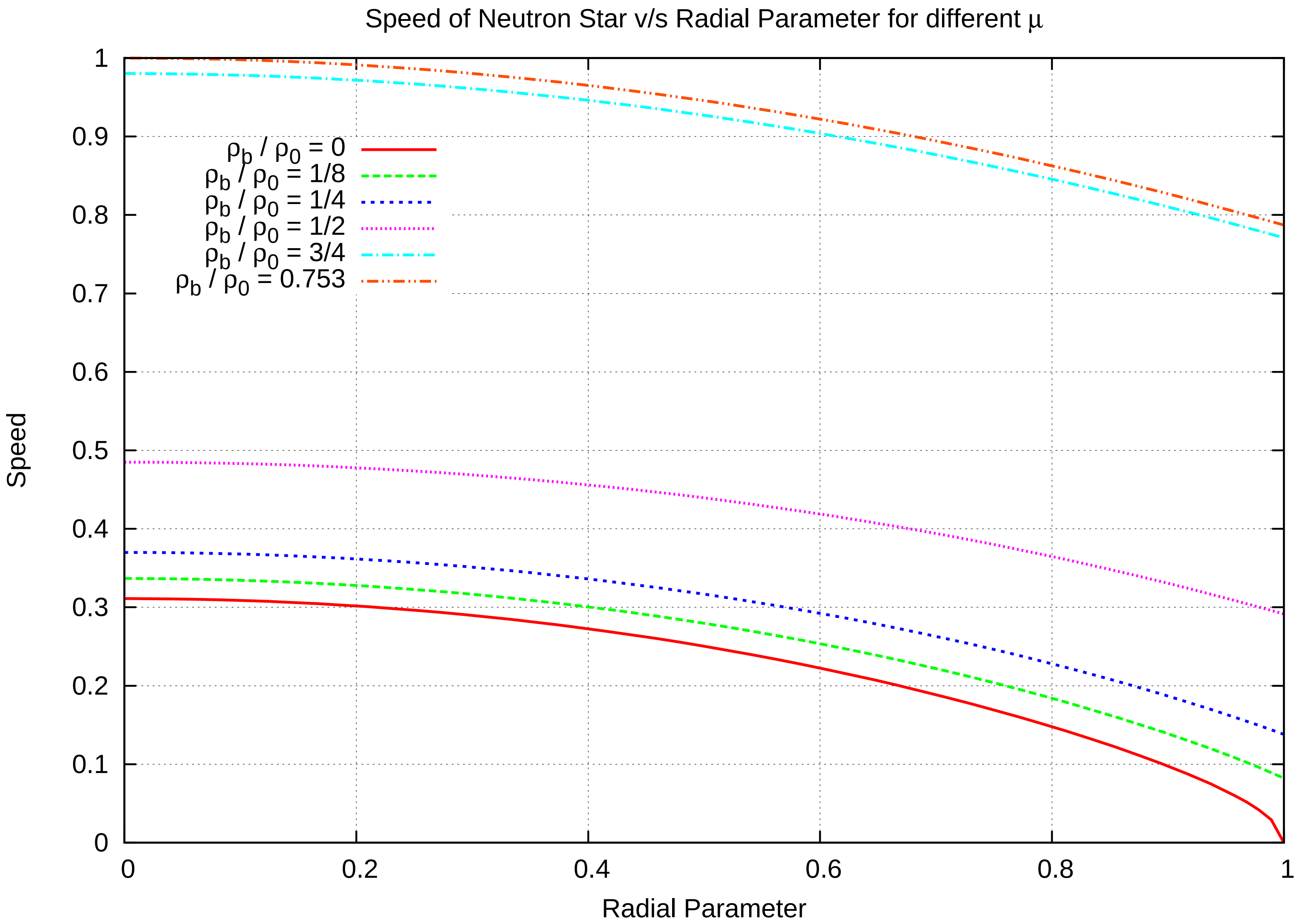
Eliminate r in favor of ρ.



Stiffer EOS for self-bound models

Tolman VII sound speed inside fluid

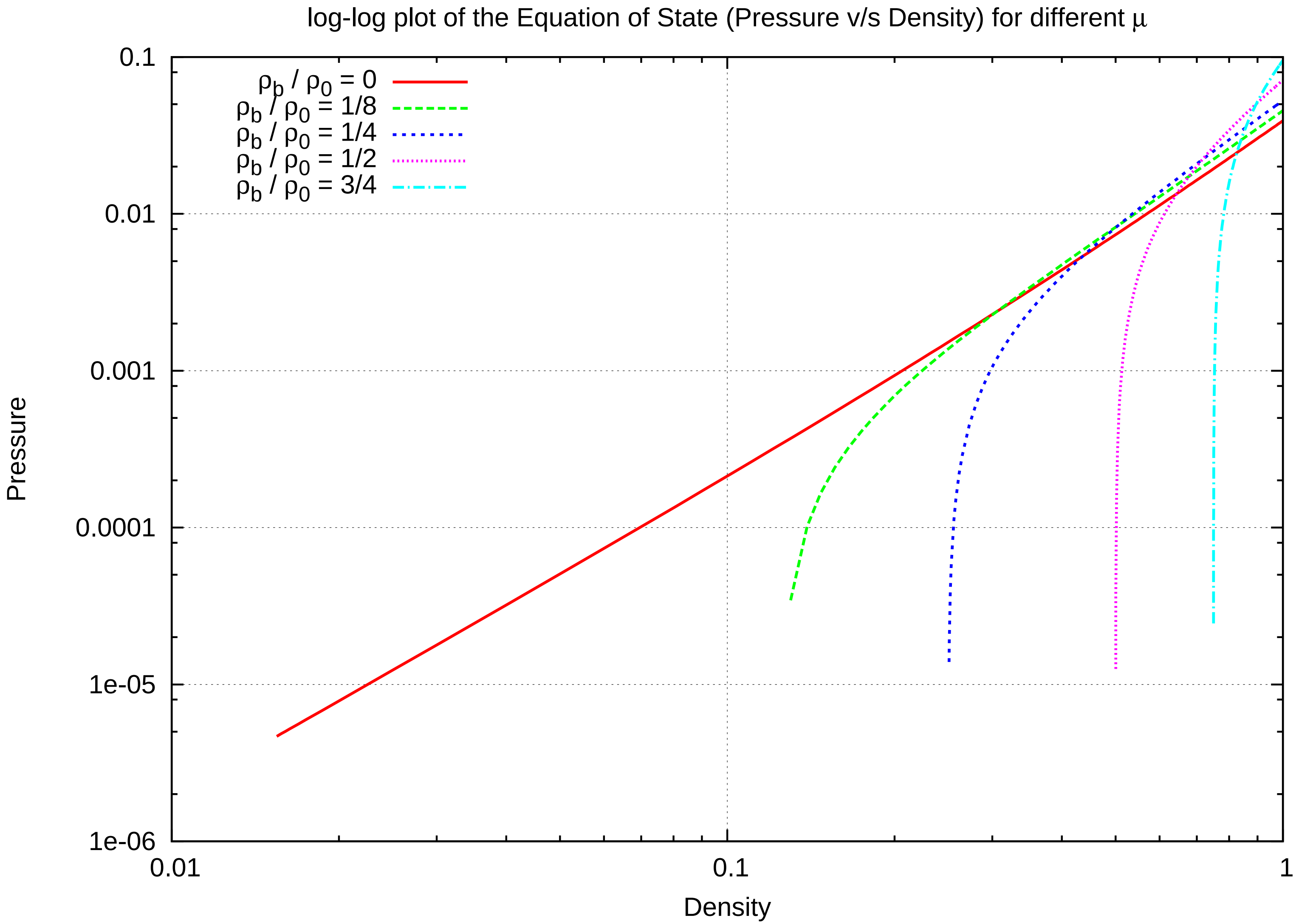
Sound speed from EOS: $v_s = \sqrt{\frac{dP}{d\rho}}$



Causal Limit ($\rho_b/\rho_0 = .7530$)

Tolman VII EOS (re-visited)

Log-Log plot and comparison with polytropic equation of state $P = \text{const}\,\rho^{\gamma}$



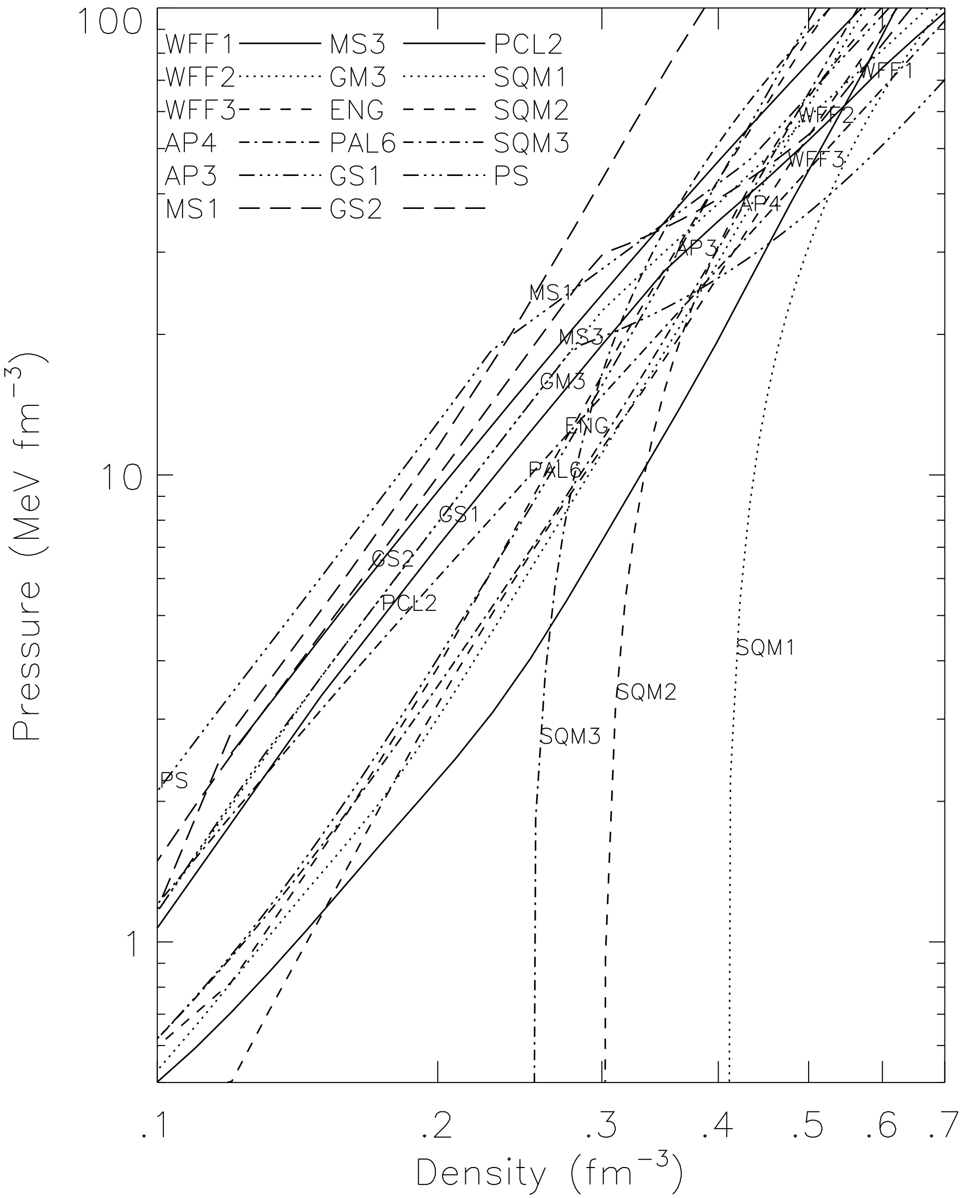
Fit the Natural solution curve to a sequence of polytropes $P = K\rho^{\gamma}$

Self-bound cases can be fit to $P = K(\rho - \rho_c)^{\gamma}$

Tolman VII EOS (re-visited)

Compare Tolman EOS with those based upon various nuclear models

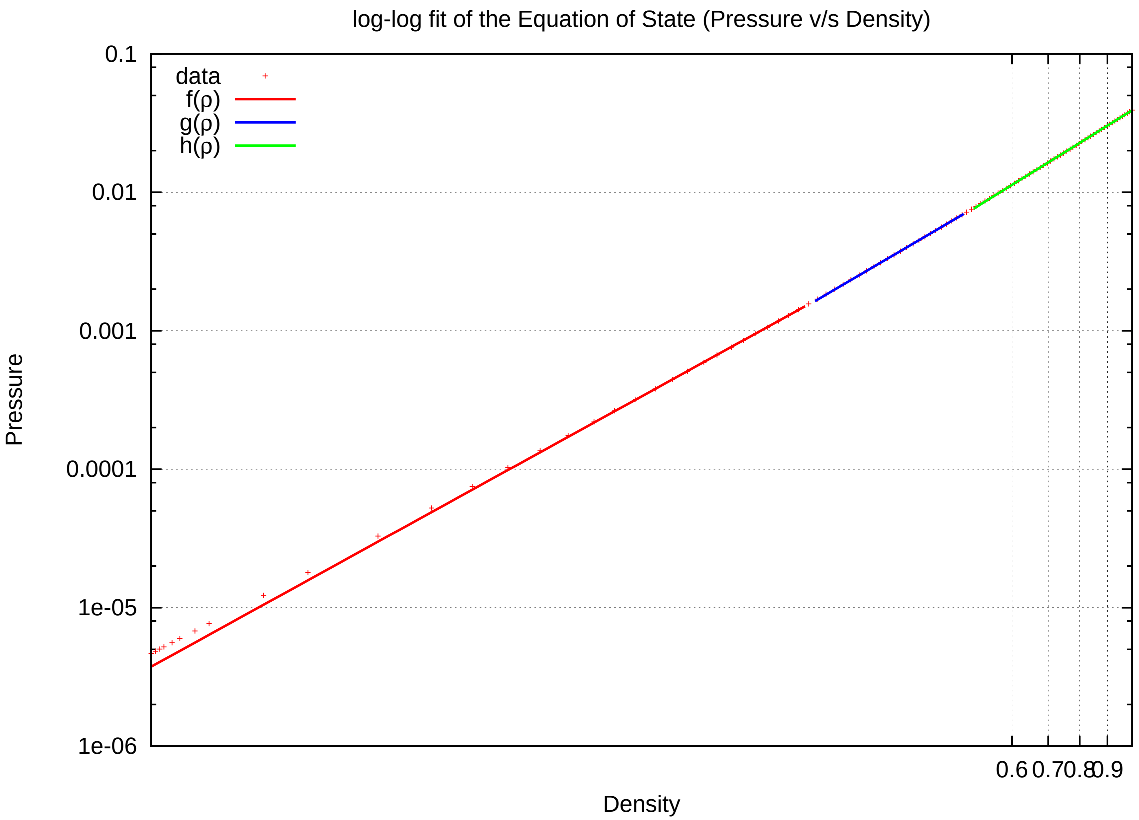
Lattimer and Prakash, Ap.J, 550, 426-442 (2001).



Tolman VII EOS (Piecewise Polytropic EOS)

$$P(\rho) \approx \begin{cases} f(\rho) = c_1 \rho^{\gamma_1} & \rho \leq 0.25 \\ g(\rho) = c_2 \rho^{\gamma_2} & 0.25 < \rho \leq 0.5 \\ h(\rho) = c_3 \rho^{\gamma_3} & 0.5 < \rho \leq 1. \end{cases}$$

$f(\rho)$	c_1	0.035
	γ_1	2.156 ± 0.004
$g(\rho)$	c_2	0.036
	γ_2	2.292 ± 0.004
$h(\rho)$	c_3	0.039
	γ_3	2.429 ± 0.002

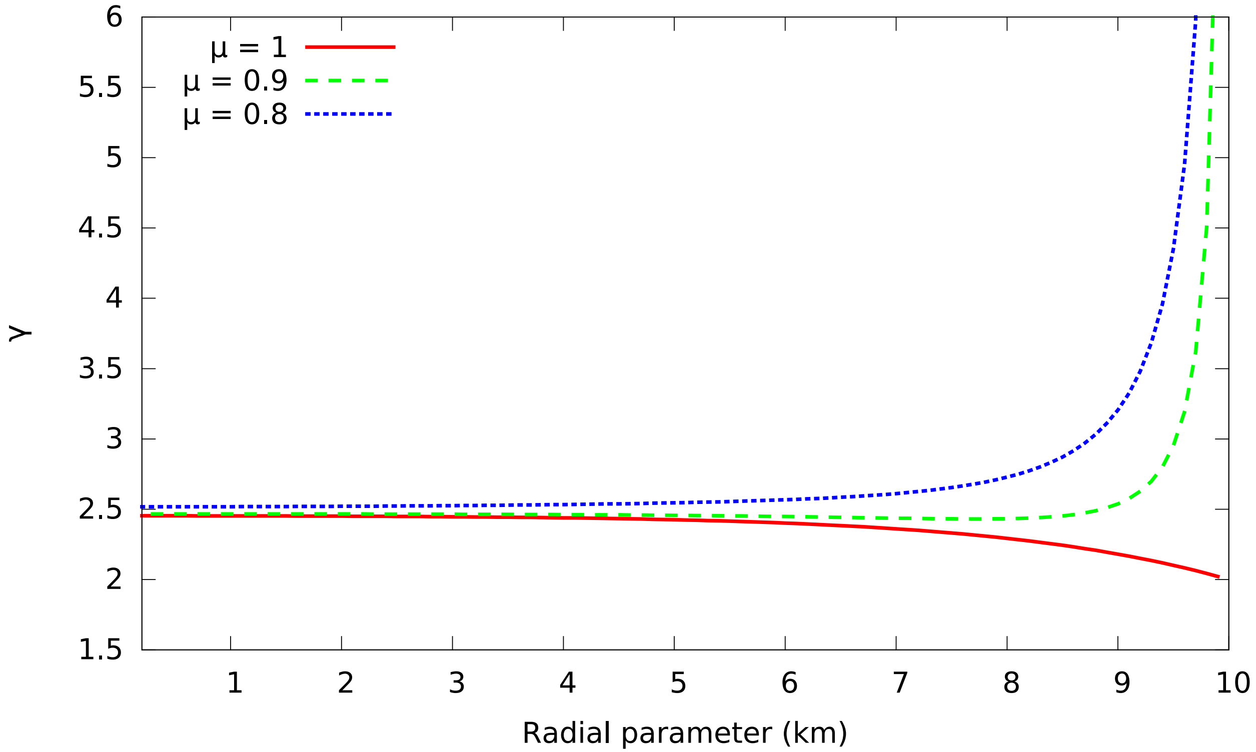


J.S. Read, B.D. Lackey, B.J. Owen, J.L. Friedman, Phys Rev D. 79.124032 (2009)

Stiffness of Different EOS

Compute adiabatic index γ for both natural and self bound cases

$$\gamma = \frac{d \log p}{d \log \rho}$$

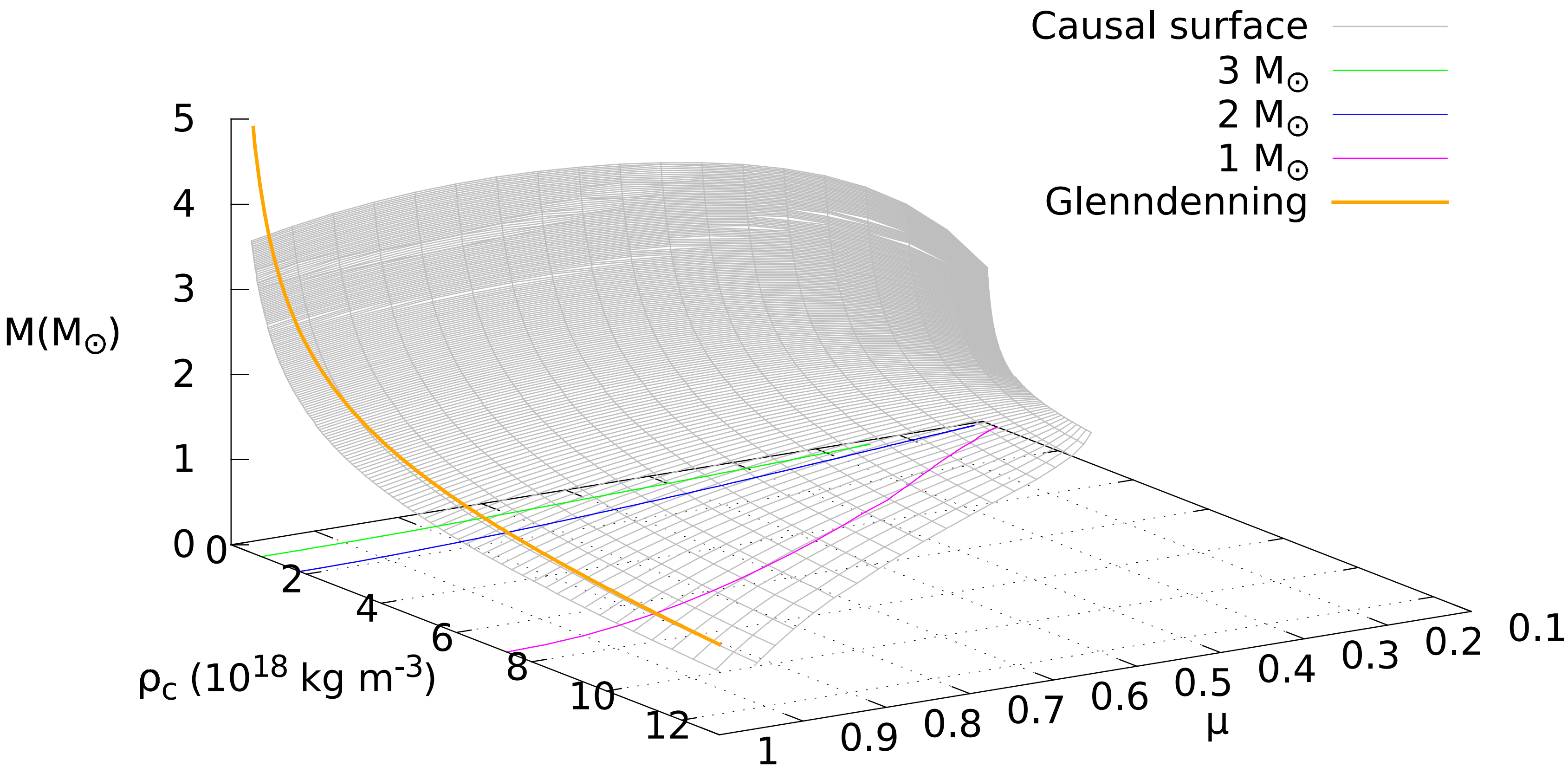


$$r_b = 10\text{km} \qquad \rho_c = 10^{15}\text{g} \cdot \text{cm}^{-3}$$

Causality limits and NS masses

Assume maximum sound speed at centre cannot exceed speed of light

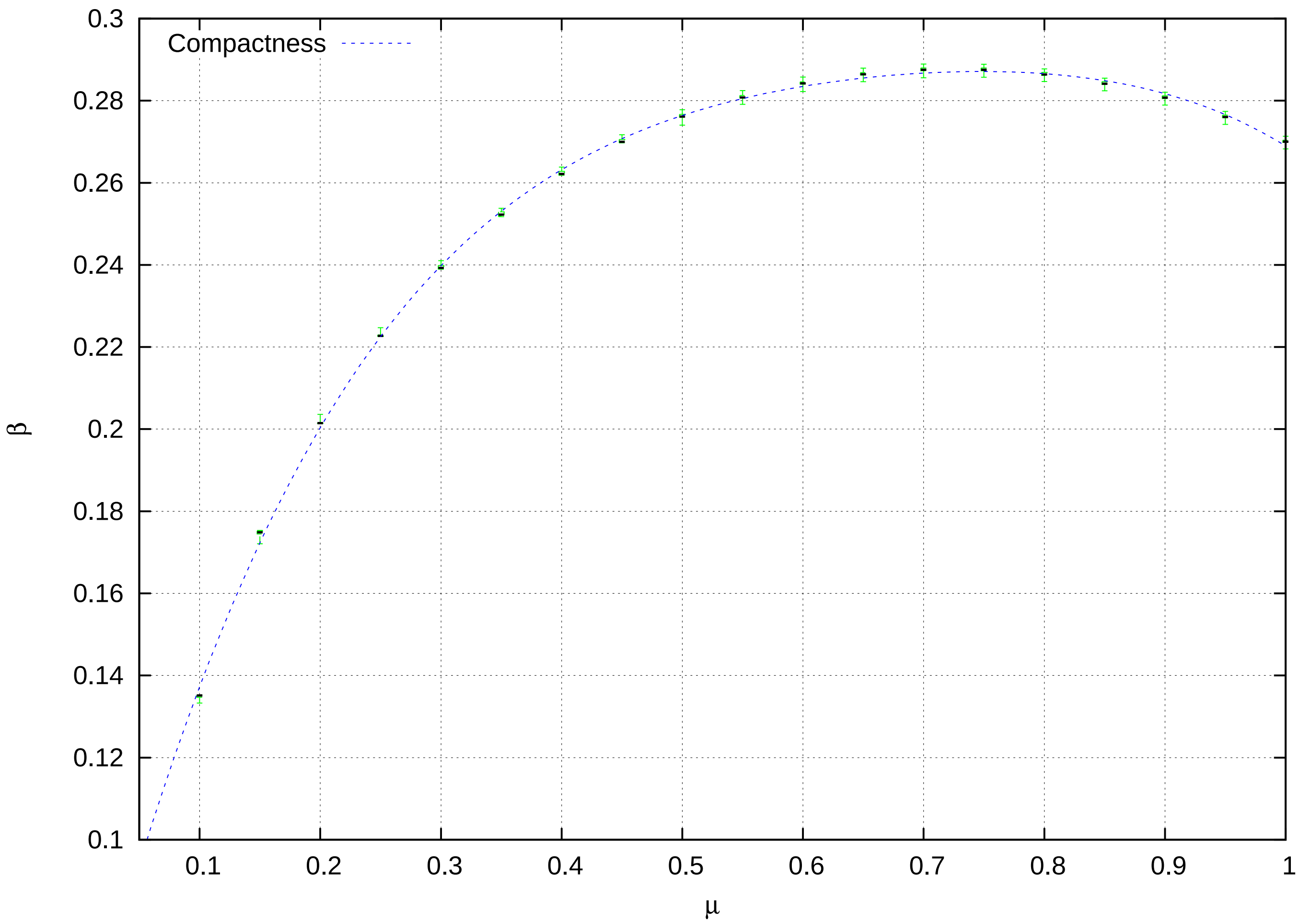
$$M = M(\rho_c, \mu)$$



All configurations below the grey surface are possible stellar configurations

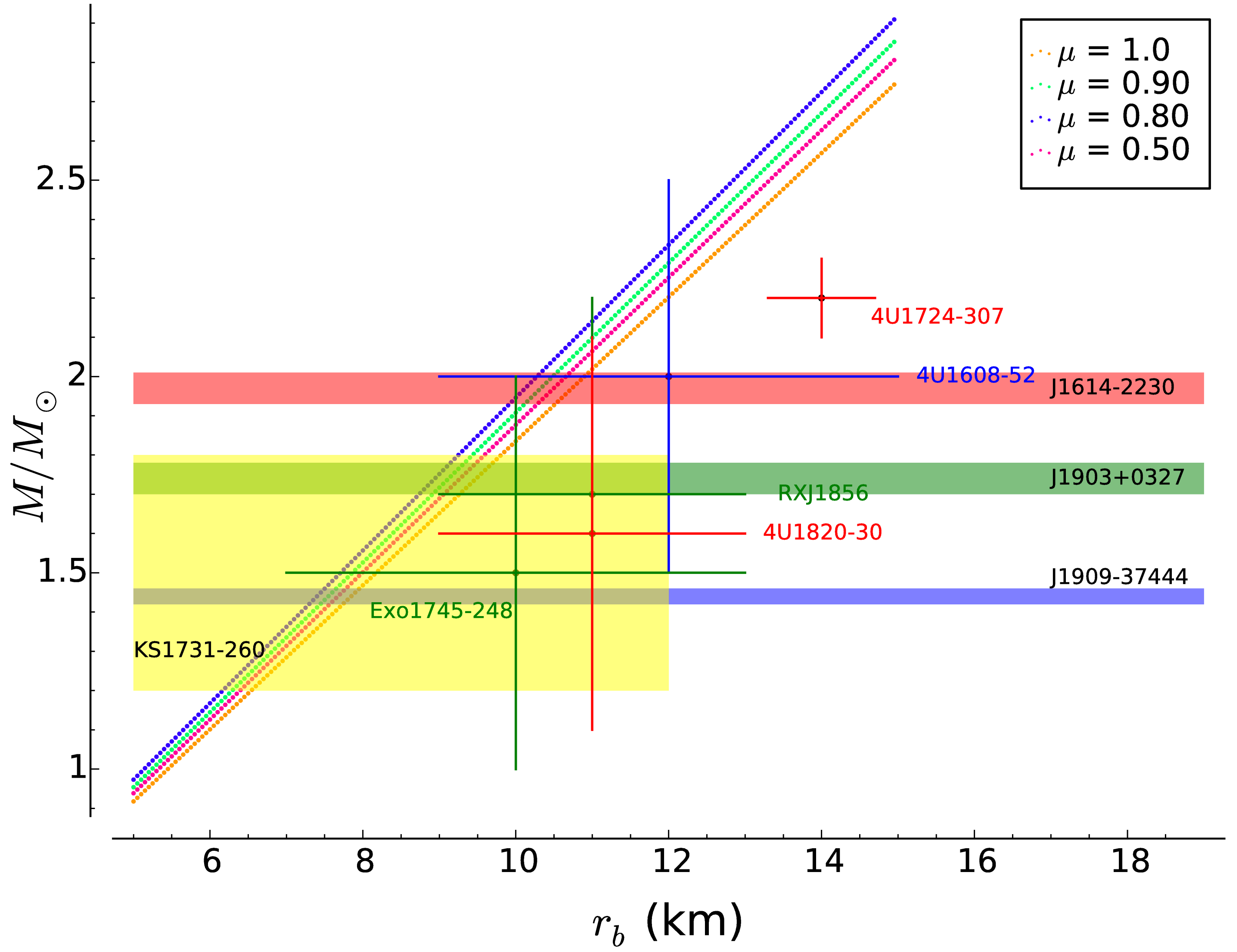
Causality and Compactness

Define compactness $\beta = M/R$ Interior Schwarzschild $\beta_{\max} = 4/9$



Tolman VII solution $\Rightarrow$ less than than maximum compactness

Comparison of Mass - Radius relation to Recent Observations



Extensions to Tolman VII

- Einstein-Maxwell equations ($q \neq 0$): (add T^{ik}_{EM} see e.g. Ivanov)

- Anisotropic pressure source ($p_\perp \neq 0$) (P and S wave seismology)

- Einstein-Born-Infeld electrodynamics (mimics S-quark EOS see e.g. H. Cuesta)

- Einstein-scalar field models (non-static interiors: Boson stars)

Use Tolman density function $\rho(r)$

Solve for λ and ν metric functions

Solve for pressure $p(r)$ and find EOS

If Physically Viable **then** What is it?