Dynamic Risk Sharing with Prepayment*

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Abstract

I analyze the role of prepayment in a dynamic risk-sharing setting with information and commitment frictions. An insurance platform contracts with a risk-averse agent with stochastic income. Part of the income can be withheld in escrow as a prepayment. I consider three endogenously incomplete markets settings with different obstacles to risk sharing: limited commitment, private information due to hidden income, and both. I show that prepayment alleviates the limited commitment problem and improves the degree of risk sharing, including possibly to full insurance depending on the model parameters; however, prepayment is ineffective in the private information settings. In the setting with both limited commitment and private information frictions, I show that private information is the binding constraint.

Keywords: risk sharing, limited commitment, private information, mechanism design, prepayment, escrow

JEL codes: D82, D86, G52

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1 Introduction

The ability of economic agents to share risk is frequently hindered by contractual and market frictions stemming from limited commitment or private information (e.g., Townsend 1994 and 2016; Ligon, Thomas and Worrall 2002). In the absence of perfect enforcement, agents may renege on an insurance arrangement when their income is high. In the absence of complete information, agents may under-report their income to avoid making contributions or pay premia. Both of these fundamental frictions cause financial markets and contracts to be endogenously incomplete and can result in imperfect risk sharing compared to the first best.

In this paper, I analyze the efficacy of a specific contractual instrument – *prepayment* (funds in escrow) for mitigating commitment, information, or both obstacles to risk sharing. I consider an infinite-horizon setting in which a risk-neutral insurance platform contracts with risk-averse agents who face idiosyncratic income fluctuations. The objective is to solve for and characterize the constrained-optimal dynamic insurance contract based on the time history of observed or reported agent income. The contract is designed to ensure voluntary participation and voluntary contributions (no reneging) at all times in the limited commitment settings and truthful reporting of the income state in the private information settings.

Unlike most of the existing literature, which focuses on a single contractual friction (e.g., see Golosov et al. 2016 for an extensive review), in this paper I analyze private information and limited commitment both separately and jointly.[1] The second distinguishing feature of my model is that the insurance platform can require the agent to make a prepayment held in escrow before the realization of the idiosyncratic income shock. This contractual instrument is particularly relevant in the context of digital financial platforms or decentralized finance (DeFi) applications where smart contracts can be used to automate the withholding of funds (programmable escrow) and to enforce payment obligations, while still being unable to observe the agents' actual income realizations.[2]

I embed the prepayment mechanism into three fully dynamic contracting frameworks: (i) limited commitment with observable income; (ii) private information (hidden income) with full commitment; and (iii) a setting featuring both limited commitment and hidden income.

[1] An exception is Atkeson (1991) who studies a setting with both moral hazard and debt repudiation frictions. Karaivanov and Townsend (2014) compute and structurally estimate a wide range of dynamic models with exogenously or endogenously incomplete markets but each setting features a single friction.

[2] For example, the maximum feasible prepayment could be lower or equal to the minimum possible income level.

The analysis yields contrasting results regarding the efficacy of prepayment, depending on the underlying contractual friction. In the setting with only limited commitment (e.g., see Thomas and Worall, 1988; Kocherlakota, 1996; Ligon, Thomas and Worrall, 2002), I demonstrate that prepayment can increase the efficiency and degree of consumption smoothing in the optimal risk-sharing contract. By collecting funds ex-ante, the contract with prepayment lowers the agent's outside option (the value of autarky) in the high-income state. I prove that, if the maximum feasible prepayment is sufficiently large, then the first-best allocation (full insurance) is implementable, mitigating the limited commitment friction entirely. Even with a small feasible prepayment, the prepayment mechanism strictly improves welfare by relaxing the limited commitment constraint.

In contrast, I show that prepayment is ineffective in the setting with private information (hidden income, e.g., as in Townsend, 1982 or Thomas and Worrall, 1990). In this framework the binding constraint is the agent's incentive to misreport high income as low income. Since, by definition, the prepayment is collected before the realization of the income state and thus is non-contingent, it does not alter the agent's marginal incentives regarding the truth-telling constraint. Consequently, the constrained-optimal contract in the hidden income model with prepayment is identical to that in the standard model without prepayment.

Finally, I analyze the setting in which there are both hidden Income and limited commitment frictions (HILC). I first prove that the truth-telling constraint is stricter than the limited commitment constraint. That is, a contract that induces a high-income agent to report truthfully rather than claiming low income automatically satisfies the agent's limited commitment constraints. Since the private information friction is the binding constraint and because, as argued before, prepayment does not alleviate the private information problem, allowing prepayment provides no welfare gain in the HILC setting with both information and commitment frictions.

The paper contributes to the theoretical literature on recursive contracts and optimal risk sharing in dynamic settings (e.g., see Townsend, 1982; Thomas and Worrall, 1988 and 1990; Kocherlakota, 1996; Cole and Kocherlakota, 2001; and the review in Golosov et al., 2016). By considering prepayment as part of the contract design, the analysis also relates to Rampini and Viswanathan (2010) who study the role of collateral in financial risk management and to Karaivanov et al. (2023) on the use of escrow in a digital safety net platform. The paper's results suggest that while digital escrow technologies can mitigate enforcement problems (strategic default), they may not be effective in addressing frictions stemming from private information.

The prepayment mechanism analyzed here has a possible practical application in the design of digital decentralized finance (DeFi) platforms. Smart contracts—self-executing code running on digital platforms—enable credible commitment to future transfers by locking funds in escrow ex-ante (e.g., see Szabo, 1996; Cong and He, 2019; Karaivanov et al. 2023), as assumed and modelled here. By automating the computation and execution of state- and history-contingent transfers, these technologies can further significantly reduce enforcement and verification costs (Catalini and Gans, 2020). However, while smart contracts can enforce payments programmatically, thus addressing commitment frictions, they remain dependent on external mechanisms such as "oracles" to verify off-platform income realizations or other relevant data. The paper's results contribute to formalize the limits of these technologies by comparing and contrasting commitment/enforcement with information frictions.

The remainder of the paper is organized as follows. Section 2 outlines the dynamic risk-sharing model. Section 3 presents the main theoretical results for settings with different contractual frictions as well as computed numerical examples and comparisons of consumption smoothing for the different scenarios. Section 4 concludes.

2 Model

I analyze the optimal dynamic insurance arrangement between an insurance platform ("the insurer") and a large number (continuum) of ex-ante identical risk-averse agents. The agents have preferences represented by the strictly increasing and strictly concave utility of consumption function $u(c)$ and discount factor $\beta \in (0, 1)$.

Every period, t an agent draws income y from a discrete stochastic process taking values $\{y_1, ..., y_S\}$ with respective probabilities $Prob(y = y_i) = \pi_i, \ \forall i = 1, ..., S$. The income realizations are i.i.d. over time. Call expected income

$$\bar{y} = E(y) = \sum_{i=1}^{S} \pi_i y_i$$

The agents contract with a competitive risk-neutral insurer over an infinite time horizon. The constrained-optimal risk-sharing contract maximizes agent's discounted utility subject to commitment or incentive-compatibility constraints, as specified below.

The insurer is conceptualized as a (digital/programmable) platform that is able to withhold or require a *prepayment* of up to amount $\bar{f}$ from the agent. However, the insurance platform does not necessarily observe the agent's income, depending on the contractual frictions present (see below for details). The maximum feasible prepayment $\bar{f}$ is taken as exogenous here and the analysis is performed for any $\bar{f} \geq 0$. In practice $\bar{f}$ could depend on the contractual setting.

The platform provides insurance via contingent transfers, τ which are either contributions/premia from the agent or payouts/indemnities to the agent and, in general, depend on the history of the agent's observed or reported income. The platform can also borrow or save externally at the gross interest rate $R = \beta^{-1}$, i.e., unlike in Kocherlakota (1996), there is no resource constraint relating total payouts and contributions across agents in a given period. We can thus think of the insurer as contracting separately with each agent, while keeping track of the agent's income history/reports, e.g., as in Karaivanov and Townsend (2014).

The outside option for the agent has value V^{out}. Most results that follow assume that the outside option equals the agent's *autarky* value whereby the agent simply consumes his or her own income each period yielding present-value utility,

$$V^{out} = \frac{E(u(y))}{1-\beta} \tag{OO}$$

where $E(u(y))$ denotes the expected utility from consuming one's income every period.

The insurer is assumed to be able to fully commit to the contract. On the agent's side, I consider the following commitment and/or information frictions:

- **limited commitment** (e.g., Thomas and Worrall, 1988): the agent may decide to quit the risk-sharing contract or renege on an insurance premium due. The contract must provide an incentive, i.e., sufficiently high present-value utility for the agent to participate and not quit at any period, as well as not to renege on a due contribution (e.g., when the agent has high income). There is no private information problem.

- **private information** (hidden income, e.g., Townsend, 1982; Thomas and Worrall, 1990): the agent's income is unobservable to the insurer and is self-reported by the agent. The contract provides incentives for truth-telling. There is no commitment problem.

- **both limited commitment and private information**: both the limited commitment and hidden income frictions are simultaneously present. The optimal insurance contract must provide incentives for staying in, not reneging, and reporting truthfully in all time periods and for all income histories.

For most of the analysis, in order to obtain analytical results, I assume that there are two possible income realizations ($S = 2$) – "low" income, y_L and "high" income, y_H where $y_H > y_L > 0$, occurring with respective probabilities $\pi_L \in (0, 1)$ and $\pi_H = 1 - \pi_L$.

3 Results

3.1 Limited Commitment

I first analyze the role of prepayment in a setting in which the only contracting friction is limited commitment by the agent. I allow for both ex-ante and ex-post commitment problems. That is, the agent could quit if her present-value utility (captured by the promised utility state in the contract) falls below the autarky value, V^{out}. The agent can also consider reneging on a due insurance premium (contribution) after the income is realized if reneging and going to autarky dominates staying in the contract. The optimal contract addresses both these issues by ensuring that, at any time and after any income history, the agent does not want to leave the contract or renege on a premium. The insurer observes the agent income perfectly.

Formally, following the literature (Spear and Srivastava, 1987; Thomas and Worrall, 1988; Karaivanov and Townsend, 2014), I write the contracting problem as a dynamic program with state variable *promised utility*, w, that is, the agent's present-value discounted utility from the current period onward. The timing of events within each model time period t is: (i) the agent begins with current promised utility state $w_t = w$; (ii) income $y_t = y_i$ is realized, implying transfer τ_i (positive or negative); (iii) the agent either (a) pays a premium or receives an indemnity, depending on the sign of τ_i, and enters next period with promised utility $w_{t+1} = w_i$, or (b) reneges on the arrangement by consuming $c_t = y_i$ and continues in autarky thereafter, with present-value V^{out}.

Problem LC

$$\Pi(w) = \max_{\{\tau_i, w_i\}} \quad \pi_H(-\tau_H + \beta\Pi(w_H)) + \pi_L(-\tau_L + \beta\Pi(w_L))$$

$$\text{s.t.} \quad \pi_H(u(y_H + \tau_H) + \beta w_H) + \pi_L(u(y_L + \tau_L) + \beta w_L) = w \qquad \text{(PK)}$$

$$u(y_i + \tau_i) + \beta w_i \geq u(y_i) + \beta V^{out}, \quad i = L, H \qquad \text{(LC)}$$

$$w_i \geq V^{out}, \ \forall i \quad \text{[continued participation]}$$

for all $w \geq V^{out}$ (ex-ante participation), with w_0 given. In Problem LC, V^{out} is the agent's outside option, the autarky value from (OO), and the insurer can borrow and save at gross interest $R = \beta^{-1}$. The choice variables are the insurance transfers τ_i, which are positive when the agent receives an insurance payout/indemnity and negative when the agent pays a net premium/contribution, and the future promised utility w_i in each income state i. The promised utility state variable w captures the history of income realizations and determines the present value of future transfers to or from the agent. In a digital platform, this could be represented by a token balance in the platform's code.

Constraint (PK) is the promise-keeping constraint, ensuring that the insurer delivers the contracted present value utility w to the agent. The limited commitment constraints (LC) ensure that the agent has no incentive to renege and walk away after any income realization. The left hand side of (LC) is the value of staying in the contract for income state $i = L, H$, while the right hand side is the value of going to the outside option of autarky forever with income y_i in hand. The final constraint, on future promises w_i, ensures the continued participation by the agent who would be better off to switch to autarky if the in-contract utility value w_i falls below V^{out} in some history.

I set the initial promised utility, w_0 so that the insurance platform breaks even on the contract ex-ante, in expectation, that is, w_0 solves $\Pi(w_0) = 0$. However, in any period, $t = 1, 2, ..., +\infty$, the insurer is allowed to have a surplus or deficit.

Denote by the superscript "*lc*" the contract elements / policy functions, (τ_i^{lc}, w_i^{lc}) for $i = \{L, H\}$ in Problem LC – each of them a function of the current state w. Let also c_i^{lc} denote the in-contract consumption in state i,

$$c_i^{lc} = y_i + \tau_i^{lc}, \;\; \forall i$$

Problem LC is well-known from the literature, e.g., see Thomas and Worrall, (1988) or Ljungqvist and Sargent (2018). The main results and contract properties are summarized in Lemma 1. Note that the lower bound V^{out} on promised utility does not affect the solution properties since promised utility either stays the same or is raised at optimum (see Lemma 1). I then focus on adding prepayment to Problem LC and analyze its effect on the constrained-optimal insurance contract.

Lemma 1. *The optimal contract solving Problem LC has the following properties:*
(a) $\forall w \in [V^{out}, u(y_H) + \beta V^{out})$ *partial insurance obtains – constraint (LC) binds for* $i = H$ *and the optimal contract is:*

$$w_H^{lc} = u(y_H) + \beta V^{out}, \;\; u(c_H^{lc}) = (1-\beta) w_H^{lc}, \; w_L^{lc} = w, \textit{ and } c_L^{lc}(w) \textit{ solving}$$

$$\pi_L(u(c_L^{lc}) + \beta w) + \pi_H(u(c_H^{lc}) + \beta w_H^{lc}) = w$$

(b) $\forall w \geq u(y_H) + \beta V^{out}$ *full insurance obtains – the optimal contract is:*

$$w_i^{lc} = w \text{ and } u(c_i^{lc}) = (1-\beta)w, \forall i.$$

(c) *The results in (a) and (b) imply that if*

$$\frac{u(\bar{y})}{1-\beta} \geq u(y_H) + \beta V^{out}, \tag{1}$$

then by setting $w_0 = w^{fi} = \frac{u(\bar{y})}{1-\beta}$ *the platform optimally provides full insurance,* $c_i^{lc} = c^{fi} = \bar{y}$, $\forall i$, *at zero expected profits,* $\Pi(w_0) = 0$.

Conversely, if

$$\frac{u(\bar{y})}{1-\beta} < u(y_H) + \beta V^{out}, \tag{L1}$$

then full insurance is infeasible and the platform provides partial insurance.

Proof: see Appendix.

When the model parameters satisfy inequality (1) in Lemma 1(c), the platform can deliver full insurance at zero expected profits for $w_0 = w^{fi}$, i.e., equal consumption across the income states, while satisfying the limited commitment constraints. This corresponds to the first-best outcome. This case arises if the discount factor β is relatively close to 1 (the agent is patient) and/or the high income level y_H is not too large relative to the mean income $\bar{y}$. A higher degree of risk aversion also makes inequality (1) more likely to be satisfied.

Below I focus on the complementary case from part (a) of Lemma 1, where the limited commitment problem has a bite and constraint (LC) binds for the high-income state $i = H$. In this case the agent needs to be offered more than the mean income $\bar{y}$ when y_H is realized to be induced to stay in the contract, resulting in partial insurance. For $i = L$ the incentive to renege (the r.h.s. of (LC)) is lower and the agent receives a payout ($\tau_L > 0$) at the solution – see the Lemma 1 proof. Thus, constraint (LC) does not bind for $i = L$ and I remove it from the prepayment analysis below.

Note also that if $w = w_H^{lc}$ then the promise keeping constraint (PK) implies $w_L^{lc}(w) = w_H^{lc}$ and $c_L^{lc}(w) = c_H^{lc}$. This means that, upon receiving a high-income realization y_H for the first time, the agent's promised utility adjusts to w_H^{lc} and thereafter the agent obtains the constant consumption $c = c_H^{lc}$ as characterized in Lemma 1, no matter what the realized income is (high or low) – the partial insurance is temporary.

Prepayment

I next allow for prepayment in the limited-commitment Problem LC. Specifically, suppose that the insurance platform requires that before the income realization the agent prepays an amount f, up to a maximum of $\bar{f} \geq 0$, to be held in a secure escrow account. The case $\bar{f} = 0$ corresponds to the standard limited commitment setting in Problem LC. Alternatively, one can think of the platform being able to withhold up to $\bar{f}$ from agent's income. Aside from the technological ability to post or withhold prepayment, the limited commitment problem is assumed to remain as before, i.e., the agent can still decide to quit if his/her promised utility is too low or upon a high income realization.

The dynamic insurance problem with prepayment is stated in Problem LCP below, with the new additional choice variable f to be determined endogenously. The timing each period is: (i) prepayment f is collected and put in escrow; (ii) agent income is realized; (iii) transfers are made. The contract instruments (prepayment f, transfers τ_i, and next-period promised utility w_i) are optimally chosen to ensure that the agent participates voluntarily and does not renege on a premium after any income history.

Problem LCP

$$\Pi(w) = \max_{\{\tau_i, w_i\},\, f \in [0,\bar{f}]} \quad f + \pi_H(-\tau_H + \beta\Pi(w_H)) + \pi_L(-\tau_L + \beta\Pi(w_L))$$

$$\text{s.t.} \quad \pi_H(u(y_H - f + \tau_H) + \beta w_H) + \pi_L(u(y_L - f + \tau_L) + \beta w_L) = w \qquad \text{(PK)}$$

$$u(y_H - f + \tau_H) + \beta w_H \geq u(y_H - f) + \beta V^{out} \qquad \text{(LCH)}$$

$$w_i \geq V^{out}, \; \forall i$$

for $\forall w \geq V^{out}$ and where $f \in [0, \bar{f}]$ is the prepayment amount with maximum $\bar{f}$.

Theorem 1 characterizes the constrained-optimal contract with limited commitment and prepayment. Understandably I focus on case (b) in Lemma 1 when full insurance (the first best) is not achievable.[3]

Theorem 1. *Suppose full insurance is not feasible in Problem LC, i.e., inequality (L1) in Lemma 1 holds. Then, the optimal insurance contract with prepayment solving Problem LCP has the following properties:*

(a) ***full insurance****: the prepayment f relaxes the limited commitment constraint*

[3] Clearly, if full insurance is already attainable in Problem LC without prepayment then no additional efficiency gains are possible.

(LCH) and can implement full insurance, $c_i^{lcp} = \bar{y}$ at zero expected profits if:

$$\frac{u(\bar{y})}{1-\beta} \geq u(y_H - \bar{f}) + \beta V^{out} \tag{C1}$$

(b) ***improved partial insurance****: otherwise, if inequality (C1) does not hold, the prepayment relaxes the limited commitment constraint (LCH) and improves the degree of partial insurance relative to in Problem LC.*

Proof: see Appendix.

Theorem 1 shows that the ability to collect prepayment or escrow funds allows improved risk sharing and consumption smoothing relative to the standard setting without prepayment (Problem LC) in Lemma 1. Specifically, Theorem 1, part (a) shows that if the prepayment upper bound is sufficiently large (inequality C1 holds), the insurance platform can provide full insurance while only partial insurance is possible in Problem LC.

In part (b), if the maximum feasible prepayment amount is relatively lower so that full insurance cannot be attained, the ability to prepay no matter how small, still allows the optimal contract to improve on the degree of partial insurance and consumption smoothing by relaxing the limited commitment constraint through lowering the agent's value of reneging and reducing the gap between c_i^{lcp} and first-best consumption $c^{fi} = \bar{y}$ (see the Theorem 1 proof for full details).

Note that in both parts (a) and (b) of Theorem 1, the optimal insurance contract uses the prepayment f to collect all or part of the due contribution/premium for an agent realizing the high-income state, which is offset by adjusting the ex-post transfer amount τ_H^{lcp}. In the low-income state the prepayment is effectively refunded in full as part of the insurance payout τ_L^{lcp} received by the agent.

3.2 Hidden Income

I next consider a setting in which risk sharing is constrained by the presence of private information. Specifically, assume that the agent's income is unobserved by the platform and the agent must be induced to report it truthfully. I refer to this setting as "hidden income". There is no commitment problem and an insurance contract is assumed fully enforceable. I relax this assumption and consider both private information and limited commitment simultaneously in Section 3.3.

The timing is as follows: (i) the agent enters the period with current promised utility w; (ii) income state H or L is realized; (iii) the agent makes an income state *report*, i to the platform which triggers transfer τ_i and next period's promised utility w_i.

Without prepayment, this is a well-known problem, e.g., see Thomas and Worrall (1990) or Ljungqvist and Sargent (2018). They prove that only the downward truth-telling constraints bind at optimum, i.e., the insurer must design the contract so that an agent with high income y_H (who would normally have to pay a premium) reports the true state, $i = H$ instead of reporting $i = L$ (lying) and receiving transfer τ_L and promised utility w_L. The contracting problem is:

Problem HI

$$\Pi(w) = \max_{\{\tau_i, w_i\}} \; \pi_H(-\tau_H + \beta\Pi(w_H)) + \pi_L(-\tau_L + \beta\Pi(w_L))$$

$$\text{s.t.} \quad \pi_H(u(y_H + \tau_H) + \beta w_H) + \pi_L(u(y_L + \tau_L) + \beta w_L) = w \tag{PK}$$

$$u(y_H + \tau_H) + \beta w_H \geq u(y_H + \tau_L) + \beta w_L \tag{TT}$$

Call the contract elements / policy functions (τ_i^{hi}, w_i^{hi}) – each of them a function of the state w. The agent reports income state $i = H$ or $i = L$ and receives contract (τ_i^{hi}, w_i^{hi}) where τ_i^{hi} is the transfer, either a contribution (premium) or an indemnity (payout). The agent's consumption c_i^{hi} equals the agent's actual income y plus the transfer,

$$c_i^{hi} = y + \tau_i^{hi}, \quad \forall i$$

Lemma 2. *The optimal insurance contract solving Problem HI satisfies, $\forall w$:*

(a) $\tau_L^{hi} > \tau_H^{hi}$ *and* $w_H^{hi} > w_L^{hi}$ *– the transfer and promised utility are monotonic in the income state*

(b) $c_H^{hi} > c_L^{hi}$ *– partial insurance is provided.*

Proof: see Thomas and Worrall (1990) or Ljungqvist and Sargent (2018).

In the hidden income setting the constrained-optimal insurance contract always features partial insurance because of the information friction, expressed by the truth-telling constraint (TT). Intuitively, the first-best outcome of full insurance (equal consumption across states $c^{fi} = \bar{y}$ and equal promised utility w^{fi}) is not incentive-compatible with truth telling, as the agent would have an incentive to always report low income and collect the payout $\bar{y} - y_L > 0$ which means that the insurer cannot break even.

Prepayment

As in Section 3.1, I next analyze allowing prepayment in Problem HI, that is, an amount $f \in [0, \bar{f}]$ collected and held in escrow in advance of the income realization, where $\bar{f}$ is the escrow/prepayment upper bound.

The contracting problem with prepayment is stated in Problem HIP, with the additional choice variable f. The contract elements (prepayment f, transfers τ_i, and next-period promised utility w_i) are optimally chosen to ensure that the agent reports the income state truthfully.

Problem HIP

$$\Pi(w) = \max_{\{\tau_i, w_i\},\, f \in [0,\bar{f}]} \quad f + \pi_H(-\tau_H + \beta V(w_H)) + \pi_L(-\tau_L + \beta \Pi(w_L))$$

$$\text{s.t.} \quad \pi_H(u(y_H - f + \tau_H) + \beta w_H) + \pi_L(u(y_L - f + \tau_L) + \beta w_L) = w \qquad \text{(PK)}$$

$$u(y_H - f + \tau_H) + \beta w_H \geq u(y_H - f + \tau_L) + \beta w_L \qquad \text{(TT)}$$

Theorem 2 characterizes the optimal insurance contract with private information (hidden income) and prepayment, $(f^{hip}, \tau_i^{hip}, w_i^{hip})$.

Theorem 2. *The optimal insurance contract with prepayment solving Problem HIP has the following properties:*
(a) *the parties can always implement the no-prepayment contract* $\{\tau_i^{hi}, w_i^{hi}\}$ *from Problem HI by choosing any feasible prepayment amount* $f^{hip} \in [0, \bar{f}]$ *and setting*

$$\tau_i^{hip} = \tau_i^{hi} + f^{hip}, \ \forall i$$

(b) *the prepayment f does not improve upon the Problem HI constrained optimum, i.e., the optimal consumption in Problem HI and Problem HIP is the same.*

Proof: (see Appendix)

Theorem 2 shows that, unlike in the limited commitment setting in Section 3.1, if the obstacle to risk-sharing is private information due to hidden income, then allowing for prepayment cannot improve the degree of insurance that is achievable, that is, the solutions (optimal transfers/consumption and promised utility as functions of w) to Problem HI and Problem HIP coincide.

The intuition for this result is that there is no feasible way to relax the truth-telling constraint (TT) by use of the prepayment f since, by construction, f cannot be contingent on the subsequent income state (it is a *pre*payment) and hence it is unable to provide additional incentives relative to what already can be done via the contingent transfers τ_i and promised utility w_i.

3.3 Both Limited Commitment and Hidden Income

I now analyze a setting with both limited commitment and private information due to hidden income. I first state and characterize the dynamic contracting problem without prepayment and then introduce the technological possibility of prepayment $f \in [0, \bar{f}]$. The problem is, $\forall w \geq V^{out}$:

Problem HILC

$$\Pi(w) = \max_{\{\tau_i, w_i\}} \quad \pi_H(-\tau_H + \beta\Pi(w_H)) + \pi_L(-\tau_L + \beta\Pi(w_L))$$

$$\text{s.t.} \quad \pi_H(u(y_H + \tau_H) + \beta w_H) + \pi_L(u(y_L + \tau_L) + \beta w_L) = w \qquad \text{(PK)}$$

$$u(y_H + \tau_H) + \beta w_H \geq u(y_H + \tau_L) + \beta w_L \qquad \text{(TT)}$$

$$u(y_i + \tau_i) + \beta w_i \geq u(y_i) + \beta V^{out}, \ \forall i \qquad \text{(LC)}$$

$$w_i \geq V^{out}, \ \forall i$$

Constraint (PK) is the promise-keeping constraint, (TT) is the downward truth-telling constraint as in Problem HI, and (LC) are the limited-commitment constraints ensuring that the agent does not renege on a transfer, as in Problem LC. As in Section 3.1, the constraint $w_i \geq V^{out}$ ensures the agent's continued participation in the contract.

Theorem 3. *For any $w \geq V^{out}$ the optimal contract $\{\tau_i^{hilc}, w_i^{hilc}\}$, $i = L, H$ solving Problem HILC satisfies:*

(a) $\tau_L^{hilc} \geq 0$; with $\tau_L^{hilc} = 0$ and $w_L^{hilc} = V^{out}$ for $w = V^{out}$.

(b) the truth-telling constraint (TT) is stricter than, i.e., implies the limited-commitment constraints (LC).

Proof: see Appendix.

Intuitively, the non-negativity of the low-income state transfer τ_L in Theorem 3(a) follows from the objective to provide consumption smoothing to the risk-averse agent. In the low-income state the agent's marginal utility (MU) is high. Hence, to deliver fixed promised utility w at the lowest cost, it is optimal for the insurer to provide additional consumption in that (high MU / low income) state rather than in the high-income state where raising the agent's utility is expensive. A hypothetical contract with $\tau_L < 0$ would mean that an agent is "taxed" while s/he is consumption poor which is the opposite of insurance and would necessitate an inefficient increase in the future promised utility w_L to satisfy the limited commitment and promise keeping constraints.

Theorem 3(b) is the key result in this section, implying that the limited-commitment constraints (LC) are automatically satisfied when truth-telling (TT) and the lower bound on future promises $w_i \geq V^{out}$ are imposed. That is, the ex-post limited commitment problem (the possibility of the agent reneging on a contribution, constraints LC) does not constrain optimal risk sharing beyond the truth-telling constraint (TT).

Note, however, that the ex-ante continued participation constraints $w_i \geq V^{out}$ in Problem HILC in general affect the contract compared to imposing constraint (TT) alone as in Problem HI. Indeed, existing work, e.g., Thomas and Worrall (1990), has shown that in Problem HI (without a lower bound on w_i) we have $w_L^{hi}(w) < w$ and $w_H^{hi}(w) > w$ – the promised utilities spread out and drift downward, with immiseration in the limit. Consequently, Theorem 3 implies that Problem HILC is equivalent to solving a modified Problem HI with imposing $w_i \geq V^{out}$, i.e., restricting the range of promised utilities.

I now introduce prepayment into the dynamic risk-sharing problem with both limited commitment and private information. The contracting problem is:

Problem HILCP

$$\Pi(w) = \max_{\{\tau_i, w_i\},\, f \in [0, \bar{f}]} \quad f + \pi_H(-\tau_H + \beta\Pi(w_H)) + \pi_L(-\tau_L + \beta\Pi(w_L))$$

$$\text{s.t.} \quad \pi_H(u(y_H - f + \tau_H) + \beta w_H) + \pi_L(u(y_L - f + \tau_L) + \beta w_L) = w \qquad \text{(PK)}$$

$$u(y_H - f + \tau_H) + \beta w_H \geq u(y_H - f + \tau_L) + \beta w_L \qquad \text{(TT)}$$

$$u(y_i - f + \tau_i) + \beta w_i \geq u(y_i - f) + \beta V^{out}, \ \forall i \qquad \text{(LC)}$$

$$w_i \geq V^{out}, \ \forall i$$

In Theorem 4 below I show that, as in Theorem 2 in Section 3.2., allowing prepayment in Problem HILC cannot improve on the constrained-optimal degree of insurance. Prepayment does relax the ex-post limited-commitment constraints (LC) as in Section 3.1., however, as shown in Theorem 3 part (b), these constraints are already satisfied at the optimum when (TT) is imposed. Hence relaxing the limited commitment constraints yields no efficiency gains since, as shown in Theorem 2, the prepayment f cannot relax the truth-telling constraint (TT).

Theorem 4. *The constrained-optimal insurance contract with prepayment solving Problem HILCP is equivalent to the contract solving Problem HILC.*

Proof: see Appendix.

To provide further intuition about the properties of the optimal dynamic insurance contract with both limited commitment and hidden income, call the non-negative Lagrange multipliers in Problem HILC as follows: μ on (PK), γ on (TT), λ_i on (LC), and η_i on $w_i \geq V^{out}$, for $i = L, H$. Given Theorem 3(b), constraints (LC) are slack ($\lambda_i = 0$) and thus the problem solution satisfies the following FOCs:

$$\text{low income:} \quad \frac{1}{u'(c_L)} = \mu - \frac{\gamma}{\pi_L}\frac{u'(y_H + \tau_L)}{u'(c_L)} \quad \text{and} \quad -\Pi'(w_L) = \mu - \frac{\gamma}{\pi_L} + \frac{\eta_L}{\beta\pi_L} \tag{2}$$

$$\text{high income:} \quad \frac{1}{u'(c_H)} = -\Pi'(w_H) = \mu + \frac{\gamma}{\pi_H} \tag{3}$$

I characterize further the constrained-optimal dynamic insurance contract solving Problem HILC (and, equivalently, Problem HILCP) via a numerical simulation. As in Townsend (1982), assume quadratic utility $u(c) = c - bc^2$, where the parameter b is chosen so that the function is strictly concave over the simulation consumption range, specifically, $b = 0.075$. The discount factor used in the simulation is $\beta = 0.95$ and the income levels are $y_H = 2$ and $y_L = 1$ with probabilities $\pi_L = \pi_H = 0.5$.

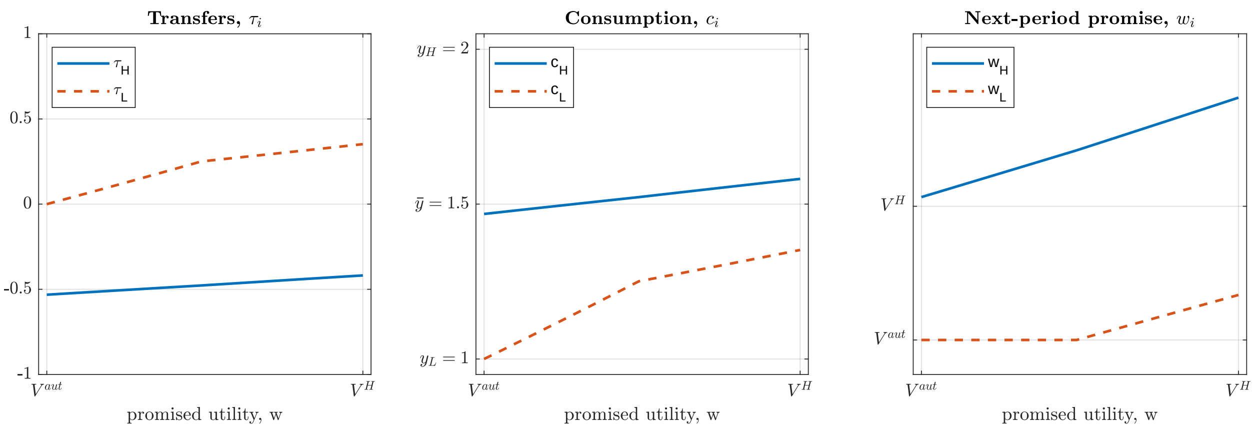
The solution to problem HILC/HILCP is displayed on Figure 1, for the state space $[V^{out}, V^H]$ where $V^H \equiv u(y_H) + \beta V^{out}$ is the value w_H^{lc} from Problem LC (see Lemma 1). The optimal contract has the following properties, $\forall w \in [V^{out}, V^H]$:

- optimal consumption satisfies $c_H^{hilc}(w) > c_L^{hilc}(w)$ – there is partial insurance. This result follows directly from the first-order conditions (2) and (3) given $\gamma > 0$ (TT binds), $u' > 0$, and $\pi_i > 0$.

- the low-income transfer, τ_L^{hilc} is non-negative and increases in w from 0 at $w = V^{out}$ to about 0.35 at $w = V^H$ (see Fig. 1). The high-income transfer, τ_H^{hilc} is negative (the agent pays a premium) and strictly increasing in w.

- consumption in the low income state, $c_L^{hilc}(w)$ equals $y_L = 1$ at $w = V^{out}$ and strictly increases in w for $w > V^{out}$, with a kink at the value $\hat{w}$ where the bound on promised utility $w_L \geq V^{out}$ stops binding (see the right panel). At that value the multiplier η_L in (2) becomes 0.

Figure 1: Problem HILC/HILCP Solution Properties



Notes: The figure displays simulation results for preferences $u(c) = c - bc^2$ with $b = 0.075$ and discount factor $\beta = 0.95$. The output levels are $y_H = 2$ and $y_L = 1$, with probabilities $\pi_L = \pi_H = 0.5$. The solution to Problem HILC/HILCP is illustrated: the constrained-optimal state-contingent transfers (left panel), consumption (middle panel), and future promises (right panel), each as function of the current promised value w over the range $V^{out} = 26.25$ to $V^H \equiv u(y_H) + \beta V^{out} = 26.64$.

- consumption in the high-income state, $c_H^{hilc}(w)$ is strictly increasing in w starting from a value lower than $\bar{y} = E(y)$ at $w = V^{out}$.
- next-period promised utility, $w_i^{hilc}(w)$ satisfies:
  - $w_L^{hilc}(w) = V^{out}$ for low values of w (the lower bound on w_L binds and $\eta_L > 0$ in FOC (2)), up to a threshold $\hat{w} > V^{out}$. For $w > \hat{w}$, $w_L^{hilc}(w) > V^{out}$ and $w_L^{hilc}(w)$ is strictly increasing in w.
  - $w_H^{hilc}(w)$ strictly increasing in w for all $w \in [V^{out}, V^H]$.

3.4 Comparing consumption smoothing for different contractual frictions

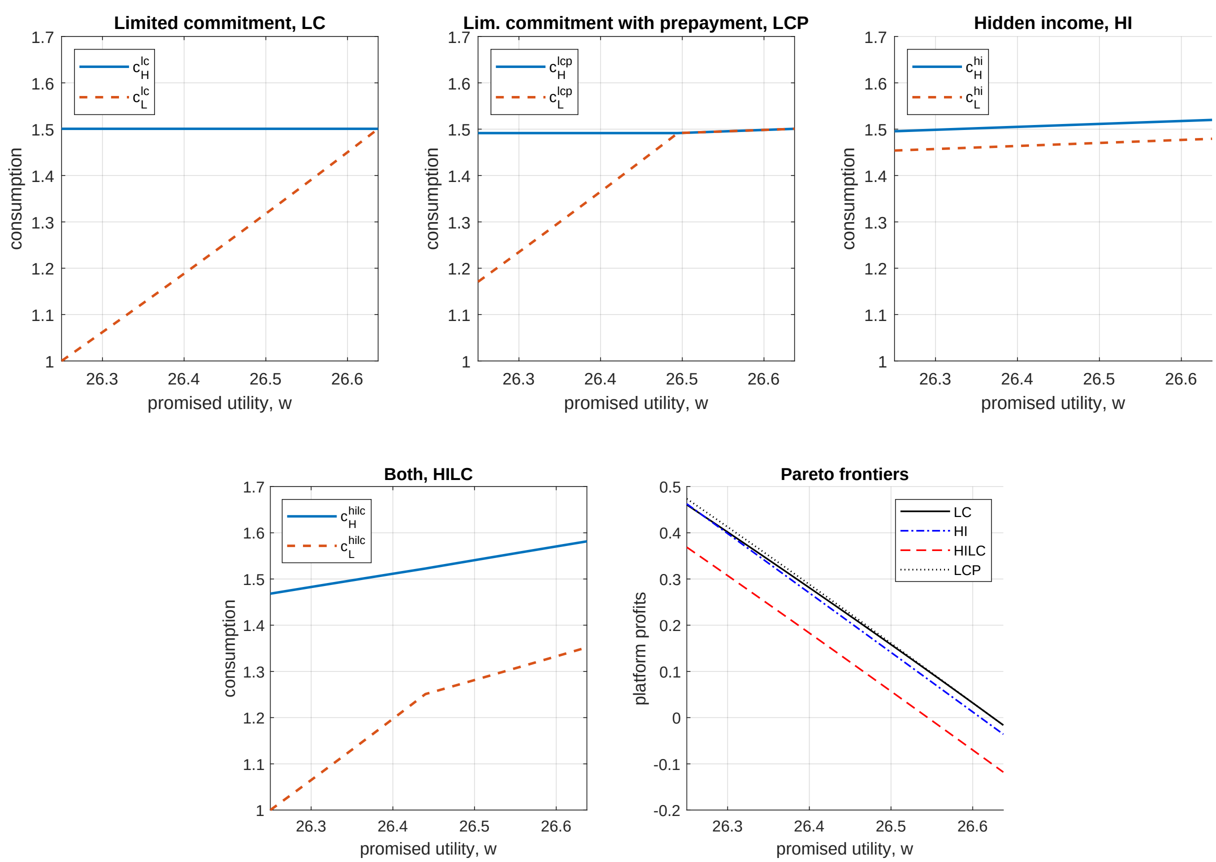
Next, I numerically compute the solutions to all Problems LC, LCP, HI(P) and HILC(P) and use the simulated policy functions to illustrate the level of consumption smoothing attained in the different contractual settings. Figure 2 illustrates the constrained-optimal consumption levels, c_H and c_L, as functions of the current promised utility w on the interval V^{out} to $V^H = u(y_H) + \beta V^{out}$.

In Problem LC (the top-left panel of Figure 2), as shown in Lemma 1, if the agent obtains high income, then his/her consumption c_H^{lc} (the solid line) is independent of the promised utility value w, while low-income consumption c_L^{lc} (the dashed line) is strictly increasing in

w. Intuitively, the insurer collects funds in the low marginal utility (high-income) state H and uses them to insure the agent against the low-income state L, see also Figure 3 below.

The top-centre panel of Figure 2 (Problem LCP) illustrates the effect of introducing prepayment f in the limited commitment setting. Specifically, we see that low-income state consumption, c_L^{lcp} for any given w is higher relative to in Problem LC (i.e., better smoothing is provided) and there is a sub-interval of promised utility values, from $u(y_H - \bar{f}) + \beta V^{out}$ to $V^H = u(y_H) + \beta V^{out}$ in which full insurance, $c_H^{lcp} = c_L^{lcp}$, is attained – see Theorem 1.

Figure 2: Risk sharing with different frictions – comparison



Notes: The figure displays simulation results for preferences $u(c) = c - bc^2$ with $b = 0.075$ and discount factor $\beta = 0.95$. The output levels are $y_H = 2$ and $y_L = 1$, with probabilities $\pi_L = \pi_H = 0.5$. The figure panels display optimal consumption as function of the promised utility state w w over the range $V^{out} = 26.25$ to $V^H = u(y_H) + \beta V^{out} = 26.64$ respectively in problems LC, HI and HILC. Each simulation is initialized at the initial promised utility w_0 that attains zero ex-ante expected profits for the insurer, $\Pi(w_0) = 0$.

The hidden income setting (HI), displayed in the top-right panel of Figure 2, always features a differential between consumption in the two income states ($c_H^{hi} - c_L^{hi} > 0$). This is required in order to provide incentives for truth-telling – see Lemma 2. Unlike in the limited commitment setting, the platform can also provide good insurance (relatively high c_L^{hi} close to $\bar{y} = 1.5$) even at low promised utility values w, since without a commitment problem the future promised utility w_L^{hi} can be reduced to below V^{out}. This incentive device is not available in the LC and HILC settings as the agent would quit the contract in such case.

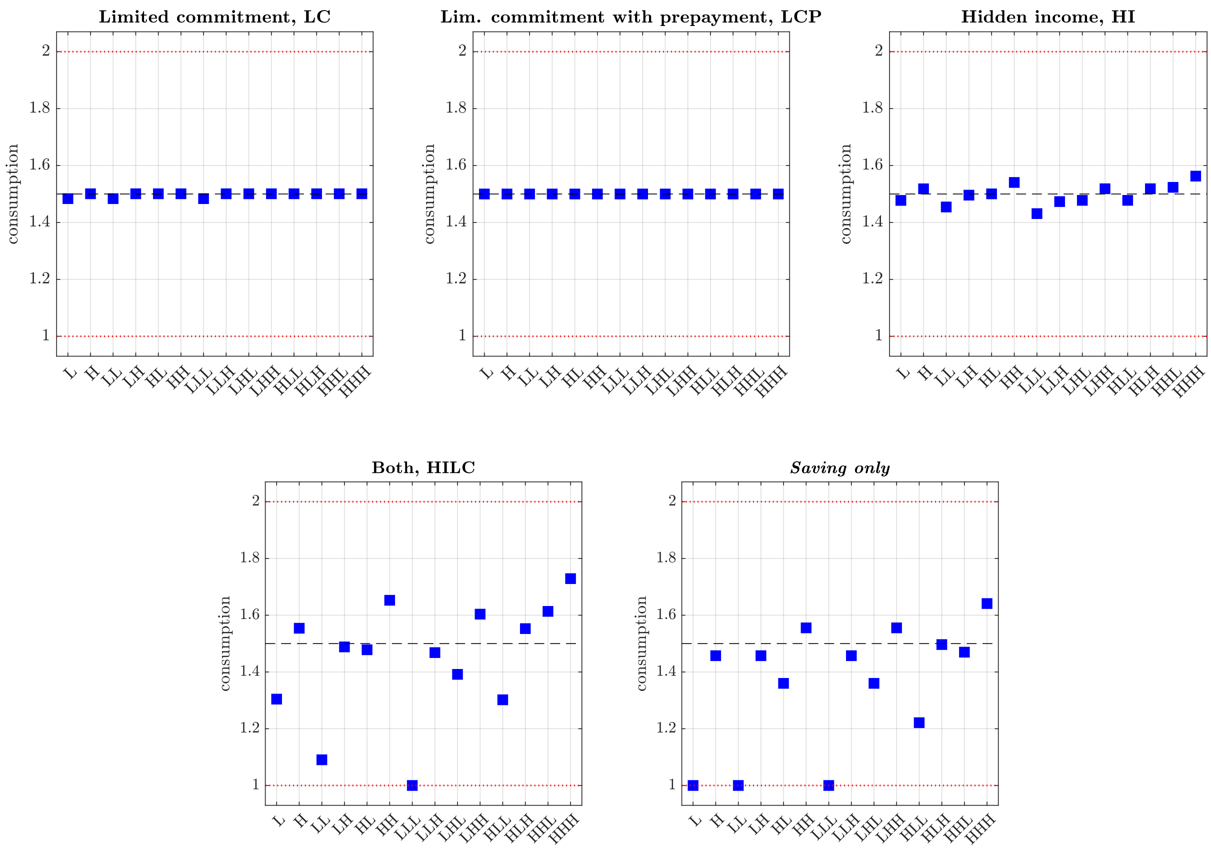
The contract with both limited commitment and private information frictions (Problem HILC), displayed in the bottom-left panel of Figure 2, combines characteristics from both the HI and LC optimal contracts. Specifically, the agent is 'taxed' in the high-income state H to insure against the low-income state, as in Problem LC, and the consumption functions c_H^{hilc} and c_L^{hilc} are strictly increasing in w, as in Problem HI, albeit with a larger gap between them because the use of future promised utility as an incentive device is constrained by the participation lower bound $w_i^{hilc} \geq V_{aut}$.

Lastly, the bottom-right panel of Figure 2 displays the Pareto frontiers computed for each setting, namely plotting the present-value discounted profit for the insurer, $\Pi(w)$ vs. the ex-ante expected present-value utility, w for the agent in the respective optimal contracts. For the chosen simulation parameters, the LC and HI frontiers lie close to each other while the LCP frontier extends slightly rightward of the LC frontier. Notably, the HILC setting where both the commitment and information frictions are present, delivers significantly lower expected utility for any fixed level of platform profits.

Figure 3 illustrates further the mechanics of consumption smoothing achieved under the various considered frictions by plotting consumption for different income time histories. The displayed consumption values are computed for the initial promise w_0 (potentially different for each setting) at which the insurer breaks even ex-ante, i.e., $\Pi(w_0) = 0$. Specifically, Figure 3 plots the optimal consumption level for all possible income histories of lengths 1, 2 or 3 time periods. For example, the history labelled "L" on the horizontal axis means low income in period 1 (history of length 1), history "HH" means high income in periods 1 and 2 (history of length 2), and history "LLH" means low income in periods 1 and 2 and high income in period 3 (history of length 3), and so on. As a benchmark, note that the first-best outcome (full insurance) corresponds to consuming $c_i^{fi} = \bar{y} = 1.5$ in each history, while autarky (consumption equal to current income, $c_i = y_i, \forall i$) would correspond to consuming $y_H = 2$ whenever current income is high (i.e., in histories H, LH, HH, LLH, LHH, HLH, HHH) and consuming $y_L = 1$ otherwise.

Figure 3 shows that the limited commitment settings achieve the highest level of smoothing, with the prepayment (panel LCP in the top-centre) narrowing the gap between the high

Figure 3: Risk sharing with different frictions – consumption



Notes: The figure displays simulation results for preferences $u(c) = c - bc^2$ with $b = 0.075$ and discount factor $\beta = 0.95$. The output levels are $y_H = 2$ and $y_L = 1$, with probabilities $\pi_L = \pi_H = 0.5$. Each figure panel displays the optimal consumption level for each possible income history of length 1, 2 or 3 periods indicated on the horizontal axis (for example, history LHL denotes state L in period 1, H in period 2, and L in period 3) for Problems LC, LCP, HI, and HILC. Each simulation is initialized at the promised utility w_0 which attains zero ex-ante expected profits for the insurer, $\Pi(w_0) = 0$. The solutions are also compared (in the right-most panel) to a *Saving Only* setting (see footnote 6). The dashed line in all panels corresponds to the full-insurance (first-best) consumption level $\bar{y} = 1.5$; the dotted red lines correspond to the y_L and y_H income levels.

and low consumption values and achieving close to full insurance.[4] The history-dependent consumption values vary sizably more in the hidden income setting HI (the top-right panel of Fig. 3), where notably histories with consecutive low or consecutive high incomes require a larger spread in consumption to maintain truth-telling incentives.

When both the commitment and information frictions are present (see panel "Both,

[4]The reason why c_L^{lc} and c_H^{lc} are numerically close in the simulation is that the break-even initial promised utility w_0 is very close to the upper bound of the state space V^H, see the Pareto frontier in Figure 2.

HILC" in the bottom row of Fig. 3), the degree of consumption smoothing across income state histories is significantly worse, explaining the previous Pareto frontier result in Fig. 2. The last panel (bottom row, right) compares the LC, LCP, HI and HILC solutions with a "saving only" setting in which the agent starts with zero savings and can accumulate, at gross rate of return R, or run down buffer-stock savings.[5] We see that in terms of state-contingent consumption the HILC setting is not dissimilar to the saving-only setting although it does provide more insurance against income histories L and LL.

4 Conclusions

This paper analyzes the role of prepayment in dynamic risk-sharing contracts with commitment or information frictions, considered both separately and jointly. I model a long-term dynamic risk-sharing contract where an insurance platform can collect or withhold ex-ante a portion of the agent's income as a prepayment. The main result is that the prepayment can be used to mitigate or resolve the limited commitment problem, but is ineffective in the settings with private information and with both private information and limited commitment.

In the limited commitment setting, the prepayment acts as a commitment device by lowering the value of the agent's outside option and thus diminishing the agent's incentive to renege on a due insurance premium when his/her income is high. This relaxes the commitment constraints, resulting in improved consumption smoothing and allowing the insurer to collect the necessary premia to fund indemnity payments when realized income is low.

In contrast, when the obstacle to risk sharing is private information (hidden income), the prepayment mechanism does not lead to an efficiency improvement, since the agent's incentive to misreport the income state depends on the utility differential between reporting high versus low income. Since the prepayment amount is fixed and collected ex-ante before the income state is realized, prepayment does not affect the agent's truth-telling constraint and cannot provide additional incentives beyond what can be achieved via state-contingent transfers and promised utility alone.

In the scenario with both limited commitment and hidden income frictions present at the same time, I prove that the private information friction dominates – that is, satisfying

[5] In this saving only (S) setting, the agent solves the following dynamic optimal savings problem:

$$V^S(a) = \max_{\{a'_i\}} \sum_{i \in L,H} \pi_i \left(u(y_i + Ra - a'_i) + \beta V^S(a'_i) \right)$$

where a is current savings, a'_i are next-period savings in income state i, and $a_0 = 0$.

the truth-telling constraint implies satisfying the limited commitment constraints. Consequently, the technological possibility of prepayment again does not improve efficiency and consumption smoothing. Numerical simulations illustrate and compare the constrained-optimal insurance contracts in all considered settings characterizing their ability to provide consumption smoothing relative to autarky, full insurance, and a saving-only setting.

While the paper's results are mainly theoretical, they offer practical implications for the design of (digital) financial and insurance platforms. A key takeaway is that technological solutions for locking funds such as smart contracts or escrow accounts are best suited for situations where the primary risk is contract repudiation or reneging on due premia. In contrast, in settings in which client income verification is costly or infeasible, requiring upfront escrow funds or prepayment would be insufficient and ineffective to improve risk sharing and instead mechanisms designed to elicit truthful revelation of unobserved states are the binding requirement.

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Appendix – Proofs

Lemma 1

(a) and (b) The solution to Problem LC has been characterized in previous work, e.g., see Ljungqvist and Sargent (2018) (hereafter LS), with properties as displayed in parts (a) and (b) of Lemma 1. Note also, that for $\tau_L^{lc} > 0$, which is the true for any $w > V^{out}$ (see LS for proof), then constraint (LC) is slack in the low income state $i = L$ since $u(y_L + \tau_L) + \beta w_L > u(y_L) + \beta V^{out}$ given $w_L \geq V^{out}$.

(c) Calling $w^{fi} = \frac{u(\bar{y})}{1-\beta}$, note that inequality (1), $\frac{u(\bar{y})}{1-\beta} \geq u(y_H) + \beta V^{out}$ is equivalent to

$$u(\bar{y}) + \beta w^{fi} \geq u(y_H) + \beta V^{out}$$

which implies that both constraints (LC) are satisfied and the insurance platform can implement full insurance at zero ex-ante expected ($\Pi(w_0) = 0$) profits by setting $w_0 = w^{fi}$ and offering the contract:

$$\tau_i^{lc} = \bar{y} - y_i \text{ (i.e., } c_i^{lc} = c^{fi} = \bar{y}) \text{ and } w_i^{lc} = w^{fi} = \frac{u(\bar{y})}{1-\beta}, \text{ for } i = L, H \tag{FI}$$

Note that profits are zero both ex-ante and in each period ($\pi_H \tau_H^{lc} + \pi_L \tau_L^{lc} = 0$).

If the opposite case holds, i.e.,

$$\frac{u(\bar{y})}{1-\beta} < u(y_H) + \beta V^{out} \tag{L1}$$

then $\forall w_0 \in [V^{out}, u(y_H) + \beta V^{out})$ partial insurance obtains with the (LC) constraint binding for income state $i = H$ – see part (a).

Finally, call $V^H = u(y_H) + \beta V^{out}$ and note that if (L1) holds then the initial promised utility level w_0^* at which $\Pi(w_0^*) = 0$ must satisfy $w_0^* < V^H$ (i.e., it is in the partial insurance range described in part (a)). To see this, call c^* the absorbing consumption level (see part Lemma 1(b)) obtained at $w = V^H$ and defined by $\frac{u(c^*)}{1-\beta} = V^H$ and note that inequality (L1) implies $c^* > \bar{y}$ by the strict monotonicity of u. But then expected consumption exceeds expected income each period and so $\Pi(V^H) < 0$, from where the monotonicity of Π implies $w_0^* < V^H$ indeed.

Theorem 1

Denote the constrained-optimal transfers solving Problem LCP by τ_i^{lcp}, $i = L, H$ and the optimal prepayment by f^{lcp}.

(a) **prepayment attains full insurance**

If inequality (C1), $\frac{u(\bar{y})}{1-\beta} \geq u(y_H - \bar{f}) + \beta V^{out}$ holds,[6] then first-best full insurance, $c_i = \bar{y}$, $\forall i$ can be attained by setting the prepayment $f^{lcp} \in [0, \bar{f}]$ so that,

$$u(y_H - f^{lcp}) + \beta V^{out} = \frac{u(\bar{y})}{1-\beta} \tag{T1}$$

Such f^{lcp} always exists given (C1), since the function $G(f) = u(y_H - f) + \beta V^{out}$ is strictly decreasing in f and since, as assumed, inequality (L1) from Lemma 1, $\frac{u(\bar{y})}{1-\beta} < u(y_H) + \beta V^{out}$ holds, i.e., $G(0) > \frac{u(\bar{y})}{1-\beta}$. Consider the following transfers τ_i^{lcp} and promises w_i^{lcp}:

$$\tau_i^{lcp} = \bar{y} - y_i + f^{lcp} \quad \text{and} \quad w_i^{lcp} = \frac{u(\bar{y})}{1-\beta} = w^{fi}$$

which implies $c_i^{lcp} = y_i - f^{lcp} + \tau_i^{lcp} = \bar{y}$.

The limited commitment constraint (LCH) in Problem LCP evaluated at $c_i^{lcp} = \bar{y}$, $w_i^{lcp} = w^{fi}$ and f^{lcp} defined in (T1) is equivalent to:

$$u(\bar{y}) + \beta w^{fi} \geq u(y_H - f^{lcp}) + \beta V^{out},$$

which is satisfied since, using (T1), both its l.h.s. and r.h.s. equal $w^{fi} = \frac{u(\bar{y})}{1-\beta}$. Constraint (PK) is also trivially satisfied at $w = w^{fi}$ and $w_i^{lcp} = w^{fi}$. Therefore, full insurance at zero expected profits is indeed implementable via the contract described above, starting from $w_0 = w^{fi} = \frac{u(\bar{y})}{1-\beta}$ (see also Lemma 1(b)).

Note that there is an indeterminacy of the prepayment and the transfers – any feasible values $(\hat{\tau}_i, \hat{f})$ with $\hat{f} \geq f^{lcp}$ which satisfy $\hat{\tau}_i - \hat{f} = \bar{y} - y_i$ achieve the same consumption and respect constraint (LCH).

(b) **prepayment improves partial insurance**

If the maximum prepayment $\bar{f}$ is relatively low so that inequality (C1) does not hold, then the insurer cannot implement the first-best full insurance, $c_i = \bar{y}$, since even requiring the maximum prepayment $f^{lcp} = \bar{f}$ would make the agent renege when income is y_H (remember,

[6] A simple sufficient condition for (C1) is $\bar{f} \geq y_H - \bar{y}$.

Theorem 1 assumes that condition (L1) from Lemma 1 holds).

However, note that the prepayment f still relaxes constraint (LCH) for any $\bar{f} > 0$. Indeed, by setting $f^{lcp} = \bar{f}$, constraint (LCH) becomes

$$u(y_H - \bar{f} + \tau_H) + \beta w_H \geq u(y_H - \bar{f}) + \beta V^{out}$$

Setting the prepayment f at its maximum possible value $\bar{f}$ therefore lowers the right hand side (the agent's outside option), while the constraint can be always satisfied by adjusting the transfer τ_H in the left hand side as needed.

Therefore (see Lemma 1), for any promised utility level $w \in [V^{out}, u(y_H - \bar{f}) + \beta V^{out})$, the constrained-optimal contract has:

$$w_H^{lcp} = u(y_H - \bar{f}) + \beta V^{out} \text{ and } u(c_H^{lcp}) = (1-\beta) w_H^{lcp}$$

which implies $c_H^{lcp} > c_H^{fi} = \bar{y}$, as in Problem LC Lemma 1(b) – i.e., partial insurance.

Comparing with Lemma 1, note however that $w_H^{lcp} < w_H^{lc}$ and so $c_H^{lcp} < c_H^{lc}$, that is, in Problem LCP consumption in the high income state c_H^{lcp} is closer to the first-best level $c_H^{fi} = \bar{y}$ compared to its respective value c_H^{lc} in Problem LC. Similarly, c_L^{lcp} is determined from the promise-keeping constraint (PK) as in Lemma 1 and, because $c_H^{lcp} < c_H^{lc}$, it satisfies $c_L^{lcp} > c_L^{lc}$, that is, c_L^{lcp} is also closer to $c_L^{fi} = \bar{y}$. Therefore the degree of partial insurance in Problem LCP is strictly improved by the prepayment, as the gap between consumption in the high and low income states shrinks. See also Figure 2 in Section 3.4 for a numerical example illustration. ■

Theorem 2

Part (a) is immediately verified by comparing problems HI and HIP.

To prove part (b), suppose that there exists a contract $(f^{**}, \tau_i^{**}, w_i^{**})$ that is different from the Problem HI solution $(0, \tau_i^{hi}, w_i^{hi})$ and achieves higher objective value in Problem HIP. Then consider the contract:

$$\tilde{f} = 0, \;\; \tilde{\tau}_i = \tau_i^{**} - f^{**}, \;\text{ and }\; \tilde{w}_i = w_i^{**}$$

Note that the objective value in Problem HIP (and HI) remains the same since

$$f^{**} + \pi_H(-\tau_H^{**}) + \pi_L(-\tau_L^{**}) = 0 + \pi_H(-(\tau_H^{**} - f^{**})) + \pi_L(-(\tau_L^{**} - f^{**}))$$

Constraints (PK) and (TT) are also unchanged since

$$\tau_i^{**} - f^{**} = \tilde{\tau}_i - 0 \text{ for } i = L, H.$$

Therefore, the contract $(\tilde{f}, \tilde{\tau}_i, \tilde{w}_i) = (0, \tilde{\tau}_i, \tilde{w}_i)$ attains the same objective value as contract $(f^{**}, \tau_i^{**}, w_i^{**})$, i.e., by assumption higher than the value from $(0, \tau_i^{hi}, w_i^{hi})$ and is feasible for Problem HI since it has no prepayment ($\tilde{f} = 0$) and satisfies all constraints. However, this contradicts the optimality of contract $(0, \tau_i^{hi}, w_i^{hi})$ solving Problem HI and hence such $(f^{**}, \tau_i^{**}, w_i^{**})$ does not exist. ■

Theorem 3

(a) Call $c_i = y_i + \tau_i$ for $i = L, H$. I first show that at the lower bound $w = V^{out}$ the following holds:

$$\tau_L = 0 \text{ and } w_L = V^{out} \tag{*}$$

To show (*), note first that the (LC) constraints require:

$$u(c_i) + \beta w_i \geq u(y_i) + \beta V^{out} \tag{4}$$

Multiplying each constraint $i = L, H$ by π_i and adding up, we obtain

$$\pi_H(u(c_H) + \beta w_H) + \pi_L(u(c_L) + \beta w_L) \geq E(u(y)) + \beta V^{out} = V^{out} \tag{5}$$

where the last equality follows from the definition of V^{out} in (OO).

At $w = V^{out}$ the promise-keeping constraint (PK) requires:

$$\pi_H(u(c_H) + \beta w_H) + \pi_L(u(c_L) + \beta w_L) = V^{out} \tag{6}$$

Using (4) – (6), we see that we must have, $\forall i$:

$$u(c_i) + \beta w_i = u(y_i) + \beta V^{out} \tag{7}$$

By standard arguments (e.g., Thomas and Worrall, 1990) the truth-telling constraints (TT) is not slack $\forall w \geq V^{out}$, hence we can use (TT) and (PK) to solve for w_L and w_H:

$$\beta w_L = w - \pi_H u(y_H + \tau_L) - \pi_L u(c_L) \tag{8}$$

$$\beta w_H = w + \pi_L u(y_H + \tau_L) - u(c_H) - \pi_L u(c_L) \tag{9}$$

Using (9) evaluated at $w = V^{out}$ and $(1-\beta)V^{out} = E(u(y)) = \pi_H u(y_H) + \pi_L u(y_L)$, equation (7) for $i = H$ then implies:

$$\begin{aligned} u(c_H) + V^{out} + \pi_L u(y_H + \tau_L) - u(c_H) - \pi_L u(c_L) = u(y_H) + \beta V^{out} &\iff \\ \pi_H u(y_H) + \pi_L u(y_L) + \pi_L u(y_H + \tau_L) - \pi_L u(c_L) = u(y_H) &\iff \\ \pi_L(u(y_H + \tau_L) - u(y_L + \tau_L)) = \pi_L(u(y_H) - u(y_L)) & \end{aligned} \tag{10}$$

By the strict concavity of u, equation (10) implies τ_L=0 i.e., $c_L = y_L$ and hence, from (7), $w_L = V^{out}$ which proves (*) as claimed. □

Next, note that at $w = V^{out}$ where $\tau_L = 0$ and $w_L = V^{out}$ as shown above, the (LC) for $i = L$ is redundant and $\lambda_L = 0$. The FOC with respect to τ_L at $w = V^{out}$ then implies:

$$\mu(V^{out}) = \frac{1}{u'(y_L)} + \frac{\gamma}{\pi_L} \frac{u'(y_H)}{u'(y_L)}$$

or, since $\gamma \geq 0$ and u is strictly increasing,

$$\mu(V^{out}) \geq \frac{1}{u'(y_L)} \tag{11}$$

Next, I prove that $\tau_L < 0$ (i.e., asking the agent to pay in the low income state) cannot be optimal for any $w > V^{out}$ using a variational argument. Start with a hypothetical contract that taxes the agent in the low-income state (i.e., has $\tau_L < 0$). I will show that the value of the objective would increase if the insurer raised τ_L and adjusted the continuation utility w_L to keep constant the agent's total value from reporting $i = L$ which equals $V_L = u(c_L) + \beta w_L$.

Note first that if $\tau_L < 0$ we would have $c_L = y_L + \tau_L < y_L$ which implies, from the $i = L$ limited commitment constraint, $w_L > V^{out}$. Consider a small increase in the transfer $d\tau_L = \epsilon > 0$ and reduce the promised utility by $dw_L = -\frac{u'(c_L)\epsilon}{\beta}$ to keep the agent's total value $V_L = u(y_L + \tau_L) + \beta w_L$ constant and thus ensuring that constraint (PK) and constraint (LC) for $i = L$ remain satisfied. Keep τ_H and w_H unchanged, thus constraint (LC) for $i = H$ is unaffected. Note that constraint (TT) remains satisfied, since its l.h.s. does not change while it r.h.s. changes by

$$u'(y_H + \tau_L)\epsilon - \beta \frac{u'(c_L)\epsilon}{\beta} = \epsilon(u'(y_H + \tau_L) - u'(y_L + \tau_L)) < 0$$

by the strict concavity of u and $y_H > y_L$. Therefore the modified contract $(\tau_L + d\tau_L, w_L - dw_L, \tau_H, w_H)$ is feasible for Problem HILC.

Now look at the effect on insurer's profits from the variation:

$$d\Pi = \pi_L \left(\underbrace{-\epsilon}_{\text{higher payout}} + \underbrace{\beta \Pi'(w_L) dw_L}_{\text{change in future value}} \right)$$

By the Envelope Theorem, $\Pi'(w) = -\mu(w)$ where $\mu(w)$ is the marginal cost of providing agent utility in state w (the multiplier on the PK constraint). We thus obtain:

$$d\Pi = \pi_L \left[-\epsilon - \beta\mu(w_L) \left(-\frac{u'(c_L)\epsilon}{\beta} \right) \right] = \pi_L \epsilon \left[\mu(w_L) u'(c_L) - 1 \right]$$

To prove that the variation increases profit (i.e., $\tau_L < 0$ is suboptimal), it therefore remains to show that:

$$\mu(w_L) > \frac{1}{u'(c_L)} \tag{12}$$

By standard arguments, e.g., Thomas and Worrall (1990) or Ljungqvist and Sargent (2018), since the utility function u is strictly concave and the insurer is risk-neutral, the value function $\Pi(w)$ is strictly concave and thus the insurer's marginal cost $-\Pi'(w) = \mu(w)$ is strictly increasing in w. Therefore, using (11), $w_L > V^{out}$ implies:

$$\mu(w_L) > \mu(V^{out}) \geq \frac{1}{u'(y_L)}$$

In addition, $\tau_L < 0$ implies $c_L < y_L$ and, because u is concave, $u'(c_L) > u'(y_L)$, that is,

$$\frac{1}{u'(y_L)} > \frac{1}{u'(c_L)}$$

Combining the above inequalities, we obtain:

$$\mu(w_L) > \frac{1}{u'(y_L)} > \frac{1}{u'(c_L)}$$

which implies that inequality (12) indeed holds, and hence the resulting change in profits $d\Pi$ is strictly positive if the insurer increases τ_L starting from $\tau_L < 0$ while reducing future promises.

Intuitively, if $\tau_L < 0$ the agent's high marginal utility in the low income state ($c_L < y_L$) makes increasing current consumption a more efficient way to deliver utility to the agent than future promises (as future income could also be high). We conclude that, in the Problem HILC solution, $\tau_L < 0$ is impossible and therefore $\tau_L \geq 0$ for all $w \geq V^{out}$, with $\tau_L = 0$ for $w = V^{out}$. □

(b) Note that, since the contract requires $w_L \geq V^{out}$, then for $\tau_L \geq 0$ (with strict inequality for $w > V^{out}$) as shown in part (a), we have

$$u(y_H + \tau_L) + \beta w_L \geq u(y_H) + \beta V^{out} \tag{13}$$

Inequality (13) implies that the truth-telling constraint (TT)

$$u(y_H + \tau_H) + \beta w_H \geq u(y_H + \tau_L) + \beta w_L$$

is stricter than, and therefore implies, the limited commitment constraint (LC) for $i = H$,

$$u(y_H + \tau_H) + \beta w_H \geq u(y_H) + \beta V^{out}$$

The (LC) constraint for $i = L$,

$$u(y_L + \tau_L) + \beta w_L \geq u(y_L) + \beta V^{out}$$

is directly implied by $\tau_L \geq 0$ and $w_L \geq V^{out}$. Therefore, satisfying the truth-telling constraint (TT) in Problem HILC ensures that the limited commitment constraints (LC) are always satisfied. ■

Theorem 4

It is easy to show that the Problem HILC solution can be implemented in Problem HILCP in an analogous way as done for Problem HIP in Theorem 2, part (a) in Section 3.2. Once again the prepayment f cannot improve the degree of insurance relative to Problem HILC since the limited commitment constraints (LC) do not bind at the optimum and the prepayment cannot relax the truth-telling constraint (TT) as shown in Theorem 2, part (b).