Supplementary Document for the Manuscript entitled "Estimating Functional Single Index Models with Compact Support"

S1 Real Data Application

S1.1 Effect of the Order of the B-spline Basis

Figure S1 displays the estimated index function $\hat{\beta}(t)$ and link function $\hat{g}(\cdot)$ with different orders of B-spline. Figure S1 shows that a higher order of B-spline functions leads to a less sparse estimator of the index function $\beta(t)$ and not many differences in estimating the link function $g(\cdot)$. This is probably due to the fact that different estimates of $\beta(t)$ result in a very similar estimate of the score $\int \beta(t)X(t)dt$. In this case, we propose to use cross-validation to choose the optimal order of B-spline functions along with other parameters in the model. Table S1 shows the optimal choice of λ, γ and bandwidth given different orders of the B-spline. It shows that the mean squared prediction error (MSPE) is minimized when the order of the B-spline basis is 4. Note that given the order of the B-spline basis, the optimal values of the sparse parameter λ, smoothing parameter γ and the bandwidth h are still chosen from a three-dimensional grid: $\{10^2, 10^4, 10^6, 10^8\} \times \{10^1, 10^3, 10^5, 10^7\} \times \{0.3, 0.5, 0.8\}$ by 10-fold cross-validation. For instance, with order 4 B-spline, the optimal choice of λ, γ and h are selected to be 10^6, 10^3 and 0.5, which gives a value of MSPE

461.61. Similarly, with order 8 B-spline, the optimal choice of λ, γ and h are selected to be 10^7, 10^5 and 0.5, respectively.

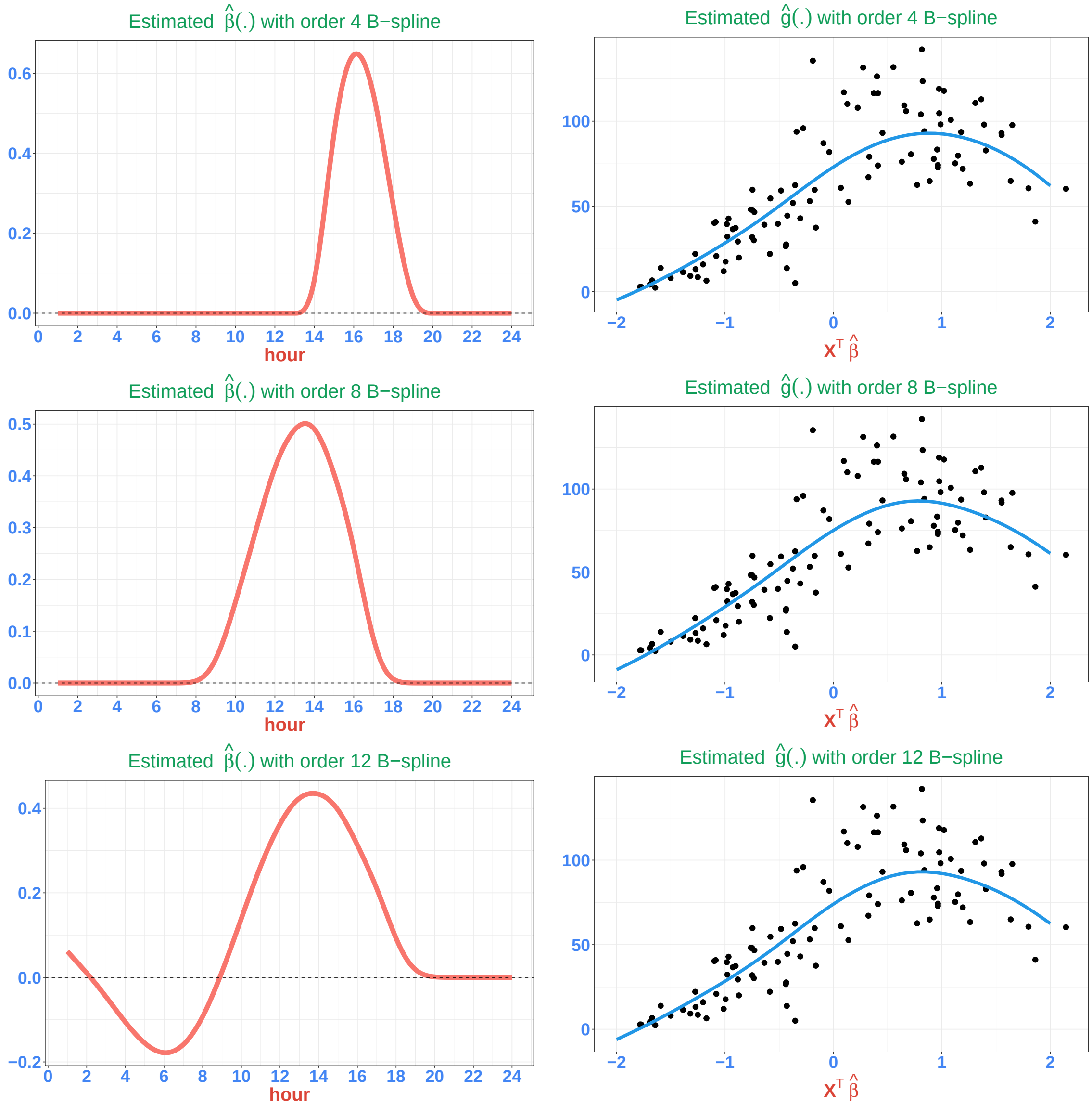

Figure S1: Estimated index function $\hat{\beta}(t)$ and link function $\hat{g}(\cdot)$ using different orders of B-spline basis functions.

Table S1: The mean squared prediction error (MSPE) using different orders of B-spline in the 10-fold cross-validation procedure. The table also displays the optimal values of the sparse parameter λ, smoothing parameter γ and the bandwidth h when using different orders of B-spline.

Order of B-spline Basis	$\log 10(\lambda)$	γ	h	MSPE
4	6	1000	0.50	461.61
8	7	10000	0.50	475.03
12	5.5	10000	0.50	520.52

S2 Simulation Results

Here we show the simulation results when the noise-to-signal ratio is set to be 20%. Details on the simulation settings can be found in Section 4 in the main manuscript.

Table S2: The prediction mean square error (PMSE), the integrated error, ISE_1 and ISE_0, defined on the nonzero subinterval and zero subintervals of the true index function $\beta(t)$ respectively on the test dataset for 100 simulation repetitions with 20% signal-to-noise ratio. Here, the true $\beta(\cdot)$ is defined in (9) in the main manuscript and the true $g(x) = \sin(4x)$. Five different methods are used: the proposed compact functional single index model(cFuSIM), the traditional functional single index model(FSIM), the locally sparsed functional linear model(Slos), the functional linear regression model(FLR) and the functional nonparametric method(NPFM).

methods	PMSE(10^{-4})	ISE_1(10^{-4})	ISE_0(10^{-4})
cFuSIM	49.0(13.8)	14.0(17.0)	4.3(9.2)
FSIM	58.0(13.6)	30.0(22.5)	40.0(23.1)
Slos	4003.5(517.8)	3162.2(3887.4)	1390.5(2159.6)
FLR	3964.8(500.7)	1819.3(1794.2)	4046.9(2881.7)
NPFM	3407.7(266.8)		

Table S3: The prediction mean square error (PMSE), the integrated error, ISE_1 and ISE_0, defined on the nonzero subinterval and zero subintervals of the true index function $\beta(t)$ respectively on the test dataset for 100 simulation repetitions with 20% signal-to-noise ratio. Here, the true $\beta(\cdot)$ is defined in (9) in the main manuscript and the true $g(x) = x$. Five different methods are used: the proposed compact functional single index model(cFuSIM), the traditional functional single index model(FSIM), the locally sparsed functional linear model(Slos), the functional linear regression model(FLR) and the functional nonparametric method(NPFM).

methods	PMSE(10^{-4})	$\text{ISE}_1(10^{-4})$	$\text{ISE}_0(10^{-4})$
cFuSIM	11.6(1.8)	18.0(7.7)	3.8(8.4)
FSIM	11.9(1.6)	17.1(7.7)	40.5(13.9)
Slos	11.7(1.6)	17.4(9.9)	4.5(5.6)
FLR	19.8(2.9)	180.2(8.8)	287.2(12.3)
NPFM	226.9(31.8)		

Table S4: The prediction mean square error (PMSE), the integrated error and ISE_1 defined on the nonzero subinterval $[0, 1]$ the true index function $\beta(t)$ respectively on the test dataset for 100 simulation repetitions with 20% signal-to-noise ratio. Here, the true $\beta(\cdot)$ is defined in (10) in the main manuscript and the true $g(x) = \sin(4x)$. Five different methods are used: the proposed compact functional single index model(cFuSIM), the traditional functional single index model(FSIM), the locally sparsed functional linear model(Slos), the functional linear regression model(FLR) and the functional nonparametric method(NPFM).

methods	PMSE(10^{-4})	$\text{ISE}_1(10^{-4})$
cFuSIM	51.0(12.8)	14.9(14.8)
FSIM	48.7(13.5)	15.2(13.8)
Slos	455.0(156.6)	792.8(14.6)
FLR	574.2(139.3)	577.5(1755.8)
NPFM	911.4(250.9)	

Table S5: The prediction mean square error (PMSE), the integrated error and ISE_1 defined on the nonzero subinterval $[0,1]$ the true index function $\beta(t)$ respectively on the test dataset for 100 simulation repetitions with 20% signal-to-noise ratio. Here, the true $\beta(\cdot)$ is defined in (10) in the main manuscript and the true $g(x) = x$. Five different methods are used: the proposed compact functional single index model(cFuSIM), the traditional functional single index model(FSIM), the locally sparsed functional linear model(Slos), the functional linear regression model(FLR) and the functional nonparametric method(NPFM).

methods	PMSE(10^{-4})	ISE_1(10^{-4})
cFuSIM	13.6(2.1)	136.2(40.4)
FSIM	13.6(2.0)	133.6(38.2)
Slos	12.0(2.1)	805.3(137.3)
FLR	12.7(2.2)	37.7(2.5)
NPFM	231.2(39.2)	