Supplementary File to the Manuscript "Finding Common Modules in a Time-Varying Network with Application to the Drosophila Melanogaster Gene Regulation Network"

Proof of Theorem 1

First we formalize the notations that are used in the following proof. For any label $\boldsymbol{e} = (e_1, \ldots, e_n)$ on a graph with adjacency matrix A, define the $K \times K$ matrix $O(\boldsymbol{e})$ by

$$O_{kl}(\boldsymbol{e}) = \sum_{ij} A_{ij} I\{e_i = k, e_j = l\}, \tag{S1}$$

where $I(z_1, z_2) = 1$ if both z_1 and z_2 are true. Define

$$O_k(\boldsymbol{e}) = \sum_l O_{kl}(\boldsymbol{e}).$$

In the following proof, we would often suppress the argument $\boldsymbol{e}$ for brevity. It is easy to show that the modularity in (10) can also be written as

$$\mathcal{Q}(\boldsymbol{e}, \mathcal{G}) = \frac{1}{2\bar{m}} \sum_{l=1}^{S} \sum_{k=1}^{K} \left(O_{kk}^l - \frac{(O_k^l)^2}{\sum_{kh} O_{kh}^l} \right),$$

where $O_{kh}^l = \sum_{ij} A_{ij}(t_l) I\{e_i = k, e_j = h\}$. We have

$$\begin{aligned}
& \mathbb{E}(O_{kh}^l | \boldsymbol{c}) \\
= \ & \mathbb{E}(\sum_{ij} A_{ij}(t_l) I(e_i = k, e_i = h) | \boldsymbol{c}) \\
= \ & \sum_{ij} \sum_{ab} \theta_{ab}(t_l) I(e_i = k, c_i = a) I(e_j = h, c_j = b).
\end{aligned}$$

Define $\mathcal{H} = (H^1, \ldots, H^S)$, where

$$H^l(R(\boldsymbol{e})) = n^2 R(\boldsymbol{e}) \Theta(t_l) R^T(\boldsymbol{e}),$$

and $R(\boldsymbol{e})$ is a $K \times K$ matrix with the ab-th entry

$$R(\boldsymbol{e})_{ab} = \frac{1}{n} \sum_{i=1}^{n} I(e_i = a, c_i = b).$$

Therefore we have $H_{kh}^l = \mathbb{E}(O_{kh}^l(\boldsymbol{e}) | \boldsymbol{c})$. Define $K \times K$ matrix $V(\boldsymbol{e})$, where

$$V(\boldsymbol{e})_{ab} = \frac{\sum_{i=1}^{n} I(e_i = a, c_i = b)}{\sum_{i=1}^{n} I(c_i = b)}.$$

Consider a community label $\boldsymbol{e} = (e_1, \ldots, e_n)$ for a time-varying network $\mathcal{G} = (G(t_l), l = 1, \ldots, S)$ from the TSBM model. Define $\mathcal{O}(\boldsymbol{e}) = (O^l(\boldsymbol{e}), l = 1, \ldots, S)$, where $O^l(\boldsymbol{e})$ is (S1) calculated using $G(t_l)$ and $\boldsymbol{e}$. Further, define

$$\mathcal{L}(\mathcal{O}(\boldsymbol{e})) = \frac{1}{S}\sum_{l=1}^{S} L(O^l(\boldsymbol{e})),$$

where

$$L(O^l(\boldsymbol{e})) = \sum_{k=1}^{K}\left(O^l(\boldsymbol{e})_{kk} - \frac{(O^l(\boldsymbol{e})_k)^2}{\sum_{kh} O^l(\boldsymbol{e})_{kh}}\right).$$

Showing the $\hat{\boldsymbol{e}}$ that maximizes the $\mathcal{Q}(\boldsymbol{e}, \mathcal{G})$ is consistent is equivalent to showing the $\hat{\boldsymbol{e}}$ that maximizes the $\mathcal{L}(\mathcal{O}(\boldsymbol{e}))$ is consistent. We show consistency by showing that there exists $\delta_S \to 0$, such that

$$P\left(\max_{\boldsymbol{e}:\ ||V(\boldsymbol{e})-V(\boldsymbol{c})||_1 \geq \delta_S} \mathcal{L}(\mathcal{O}(\boldsymbol{e})) < \mathcal{L}(\mathcal{O}(\boldsymbol{c}))\right) \to 1, \quad \text{as} \quad S \to \infty.$$

Here $||R||_1 = \sum_{kl} |R_{kl}|$.

We have

$$\begin{aligned}
& \max_{\boldsymbol{e}} |\mathcal{L}(\mathcal{O}(\boldsymbol{e})) - \mathcal{L}(\mathcal{H}(R(\boldsymbol{e})))| \\
\leq \ & \max_{\boldsymbol{e}} \sum_{ij} \left| \frac{1}{S}\sum_{l=1}^{S}(A_{ij}(t_l) - \theta_{c_i,c_j}(t_l)) - \frac{1}{S}\sum_{l=1}^{S}\left(\frac{d_i(t_l)d_j(t_l)}{2m(t_l)} - \frac{E(d_i(t_l))E(d_j(t_l))}{2E(m(t_l))}\right)\right| \delta(e_i, e_j) \\
\leq \ & \sum_{ij}\left|\frac{1}{S}\sum_{l=1}^{S}(A_{ij}(t_l) - \theta_{c_i,c_j}(t_l))\right| + \sum_{ij}\left|\frac{1}{S}\sum_{l=1}^{S}\left(\frac{d_i(t_l)d_j(t_l)}{2m(t_l)} - \frac{E(d_i(t_l))E(d_j(t_l))}{2E(m(t_l))}\right)\right|.
\end{aligned}$$

With Chebychev's inequality, we have

$$P\left(\left|\sum_{l=1}^{S} A_{ij}(t_l) - \sum_{l=1}^{S}\theta_{c_ic_j}(t_l)\right| \geq S\epsilon\right) \leq \frac{var(\sum_{l=1}^{S} A_{ij}(t_l))}{S^2\epsilon^2}.$$

Furthermore,

$$
\begin{aligned}
& var(\sum_{l=1}^{S} A_{ij}(t_l)) = \sum_{l=1}^{S} var(A_{ij}(t_l)) + 2\sum_{s<r} cov(A_{ij}(t_s), A_{ij}(t_r)) \\
& = \sum_{l=1}^{S} \theta_{c_ic_j}(t_l)(1-\theta_{c_ic_j}(t_l)) + 2\sum_{l=2}^{S} cov(A_{ij}(t_1), A_{ij}(t_l)) + \dots \\
& +2\sum_{l=S-1}^{S} cov(A_{ij}(t_{S-2}), A_{ij}(t_l)) + 2cov(A_{ij}(t_{S-1}), A_{ij}(t_S)) \\
& \leq \frac{S}{4} + \frac{1}{2}\left[(\alpha+\alpha^2+\cdots+\alpha^{S-1}) + \ldots + (\alpha+\alpha^2)+\alpha\right] \\
& \leq \frac{S}{4} + \frac{S}{2(1-\alpha)}.
\end{aligned}
$$

Here we used the fact that $\theta_{c_ic_j}(t_l)(1-\theta_{c_ic_j}(t_l)) \leq 1/4$, $l = 1, \ldots, S$. Now we have

$$
\sum_{ij} \left| \frac{1}{S} \sum_{l=1}^{S} (A_{ij}(t_l) - \theta_{c_i,c_j}(t_l)) \right| \to 0 \text{ as } S \to \infty.
$$

With similar arguments, we can show

$$
\sum_{ij} \left| \frac{1}{S} \sum_{l=1}^{S} \left(\frac{d_i(t_l)d_j(t_l)}{2m(t_l)} - \frac{E(d_i(t_l))E(d_j(t_l))}{2E(m(t_l))} \right) \right| \to 0 \text{ as } S \to \infty.
$$

Therefore $\mathcal{L}(\mathcal{O}(\boldsymbol{e}))$ is uniformly close to $\mathcal{L}(\mathcal{H}(R))$, i.e., there exists $\epsilon_S \to 0$ such that

$$
P\left(\max_{\boldsymbol{e}} |\mathcal{L}(\mathcal{O}(\boldsymbol{e})) - \mathcal{L}(\mathcal{H}(R))| < \epsilon_S\right) \to 1 \text{ as } S \to \infty. \tag{S2}
$$

To show that there exists $\delta_S \to 0$, such that

$$
P\left(\max_{\boldsymbol{e}:\ ||V(\boldsymbol{e})-V(\boldsymbol{c})||_1 \geq \delta_S} \mathcal{L}(\mathcal{O}(\boldsymbol{e})) < \mathcal{L}(\mathcal{O}(\boldsymbol{c}))\right) \to 1 \text{ as } S \to \infty,
$$

we first show that $\mathcal{L}(\mathcal{H}(R)))$ is uniquely maximized over $\{R : R \geq 0, R^T\mathbf{1} = \pi\}$ by $\mathbb{S} = R(\boldsymbol{c})$. If $\mathcal{L}(\mathcal{O}(\boldsymbol{e}))$ is maximized by the true label $\boldsymbol{c}$, then $\mathcal{L}(\mathcal{H}(R))$ should be maximized by the true assignment $\mathbb{S}$.

The following equation is true,

$$
\sum_{k} \left(H_{kk}^{t_l} - \frac{(H_k^{t_l})^2}{H_0^{t_l}} \right) + \sum_{k\neq h} \left(H_{kh}^{t_l} - \frac{H_k^{t_l} H_h^{t_l}}{H_0^{t_l}} \right) = 0, \text{ for } l = 1, \ldots, S,
$$

where $H_k^{t_l} = \sum_h H_{kh}^{t_l}$ and $H_0^{t_l} = \sum_k H_k^{t_l}$. Define

$$\Delta_{kh} = \begin{cases} 1 & \text{if } k = h, \\ 0 & \text{if } k \neq h. \end{cases}$$

We have

$$\begin{aligned}
\mathcal{L}(\mathcal{H}(R)) &= \frac{1}{2N}\sum_{l=1}^{S}\sum_{kh}\Delta_{kh}\left(H_{kh}^{t_l} - \frac{H_k^{t_l}H_h^{t_l}}{H_0^{t_l}}\right) \\
&= \frac{n^2}{2S}\sum_{l=1}^{S}\sum_{kh}\Delta_{kh}\left(\sum_{ab}\theta_{ab}(t_l)R_{ka}R_{hb} - \frac{(\sum_{as}\theta_{as}(t_l)R_{ka}\pi_s)(\sum_{br}\theta_{br}(t_l)R_{hb}\pi_r)}{H_0^{t_l}}\right) \\
&= \frac{n^2}{2S}\sum_{l=1}^{S}\sum_{kh}\sum_{ab}\Delta_{kh}R_{ka}R_{hb}\left(\theta_{ab}(t_l) - \frac{(\sum_{as}\theta_{as}(t_l)\pi_s)(\sum_{br}\theta_{br}(t_l)\pi_r)}{H_0^{t_l}}\right) \\
&\leq \frac{n^2}{2S}\sum_{l=1}^{S}\sum_{kh}\sum_{ab}\Delta_{ab}R_{ka}R_{hb}\left(\theta_{ab}(t_l) - \frac{(\sum_{as}\theta_{as}(t_l)\pi_s)(\sum_{br}\theta_{br}(t_l)\pi_r)}{H_0^{t_l}}\right) \\
&= \frac{n^2}{2S}\sum_{l=1}^{S}\sum_{ab}\Delta_{ab}\pi_a\pi_b\left(\theta_{ab}(t_l) - \frac{(\sum_{as}\theta_{as}(t_l)\pi_s)(\sum_{br}\theta_{br}(t_l)\pi_r)}{H_0^{t_l}}\right) \\
&= \mathcal{L}(\mathcal{H}(\mathbb{S}))
\end{aligned}$$

The inequality is true because of the conditions in Theorem 1. Next we need to show that $\mathbb{S}$ is the unique maximizer of $\mathcal{L}(\mathcal{H}(\mathbb{S}))$. This is true by Lemma 3.2 in Bickel and Chen (2009) by observing that the inequality $\mathcal{L}(\mathcal{H}(R)) \leq \mathcal{L}(\mathcal{H}(\mathbb{S}))$ only holds when $\Delta_{kh} = \Delta_{ab}$ for $R_{ka}R_{hb} > 0$ and Δ does not have two identical columns.

Now we have shown $\mathcal{L}(\mathcal{H}(R))$ is uniquely maximized by $\mathbb{S}$. By the continuity of $L(.)$ in the neighborhood of $\mathbb{S}$, there exists $\delta_S \to 0$, such that

$$\mathcal{L}(\mathcal{H}(R(\mathbf{c}))) - \mathcal{L}(\mathcal{H}(R(\mathbf{e}))) > 2\epsilon_S \;\text{ for }\; ||V(\mathbf{e}) - V(\mathbf{c})||_1 \geq \delta_S.$$

Here we used the fact that

$$\begin{aligned}
||R(\boldsymbol{e}) - \mathbb{S}||_1 = \sum_{ab}|\pi_b V_{ab}(\boldsymbol{e}) - \pi_b V_{ab}(\boldsymbol{c}))| &\geq (\min_b \Pi_b) \times \sum_{ab}|V_{ab}(\boldsymbol{e}) - V_{ab}(\boldsymbol{c})| \\
&= (\min_b \pi_b) \times ||V(\mathbf{e}) - V(\mathbf{c})||_1.
\end{aligned}$$

Thus, with (S2), we have

$$
\begin{aligned}
&P\left(\max_{\boldsymbol{e}:\ ||V(\boldsymbol{e})-V(\mathbf{c})||_1\geq\delta_S}\mathcal{L}\left(\mathcal{O}(\boldsymbol{e})\right)<\mathcal{L}\left(\mathcal{O}(\boldsymbol{c})\right)\right)\\
&\geq P\Bigg(\left|\max_{\boldsymbol{e}:\ ||V(\boldsymbol{e})-V(\mathbf{c})||_1\geq\delta_S}\mathcal{L}\left(\mathcal{O}(\boldsymbol{e})\right)-\max_{\boldsymbol{e}:\ ||V(\boldsymbol{e})-V(\boldsymbol{c})||_1\geq\delta_S}\mathcal{L}(\mathcal{H}(\boldsymbol{e}))\right|<\epsilon_S,\\
&\qquad\left|\mathcal{L}\left(\mathcal{O}(\boldsymbol{c})\right)-\mathcal{L}(\mathcal{H}(\boldsymbol{c}))\right|<\epsilon_S\Bigg)\to 1.
\end{aligned}
$$

This implies that

$$P(||V(\hat{\boldsymbol{c}})-V(\boldsymbol{c})||_1\leq\delta_S)\to 1,$$

where $\hat{\boldsymbol{c}}=\arg\max_{\boldsymbol{e}}\mathcal{L}\left(\mathcal{O}(\boldsymbol{e})\right)$ is the estimator. Since

$$
\begin{aligned}
\frac{1}{n}|\boldsymbol{e}-\boldsymbol{c}|=\frac{1}{n}\sum_{i=1}^{n}I(c_i\neq e_i)=\sum_k\pi_k(1-V_{kk}(\boldsymbol{e})) &\leq \sum_k(1-V_{kk}(\boldsymbol{e}))\\
&= \frac{1}{2}\left(\sum_k(1-V_{kk}(\boldsymbol{e}))+\sum_{k\neq l}V_{kl}(\boldsymbol{e})\right)\\
&= \frac{1}{2}||V(\boldsymbol{e})-V(\boldsymbol{c})||_1,
\end{aligned}
$$

we have established the consistency property of the estimator.

Gene Community Membership

Group 1: Su(z)12, ben, cathD, CG33205, cher, kay, Ptpmeg, bnl, 140up, bib, betaTub56D, chico, PhKgamma, sktl, Ca-alpha1D, ash1, Oseg1, sle, chif, wls, Ddx1, Sulf1, CG3987, Uba1, G-salpha60A, Ice, Dok, Glycogenin, Pak, Lim1, up, Ark, Psn, Appl, tub, ics, Rho1, CG8247, CG9445, Hs2st, ALiX, Snap, vap, hyd, trc, ird5, Syx1A, Tab2, ImpE3, Tak1, 14-3-3zeta, Mlc1, shn, par-6, ast, Dscam, chinmo, CG13850, aft, grp, Ggamma1, bt, Mhc, Msp-300, pdm2, Idgf2, uzip, Prm, l(2)37Cc

Group 2: emp, CG2678, pbl, cactin, mus304, Pvr, msk, Tm1, CG18369, sec23, Pde8, cos, Dhc64C, Not1, nej, Es2, mbl, sav, lola, scrib, cnk, Yp1, CG17739, B4, inaC, Sobp, Spred, sbb, xmas-2, pot, chp, Su(H), hep, mmy, mri, Dad, CG34417, RhoL, twi, LanA, dei, Orct, Pka-C3, raw, gbb, pn, Set2, dah, Jafrac2, esg, tum, p53, cbt, CG4945, Dcp-1, tutl, ash2, Lis-1, twf, rdx, alph, mbo, sced, ninaE, wee, blow, Hph, fs(1)M3, osa, kkv, Hr46, eff, Pink1, pav, cort, CG9104, LCBP1, bcd, slam, pio, dos, Dredd

Group 3: crc, Rab5, Rab7, Arc42, RhoGAPp190, btl, Cyt-c-d, emc, tsh, Akap200, Abi, robo, stumps, abd-A, cbx, Apc2, Akt1, CG34379, tup, G-ialpha65A, Pkn, boss, wrapper, Rack1, G-oalpha47A, Khc, CG10641, noc, Scgdelta, sw, qkr58E-3, mfas, mam, capu, emb, sty, wg, LanB1, LanB2, oaf, Dys, mask, pnr, mbt, morgue, Src42A, peb, Eip63E, svr, zfh2, eve, th, enok, Hr78, Cks30A, eIF-4a, Sh, fog, ksr, CG12896, qm, sli

Group 4: pros, Mmp2, bowl, fd59A, tra2, lva, Sema-2a, Eip74EF, eg, MED11, CdsA, Hmgs, spir, qua, spen, CG3075, cas, Vhl, CG32486, Lar, MED4, siz, dsx, foxo, rad50, cp309, eya, scyl, pnut, phyl, tow, rux, Mkp3, sno, jagn, Trl, CadN, gro, klar, inv, shot, puc, Atg6, Pk61C, FKBP59, Eip71CD, Hem, S6kII, wupA, dsf, Optix, Sxl, sas, cic, rhea, lilli, toe, frc, Krn, brk, cact, dally, Ank2, ftz-f1, how, Chc, sgl, by, Nf1, l(2)gl, bs, gl, ex, shrb, run, pnt, dnc, Fas3, zfh1, par-1, rg, apt, yellow-f, ken, cg, fw, alpha-Spec, Nrx-IV, chn, RhoGAP68F, CG32082, elav, lmd, disco, Sema-1b, Aph-4, CG9769, Idgf4, Mbs, opa, bnb, Apc, stil, Cip4, dib, Idgf1, caup, fru, Rheb, tou, Pgant35A, Tl, tkv, CG4500, CG8216, eIF-4E, ecd, Antp, Moe, msi, baz, Atpalpha, Cct1, Mef2, Mitf, poe

Group 5: neur, MYPT-75D, tll, Su(var)2-10, CycB, ena, Dat, dome, sqd, tud, sina, l(2)efl, d4, Pcaf, Iswi, Abd-B, ssh, car, xl6, CtBP, kuk, fs(2)ltoPP43, sls, Myo31DF, Sur, Rpd3, dpld, bhr, dom, lwr, l(3)mbt, tld, Myo61F, tin

Group 6: W, toy, tamo, caps, CG9520, MED24, tral, sns, santa-maria, Syb, ofs, Cyp1, bap, cnc, acj6, Ef1gamma, Tina-1, Vps28, ey, ea, Dlic2, CG34450, Crk, scra, Chi, dco, RpL30, gkt, CG6372, CSN4, Cp1, Sin3A, exu, Wnt5, sprt, sar1, drk, LIMK1, glec, NetB, CG14995, Pen, CG17470, SelG, Atx2, chrw, 42339

Group 7: e, lin, tra, ttk, CG6416, Jra, Btk29A, Mmp1, sll, Lac, Arp66B, Lcp65Ag1, hdc, Axn, CG8149, yrt, Sry-delta, Alh, hkl, ry, dpn, E2f, sec13, Alk, ix, rib, hkb, beta4GalNAcTA, Mtl, CG9509, CkIIalpha, Patj, E(bx), dsd, Ephrin, nmo, vlc, fl(2)d, jing, kis, sgg, odd, bl, drl, Bap60, pelo, meso18E, t, kuz, Rbp9, Idgf5, bun, ytr, E2f2, SelD, PebIII, flw, brat, snf, cta, Traf6, Pop2, trio, her, prod, stg

Group 8: rdo, da, hpo, tok, spn-A, Borr, CG16719, ine, bon, cin, Mer, jumu, glu, CycE, esc, alien, Dl, Stat92E, CycA, Eip63F-1, lic, E(z), Caf1, dmt, thr, asf1

Group 9: d, Galpha73B, BicD, srp, fred, Src64B, msn, dup, mib1, dlg1, dock, l(1)G0289, Abl, Sb, sax, brm, knrl, CG2681, Atg5, CG3829, Cka, form3, chic, wal, bigmax, NAT1, sdt, fal, Pk92B, Dap160, l(2)gd1, spin, dpp, CG17221, Ald, 18w, Nrt, Fmr1, wus, jar, dj, CG1942, mth, trn, pyd, shg, loco, Gtp-bp, crol, fax, sn, ap, N, prd, Hs6st, Pvf1

Group 10: flfl, ct, Dp, ppan, CG12301, fs(1)K10, Pi3K92E, amx, drm, CG1244, Taf4, v, dl, numb, Rca1, hyx, pcm, Arc-p34, EcR, Dif

Gene Ontology Enrichment Analysis

Using the WebGestalt (web-based gene set analysis toolkit), we find the biological processes that are significantly enriched in Groups 1-10. For each group, we select the top 30 enriched biological processes, i.e., select the 30 biological processes with smallest p-values. There are many overlaps in the enriched biological processes. For example, all groups are enriched in *developmental process* and *system development*. Merging the selected enriched biological processes from Groups 1-10 leads to a list of 86 biological processes. We then create a 86×10 matrix with the (i, j)-cell equals the p-value that measures the enrichment of the i-th biological process in the j-th gene group.

When creating the heat map, we leave out the following biological processes: *developmental process*, *multicellular organismal process*, *single-organism process*, *single-multicellular organism process*, *cellular developmental process*, *anatomical structure development*, *multicellular organismal development*. These general processes are parents (or ancestors) to most of the processes in the heat map, and they are enriched in all ten groups .

Community Findings from Method 2

The following table compares the community finding results from the proposed method and Method 2.

	1	2	3	4	5	6	7	8	9
1	48	1	2	6	1	10	1	0	0
2	3	77	1	0	0	1	0	0	0
3	0	0	60	2	0	0	0	0	0
4	1	0	83	35	2	0	5	0	0
5	0	0	0	0	34	0	0	0	0
6	1	11	1	0	0	34	0	0	0
7	23	0	0	0	0	1	42	0	0
8	4	0	0	0	0	0	0	22	0
9	3	0	13	0	0	38	2	0	0
10	2	0	0	0	0	0	0	0	18

Table S1: The rows are community findings from the proposed method (10 groups) and the columns are community findings from Method 2 (9 groups).