# Supplementary Document for "Estimating Truncated Functional Linear Models with a Nested Group Bridge Approach"

## A: Penalized B-Splines Method

Let $\mathcal{S}_{dM}$ be the linear space spanned by the B-spline basis functions $\{B_k(t) : k = 1, \ldots, M + d\}$ with degree $d$ and $M + 1$ equally spaced knots defined on $[0, T]$. The penalized B-splines estimator of $\beta(t)$ proposed by Cardot et al. (2003) is the one in $\mathcal{S}_{dM}$ which is defined as

$$\hat{\beta}_{PS}(t) = \sum_{k=1}^{M+d} \hat{b}_k B_k(t) = \hat{\boldsymbol{b}}^{\mathrm{T}} \boldsymbol{B}(t) \tag{S.1}$$

where $\hat{\boldsymbol{b}}$ minimizes the penalized least squares

$$\frac{1}{n} \sum_{i=1}^{n} \left( Y_i - \sum_{k=1}^{M+d} b_k \int_0^T X_i(t) B_k(t) \, \mathrm{d}\, t \right)^2 + \kappa \left\| \boldsymbol{b}^{\mathrm{T}} \boldsymbol{B}^{(m)} \right\|_2^2, \tag{S.2}$$

with smoothing parameter $\kappa > 0$. The tuning parameter $\kappa$ can be chosen by cross validation, AIC or BIC.

## B. Effect of the Group Bridge Parameter $\gamma$

We conduct a simulation study to numerically investigate the effect of the group bridge parameter $\gamma$. The setting is the same as scenario III with the functional covariates generated by a linear combination of B-spline basis functions, the signal-to-noise ratio 2 and the sample size $n = 100$. We can observe that the results are similar when $0 < \gamma < 1$, and they are better than the results based on $\gamma = 1$.

Table S.1: Investigation of effect of the group bridge parameter $\gamma$ on $\hat{\delta}$ and mean integrated squared errors (ISE) of estimators for $\beta(t)$. The results are obtained based on 200 simulation replications with the corresponding Monte Carlo standard deviations included in parentheses.

| $\gamma$ | $\hat{\delta}$ | True $\delta$ | ISE ($\times 10^{-2}$) |
|---|---|---|---|
| 0.2 | 0.48 (0.11) | 0.50 | 1.50 (1.32) |
| 0.5 | 0.50 (0.09) | 0.50 | 1.36 (1.05) |
| 0.8 | 0.51 (0.06) | 0.50 | 1.38 (1.03) |
| 1 | 0.66 (0.15) | 0.50 | 1.54 (1.21) |

## C. Additional Simulation Results

In Table S.2 and S.3, we display the simulation results for the smooth functional covariates discussed in Section 4. We compare the estimated coefficient curves for various methods with rough functional covariates in Figure S.1 and smooth functional covariates in Figure S.2.

Table S.2: The mean of estimators for $\delta$ based on 200 simulation replications with the corresponding Monte Carlo standard deviation included in parentheses.

| | NGR | TR (Method A) | TR (Method B) | FLiRTI | SLoS | True Value |
|---|---|---|---|---|---|---|
| Scenario I | | | | | | |
| $n = 100$ | 0.64 (0.07) | 0.46 (0.06) | 0.50 (0.09) | 0.59 (0.13) | 0.59 (0.16) | 0.50 |
| $n = 500$ | 0.63 (0.04) | 0.49 (0.03) | 0.52 (0.05) | 0.69 (0.19) | 0.61 (0.08) | 0.50 |
| Scenario II | | | | | | |
| $n = 100$ | 0.56 (0.06) | 0.41 (0.05) | 0.42 (0.06) | 0.56 (0.16) | 0.53 (0.16) | 0.50 |
| $n = 500$ | 0.55 (0.03) | 0.43 (0.02) | 0.45 (0.04) | 0.56 (0.14) | 0.55 (0.06) | 0.50 |
| Scenario III | | | | | | |
| $n = 100$ | 0.49 (0.07) | 0.31 (0.03) | 0.35 (0.09) | 0.55 (0.20) | 0.48 (0.11) | 0.50 |
| $n = 500$ | 0.49 (0.03) | 0.30 (0.01) | 0.39 (0.07) | 0.58 (0.18) | 0.50 (0.08) | 0.50 |

NGR, our proposed nested group bridge method; TR (Method A), the truncation method that estimates $\delta$ and $\beta(t)$ simultaneously proposed by Hall and Hooker (2016); TR (Method B), the truncation method that estimates $\delta$ and $\beta(t)$ iteratively (Hall and Hooker, 2016); FLiRTI, the FLiRTI method proposed by James et al. (2009); SLoS, the SLoS method proposed by Lin et al. (2017).

Table S.3: Mean integrated squared errors of estimators for $\beta(t)$ based on 200 simulation replications with the corresponding Monte Carlo standard deviation included in parentheses.

| | NGR | PS | TR (Method A) | TR (Method B) | FLiRTI | SLoS |
|---|---|---|---|---|---|---|
| Scenario I | | | | | | |
| n = 100 | 0.06 (0.06) | 0.08 (0.03) | 0.08 (0.15) | 0.07 (0.05) | 0.50 (0.30) | 0.20 (0.27) |
| n = 500 | 0.03 (0.01) | 0.04 (0.01) | 0.02 (0.02) | 0.04 (0.01) | 0.20 (0.19) | 0.03 (0.02) |
| Scenario II | | | | | | |
| n = 100 | 0.02 (0.04) | 0.05 (0.02) | 0.04 (0.03) | 0.03 (0.02) | 0.13 (0.11) | 0.07 (0.10) |
| n = 500 | 0.01 (0.00) | 0.02 (0.02) | 0.03 (0.00) | 0.01 (0.01) | 0.02 (0.04) | 0.00 (0.00) |
| Scenario III | | | | | | |
| n = 100 | 0.03 (0.05) | 0.04 (0.02) | 0.10 (0.02) | 0.08 (0.05) | 0.50 (0.50) | 0.03 (0.05) |
| n = 500 | 0.01 (0.01) | 0.01 (0.01) | 0.09 (0.01) | 0.04 (0.01) | 0.15 (0.18) | 0.01 (0.01) |

NGR, our proposed nested group bridge method; PS, the penalized B-splines method; TR (Method A), the truncation method that estimates $\delta$ and $\beta(t)$ simultaneously proposed by Hall and Hooker (2016); TR (Method B), the truncation method that estimates $\delta$ and $\beta(t)$ iteratively (Hall and Hooker, 2016); FLiRTI, the FLiRTI method proposed by James et al. (2009); SLoS, the SLoS method proposed by Lin et al. (2017).

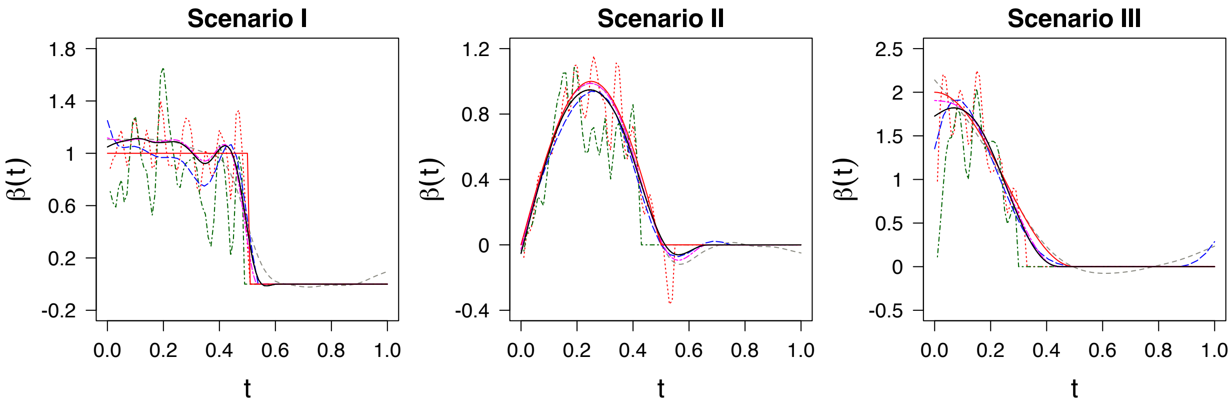


Figure S.1: Estimated coefficient functions with rough functional covariates and $n = 500$ in one randomly selected simulation replicate for various methods (——, the proposed nested group bridge method; - - - - -, the penalized B-splines method; ⋯⋯, the truncation method A; -·-·-·-, the truncation method B; — — —, the FLiRTI method; -·—·-, the SLoS method; ——, the true $\beta(t)$).

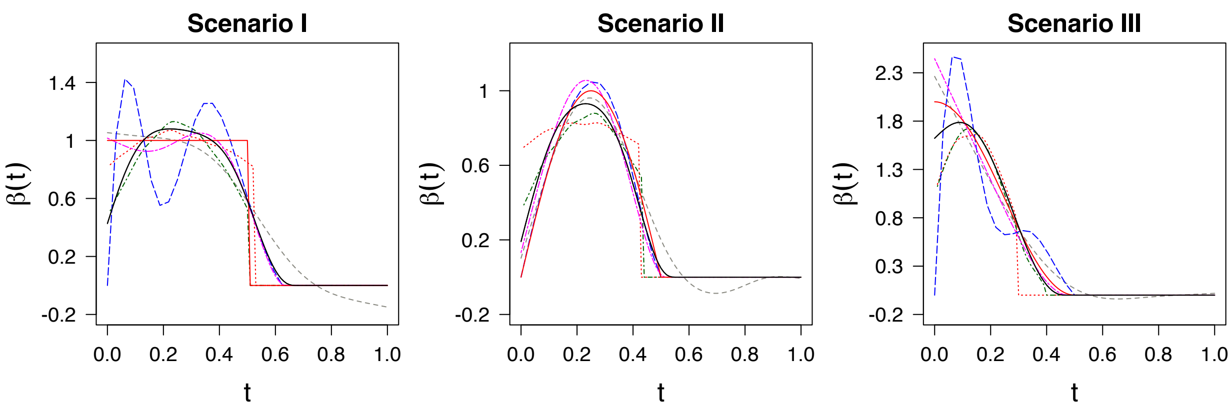


Figure S.2: Estimated coefficient functions with smooth functional covariates and $n = 500$ in one randomly selected simulation replicate for various methods (——, the proposed nested group bridge method; - - - - -, the penalized B-splines method; ⋯⋯, the truncation method A; -·-·-·-, the truncation method B; — — —, the FLiRTI method; -·—·-, the SLoS method; ——, the true $\beta(t)$).

# D. Proofs

Without of loss of generality, we assume that $T = 1$. We will first collect some remarks on notations that are used in the sequel. Boldface symbol is used to denote matrix or vector. If $0 < q < \infty$, $L^q$ is defined as the space of functions $f(t)$ over the interval $[0, 1]$ such that

$\int_0^1 |f(t)|^q \, \mathrm{d}\, t < \infty$. Two functions $g(t)$ and $f(t)$ are identified as the same if $g(t) = f(t)$ almost everywhere over $[0, 1]$ with respect to the usual Lebesgue measure. With this convention, $L^q$ is treated as a Banach space with the norm $\|f\|_q = (\int_0^1 |f(t)|^q \, \mathrm{d}\, t)^{1/q}$. When $q = 2$, we get the Hilbert space $L^2$ with the inner product $\langle g, f\rangle = \int_0^1 g(t)f(t) \, \mathrm{d}\, t$ and the $L^2$ norm $\| \cdot \|_2$. Since $\mathbb{R}^m$ for a positive integer $m$ is also a Hilbert space, we use the same notation $\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \boldsymbol{u}'\boldsymbol{v}$ and $\|\boldsymbol{u}\|_2 = (\boldsymbol{u}'\boldsymbol{u})^{1/2}$ to denote the inner product and the norm of vector $\boldsymbol{u}$ and $\boldsymbol{v}$. Here, $\boldsymbol{u}'$ is used to denote the transpose of $\boldsymbol{u}$. To reduce notational burden and make our presentation concise, we use $\langle f, \boldsymbol{B}\rangle$ to denote the vector $(\langle f, B_1\rangle, \langle f, B_2\rangle, \ldots, \langle f, B_M\rangle)$. The supremum norm of a function $f(t)$ is conventionally denoted by $\|f\|_\infty$ and defined as $\|f\|_\infty = \sup\{|f(t)| : t \in [0, 1]\}$. Similarly, the supremum norm of a vector $\boldsymbol{u} = (u_1, u_2, \ldots, u_m) \in \mathbb{R}^m$ is also denoted by $\|\boldsymbol{u}\|_\infty$ and defined as $\|\boldsymbol{u}\|_\infty = \max\{|u_i| : i = 1, 2, \ldots, m\}$. The operator norm of a linear operator $\Lambda$ on a Hilbert space $\mathcal{H}$, is traditionally denoted by $\|\Lambda\|$ and defined as $\|\Lambda\| = \sup\{\|\Lambda f\|_2 : f \in \mathcal{H}, \|f\|_2 = 1\}$. Here, $\mathcal{H}$ could be $L^2$ or $\mathbb{R}^m$.

As our estimator is based on B-spline basis, before we dive into proofs of the theorems, we briefly discuss some basic properties of B-spline basis that are used in our proofs. A detailed treatment of B-spline can be found in de Boor (2001). The B-spline basis has a local support property, which means each B-spline basis function is nonzero over no more than $d+1$ adjacent subintervals. Also, each B-spline basis function is non-negative and they form a partition of unity, that is, $\sum_{j=1}^{M+d} B_j(t) = 1$ for all $t \in [0, 1]$. Assume $\beta_0(t)$ satisfies condition *C.2*, according to Theorem XII(6) of de Boor (2001), there exists some $\beta_s(t) = \sum\limits_{j=1}^{M+d} b_{sj}B_j(t) = \boldsymbol{B}'\boldsymbol{b}_s$ with $\boldsymbol{b}_s = (b_{s1}, \ldots, b_{s,M+d})'$, such that $\|\beta_s - \beta_0\|_\infty \leq C_0 M^{-p}$ for some positive constant $C_0$ and $p$. Define $b_{0j} = b_{sj} I_{(j \leq J_1)}$ and put $\beta_{0s}(t) = \sum\limits_{j=1}^{M+d} b_{0j}B_j(t) = \boldsymbol{B}'\boldsymbol{b}_0$, where $\boldsymbol{b}_0 = (b_{01}, \dots, b_{0,M+d})'$. It is easy to see that $b_{0j} = 0$ if the support of $B_j(t)$ is contained in $[\delta_0, 1]$. It is obvious that $\|\beta_{0s} - \beta_0\|_\infty \le C_1 M^{-p}$ for some positive constant $C_1$.

The following lemmas are established to prove the theorems in Section 3.

**Lemma 1.** *If* C.1 *and* C.3 *hold, then for some positive constants $C_{\rho_1}$ and $C_{\rho_2}$,*

$$P(C_{\rho_1}\kappa n/M < \rho_{\min}(\boldsymbol{U}'\boldsymbol{U} + n\kappa\boldsymbol{V}) \le \rho_{\max}(\boldsymbol{U}'\boldsymbol{U} + n\kappa\boldsymbol{V}) < C_{\rho_2}n/M) \to 1, \tag{S.3}$$

*where $\rho_{\min}$ and $\rho_{\max}$ denote the smallest and largest eigenvalues of a matrix, respectively.*

*Proof.* This is a consequence of Lemma 6.1 and 6.2 of Cardot et al. (2003). ■

**Lemma 2.** $\sup_{i,j}\ |v_{ij}| = O(M^{2m-1})$.

*Proof.* Let $B_{jd}$ denote the $j$th normalized B-spline defined on $[0, 1]$ with degree $d$ and $M+1$ equispaced knots $0 = t_0 < t_1 < ... < t_M = 1$, $j = 1, ..., M + d$. The knots divide $[0, 1]$ into $M$ subintervals with equal length $\Delta = 1/M$. Now consider $B_{d+1,d}$, $B_{d+2,d}$, ... , and $B_{M,d}$ that are positive on $d + 1$ such subintervals. Let $B'_{jd}$ and $B''_{jd}$ denote the first and second derivatives of $B_{jd}$ respectively. Then it follows from X(8) of de Boor (2001) that

$$\begin{aligned} B'_{jd}(t) &= \frac{1}{\Delta}(B_{j-1,d-1}(t) - B_{j,d-1}(t)) \\ &= M(B_{j-1,d-1}(t) - B_{j,d-1}(t)), \qquad j = d+1, ..., M, \end{aligned} \tag{S.4}$$

Since $0 \leq B_j(t) \leq 1$, $|B_{jd}^{'}| \leq M$. By taking derivative of (S.4), we have

$$\begin{aligned} B_{jd}^{''}(t) &= \frac{1}{\Delta}(B_{j-1,d-1}^{'}(t) - B_{j,d-1}^{'}(t)) \\ &= \frac{1}{\Delta^2}(B_{j-2,d-2}(t) - 2B_{j-1,d-2}(t) + B_{j,d-2}(t)) \\ &= M^2(B_{j-2,d-2}(t) - 2B_{j-1,d-2}(t) + B_{j,d-2}(t)). \end{aligned}$$

and hence $|B_{jd}^{''}| \leq 2M^2$. Then we can deduce that $|B_{jd}^{(m)}| \leq C_m M^m$, where $C_m$ is some constant depending on $m$. Since $|B_{jd}^{(m)}| \geq 0$ on at most $d+1$ subintervals, $\left\|B_{jd}^{(m)}\right\|_2 \leq 2C_m(d+1)^{1/2}\Delta^{1/2}M^m$. This further implies that

$$\sup_{i,j} |v_{ij}| = \sup_{i,j} |\langle B_{id}^{(m)}, B_{jd}^{(m)}\rangle| \leq \sup_{i,j} \left\|B_{id}^{(m)}\right\|_2 \left\|B_{jd}^{(m)}\right\|_2 \leq 4C_m^2(d+1)M^{2m-1}, \tag{S.5}$$

which yields the conclusion of the lemma. ■

Let $\ell(\boldsymbol{b}) = n^{-1}(\boldsymbol{Y} - \boldsymbol{U}\boldsymbol{b})^{'}(\boldsymbol{Y} - \boldsymbol{U}\boldsymbol{b}) + \kappa \boldsymbol{b}^{'}\boldsymbol{V}\boldsymbol{b}$. We can write $\ell(\boldsymbol{b})$ as

$$\begin{aligned} \ell(\boldsymbol{b}) &= \frac{1}{n}\sum_{i=1}^{n}\Big(\langle \beta, X_i\rangle - \langle \boldsymbol{B}^{'}\boldsymbol{b}, X_i\rangle + \varepsilon_i\Big)^2 + \kappa \boldsymbol{b}^{'}\boldsymbol{V}\boldsymbol{b} \\ &= \frac{1}{n}\sum_{i=1}^{n}\Big(\langle \beta - \boldsymbol{B}^{'}\boldsymbol{b}, X_i\rangle^2 + 2\varepsilon_i\langle \beta - \boldsymbol{B}^{'}\boldsymbol{b}, X_i\rangle + \varepsilon_i^2\Big) + \kappa \boldsymbol{b}^{'}\boldsymbol{V}\boldsymbol{b} \\ &= \langle \Gamma_n(\beta - \boldsymbol{B}^{'}\boldsymbol{b}), \beta - \boldsymbol{B}^{'}\boldsymbol{b}\rangle + \frac{2}{n}\sum_{i=1}^{n}\varepsilon_i\langle \beta - \boldsymbol{B}^{'}\boldsymbol{b}, X_i\rangle + \frac{1}{n}\sum_{i=1}^{n}\varepsilon_i^2 + \kappa \boldsymbol{b}^{'}\boldsymbol{V}\boldsymbol{b}, \end{aligned}$$

where $\Gamma_n$ is the empirical version of the covariance operator $\Gamma$ of $X$, and is defined by

$$(\Gamma_n x)(v) = \frac{1}{n}\sum_{i=1}^{n}\int_0^1 X_i(v)X_i(u)x(u)\,\mathrm{d}\,u.$$

Let $\boldsymbol{H}$ be the $(M+d)\times(M+d)$ matrix with element $h_{i,j} = \langle \Gamma_n B_i, B_j\rangle$. Then the gradient of $\ell$ with respect to $\boldsymbol{b}$ is

$$\nabla\ell(\boldsymbol{b}) = 2\boldsymbol{H}\boldsymbol{b} - 2\langle \Gamma_n\beta, \boldsymbol{B}\rangle - \frac{2}{n}\sum_{i=1}^{n}\varepsilon_i\langle X_i, \boldsymbol{B}\rangle + 2\kappa\boldsymbol{V}\boldsymbol{b}$$

and the Hessian is

$$\nabla^2\ell(\boldsymbol{b}) = 2\boldsymbol{H} + 2\kappa\boldsymbol{V}.$$

At the point $\boldsymbol{b} = \boldsymbol{b}_0$, the gradient of $\ell$ can be written as

$$\begin{aligned}
\nabla\ell(\boldsymbol{b}_0) &= 2\boldsymbol{H}\boldsymbol{b}_0 - 2\langle \Gamma_n\beta, \boldsymbol{B}\rangle - \frac{2}{n}\sum_{i=1}^{n}\varepsilon_i\langle X_i, \boldsymbol{B}\rangle + 2\kappa\boldsymbol{V}\boldsymbol{b}_0 \\
&= 2\langle \Gamma_n\boldsymbol{B}, \beta_{0s}\rangle - 2\langle \Gamma_n\beta, \boldsymbol{B}\rangle - \frac{2}{n}\sum_{i=1}^{n}\varepsilon_i\langle X_i, \boldsymbol{B}\rangle + 2\kappa\boldsymbol{V}\boldsymbol{b}_0 \\
&= 2\langle \Gamma_n\beta_{0s}, \boldsymbol{B}\rangle - 2\langle \Gamma_n\beta, \boldsymbol{B}\rangle - \frac{2}{n}\sum_{i=1}^{n}\varepsilon_i\langle X_i, \boldsymbol{B}\rangle + 2\kappa\boldsymbol{V}\boldsymbol{b}_0 \\
&= 2\langle \Gamma_n(\beta_{0s}-\beta), \boldsymbol{B}\rangle - 2\langle\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i X_i, \boldsymbol{B}\rangle + 2\kappa\boldsymbol{V}\boldsymbol{b}_0.
\end{aligned} \tag{S.6}$$

In other words, for each $j = 1, 2, \ldots, M+d$,

$$\frac{\partial\ell(\boldsymbol{b}_0)}{\partial b_{0j}} = 2\langle \Gamma_n(\beta_{0s}-\beta), B_j\rangle - 2\langle\frac{1}{n}\sum_{i=1}^{n}\varepsilon_i X_i, B_j\rangle + 2\kappa V_j\boldsymbol{b}_0, \tag{S.7}$$

where $V_j$ represents the $j$th row of $\boldsymbol{V}$.

Below we first provide bounds of $\frac{\partial \ell(\boldsymbol{b}_0)}{\partial b_{0j}}$ and $\nabla^2 \ell(\boldsymbol{b})$ in Lemma 3 and 4, respectively.

**Lemma 3.** *For any $\epsilon > 0$, there exists a constant $C_2$ such that*

$$P\left(\sup_j \left|\frac{\partial \ell(\boldsymbol{b}_0)}{\partial b_{0j}}\right| \leq C_2 M^{-1/2} n^{-1/2}\right) > 1 - \epsilon \tag{S.8}$$

*holds for all sufficiently large $M$ and $n$.*

*Proof.* Below we will develop bounds for each term in (S.7). In the first term, $\Gamma_n$ converges to $\Gamma$ almost surely according to Proposition 1 in Dauxois et al. (1982). Thus, $\|\Gamma_n\|$ converges to $\|\Gamma\|$ almost surely, since the operator norm is continuous. Recall that, the function $\beta_{0s}$ is chosen to satisfy $\|\beta_{0s} - \beta\|_\infty \leq C_1 M^{-p}$, where $C_1$ is a positive constant. This implies that $\|\beta_{0s} - \beta\|_2 = \sqrt{\int_0^1 (\beta_{0s}(t) - \beta(t))^2 \, \mathrm{d}\, t} \leq \sqrt{\int_0^1 (C_1 M^{-p})^2 \, \mathrm{d}\, t} = C_1 M^{-p}$. We know that each B-spline basis function is nonzero over no more than $d+1$ adjacent subintervals. Also, each B-spline basis function is non-negative and the basis functions form a partition of unity. The two properties together imply that

$$\|B_j\|_2^2 = \int_0^1 B_j^2(t) \, \mathrm{d}\, t \leq (d+1) M^{-1}. \tag{S.9}$$

Applying Cauchy-Schwarz inequality and (S.9) yields

$$\sup_j |\langle \Gamma_n(\beta_{0s} - \beta), B_j \rangle| \leq \sup_j \|\Gamma_n\| \, \|\beta_{0s} - \beta\|_2 \, \|B_j\|_2 \leq C_1 (d+1)^{1/2} M^{-p-1/2} \|\Gamma_n\|. \tag{S.10}$$

Since $\|\Gamma_n\|$ converges to $\|\Gamma\|$ almost surely and hence in probability, we conclude that, for any given $\epsilon > 0$, there is a constant $\rho_1(\epsilon)$ depending on $\epsilon$, such that for all sufficiently large $n$ and $M$,

$$P\left(\sup_j |\langle \Gamma_n(\beta_{0s} - \beta), B_j\rangle| \leq \rho_1(\epsilon) M^{-p-1/2}\right) > 1 - \epsilon. \tag{S.11}$$

For the second term, by Condition *C.1* and CLT (Aldous, 1976), $\sqrt{n}\left(\frac{1}{n}\sum_{i=1}^n \varepsilon_i X_i\right)$ converges to a Gaussian random element in $L^2([0,1])$ in distribution, whose mean is 0. This implies that, for any given $\epsilon > 0$, there is a constant $\rho_2(\epsilon)$ which only depends on $\epsilon$, such that for sufficiently large $n$,

$$P\left(\sqrt{n}\left\|\frac{1}{n}\sum_{i=1}^n \varepsilon_i X_i\right\|_2 < \rho_2(\epsilon)\right) > 1 - \epsilon. \tag{S.12}$$

Each $X_i$ is a mapping from sample space $\Omega$ to the space $L^2([0,1])$. Specifically, we let $X_i^\omega \in L^2([0,1])$ be the image of the sample $\omega \in \Omega$ under the mapping $X_i$. We then denote $\Omega_\epsilon \subset \Omega$ the set of $\omega$ that makes $\left\|\frac{1}{n}\sum_{i=1}^n \varepsilon_i X_i^\omega\right\|_2 < \rho_2(\epsilon) n^{-1/2}$ hold. Thus, $P(\Omega_\epsilon) > 1 - \epsilon$. Then by Cauchy-Schwarz inequality, on $\Omega_\epsilon$, it holds

$$\sup_j \left|\langle \frac{1}{n}\sum_{i=1}^n \varepsilon_i X_i^\omega, B_j\rangle\right| \leq \sup_j \left\|\frac{1}{n}\sum_{i=1}^n \varepsilon_i X_i^\omega\right\|_2 \|B_j\|_2 \leq \rho_2(\epsilon)(d+1)^{1/2} n^{-1/2} M^{-1/2}. \tag{S.13}$$

For the third term, according to lemma 2, we first have $\sup_{ij} |v_{ij}| \leq 4C_m^2(d+1)M^{2m-1}$ and the number of non-zero elements in each row of $\boldsymbol{V}$ is at most $2d+1$. For $\boldsymbol{b}_0$, we have $\|\boldsymbol{b}_0\|_\infty \leq C_3$ for some positive constant $C_3$ that does not depend on $M$. This conclusion can be derived from the fact that $\beta_{0s}(t) = \boldsymbol{B}'(t)\boldsymbol{b}_0$, continuity of $\beta(t)$ and the discussion on page 145-149 of de Boor (2001). Given these bounds, we have $\sup_j |V_j \boldsymbol{b}_0| \leq 4C_3 C_m^2(d+1)(2d+1)M^{2m-1}$.

Combining this result with (S.11) and (S.13), and using Condition *C.3*, we deduce (S.8). ■

We write $u = \Theta_P(v)$ if $u/v$ is bounded away from 0 and $\infty$ with probability tending to one. The next result concerns the order of $\nabla^2\ell(\boldsymbol{b})$.

**Lemma 4.** *$\nabla^2\ell(\boldsymbol{b})$ is positive-definite and $\|\nabla^2\ell(\boldsymbol{b})\| = \Theta_P(M^{-1})$. Furthermore, $\sup_{ij}|\frac{\partial^2\ell(\boldsymbol{b})}{\partial b_i \partial b_j}| = O_P(M^{-1})$.*

*Proof.* By Condition *C.4*, $\rho_{\min}(\boldsymbol{H}) = \Theta_P(M^{-1})$ and $\rho_{\max}(\boldsymbol{H}) = \Theta_P(M^{-1})$. Since $\rho_{\max}(\kappa\boldsymbol{V}) = O(\kappa M^{-1})$ and $\kappa = o(1)$ according to the condition *C.3*, we then have $\|\nabla^2\ell(\boldsymbol{b})\| = \Theta_P(M^{-1})$.

By Cauchy-Schwarz inequality and (S.9), we have $\sup_{ij}|h_{ij}| = \sup_{ij}|\langle\Gamma_n B_i, B_j\rangle| \leq \|\Gamma_n\|\|B_i\|_2\|B_j\|_2 = O_P(M^{-1})$. Combining this result with (S.5), and using the Condition *C.3*, we conclude that $\sup_{ij}|\frac{\partial^2\ell(\boldsymbol{b})}{\partial b_i \partial b_j}| = O_P(M^{-1})$. ■

**Lemma 5.** *Suppose* C.1 - C.5 *hold. Then $\|\hat{\boldsymbol{b}}_n - \boldsymbol{b}_0\|_2 = O_p(Mn^{-1/2})$.*

*Proof.* Let $\hat{\boldsymbol{b}}_n = \boldsymbol{b}_0 + \delta_n\boldsymbol{u}$ with $\|\boldsymbol{u}\| = 1$. Therefore, it is sufficient to show that $\delta_n = O_p(Mn^{-1/2})$.

Denote $D(\boldsymbol{u}) = Q(\boldsymbol{b}_0 + \delta_n\boldsymbol{u}) - Q(\boldsymbol{b}_0)$, where $Q$ is the objective function (3). Then $D(\boldsymbol{u})$ is the sum of $D_1 \equiv \ell(\boldsymbol{b}_0 + \delta_n\boldsymbol{u}) - \ell(\boldsymbol{u})$ and $D_2 \equiv \lambda\sum_{j=1}^M c_j\|\hat{b}_{nA_j}\|_1^\gamma - \lambda\sum_{j=1}^M c_j\|b_{0A_j}\|_1^\gamma$. For $D_1$, according to Lemma 3 and Lemma 4, also noting that it is a quadratic function of $\boldsymbol{b}$, by Taylor expansion, we can show that

$$\begin{aligned} D_1 &= \ell(\boldsymbol{b}_0 + \delta_n\boldsymbol{u}) - \ell(\boldsymbol{b_0}) \\ &= \delta_n\nabla'\ell(\boldsymbol{b}_0)\boldsymbol{u} + \frac{1}{2}\delta_n^2\boldsymbol{u}'\nabla^2\ell(\boldsymbol{b}_0)\boldsymbol{u} \\ &\geq \delta_n O_P(M^{-1/2}n^{-1/2}) + C_4\delta_n^2 M^{-1} \qquad \text{(S.14)} \end{aligned}$$

for some constant $C_4 > 0$ with probability tending to one. Since $b^\gamma - a^\gamma \leq 2(b-a)b^{\gamma-1}$ for $0 \leq a \leq b$, we have

$$\begin{aligned}
-D_2 &\leq 2\lambda \sum_{j=1}^{M} c_j \|b_{0A_j}\|_1^{\gamma-1} \left( \|b_{0A_j}\|_1 - \|\hat{b}_{nA_j}\|_1 \right) \\
&\leq 2\lambda \sum_{j=1}^{M} c_j \|b_{0A_j}\|_1^{\gamma-1} \left( |A_j| \|b_{0A_j} - \hat{b}_{nA_j}\|_2^2 \right) \\
&\leq 2\lambda\eta \left( \sum_{j=1}^{M} \|b_{0A_j} - \hat{b}_{nA_j}\|_2^2 \right)^{1/2} \\
&\leq 2\lambda\eta\delta_n M^{1/2}. \qquad \text{(S.15)}
\end{aligned}$$

Since $\hat{\boldsymbol{b}}_n$ minimizes $Q(\boldsymbol{b})$, we have $D_1 + D_2 \leq 0$. According to condition *C.5*, $\lambda\eta = O(n^{-1/2}M^{-1/2})$. Then

$$\delta_n O_P(M^{-1/2}n^{-1/2}) + C_4\delta_n^2 M^{-1} - O(\delta_n n^{-1/2}) \leq 0 \qquad \text{(S.16)}$$

with probability tending to one, which implies that $\delta_n = O_p(Mn^{-1/2})$. Thus the conclusion of the lemma follows. ■

**Lemma 6.** *Suppose conditions* C.1 - C.6 *hold. Then* $P\left(\hat{b}_{nA_j} = 0 \text{ for } j > J_1\right) \to 1$.

*Proof.* Define $\tilde{\boldsymbol{b}}_n = (\tilde{b}_{n1}, ..., \tilde{b}_{n,M+d})'$ by $\tilde{b}_{nk} = \hat{b}_{nk} I_{(k \leq J_1)}$, $k = 1, ..., M+d$. We have $\hat{\theta}_j^{1-1/\gamma} = \lambda\gamma c_j^{1-1/\gamma} \|\hat{b}_{nA_j}\|_1^{\gamma-1}$. The Karush-Kuhn-Tucker condition for (6) implies

$$2(\boldsymbol{Y} - \boldsymbol{U}\hat{\boldsymbol{b}}_n)' U_k - 2n\kappa\hat{\boldsymbol{b}}_n' V_k = \sum_{j=1}^{\min\{k,M\}} n\hat{\theta}_j^{1-1/\gamma} c_j^{1/\gamma} \text{sgn}(\hat{b}_{nk}), \qquad \hat{b}_{nk} \neq 0,$$

where $U_k$ is the $k$th column of $\boldsymbol{U}$ and $V_k$ is the $k$th column of $\boldsymbol{V}$. Observe that $\|\hat{b}_{nA_j}\|_1 -$

$\|\tilde{b}_{nA_j}\|_1 = \sum_{k=\max\{j,J_1+1\}}^{M+d} |\hat{b}_{nk}|$ and $(\hat{b}_{nk} - \tilde{b}_{nk})\mathrm{sgn}(\hat{b}_{nk}) = |\hat{b}_{nk}| I_{(k \geq J_1+1)}$. Thus

$$
\begin{aligned}
2(\boldsymbol{Y} - \boldsymbol{U}\hat{\boldsymbol{b}}_n)'\boldsymbol{U}(\hat{\boldsymbol{b}}_n - \tilde{\boldsymbol{b}}_n) &= 2n\kappa \sum_{k=1}^{M+d} (\hat{b}_{nk} - \tilde{b}_{nk})\hat{\boldsymbol{b}}_n' V_k + n\lambda\gamma \sum_{j=1}^{M} \sum_{k=\max\{j,J_1+1\}}^{M+d} c_j \|\hat{b}_{nA_j}\|_1^{\gamma-1} |\hat{b}_{nk}| \\
&= 2n\kappa \sum_{k=1}^{M+d} (\hat{b}_{nk} - \tilde{b}_{nk})\hat{\boldsymbol{b}}_n' V_k + n\lambda\gamma \sum_{j=1}^{M} c_j \|\hat{b}_{nA_j}\|_1^{\gamma-1} (\|\hat{b}_{nA_j}\|_1 - \|\tilde{b}_{nA_j}\|_1).
\end{aligned}
$$

Since $\gamma b^{\gamma-1}(b-a) \leq b^\gamma - a^\gamma$ for $0 \leq a \leq b$, for $j \leq J_1$ , we have

$$
\gamma \|\hat{b}_{nA_j}\|_1^{\gamma-1} (\|\hat{b}_{nA_j}\|_1 - \|\tilde{b}_{nA_j}\|_1) \leq \|\hat{b}_{nA_j}\|_1^\gamma - \|\tilde{b}_{nA_j}\|_1^\gamma,
$$

Observe that $\|\tilde{b}_{nA_j}\|_1 = 0$ for $j > J_1$. Thus

$$
\begin{aligned}
&2(\boldsymbol{Y} - \boldsymbol{U}\hat{\boldsymbol{b}}_n)'\boldsymbol{U}(\hat{\boldsymbol{b}}_n - \tilde{\boldsymbol{b}}_n) \qquad \text{(S.17)} \\
&\leq 2n\kappa \sum_{k=1}^{M+d} (\hat{b}_{nk} - \tilde{b}_{nk})\hat{\boldsymbol{b}}_n' V_k + n\lambda \sum_{j=1}^{J_1} c_j (\|\hat{b}_{nA_j}\|_1^\gamma - \|\tilde{b}_{nA_j}\|_1^\gamma) + n\lambda\gamma \sum_{j=J_1+1}^{M} c_j \|\hat{b}_{nA_j}\|_1^\gamma.
\end{aligned}
$$

By the optimality of $\hat{\boldsymbol{b}}_n$, we have

$$
\begin{aligned}
&\|\boldsymbol{Y} - \boldsymbol{U}\hat{\boldsymbol{b}}_n\|_2^2 + n\kappa \hat{\boldsymbol{b}}_n' \boldsymbol{V} \hat{\boldsymbol{b}}_n + n\lambda \sum_{j=1}^{M} c_j \|\hat{b}_{nA_j}\|_1^\gamma \\
\leq &\|\boldsymbol{Y} - \boldsymbol{U}\tilde{\boldsymbol{b}}_n\|_2^2 + n\kappa \tilde{\boldsymbol{b}}_n' \boldsymbol{V} \tilde{\boldsymbol{b}}_n + n\lambda \sum_{j=1}^{M} c_j \|\tilde{b}_{nA_j}\|_1^\gamma. \qquad \text{(S.18)}
\end{aligned}
$$

It follows from (S.17) and (S.18) that

$$
\begin{aligned}
&2(\boldsymbol{Y}-\boldsymbol{U}\hat{\boldsymbol{b}}_n)'\boldsymbol{U}(\hat{\boldsymbol{b}}_n-\tilde{\boldsymbol{b}}_n)+(1-\gamma)n\lambda\sum_{j=J_1+1}^{M}c_j\|\hat{b}_{nA_j}\|_1^\gamma \\
\leq& n\lambda\sum_{j=1}^{M}c_j\|\hat{b}_{nA_j}\|_1^\gamma - n\lambda\sum_{j=1}^{M}c_j\|\tilde{b}_{nA_j}\|_1^\gamma + 2n\kappa\sum_{k=J_1+1}^{M+d}\hat{\boldsymbol{b}}_n' V_k\hat{b}_{nk} \\
\leq& \|\boldsymbol{Y}-\boldsymbol{U}\tilde{\boldsymbol{b}}_n\|_2^2-\|\boldsymbol{Y}-\boldsymbol{U}\hat{\boldsymbol{b}}_n\|_2^2+n\kappa\tilde{\boldsymbol{b}}_n'\boldsymbol{V}\tilde{\boldsymbol{b}}_n-n\kappa\hat{\boldsymbol{b}}_n'\boldsymbol{V}\hat{\boldsymbol{b}}_n+2n\kappa\sum_{k=1}^{M+d}(\hat{b}_{nk}-\tilde{b}_{nk})\hat{\boldsymbol{b}}_n' V_k \\
=& \|\boldsymbol{U}(\hat{\boldsymbol{b}}_n-\tilde{\boldsymbol{b}}_n)\|_2^2+2(\boldsymbol{Y}-\boldsymbol{U}\hat{\boldsymbol{b}}_n)'\boldsymbol{U}(\hat{\boldsymbol{b}}_n-\tilde{\boldsymbol{b}}_n)+n\kappa(\hat{\boldsymbol{b}}_n-\tilde{\boldsymbol{b}}_n)'\boldsymbol{V}(\hat{\boldsymbol{b}}_n-\tilde{\boldsymbol{b}}_n).
\end{aligned}
$$

Consequently,

$$
(1-\gamma)n\lambda\sum_{j=J_1+1}^{M}c_j\|\hat{b}_{nA_j}\|_1^\gamma \leq (\hat{\boldsymbol{b}}_n-\tilde{\boldsymbol{b}}_n)'(\boldsymbol{U}'\boldsymbol{U}+n\kappa\boldsymbol{V})(\hat{\boldsymbol{b}}_n-\tilde{\boldsymbol{b}}_n). \tag{S.19}
$$

By (S.3) and condition *C.3*,

$$
(1-\gamma)n\lambda\sum_{j=J_1+1}^{M}c_j\|\hat{b}_{nA_j}\|_1^\gamma \leq O_p(nM^{-1})\|\hat{\boldsymbol{b}}_n-\tilde{\boldsymbol{b}}_n\|_2^2. \tag{S.20}
$$

Given $|A_j| = M+d-j+1$ and $\boldsymbol{b}^{(0)}$, which can be obtained by the penalized B-splines method (Cardot et al., 2003), the constants $c_j = |A_j|^{1-\gamma}/\|b_{A_j}^{(0)}\|_2^\gamma$ can be scaled so that $\min_{j\leq J} c_j \geq 1$ and

$$
\sum_{j=J_1+1}^{M}c_j\|\hat{b}_{nA_j}\|_1^\gamma \geq \left(\sum_{j=J_1+1}^{M}\|\hat{b}_{nA_j}\|_1\right)^\gamma \geq \|\hat{\boldsymbol{b}}_n-\tilde{\boldsymbol{b}}_n\|_1^\gamma \geq \|\hat{\boldsymbol{b}}_n-\tilde{\boldsymbol{b}}_n\|_2^\gamma. \tag{S.21}
$$

If $\|\hat{b}_{nA_{J_1+1}}\|_2>0$ which is equivalent to $\|\hat{\boldsymbol{b}}_n-\tilde{\boldsymbol{b}}_n\|_2>0$, combination of (S.20) and (S.21) yields

$$(1-\gamma)n\lambda \le O_p(nM^{-1})\|\hat{\boldsymbol{b}}_n - \tilde{\boldsymbol{b}}_n\|_2^{2-\gamma}.$$

Together with Lemma 5 and the fact that $\|\hat{\boldsymbol{b}}_n - \tilde{\boldsymbol{b}}_n\|_2 \le \|\hat{\boldsymbol{b}}_n - \boldsymbol{b}_0\|_2$, this implies that $(1-\gamma)n\lambda \le O_p(M^{1-\gamma}n^{\gamma/2})$. Now, by condition *C.6*,

$$P(\|\hat{b}_{nA_{J_1+1}}\|_2 > 0) \le P\left(\frac{\lambda}{M^{1-\gamma}n^{\gamma/2-1}} \le \; O_p(1)\right) \to 0.$$

Then the conclusion of the lemma follows. ■

*Proof of Theorem 1.* By Lemma 5, $\|\hat{\beta}_n - \beta_{0s}\|_2^2 \le \|\hat{\boldsymbol{b}}_n - \boldsymbol{b}_0\|_2^2 \sum_{j=1}^{M+d}\int_0^1 B_k^2(t)dt = O_p(M^2 n^{-1})$. Since $\|\beta_0 - \beta_{0s}\|_\infty = O(M^{-p})$, $\|\beta_{0s} - \beta_0\|_2 \le O(M^{-p})$. Thus $\|\hat{\beta}_n - \beta_0\|_2 \le \|\hat{\beta}_n - \beta_{0s} + \beta_{0s} - \beta_0\|_2 \le \|\hat{\beta}_n - \beta_{0s}\|_2 + \|\beta_{0s} - \beta_0\|_2 = O_p(Mn^{-1/2} + M^{-p})$. ■

*Proof of Theorem 2.* (i) We know that $\delta_0 \in [t_{J_1-1}, t_{J_1})$. By the compact support property of B-spline basis functions, for all $t \in [t_{J_1}, 1]$, $\hat{\beta}_n(t) = \sum_{j=1}^{M+d} \hat{b}_{nj}B_j(t) = \sum_{j=J_1+1}^{M+d} \hat{b}_{nj}B_j(t)$. If $\hat{b}_{nA_{J_1+1}} = 0$, then $\hat{\beta}_n(t) = 0$ on $[t_{J_1}, 1]$. Thus by Lemma 6, $P\left(\hat{\beta}_n(t) = 0 \text{ on } [t_{J_1}, 1]\right) \ge P\left(\|\hat{b}_{nA_{J_1+1}}\|_2 = 0\right) \to 1$. Therefore, given $0 < \zeta_1 < 1 - \delta_0$, for $M$ sufficiently large, $\delta_0 + \zeta_1 > t_{J_1}$. Then $P\left(\hat{\beta}_n(t) = 0 \text{ on } [\delta_0 + \zeta_1, 1]\right) \ge P\left(\hat{\beta}_n(t) = 0 \text{ on } [t_{J_1}, 1]\right) \ge P\left(\|\hat{b}_{nA_{J_1+1}}\|_2 = 0\right) \to 1$.

(ii) We first argue that $P(\hat{b}_{nA_{J_1-d}} = 0) \to 0$. To see that, for some fixed $\zeta_2 > 0$, $\beta_0(t) \ne 0$ for some $t \in (\delta_0 - \zeta_2, \delta_0)$. Since $\|\beta_s - \beta_0\|_\infty = O(M^{-p})$, for sufficiently large $M$, there is some $K$ such that $t_K \ge \delta_0 - \zeta_2$. We also have $|\hat{b}_{nK}| \ne 0$ with probability tending to one, which further implies that $P(\delta_0 - \zeta_2 \le t_K \le \hat{\delta}_n) \to 1$. On the other hand, from Lemma 6 we deduce that $P(\hat{\delta}_n \leq \delta_0 + \zeta_2) \to 1$. Therefore, together we obtain the claim that $\hat{\delta}_n$ converges to $\delta_0$ in probability, by noting that $\zeta_2 > 0$ is arbitrary. ■

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