More about Continuous Distribution Models

(and Connections with the discrete models of Ch 3)

Binomial and Poisson Relationship

Recall that the Binomial RV modeled probabilities for the number of successes in n Bernoulli trials. (e.g 10 coin tosses,P(H)=.5, X=number of heads).

A slight variant of this model also gives rise to Poisson probabilities. This is a continuous time model, whereas the sequence of Bernoulli trials treats time as discrete. To describe this analogy, we need to start with a Poisson Process (p 137).

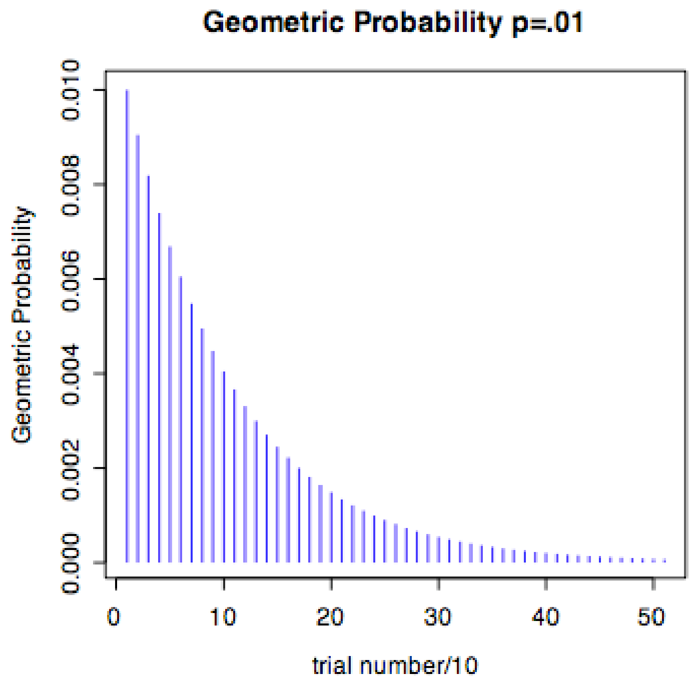
Suppose we are watching a process, such as a paper manufacturing process, which produces paper on a continuous roll. Every once in a while, there is a fault in the paper. If we start watching the process at time 0, and watch it during the time interval (0,T), then the number of faults observed, call it N(T), could be 0,1,2, ... If faults occur at the rate of λ per minute, we would expect the number of faults in (0,T) to have a Poisson distribution with mean λT.

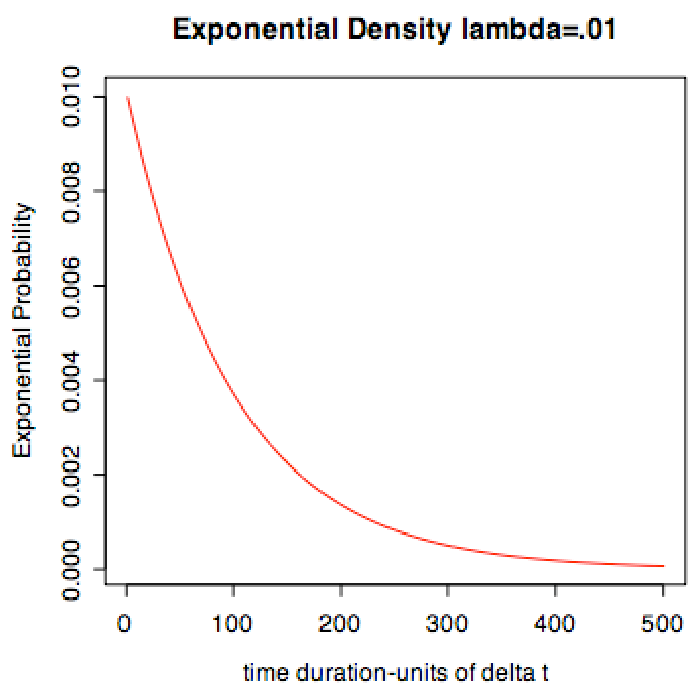
Another way to describe this manufacturing process is to break up time into a sequence of small intervals of length Δt. In fact in (0,T) you have $T/\Delta t$ such intervals, and lets suppose this in an integer $n= T/\Delta t$ .If Δt is small, one would expect 0 faults in most of the intervals, 1 fault in a few intervals, and no intervals with more than 1 fault. This specification is very close to the Poisson Process described on p 137. But if you think of each interval's result (0 or 1) as a Bernoulli trial, then the number of events that occur during (0,T), call it X(T), should have a Binomial distribution. In other words, the random variables X(T) and N(T) should have very similar probabilities. At the same time, N(T) has a Poisson distribution with mean λT, while X(T) has a Binomial distribution with parameters $n= T/\Delta t$ and $p=\lambda\Delta t$ (and also has mean $np=\lambda T$).

So as long as Δt is small enough that at most 1 event can occur in an interval, we have two ways of describing the same process. One is a discrete time process (Binomial) and one is a continuous time process (Poisson). Note that, in this analogy, a "failure" in the discrete time process is the lack of an event (a fault in the example) whereas "success" is an interval where an event occurs.

Geometric and Exponential Relationship

In the discrete approximation to the Poisson process described above, we have a series of Bernoulli trials in which p is the probability of a "success" in each time interval of size Δt. How many time intervals would we have to wait before the first "success" occurs? This number of time intervals must have a Geometric distribution (reasoning from the Bernoulli trials setting). The continuous analog of the Geometric distribution is the Exponential distribution (p 177). It models the time until an event occurs in a process like the Poisson process. See the plots of both Geometric and Exponential RVs below.





Evidently, the time until the first success in the sequence of Bernoulli trials has a very similar distribution as the time until the first event in the Poisson process. This analogy extends to the **Negative Binomial-Gamma relationship**. The time until the rth success in a sequence of Bernoulli trials has a very similar distribution to the time until the rth event in the Poisson Process. The time until the rth event (sometimes called the waiting time until the rth event) has a Gamma distribution. The density, mean and variance of the Gamma distribution are given on p 175 of your text. Just as the Geometric is a special case of the Negative Binomial Distribution, the Exponential is a special case of the Gamma. If you put r=1 in the Neg. Binomial, you get the Geometric. If you put $\alpha=1$ in the Gamma, you get the Exponential.

Overview:

The Geometric, Negative Binomial, Exponential, and Gamma distributions have links that you should be aware of, since knowing these links will help you to realize which model (if any) applies to a given situation. There are also close links between the Normal, Gamma, Poisson and Binomial which I will outline in a future lecture.