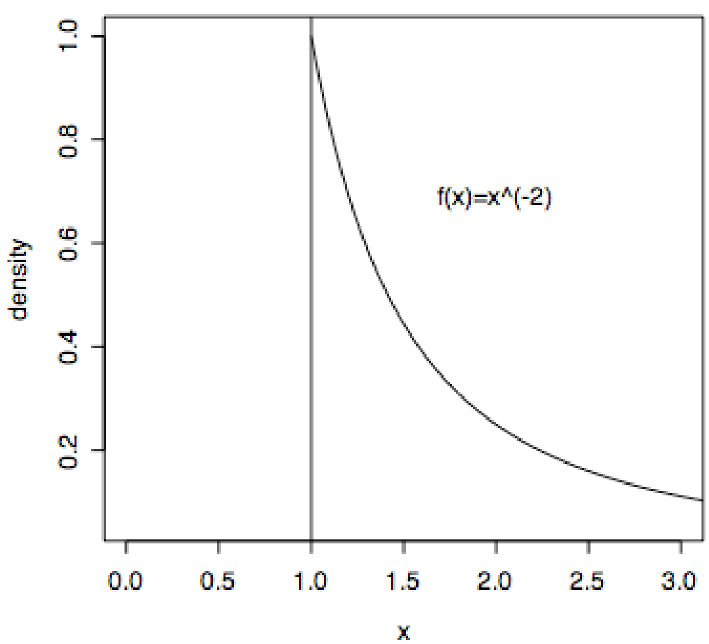
Some worked exercises from Ch 4:

Ex 10 p 151

a) It looks like this density depends on two parameters – how are we going to plot it for general k and θ? But note that we can re-write the density as a constant times $(x/\theta)^{-(k+1)}$. And since the constant must always be such that the integral is 1, we don't have to worry about it when we are sketching it for shape – if we plot the $(x/\theta)^{-(k+1)}$, the function f() will be the same if we were to scale the axis properly. But since we are to "sketch" the graph, we will not worry about this scaling. We can wlog let θ=1.

Of course, it is conceivable that the function might depend qualitatively on k. But if we look at the function, it is obvious it is a sort of exponential decline, so any value of k will do for this part, so we will choose k=1. (Note we are told k>0 so the exponent will never be negative or 0.)

Note another condition on f(x) is that $x \geq \theta$. So we must draw f(x) over the range $(\theta, +\infty)$, and in our case this will be $(1, +\infty)$. The c is just the constant necessary to make the integral 1. In this case $c = k/\theta = 1$, so the scale shown is correct.



b) integrating x^{-2} from 1 to ∞ is $(-\infty^{-1}) - (-1^{-1}) = 1$

c) integrating the general density expression from θ to b, $\int_{\theta}^{b} \frac{k\theta^{\kappa}}{x^{k+1}} dx = -(\theta/b)^{k}+1$

d) $P(a \leq X \leq b) = -(\theta/b)^{k}+1 - [-(\theta/a)^{k}+1] = (\theta/a)^{k} - (\theta/b)^{k}$ as long as $\theta \leq a \leq b$

Ex 24 p 160

Pareto density again!

a) $E(X) = \int_\theta^\infty x \frac{k\theta^\kappa}{x^{k+1}} dx = \int_\theta^\infty \frac{k\theta^\kappa}{x^k} dx$ $=k\theta^k/(-k+1)[\,(0)-(\theta^{-k+1})]=k\theta/(k-1)$ for $k>1$

b) E(X) is proportional to (integral of 1/x) which is ∞.

c) d) are like a) and b)

e) turns out that the integral converges as long as $n<k$. Note that the lower limit of the expected value integral is $\theta > 0$, so the integrand does not blow up at this lower limit.

Ex 38 p 172

This is an easy one. 95% of a normal distribution is within 2 SDs of the mean.
So $\sigma = .05°$.
(The 2 is approximate – if you use the tabulated z=1.96 for a left area =.975, the σ is slightly larger.)

Ex 43 p 172

$X \sim N(\mu=12, \sigma=3.5)$. Want c such that $P(X< c-1)=0.99$ (all weights are in lbs).

The 99th percentile of the N(0,1) dist is 2.33 so we want $z=(c-1-12)/3.5 = 2.33$
Hence $c=2.33*3.5+13=21.155$ lbs.

Note: Always check your answer for "reasonable-ness". If the parcel weight distribution has mean 12 and SD 3.5, then 21 is more than 2 SDs above the mean, so the vast majority of parcels will weight less, so 99% seems plausible.

Ex 52 p 173

If $X \sim N(\mu,\sigma)$, then $z=(x-\mu)/\sigma \sim N(0,1)$. Note $x= z\sigma+\mu$.

Suppose z_p is such that $P(z< z_p)=p$. Since $z< z_p$ iff $z\sigma+\mu < z_p\sigma+\mu$, $P(z\sigma+\mu < z_p\sigma+\mu) = P(z< z_p) = p$, and so $P(x< z_p\sigma+\mu) = p$ which is the same as in the box on p 168.

This should be understood more than algebraically – think of expressing the pth percentile of the original X scale in terms of SDs from the mean – normal probabilities only depend on the number of SDs from the mean (and whether above or below the mean). This number of SDs **is** the z you compute from $z=(x-\mu)/\sigma$.

Exercise based on pp 178-179.

Light bulbs sold by a certain manufacturer are claimed to be exponentially distributed with mean 1000 hours. Suppose a bulb has lasted for 1000 hours. What is the probability that it last another 1000 hours?

From the memoryless property of the exponential distrbution,
P(T>2000|T>1000) = P(T>1000) = 1-exp(-.001*1000)=1-e^{-1} = 0.632.

Note: This **memoryless** property is proved on the top of p 179. The general form may be expressed as T exponential --> P(T>s+t|T>s) = P(T>t).

Ex 65 p 181

a) $-y^{1/2} < X < y^{1/2}$
b) Just note the result: $X \sim N(0,1)$ implies $X^2 \sim$ chisquared (1 df)

Note: An important further result that we will experience in Ch 5 is that if $Z_i \sim N(0,1)$ and independent then $X = \sum_{i=1}^{\nu} Z_i^2$ has a chi-squared distribution on ν degrees of freedom.

"Degrees of freedom" can be thought of as something like "number of independent sources of variation". Sometimes this is the number of observations in a sample.

98. p 200

a) n=250, p=.05 approx prob most easily obtained via Normal approx (since np and n(1-p) both >10)with mean np=12.5 and SD=$(np(1-p))^{1/2}$ = 3.45. P(#defectives ≥ 25) for the N(μ=12.5,σ) requires P(Z≥z) where z=(24.5-12.5)/3.45 = 3.48 so rqd prob is 1-.9997 = .0003 In other words, The probability that 10% or more boards were defective is .0003.
b) P(exactly 10 defectives) = P((9.5-12.5)/3.45 < Z < (10.5-12.5)/3.45) = P(-.87<Z<-.58)= .2810-.1922 = .0888
The probability of exactly 10 defectives is .0888.

100. p 200

a) cdf(x) = $F_X(x)$ = $\int_1^x \frac{3}{2}.\frac{1}{x^2}\,dx$ = (3/2)($-x^{-1}$)- (3/2)(-1^{-1}) = (3/2)(1-1/x) for 1≤x≤3 seconds.
Note: if x<1, F(x)=0 and if x>3, F(x)=1.
b) P(X≤2.5) = (3/2)(1-2/5) = 9/10 and P(X≤1.5)=(3/2)(1-2/3)=1/2 so P(1.5≤X≤2.5)=9/10-1/2 = 2/5
c) E(X)= $\int_1^3 x\frac{3}{2}.\frac{1}{x^2}\,dx = \int_1^3 \frac{3}{2}.\frac{1}{x}\,dx$ = (3/2) $\log_e(3)$ = 1.648
Note log is to base e is sometimes written ln().

Note: Sometimes the knowledge that a certain functional form is a density function can save you time by writing down an integral directly. For example, look at the gamma density on p 175. If you want to integrate $\int_0^\infty x^p e^{-x} dx$, you will know instantly that the answer is p! (as long as p is an integer, $\Gamma(p+1)$ otherwise). This is a lot simpler than doing integration by parts many times. Note also that this trick makes computing the moments of Gamma quite easy to compute. For example, if X is standard gamma with α=5, the expression for E(X) is the integral of the density of a standard gamma with α=6, but with a different constant. So E(X) = 1 . $\frac{\Gamma(6)}{\Gamma(5)}$ = 5. Of course, this particular integral is also known from the formula for E(X) = αβ where α=5 and β=1.

d)SD(X) = $\sqrt{V(X)}$
V(X)=E(X^2) – E^2(X)
E(X^2) = $\int_1^3 x^2 . \frac{3}{2} . \frac{1}{x^2} dx$ = (3/2)(3-1) =3.
So SD(X) = $(3-1.648^2)^{1/2}$ = .533

Note: Again check for reasonable-ness. The range of X is 1 to 3, so a SD of .533 seems OK.

e)h(X) =portion of reaction time that light is on
=0 if 1≤ X <1.5
=X-1.5 if 1.5≤X≤2.5
=1 if 2.5 < X <3
Therefore E(h(X)) = $\int_{1.5}^{2.5} (x-1.5)\frac{3}{2} . \frac{1}{x^2} dx + \int_{2.5}^{3} 1 . \frac{3}{2} . \frac{1}{x^2} dx$ =
(3/2)ln(2.5/1.5)-1.5*(2/5)+1/10 =.266

Note: An alternative approach is to express E(h(X)) as = 0*P(1<X<1.5) + E(X-1.5|1.5<X<2.5)*P(1.5<X<2.5) + 1*P(2.5<X<3) = 0+ $\int_{1.5}^{2.5} (x-1.5)\frac{3}{2} . \frac{1}{x^2} dx$ +1/10=.266

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112. p 202 (oops – this was an assignment question – instead I'll do 109. p 202)

109. p 202. X=ln(I_o/I_i) is defined as the current gain, and is assumed N(μ=1,σ=0.05)
a) I_o/I_i is a random variable whose natural logarithm is Normal, and this is the definition of a lognormal random variable (we omitted it p 184, but it is useful so I will mention it here).
b) Required probability is P(X>ln(2)) = P((X-1)/.05 > (ln(2)-1)/.05) = P(Z>-6.14)=1-0 =1
c) Use formulas on p 184. mean is 2.72, var is .0185.

Note: The naming of the lognormal distribution is a bit perplexing. It is not the logarithm of a normal RV, but rather it is the RV whose logarithm is normal. Its

parameters are just the parameters of the normal distribution that results after taking the logarithm. But the mean and variance of the lognormal itself are given (in terms of the mean and sd of the associated normal) by the formulas on p 184. Note the strong right skewness of the lognormal distribution. When you see a sample data distribution that has a strong right skew, often taking the log of the data will convert it to normal, and then all the strategies of the normal distribution can be used.
It is very common for data to have a lognormal distribution. For example personal income distributions are usually of this shape. Also, imagine the shape of the size of earthquakes. You have Richter size of 2 very often, 4 occasionally, 6 rarely and 8 very rarely. The distribution would certainly be very right skewed.

Although the lognormal distribution is in section 4.5, which I said we would omit, you should know about it.