Some worked exercises from Ch 5

Section 5.1

1. X is number of hoses in use on self-service island
Y full-service island

a) $P(X=1,Y=1) = .20$ (directly from the table)

b) $P(X\le 1,Y\le 1) = .10 + .04 + .20 + .08 = .42$

c) event is that both islands are in use. P(both islands are in use) = P(X = 1 or 2, Y=1 or 2) = .20 +.06 + .30 + .14 = .70

d) $P(X=x) = .16,.34,.50$ for x=0,1,2 resp (just add rows of the table)
$P(Y=y) = .24,.38,.38$ for y=0,1,2 resp (just add columns of the table)
$P(X\le 1) = .16+.34 = 0.50$

e) Either by examining the table and noting $P(X\le 1|Y=i)$ depends on i, or else by computing $P(Y\le 1) = .24 + .38 = .62$ and noting $.50 * .62 \ne .42$ (as in part b), we can conclude that X and Y are not independent. (There are other ways too – $P(X=0,Y=0) = .10 \ne P(X=0)*P(Y=0) = .16*.24 = .384$.)

12. a) $f_X(x)=\int_0^\infty xe^{-x(1+y)}dy = xe^{-x}\int_0^\infty e^{-xy}dy = xe^{-x}.x^{-1} = e^{-x}$
so $P(X>3) = e^{-3}$

b) We need to check whether $f_{X,Y}(x,y) = f_X(x).f_Y(y)$. We need $f_Y(y)$.
$f_Y(y)= \int_0^\infty xe^{-x(1+y)}dx = (1+y)^{-2}$ (since we know $\int_0^\infty (1+y)xe^{-x(1+y)}dx = (1+y)^{-1}$)
and $f_X(x) . f_Y(y) = e^{-x}(1+y)^{-2}$ which is not the same as $f_{X,Y}(x,y) = xe^{-x(1+y)}$.
(Not that it cannot be, but that it is not so for all x,y). So X and Y are not independent.

c) We want P(X>3 or Y>3 or both) = $P(\{X>3\}\cup\{Y>3\}) = 1-P(X\le 3$ AND $Y\le 3)$
But this requires a double integral to evaluate and we will skip the rest – it turns out to be 0.300.

15. a) Using the hint $F_Y(y) = P(Y\le y)$.

Now to express $\{Y\le y\}$ in terms of the X_i, we need to think about the network and when it will fail – when $Y\le y$, the network has failed by time y, and this will happen if either $X_1\le y$ or if both $X_2\le y$ and $X_3\le y$, or both these conditions occur. So
$\{Y\le y\}$ is the same as $\{X_1\le y\}\cup(\{X_2\le y\}\cap\{X_3\le y\})$
and
$P(Y\le y) = P(X_1\le y) + P(X_2\le y \text{ and } X_3\le y) – P(X_1\le y \text{ and } X_2\le y \text{ and } X_3\le y)$

But for any $Y \sim \text{expo}(\lambda)$, $P(Y\le y) = 1-e^{-\lambda y}$, so using independence of the X_i, we have

$P(Y\le y) = (1-e^{-\lambda y}) + (1-e^{-\lambda y})^2 - (1-e^{-\lambda y})^3$ for any $y\ge 0$

Now the density is obtained by differentiating $F_Y(y)$ ($=P(Y\le y)$).

$f_Y(y) = \lambda e^{-\lambda y} + 2(1-e^{-\lambda y})\lambda e^{-\lambda y} - 3(1-e^{-\lambda y})^2 \lambda e^{-\lambda y}$

$= \lambda e^{-\lambda y}(1+2(1-e^{-\lambda y})-3(1-e^{-\lambda y})^2)$

$= \lambda e^{-\lambda y}(-2 e^{-\lambda y} +6 e^{-\lambda y} - 3 e^{-2\lambda y})$

$= \lambda e^{-\lambda y}(4e^{-\lambda y}-3 e^{-2\lambda y}) = 4\lambda e^{-2\lambda y}-3\lambda e^{-3\lambda y}$ for $y \geq 0$

b) We need to integrate $y^* f_Y(y)$ over $(0,\infty)$

This is simplified if we realize, for any expo X with parameter θ, $E(X)= \theta^{-1}$

so $\int_0^\infty y e^{-\theta y} dy = \theta^{-1}$. We can use this for $\theta = 2\lambda$ or 3λ.

$E(Y) = (4/2).(2\lambda)^{-1} - (3\lambda)^{-1} =(2/3)^* \lambda^{-1}$

For example, if each component had mean 1 (1 hour say), then the average time the systems lasts is only 2/3 hours.

Section 5.2 – Covariance and Correlation

Ex 28 p 225 – This is a useful result – you should be aware of it. It is not so important to prove it although it is quite simple. The double integral separates into the product of two single integrals since $f_{X,Y}(x,y) = f_X(x).\ f_Y(y)$ when X and Y are independent.

Ex 35 p 225 – This is another useful result, especially the b) and c) parts. It follows from the rules of the E() operator (p 116).