

Notices: No office hours for KLW on Friday, Oct 5. But still 12-1 for Nathoo (10543).

Reminder: Assignment due Oct 10 on exercises in Ch 4: 4.1-3, 4.2-8, 4.3-4, 4.4-3 (interpretation only), 4.5-4, 4.6-6, 4.7-8

Today: comment on self-test #3, more from Ch 4 on Markov Chains.

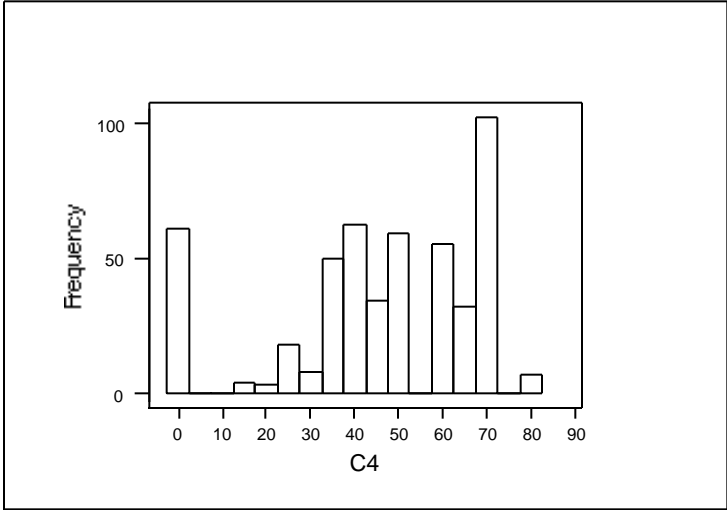
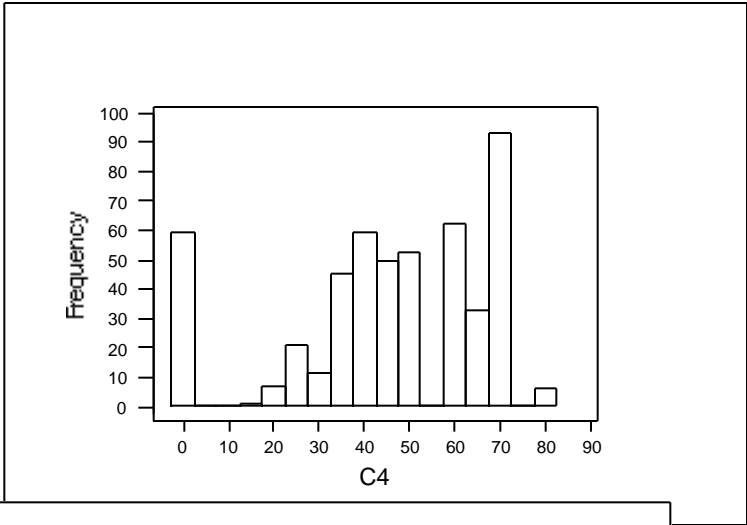
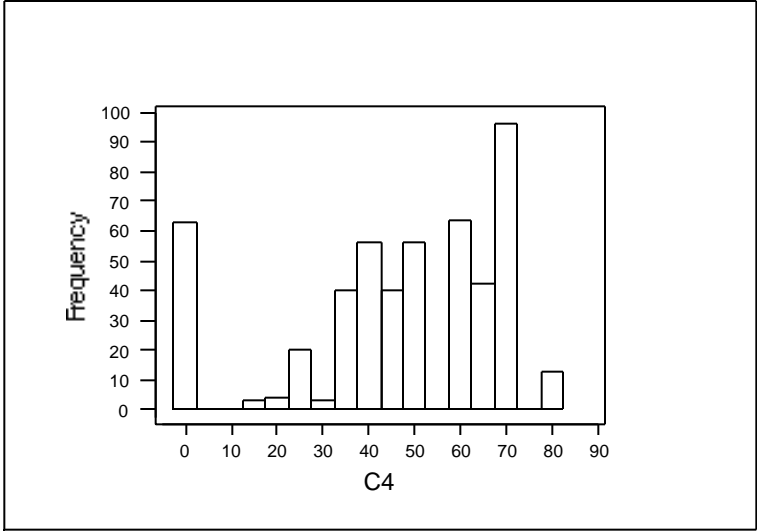
Sent e-mail re revised Q3 from self-test. Problem should have said:

“3. A game show asks you to select from a bin a random sample of 12 coloured balls, from a large bin with an equal number of balls of each of the colours: red, blue, yellow and green. ....”

Some students recognized the multinomial as the underlying distribution of the vector outcomes. However, the prize from the game was the product of the colour frequencies in the sample, and this distribution is not known. But it can be simulated.

Here is the MINITAB program:

```
Gmacro                # starts every macro file
multinom              # name relates to multinomial sampling context
brief 0               # suppress output
do k1=1:500           # starts a do-loop
set c1                # puts 1,2,3,4 in a column
1:4
end
sample 12 C1 C2;      #samples 12 times with replacement from {1,2,3,4}
replacement.         # and puts result in C2
tally c2;            # computes the frequency distribution in c2
store c1 c3.         # and stores it in c3 (freq of k in c3(k), k=1,2,3,4).
let c4(k1)=c3(1)*c3(2)*c3(3)*c3(4) # computes the payoff
enddo                # end of do-loop
brief 2              #allow output
code ('*')0 c4 c4 # fixes problem that tally treats 0 frequency as missing
#                   # value, which MINITAB records as *. Changes * to 0.
hist c4              # graph frequency distribution of payoff
mean c4              # mean payoff
stdev c4             # SD of payoff
endmacro             #ends every macro (in MINITAB)
```



	Simulation 1	Simulation 2	Simulation 3
MEAN	45.5	45.9	46.7
SD	22.1	22.5	22.9
Proportion of zeros	0.126	0.120	0.126

Back to Ch4: MARKOV CHAINS

Recall:

MC  $\{X(n): n=0,1,2,3,\dots\}$  is defined by

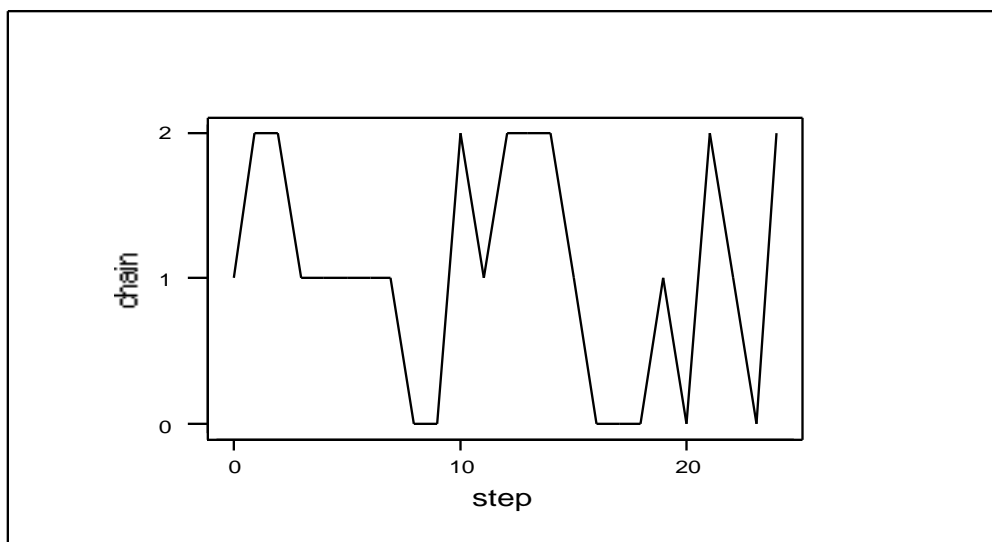
1. STATE SPACE: values that  $X(n)$  can take
2. STARTING STATE:  $X(0)$
3. 1-STEP TRANSITION MATRIX: a matrix of elements  $P=[p_{ij}]$  where

$p_{ij}$  is the probability that, given the chain is in state  $i$  at some step  $n$ , it will be in  $j$  at the next step.

Suppose we are simulating occupancy in a barber shop with only 1 chair and waiting room for 1 person only. Number of customers in the shop is 0,1,2. Let the state space be  $\{0,1,2\}$  and suppose there is one customer at time 0. What would a reasonable transition matrix be? Suppose time steps are 15 minutes.

	0	1	2
0	.5	.3	.2
1	.3	.5	.2
2	.0	.5	.5

What would a chain of 25 steps look like?



This chain has been produced by a MINITAB macro called mchain:

Gmacro

MChain

```
# Here is a program that will simulate a Markov Chain: To adapt
# it to any Markov chain, just specify the size of the state space, k1,
# the transition matrix, c1-ck1, the number of time points to
# simulate, k4, and the start state, k2. The program produces a plot
# of the k4 values of the chain, and also an estimate of the steady
# state distribution.
```

```
#
```

```
# To use this program, just click on OTHER FILES - EXECUTE MACRO
# under the FILE menu. If the program is on a disk, you will
# be able to direct MINITAB to it using the mouse.
```

```
#
```

```
# We begin with the initialization
```

```
#
```

```
let k4=25 # length of chain to be simulated
```

```
let k2=1 # set start state for Markov chain
```

```
let k1=3 # set size of state space of Markov chain
```

```
read c1-ck1 # read in k1 by k1 transition matrix
```

```
.5 .3 .2
```

```
.3 .5 .2
```

```
.0 .5 .5
```

```
end
```

```
noecho # we don't need to see the program while it is executing
```

```
copy c1-ck1 m1 # we want the columns to be the distributions
```

```
trans m1 m2 # to generate so have to transpose transition
```

```
copy m2 c1-ck1 # matrix (See rand & discrete below)
```

```
let k6=k2 # we will want to change this "from" state at each loop
```

```
Let k1=k1-1 # We want our states to be 0,1,2,...,k1-1
```

```
set c100 # the column that makes up part of the table used
```

```
# to define the discrete probability distribution
```

```
0:k1
```

```
end
```

```
Let k4=k4-1 # We already have one of the k4 values from k2
```

```
set c200 # this stores resulting chain
```

```
# We just fill it with any numbers for now
```

```

1:k4
end
name c200 'chain'
do k5=1:k4
  rand 1 c101; # This uses the ck6 probabilities to generate a value
  discrete c100 ck6. # c100 is just 0,1,2,3,...,m-1 for m state MC
  let c200(k5)=c101(1) # capture generated state
  let k6=c101(1)+1 # record result for input to next loop
# Note that c1 holds probs for transition from state 0, c2 from state 1 ...
enddo
set c201 # this sets up the time variable in plot
0:k4
end
name c201 'step'
stack k2 c200 c200 # puts start value at beginning of chain
print c201 c200
table c200; # this just computes the proportion of time spent in
# each state. Compare with the steady state
# probabilities
colp. # column percents subcommand to table
plot c200*c201; # Plot of Markov Chain
connect. # joins the points by lines
# To get the output, just copy and paste from the MINITAB
# Session window to a word processor, OR, select a portion of the
# Session window and under the File menu select "Print Selection".
# The graph can be printed either by COPY GRAPH in the EDIT menu, and
# then paste to a word processor, OR by "Print Window" under the
# File menu - but for either of these, one has to find the graph at
# the bottom of the WINDOW menu.
endmacro

```

What would happen if we ran this chain for a large number of steps?

Simulate: 1000 steps → .28, .44, .28 as the proportions in each state.

OR

Calculate: Method 1 – high power of transition matrix:

MTB > read 3 3 m1

DATA> .5 .3 .2

DATA> .3 .5 .2

DATA> 0 .5 .5

3 rows read.

MTB > mult m1 m1 m2

MTB > mult m2 m2 m3  
 MTB > mult m3 m3 m4  
 MTB > prin m1-m4

Data Display

Matrix M1

0.5 0.3 0.2  
 0.3 0.5 0.2  
 0.0 0.5 0.5

Matrix M2

0.34 0.40 0.26  
 0.30 0.44 0.26  
 0.15 0.50 0.35

Matrix M3

0.2746 0.4420 0.2834  
 0.2730 0.4436 0.2834  
 0.2535 0.4550 0.2915

Matrix M4

0.267913 0.446391 0.285696  
 0.267911 0.446394 0.285696  
 0.267721 0.446518 0.285761

Read off .27, .45, .28 as long run probabilities of being in state 0,1,2.

Calculate – Method 2: See Theorem 4.6.3 p 171 and examples following:

$\mathbf{P} =$  and  $\mathbf{1} = 1$

$$\begin{array}{r} 0.5 \ 0.3 \ 0.2 \\ 0.3 \ 0.5 \ 0.2 \\ 0.0 \ 0.5 \ 0.5 \end{array} \rightarrow \begin{array}{l} {}_1 .5 + {}_2 .3 + {}_3 0 = 1 \\ {}_1 .3 + {}_2 .5 + {}_3 .5 = 2 \\ {}_1 .2 + {}_2 .2 + {}_3 .5 = 3 \end{array}$$

But these equations are dependent (sum to 1 on each side) – to solve need another equation. Use  $x + y = 1$

This will produce the same result as the previous method. .45, .27, .28 (rounded).

Suppose we want to know how much money the barber makes.

He charges approx \$10 per 15 minutes as long as there is someone in the shop.

But  $.45 + .28 = .73$

so average earning is \$7.30 per 15 minutes or \$29.20/hr.

Exercise 4.2-5

Cell multiplication: a cell either divides into two, or dies. Track population size at each generation.  $P(\text{divide}) = 0.4$  and  $P(\text{die}) = 0.6$

Start with 2 cells. How many at next generation?

0,2,4 with probability  $.6^2, 2*.6*.4, .4^2$

What about start with  $i$  cells?

How many cells double?  $X = \text{Bin}(i, 0.4)$  and resulting number of cells is  $2X$ .

So  $P(i \rightarrow j | \text{at } i) = \text{binomial probability from } \text{Bin}(i, 0.4) \text{ at } k=j/2 \text{ for } j \text{ even, } 0 \text{ otherwise.}$

For example,  $i=2, P(\text{next generation size is } 1) = 2C1 .4^1 .6^1$  as above.

Note importance of recognizing Binomial context to solve general case.

## Section 4.7:

### Absorbing States and First Passage Times

Example: Queuing system: Suppose capacity of queue is 10. How long till Capacity first reached? First Passage time.

Could also be time to “absorption” if system shuts down at full capacity.

Absorbing states and nonabsorbing states. If  $i$  is absorbing state, what is  $p_{ij}$  ?

$Q$  is transition matrix among non-absorbing states (rows will not all sum to 1)

Theorem 4.7-2 gives equations for mean times to absorption. See examples 4.7-4 and 4.7-5.