STAT 280 October 10, 2001

A brief coverage of Section 4.7 Absorbing States:

A state j is an absorbing state if, once visited, is stuck there.
There may be more than one absorbing state, and so it is sometimes usueful to be able to refer to the set of absorbing states in a State Space, $\{j: j \varepsilon A\}$. Often we are interested in finding the distribution of absorption times (the number of steps the MC will have before an absorbing state is visited}

e.g. S={0,1,2,3,4} and {4} is an aborbing state. Suppose X(0)=0. Outcome might be:

0, 1, 3, 2, 1, 3, 1, 3, 2, 3, 1, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, 4, ……

In this case the absorption time is 11 since X(11)=4 and X(i) is not 11 for any i < 11.

First suppose there is only one absorbing state j. Then at any time k, if the chain has visited j so far, it will remain at j. So the chance that the chain is already absorbed by time k is the same as the chance that it is in state j after k steps. (We have assumed that the chain starts at i at time 0. ) In symbols,

$P(\text{absorption time } T \leq k \mid \text{start at i at time 0}) = P^{(k)}(i\text{->}j)$

where the right side is the (i,j) element of the k-step transition matrix.

It is easy to generate these k-step transition probabilities using matrix multiplication (via computer) so we can actually work these absorption probabilities numerically for any finite k, as long as the starting state i and the one-step transition matrix is known.
This method generates a cdf, and by differencing we can get the pdf, and the mean time to absorption. This is the numerical approach.

If we only need the mean time to absorption, there is another way to do it – solving a system of equations. See Theorem 4.7-2 on p 180-181 for details, but the ingredient needed for this is the one-step transition matrix Q on the non-absorbing states. This will not be a proper transition matrix since its rows will not necessarily add to 1 (possibly less than 1 if the state i corresponding to row i can move to an absorbing state in one step.)

The solutions $\mu_1, \mu_2, \mu_3, \ldots$ are the mean absorption times when the chain has been started in state i=1,2,3 ….respectively.

This absorbing state technique can allow computation of the distribution (or mean time) to a first passage from a starting state i to a different state j. Assume that there are no absorbing states (otherwise the mean time would be infinite as long as the probability of ever reaching the absorbing state were positive). Just use the trick outlined on p 182: redefine the MC to have an absorbing state j.

For example, if the transition matrix is

.1 .3 .2 .4
0 .5 .5 0
.2 .5 0 .3
.8 .2 0 0

Draw a map of the access of states:

1-> 1
->2 -> 2
-> 3
-> 3 -> 1
-> 2
-> 4
-> 4 -> 1
-> 2

one can see that 1->2->3->4-> 1 has positive probability so this chain will always continue to visit all the states an infinitew number of times. Now suppose we want to know the time until the chain first gets to state 4. Redefine the transition matrix as:

.1 .3 .2 .4
0 .5 .5 0
.2 .5 0 .3
0 0 0 1

The time to absorption in this MC is the same as the first passage time to 4 in the original MC.

When there is more than one absorbing state, there is another question of the relative probabilities of the chain ending up in the various absorbing states. For example, in the gamblers ruin problem, it would be interesting to know the probability of ruin (and of the gambler reaching his target). So ruin and target are two absorbing states.

Again, there are two methods for computing these absorption probabilities. Multiplying the transition matrix (as in Example 4.7-7) or solving a system of equations (Thm 4.7-3). The latter approach also results in a matrix formula for the mean absorption times (Thm 4.7-4).

Simulation of the Gamblers Ruin Process:

The simple approach to this is to revert to our random walk program:

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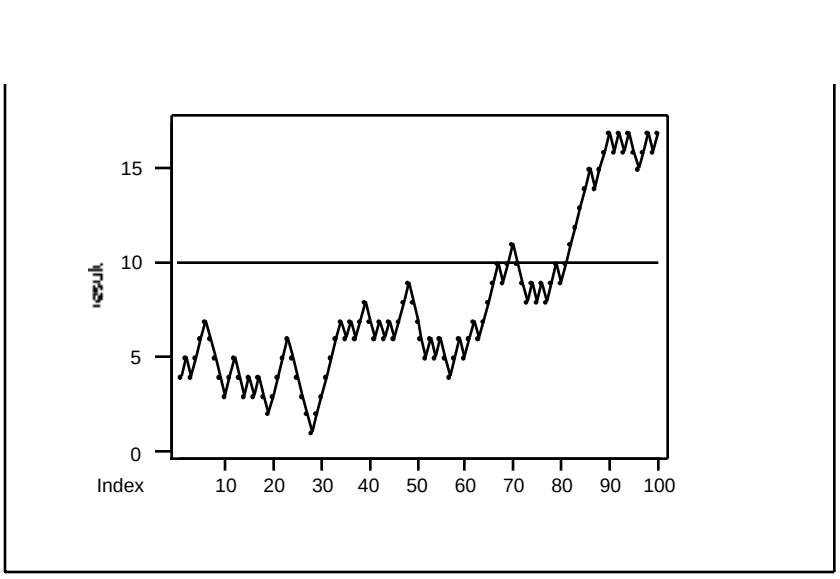
```
gmacro
rw.mac
```




```
# To run this program, just type %rw in the MINITAB session window
#
erase c1 c2 c3            #start clean
rand 100 c1;              # generate those fair coin tosses
bern 0.5.
let c2=2*c1-1             # turn the 0-1 indicators into steps
parsum c2 c3               # partial sums (times series of results)
name c3 'net_walk'
tsplot c3;                # plot in order of steps
line 0 0 100 0.
Endmacro
```

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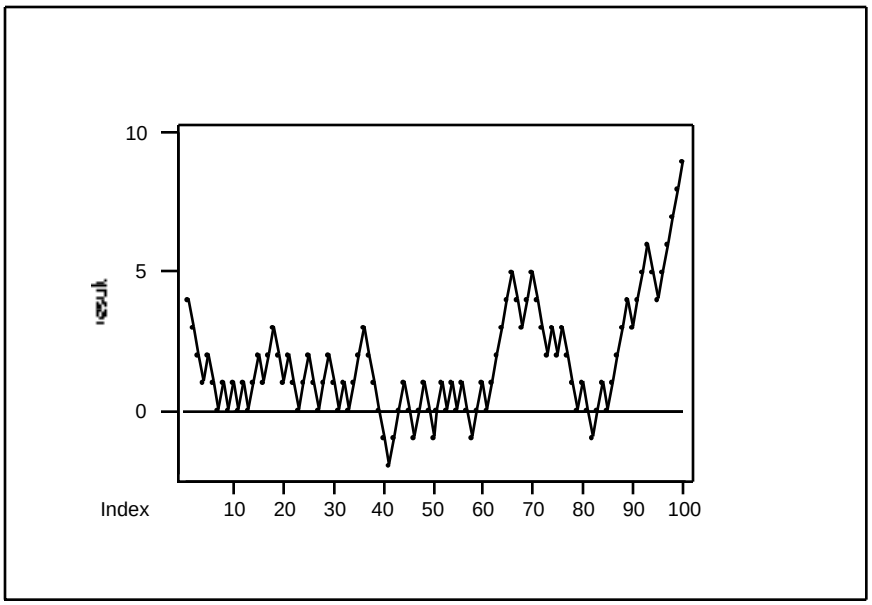
The result will allow us to see whether the gambler would have achieved his objective before being ruined. For example:

In this case, if the gamblers target was 10, and if 0 represents ruin, then the gambler succeeds on the 67$^{th}$ game. In another simulation, here is what happened:



In this case, the gambler was ruined about the $8^{th}$ game.

From repeated use of this program, we could determine the probability of ruin and the distribution of the time until absorption (ruin or target) or the conditional distribution of time to absorption given the gambler is ruined.

It turns out in this case, in which the gambler starts with $5 and aims for a target of $10, that the probability is .5 of each outcome. It can also be shown that if the gamblers target were 15, and he started out with 5, his probability of ruin would be 10/15.

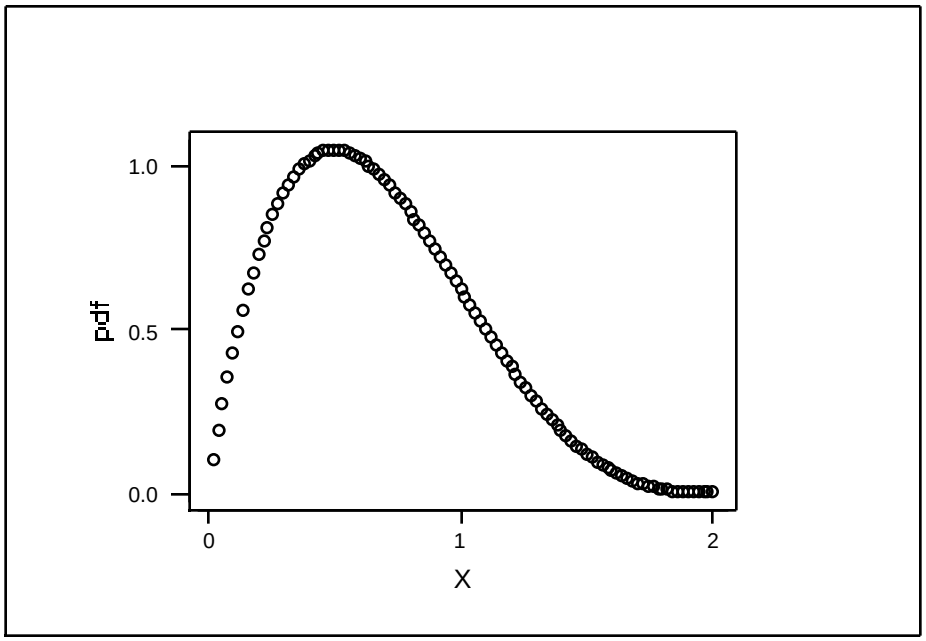
Chapter 5: Continuous Random Variables and Continuous Distributions.

From probability mass functions to probability density functions.

pmf -> pdf

| x | P(X=x) |
|---|---|
| 0 | .2 |
| 1 | .6 |
| 2 | .2 |

What if X can take any value in the interval 0 -> 2? How do we specify the values that are more or less likely to occur in a sample?




This plot shows relative frequency per unit (density). Ordinates are not probabilities. Integrals of the function over an interval (a,b) is the probability that X is in (a,b).

pdf does show that .5 is more likely than 1.5 however.

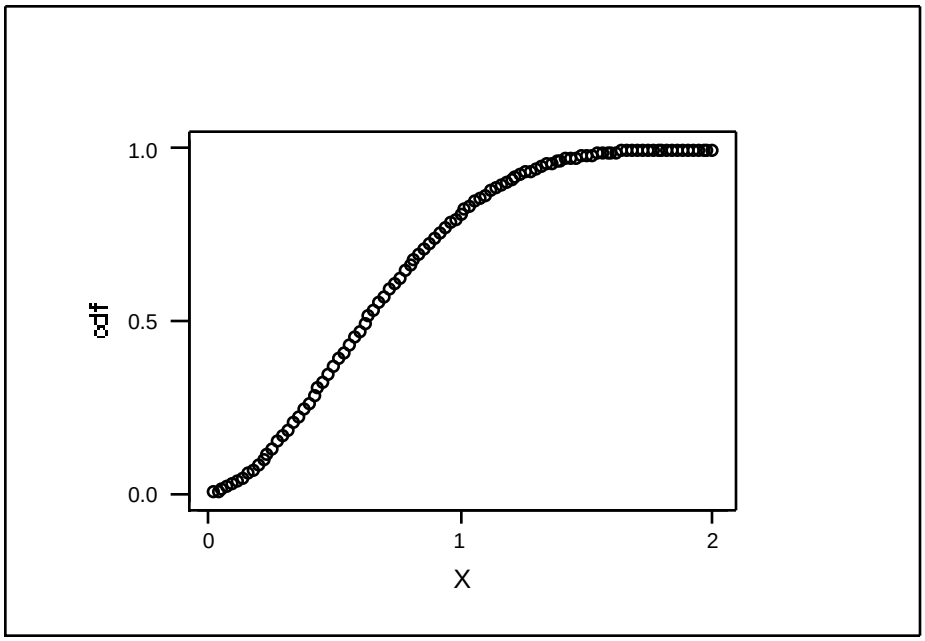
Is integral over whole domain of pdf =1 ?

Continuous Uniform density? U(0,1)

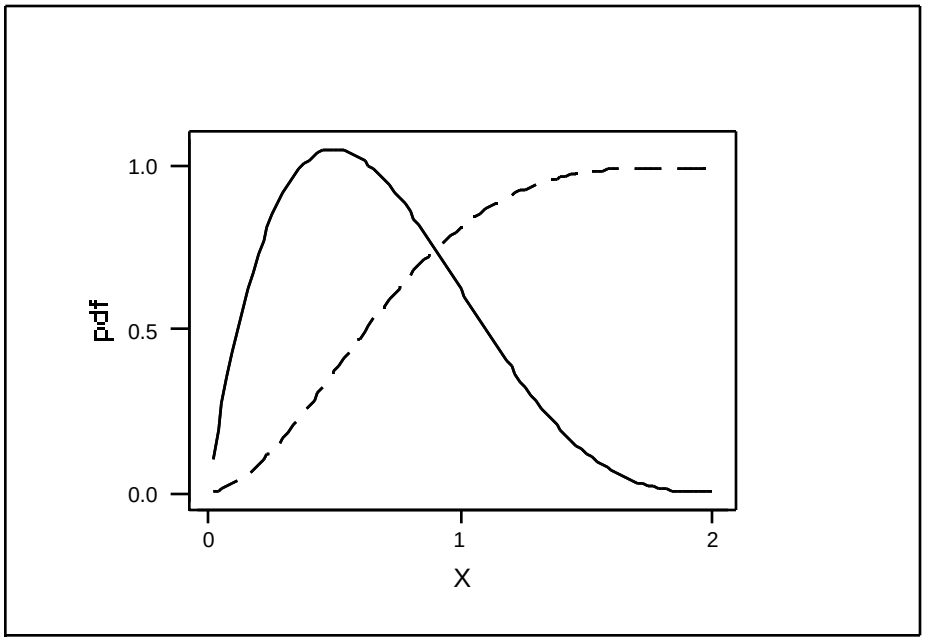
CDF of X : F(x) integral from minimum (or – inf) to x.

F(+inf) = 1

In example above F(2)=1.




Note relationship of pdf and cdf.




Expected value:  average for population.

Section 5.3

Simulating Continuous RVs

A General Method when F() is known:

See p 5.3

$P(F(X)<t) = P(X<F^{-1}(t)) = F(F^{-1}(t)) = t$

Therfore F(X) is uniform!

The rest of the lecture concerned the normal and exponential distribution and simulations from them.

Assignment by e-mail: