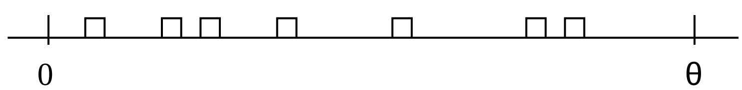
Stat 280 October 24, 2001

6.6 Methods of Moments Estimation

Method for determining estimator formula for a parameter when the pdf involving the parameter is known:

Example: Uniform $(0,\theta)$ Data $X_1, X_2, \ldots, X_7$



How to estimate θ? Max + a bit?

Max + Min?

2*Median?

Many methods, and criteria help to sort out which is best.

But sometimes optimization not possible – use ad hoc ….
Method of Moments.

$E(X) = \theta/2$ is known. But $\bar{X}$ (sample mean) estimates E(X).
Set $\bar{X} = \hat{\theta}/2$ and solve for $\hat{\theta}$
So $\hat{\theta}$ (est of θ) is $2*\bar{X}$.

See example 6.6-2 for less trivial example.

Chapter 7: Counting Processes and Queues

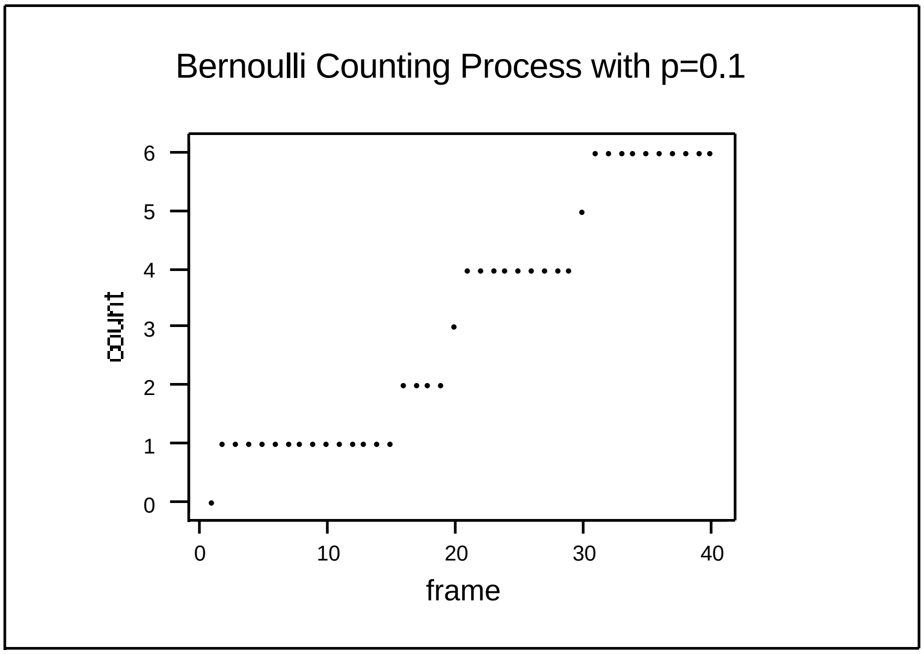
Context: Count of events as they occur. Usually continuous time.
N(t): number of events that occur during (0,t)
N(t) is a rv for each t. The distribution of N(t) usually depends on t.

Examples: phone calls, computer failures, earthquakes, traffic accidents, births, deaths, insurance claims, e-business orders, …

Simpler case introduced in Section 7.1: Discrete time, t=1,2,3,…

Time intervals such as minutes, days or months instead of continuous time. Call these time “frames”.
Text uses different notation for this particular discrete time model:
X(n) instead of N(t)
Bernoulli Counting Process: each frame has event with prob p

X(n) has Bin(n;p) distribution.



In what frames did “events” occur? How many frames does it take for a change in X(n)? The number of frames from one event to the next event has a geometric distribution and this explains Theorem 7.1-3, p 274.

(Note time = number of frames x frame width)

Suppose we are actually observing a continuous time process, like a telephone exchange. An event is a call.

Calls arrive at rate λ per minute. Suppose frame is 1 minute. Then the p in the Bernoulli counting process is also λ. But note that λ must be less than 1 in this model – the frame-based approximation to the continuous time process will not be useful if events occur more than once per frame, on average. If λ were 5 per minute, we would need a smaller frame, say 1 second, and then the p would be 1/12 so that in 60 seconds we would still get 5 events per minute, on average.

Is this a Markov chain? Yes, because it has the Markov property.
Informally: Future indept of past given present.

What is transition matrix? See p 269

Row associated with state k, k=0,1,2,… representing count

0,0,0…,1-p, p, 0,0,0 where the 1-p is in the kth position, k=0,1,2,…

P[X(n)=k|X(n-1)=k] =1-p for n=2,3,4,… and k=0,1,2,…. And

P[X(n)=k+1|X(n-1)=k] =p for n=2,3,4… and k=0,1,2,….

P[X(n)=k+2|X(n-1)=k] = ? for n=2,3,4… and k=0,1,2,….

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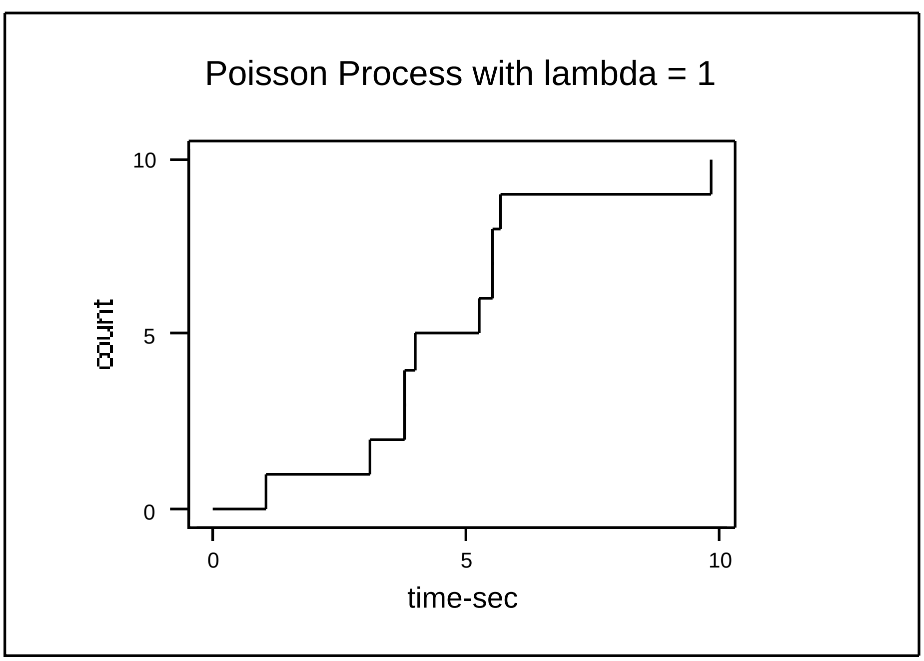
Poisson Process: Extend Bernoulli Counting process by considering continuous time (in which frame size is 0 (or dt)) Call count N(t).

(Vertical lines should be omitted since change instantaneous).
Note that rate of 1 per second is a constant average rate, but that at any time t, the number of events that occurs is N(t) and N(t)/t is not constant.



P(N(t)=x) = see Poisson with mean λt on p 277.

λ is the average rate of events per unit time.

Note conditions for Poisson to apply:

i) independent increments
ii) stationary increments
iii) one event at a time
iv) prob of event in dt is λ dt

can be made rigorous and then i)-iv) IMPLY Poisson.

These conditions can be used for a mental check of Poisson context.

Example 1: telephone calls at SFU switchboard 291-3111
Example 2: students arriving at B lot from 9-10 am Wed.
Example 3: number of customers arriving at Safeway 9-10 am Wed.

Time between events: "interarrival times"

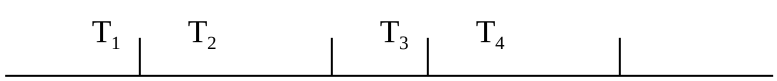
Note from Poisson pmf,

P(N(t)=0) = $e^{-\lambda t}$.   Let T be the time until the first event.

Since {N(t)=0} is the same event as {T>t}

P(T>t) = $e^{-\lambda t}$ and P(T$\leq$t) = 1-$e^{-\lambda t}$

This proves T is exponential. In fact can show (p 278) that all the interarrival times have this same exponential distribution.



In fact, {T_i: i= 1,2,3,… } are IID exponential (λ) (mean is λ^{-1})

Poisson Process with rate parameter λ (i.e. mean λt) has
Interarrival time that are exponential with rate parameter λ (i.e. mean λ^{-1})

What about waiting times for the Poisson Process? Gamma. (p 281)

Note that $\{W_n>t\}$ is the same event as $\{N(t) \leq n-1\}$
CDF of W_n can be defined in terms of sum of Poisson probs (p 281).
Differentiate CDF of Wn to get Gamma density.
Note: This representation of CDF of Gamma is a finite series.

How can T_1 have the same distribution as T_2 ?

Memoryless property of exponential. (p 239)

$P(T_1>t+s|T_1>s) = P(T_1>t)$

Problems to try 7.2-4, 7.1-4