STAT 280 Nov. 21, 2001

Today: Sections (Apparently unrelated topics)
9.1 Distribution of Extremes
10.1 The Reliability Function
10.2 The Hazard Rate

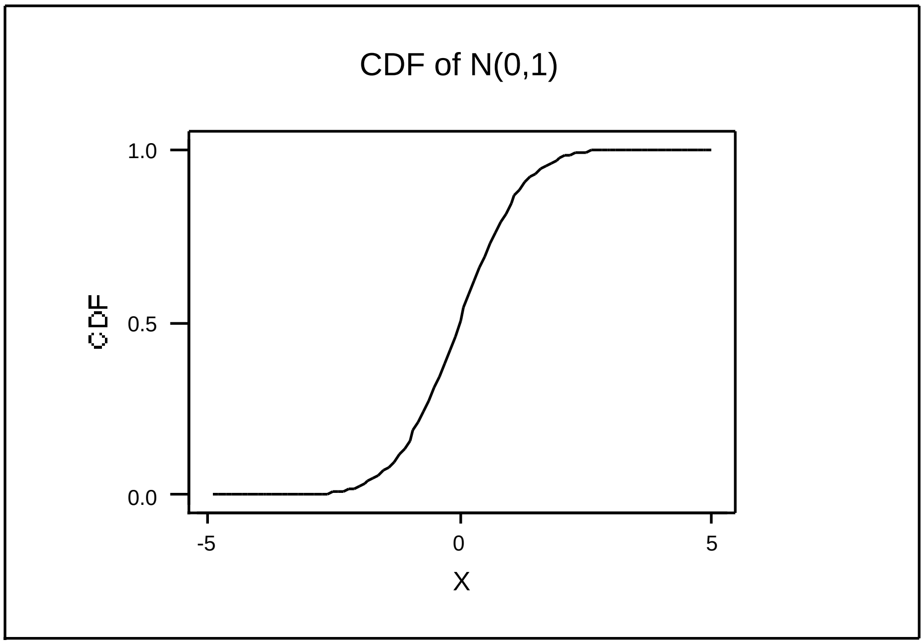
9.1 Distribution of Extremes

Consider a random sample of n observations:

$x_1, x_2, x_3, \ldots, x_n$

We model this observation as the realization of an IID sequence of random variables
$X_1, X_2, X_3, \ldots, X_n$

The sampled population may be described by its associated CDF $F_X(x) = P(X \leq x)$



e.g. X ~ N(0,1)

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     .         ..    ... .    :     .
-----+---------+---------+---------+---------+---------+-sample1

        .         .. :  :.        .        .
-----+---------+---------+---------+---------+---------+-sample2

    .    .    .  .    .  .      . .    .         .
-----+---------+---------+---------+---------+---------+-sample3

    .          :  .     .. .     .. .
-----+---------+---------+---------+---------+---------+-sample4

     .      .      . ....    .    .  .
-----+---------+---------+---------+---------+---------+-sample5
```

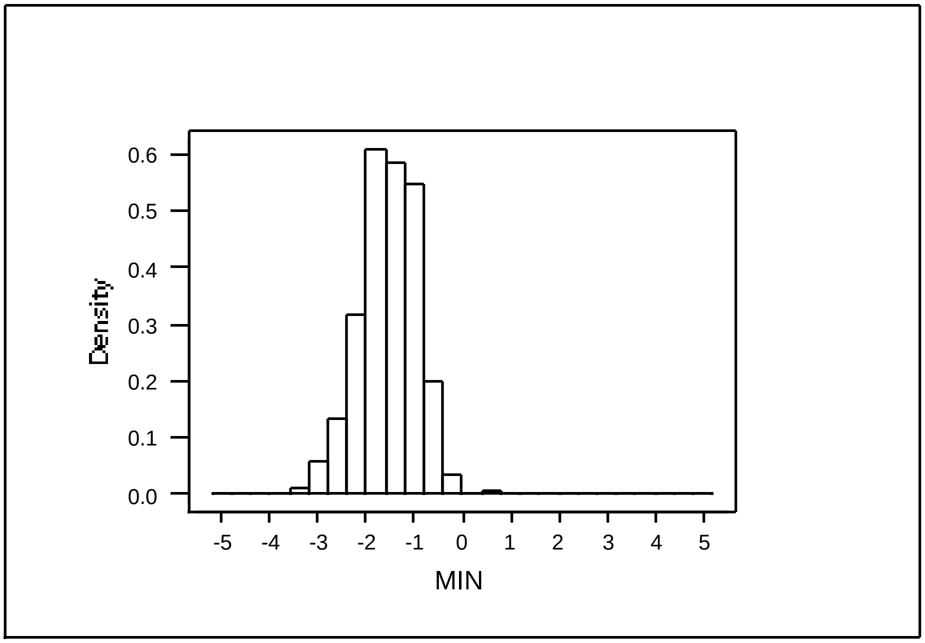
Look at the distrib'n of the minimums

```
    :. ..
-----+---------+---------+---------+---------+---------+-MIN
   -1.60     -0.80      0.00      0.80      1.60      2.40
```

We can do this for 500 samples of size 10:

```
                            :::...
                          : . ::::::
                         :: : ::::::.  : .
                        ..:::: ::::::::::::
                    :   ::::::.:::::::::::: .
                  : ..:::::::::::::::::::::::: .
           .  .  . : :.:::::::::::::::::::::::::.: ....
  ---+---------+---------+---------+---------+---------+---C11
  -3.50     -2.80     -2.10     -1.40     -0.70      0.00
```

or as a histogram:

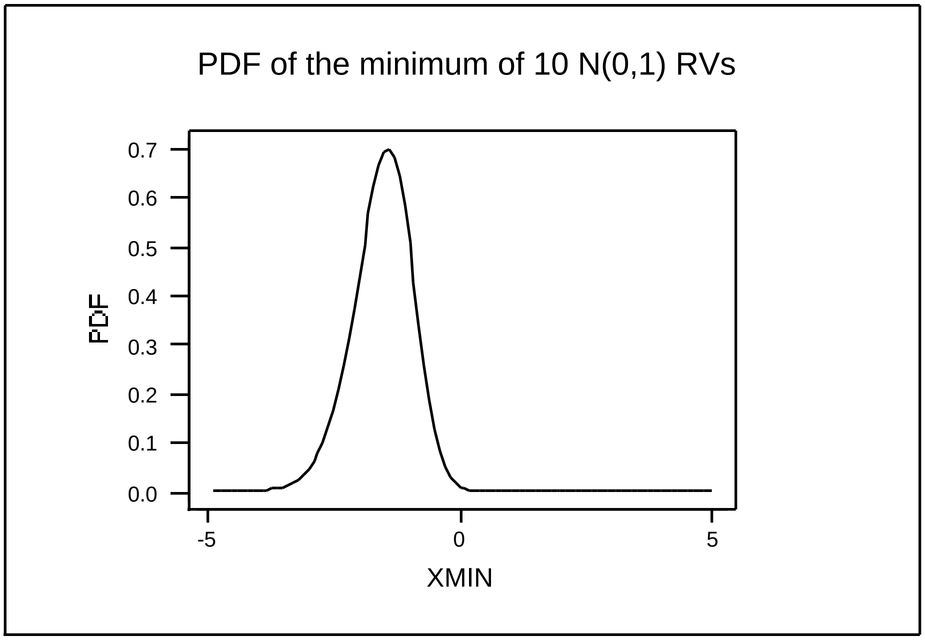


How can this distribution of the MINIMUM be computed analytically?

Let X_{MIN} = minimum of $\{X_1, X_2, X_3, \ldots, X_n\}$
$P(X_{MIN} > x) = P(\text{ all } X_i > x) = (1-F_X(x))^n$ because of independence

So, $F_{MIN}(x) = P(X_{MIN} \leq x) = 1- (1-F_X(x))^n$
and $f_{MIN}(x) = n(1-F_X(x))^{n-1} f_X(x)$

Example: Suppose X is Normal(0,1). Can't write down $F_X(x)$
but can compute it (with MINITAB), and similarly $n(1-F_X(x))^{n-1} f_X(x)$
So for n=10, and $F_X()$ cdf of N(0,1), the density of X_{MIN} is:



This is an example where we did not need to simulate the result since it was tractable.

There is a similar result for X_{MAX}.

Important special case: X exponential rate λ.

What is $n(1-F_X(x))^{n-1} f_X(x)$ when
$f_X(x) = (\lambda)e^{-\lambda x}$ and $F_X(x) = 1- e^{-\lambda x}$

$$n(1-F_X(x))^{n-1} f_X(x) = n\,(e^{-\lambda x})^{n-1}\,(\lambda)e^{-\lambda x}$$
$$= (n\lambda)\,e^{-(n\lambda)x}$$

So conclude X_{MIN} is Exponential with rate $n\lambda$

Application to Poisson processes.

Suppose Poisson Process 1 has expo interarrivals with rate λ_1 and

Poisson Process 2 has expo interarrivals with rate λ_2

Then the combined processes have interarrivals that are expo rate $(\lambda_1 + \lambda_2)$, and combined process is Poisson.

For example: Cars pass with rate 10/ minute, Trucks pass with rate 2/minute, so number of vehicles (cars or trucks) that pass is Poisson with rate 12/minute. In 5 minutes, use Poisson with mean 60.

10.1 The Reliability Function

Let T denote time to failure

$R(t) = P(T > t)$ is the reliability function of T
$= 1 - F_T(t)$

Redundant components can increase reliability.

See series and parallel redundancy p 375.

Reliability of series connection is product of reliabilities
Reliability of parallel connection - see Thm p 375

Example:

Two components in series both have reliability
$R(t) = e^{-t}$

What is reliability of circuit? e^{-2t} which is smaller.

Series circuit will tend to fail sooner than single component.

Two components in parallel: each with $R(t)=e^{-t}$

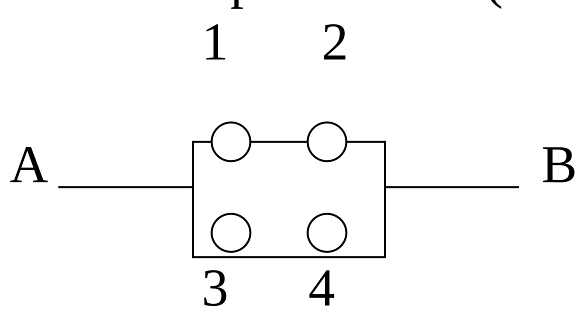
What is reliability of two-component circuit?

$1-(1-e^{-t})^2 = 2e^{-t} - e^{-2t} = e^{-t}(2-e^{-t}) > e^{-t}$ when $t>0$

Parallel redundancy is effective in improving reliability.

For example: lifetime is exponential with mean 1
R(3)=P(lifetime of circuit > 3) =
0.14 for single component
0.018 for two components in series
0.25 for two components in parallel

More complex circuit (Ex 10.1-4)



The 1,2,3,4 are rate constants for the component's lives.
Reliability of 1,2 is $e^{-(1+2)t}$ and of 3,4 is $e^{-(3+4)t}$
So, reliability of circuit is $1-(1- e^{-(1+2)t})(1- e^{-(3+4)t})$

And for any t you can work out the
P(circuit lives to age t).

10.2 Hazard Rate

Exponential duration is one ended by constant "force of mortality", ie. Each Δt has approx prob $\lambda\Delta t$ of being death interval (which is why the Bernoulli frames approach worked).

This "force of mortality" idea is defined by the hazard function: $h(t) = f_T(t)/(1-F_T(t))$

Can show this is limit as $\Delta t \to 0$ of

$P(t < T < t+\Delta t \mid T > t)$ which is the probability that

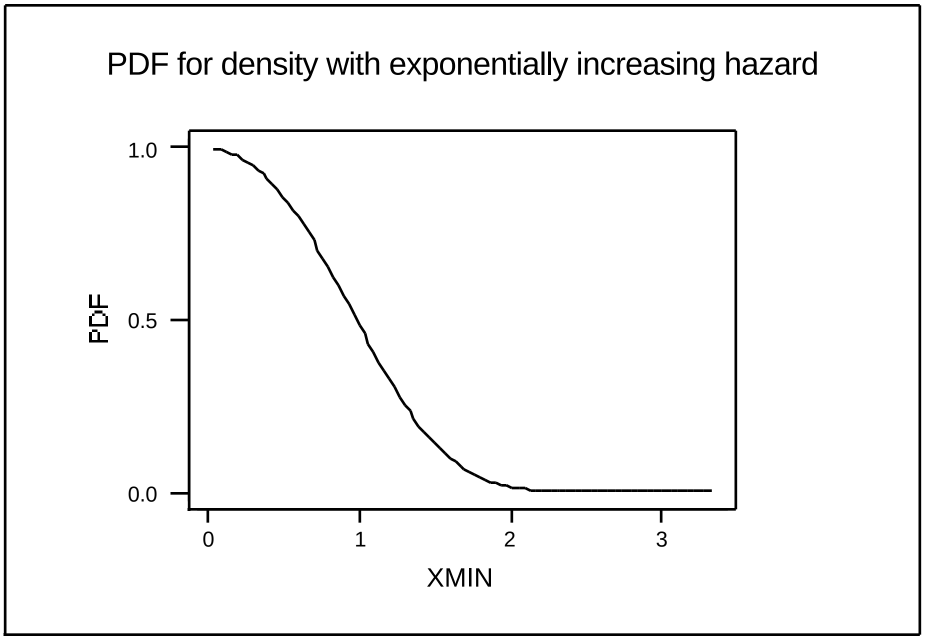
the item dies in $(t, t+\Delta t)$ given that it has survived to

age t. (See p 381)

Hazard rate for an exponential lifetime =
$\lambda e^{-\lambda t} / (1-(1- e^{-\lambda t})) = \lambda$ and does not depend on t!

Hazard rate and pdf can be computed from each other (See Thm 10.2-1).

Example of increasing hazard rate. Example 10.2-4.



cf Exponential PDF.

THE END of New Material!

Next Week – Review of Whole Course.