Stat 280 Final Examination Fall Semester 1995

Instructions: This examination is open book. Attempt all questions. Be sure to read the questions carefully, and think about your approach, before actually beginning your answer. You have 180 minutes to complete the examination. The marks for each question are noted, and total 90. This exam has 8 questions on two pages - check your copy.

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1. (12 marks)

a) During a baseball game, the batter's result is recorded as an Out, a Run, or a Walk. Experience over several games for all players in a league shows that the long run proportions of each outcome are .50, .30, and .20 respectively. In a particular game it turns out that, in 50 of the 80 "at bats", the result is "Out".
What are the mean and SD of the proportion of 80 "at bats" that would be "Out", assuming the long run proportions apply? State whatever assumptions you need to make a reasonable calculation.

Ans. (2+3 marks) Assume binomial (80;.5) for number of Outs. mean =.50
$SD=(.5*.5/80)^{1/2} = .055$

b) In 80 at bats, there were only 5 home runs in the baseball game described in part a). If the game took 4 hours to play, what particular distribution would you use to describe the time until the first home run occurred? Be sure to specify the values of any parameters in the distribution you propose.

Ans. (7 marks) Like minimum Y of 5 IID U(0,4). $P(Y>s) = (1-s/4)^5$ for $0<s<4$, =0 if $s<=0$, and =1 if $s>=4$.
Can give 4 marks if say exponential with rate parameter 1.25.

2. (8 marks) A discrete time Markov Chain has state space {0,1,2} and a one-step transition matrix P =
$$\begin{pmatrix} .2 & .3 & .5 \\ 0 & 1 & 0 \\ .5 & 0 & .5 \end{pmatrix}$$
. Also, $X_0=0$.

i) Describe the long run behavior of this chain.

Ans. (4 marks) The chain ends up in state 1 forever.

ii) Is it possible to make the chain transient by changing exactly one of the transition probabilities? Explain.

Ans. (4 marks) No. Since row sums must add to one, we can't change exactly one transition prob!

3. (8 marks) A continuous time stochastic process {X(t) t>0} takes values in the state space (0,1,2,...). When X(t)=i, the time that the process remains in i has an exponential distribution with rate parameter a, for i >0, and for i=0, the time to the next birth is exponential with rate b. When the process changes from state i to state $j \neq i$ it does so according to a certain transition matrix P.

i) Under what conditions is {X(t) t>0} a birth and death process?

(4 marks) $p_{ij} > 0$ only when j=i+-1; also, (I think 0<b<a<inf and sum over rows=1 are already implied in question, although you could give some credit for these if the expected answer is not forthcoming.).

ii) When {X(t) t>0} is a birth and death process, show how you would find its parameters in terms of a,b and P, so that you could substitute into equation (6.20) on p 275 of your text.

(4 marks) λ_v= b and μ_v=a-b

4. (12 marks) A Stochastic Process {X(n): n=0,1,2,...} takes values 0, 1, 2, 3 only. P[X(n+1)=j | X(n)=i]=1/(i+1) for j=0,1,..,i as long as i > 0, and P(X(n+1)=3 | X(n)=0) =1. Estimate the number of occasions on which the process visits state 3, over the times (0,1,2,...1000)? (Just a single number is sought here, although you should show your work as usual.).

MC with trans matrix

$$\begin{matrix} 0 & 0 & 0 & 1 \\ .5 & .5 & 0 & 0 \\ .33 & .33 & .33 & 0 \\ .25 & .25 & .25 & .25 \end{matrix}$$

Limiting Dist is x0,x1,x2,x3 with

$$\begin{aligned} x0 &= \quad .5x1+(1/3)x2+.25x3 \\ x1 &= \quad .5x1+(1/3)x2+.25x3 \\ x2 &= \quad (1/3)x2+.25x3 \\ x3 &= x0 + \quad +.25x3 \end{aligned}$$

solve x0=(3/4)x3 ; x2=(3/8)x3 ; x1=(2/3)x2+(1/2)x3=(3/4)x3
so x0+x1+x2+x3 = 1 becomes x3=(8/23) , x0=(6/23), x1=(6/23), x2=(3/23)
(Of course one only really needs x3). So the estimate is (8/23)*1000 or about 348 times.

5. (18 marks) Boating accidents occur according to a Poisson Point process at the rate of 2 per day during the month of August. Twenty-five percent of these accidents involve power boats, while seventy-five percent are with non-power boats.

i) What is the chance that the first power boat accident in August is reported before the third non-power boat accident?

Ans i) use p 222 with n=1, m=3, λ_1 =0.5 and λ_2=1.5.

$3C1\ (.25)(.75)^2 + 3C2\ (.25)^2(.75) + 3C3\ (.25)^3(.75)^0 = (.25)^3(9+9+1) = 19/64$

ii) Find an approximate probability that the tenth accident occurred before 3 full days have elapsed.

Ans ii) time to tenth is gamma(10,2) and will be approx normal(5,2.5) from p 63.
So P(z<(3-5)/2.5) = P(z<.8) = .2 approx (actually .212). They don't have tables necessarily, but should know P(z<.8) approx!

iii) Write an expression for the exact probability that the tenth accident occurs before 3 full days have elapsed.

$P(S_{10}<=3)=P(N(3)>=10)$ = sum (i=10 to inf) $P(N(3)=i)$ where $P(N(3)=i)$ is Poisson mean 6. (They should actually subst the Poisson formula.)

6. (10 marks)

i) A Poisson random variable can always be interpreted as a particular random variable X(t) occurring in a Poisson Process, {X(t): t>0}. Explain.

Ans i) A Poisson mean m can be thought of as a count in a PP at time t in which the rate parameter is m/t.

ii) Use i) to explain why the sum of Poisson random variables must be Poisson, even if the rate parameters of the component Poisson Process, are not equal.

Ans ii) A Poisson mean m1 and a Poisson mean m2 must be Poisson mean m1+m2 because we can think of it being generated by over two periods in which the rate is a constant λ, but the first period is (0,t1) where $t1=m1/\lambda$, and the second period is $t2= m2/\lambda$, so the total count would be Poisson over t1+t2.

7. (10 marks)

A Markov Chain has state space {A,B,N} and transition matrix P = (.2 .3 .5).

$$P = \begin{pmatrix} .2 & .3 & .5 \\ .0 & .2 & .8 \\ .2 & .4 & .4 \end{pmatrix}$$

i) If the Chain has been operating for a long time, what is the chance that the next three letters in the chain are BAN?

Ans. i) The chance is 0, since $p_{BA}=0$

ii) Re-do part i) for the word NAB?

we need the long run prob, at least for N.
$x_a=.2x_a+.2x_n$
$x_b=.3x_a+.2x_b+.4x_n$ and $x_a+x_b+x_n=1$ gives $x_n=6.4/11.8 = .54$
and P(NAB) = .54*.2*.3=.32

8. (12 marks)

A coin having probability p of coming up heads is successively flipped until 2 of the most recent 3 flips are heads. Let N denote the number of flips.

i) Find an expression for E(N) that can be solved using simple algebra. Hint: You will need to define a new random variable on which you can condition.
Let N* be the number of flips til first head.
$E(N|N^*)=(N^*+1)p + (N^*+2)(p)(1-p) + (N^*+2+N)(1-p)^2$

where the first term is when the first head is followed by a head, the second term is when it is followed by a tail and then a head, and the last term is when it is followed by two tails.

When two tails have occurred, it is just like starting over, since it is impossible to have success until at least two more heads are flipped.

Taking expectations, one gets a recursion for E(N) in terms of p, since E(N*)=1/p. Simplification is unnecessary for this part.

E(N) can be expressed in terms of E(N*)

ii) Solve the expression for E(N), and check whether your answer is reasonable or not.

The answer is, $E(N)=[E(N^*)+(2-p)]/[2p-p^2] = [1/p+(2-p)]/[2p-p^2]$ = 4 and 2/3 for p=1/2
=inf when p=0
=2 when p=1
all which seem right, or about right in case p=1/2.

(KLW 95/12/06)