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# Contingency in the Mathematics Classroom: Opportunities Taken and Opportunities Missed

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# Contingency in the Mathematics Classroom: Opportunities Taken and Opportunities Missed

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**Abstract:** We describe and analyze three episodes from mathematics classrooms. In each case, the teacher was confronted by a "contingent" situation that they had not anticipated or planned for yet that offered interesting and fruitful learning possibilities if pursued. In two cases, we analyze the teacher's response; in the third, we speculate how they might have responded. In each case, we propose that the teacher's ability to capitalize on these contingent situations is underpinned by their knowledge and awareness of the mathematical potential of the unexpected opportunity and by an interest in, and commitment to, mathematical enquiry.

**Résumé:** Nous présentons une description et une analyse de trois situations provenant de cours de mathématiques. Dans chacun des cas, l'enseignant a dû faire face à un événement « imprévu », donc une situation qui ne faisait pas partie du programme de la leçon, mais qui offrait des possibilités intéressantes à exploiter pour l'apprentissage. Pour deux de ces cas, nous analysons la réaction de l'enseignant, et pour le troisième cas nous imaginons comment l'enseignant aurait pu réagir. Dans les trois cas, nous estimons que sa capacité de tirer profit de telles situations dépend de son habileté à reconnaître le potentiel mathématique des occasions imprévues, et de son intérêt et de sa curiosité pour l'investigation mathématique.

## PROLOGUE

Sirotic and Zazkis (2007) reported the following exchange between a teacher and a student in a Grade 9 mathematics classroom:

Student: Is $\pi$ the only irrational number?
Teacher: No, remember, there is also $\sqrt{2}$.

Though the particular context of this exchange was not described, this issue might arise when students make various calculations related to areas or circumferences of circles or surface areas and volumes of spheres. The number $\pi$ appears in these calculations but can be entered in more than one way on a calculator. When two students use the same formulas and seemingly the same

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numbers but one enters "$\pi$" into a calculator and another uses an approximation such as 3.14, they get different answers. At this point they approach the teacher, demonstrating discrepancy and confusion. The teacher explains to the class that $\pi$ is irrational, that it has infinitely many digits in its decimal representation, and somehow several of these digits, more than can be displayed, are stored in the memory of some calculators. However, to avoid the discrepancy, and because not all calculators are sophisticated enough to have a "$\pi$" key, she suggests that the class use 3.14 in the subsequent calculations. This may have invoked the reported exchange.

Considering what was said from the perspective of logical correctness, the teacher's answer is appropriate. To show that some object is not unique, it is sufficient to identify another one of the same kind. However, this teacher's answer misses a valuable opportunity to extend the conversation about irrational numbers and their counterintuitive-at-first abundance in relation to the rational numbers. Of course, the teacher may have had a full agenda of tasks for the students, to be completed toward a test or toward meeting some prescribed learning outcome and therefore decided to postpone the discussion of irrationality to a later date. Another possibility is that this mathematics teacher was not herself aware that, in a specific sense, the irrational numbers are actually more numerous than the rationals. Such knowledge, of various alephs and of different infinities, is typically acquired in the study of Cantorian set theory. If the teacher *had* possessed the relevant content knowledge, how might she have responded to the unexpected situation?



## ON MATHEMATICAL KNOWLEDGE AND CONTINGENCY

Teachers' knowledge and its role in teaching and in teaching development have been the focus of a variety of recent studies (e.g., Adler & Ball, 2009; Leikin & Zazkis, 2010; Rowland & Ruthven, 2011). There have been attempts to describe various components or layers of such knowledge (e.g., Ball, Thames, & Phelps, 2008; Hill, Ball, & Schilling, 2008). However, it has been suggested that this kind of labeling diverts attention from the mathematical integrity of mathematics teaching (Watson, 2008).

Of course, teaching is unimaginable without teachers' knowledge of what is taught. However, the extent and the nature of such knowledge are sometimes questioned (Ruthven, 2011; Zazkis & Leikin, 2010). We suggest that one's stance regarding the mathematical knowledge needed (or essential) for teaching depends on one's perception of teaching itself. If teaching involved only attending to prescribed scenarios and delivering a predetermined curriculum, then it is likely that knowing that curriculum would suffice. However, teaching also involves attending to students' questions, anticipating some difficulties and dealing with unexpected ones, taking advantage of opportunities, making connections, and extending students' horizons beyond the immediate tasks. In short, teaching involves dealing with unpredictable, *contingent* events in the classroom. With this perspective on teaching, mathematical knowledge beyond the immediate curricular prescription is beneficial and demonstrably essential.

We interpret the notion of contingency in terms of one of four components of the Knowledge Quartet (KQ) described by Rowland, Huckstep, and Thwaites (2005). The KQ provides a way of organizing the situations in which mathematics teachers' knowledge "plays out" in the practice of teaching. Specifically, this situational knowledge of mathematics in teaching is organized into four dimensions: foundation, transformation, connection, and contingency. The last of these, contingency, is witnessed in teachers' responses to classroom events that were not anticipated or planned, usually triggered by an answer or a remark contributed by a student. Chick and Stacey (2013) also frame their article with reference to the contingency dimension of the KQ. This aspect of mathematics teacher knowledge is related to what Mason described as "knowing-to act in the moment" (e.g., Mason & Johnston-Wilder, 2004; Mason & Spence, 1999) or the ability to think on one's feet (Schon, 1987). McNair (1978–1979) likened the teacher's reactions to classroom indicators to the pilot's responses to what they see on their instrument panels and the clinician's therapeutic prescription in response to symptoms and readings. However, she commented that in the case of the teacher, "decision making becomes more complex in the sense that it is less prescribed and more dependent on the judgement of the individual" (McNair, 1978–1979, p. 26). Lampert and Ball (1999, p. 39) advised that "teachers be prepared for the unpredictable," because they will have to "figure out what is right practice in the situation" and cannot entirely depend on experts' advice on what to do.

Contingency has been considered in the context of a mathematics teacher development program by Remillard and Geist (2002), who researched the piloting of a curriculum intended for use with elementary teachers in an inquiry-group setting. The mathematics educator facilitators experienced some unanticipated, at times confrontational, reactions from the teacher participants. These unexpected participant contributions, which Remillard and Geist (2002, p. 7) called "openings in the curriculum," required the facilitators to make contingent decisions about how best to respond. Our focus is on unpredicted events of a mathematical kind. Of course, unplanned events are sometimes provoked by student behavior and other managerial issues, but these lie beyond the scope of our current mathematical focus.

Watson and Barton (2011) provided insightful examples of how teachers' mathematical knowledge informs their planning and preparing for instruction. In particular, they exemplified how mathematical modes of enquiry are enacted in planning for instruction, focusing on the potential use of particular resources. Here we attend to the "opposite" of planning—to situations that are not planned and that have the potential to take a teacher outside of their planned route through the lesson. Dealing with these situations invites deviation from a planned agenda and an act of improvisation. We contend that in the same way that experience and knowledge enable skilfull jazz improvisers in their musical work, profound mathematical knowledge underpins and informs mathematical modes of enquiry and thereby helps a teacher take advantage of emergent, contingent situations in the classroom.

Viewing student learning in mathematics classrooms from an improvisational perspective, Martin and Towers (2009, p. 1) have explored the way in which students' collective mathematical understanding constantly changes and grows as learners work together. They described how the growth of collective mathematical understanding can occur through a mechanism of coordination and interaction, which they called "improvisational coaction." In this article, we consider the actual and potential contribution of the teacher to this learning process.

## THREE STORIES

In what follows, we present three stories that exemplify such contingent situations. We shall describe what *did* happen in the classrooms, and sometimes we will speculate about what *could have* happened. In each case, like the earlier irrational number exchange, a contingent opportunity arises in the classroom on account of something said by a student. Having described these three scenarios, we proceed to consider actual or potential mathematical responses to each of them. Our focus is on the specific *mathematical* knowledge on the part of the teacher that enables these responses. We also bring into focus the more general mathematical disposition that might provoke these particular mathematical responses.

## Sequence of Numbers: Laura's Story 1

Using the ideas of Brown and Walter (1983), Laura engaged her students—prospective elementary school teachers—in a problem posing exercise. She presented the following arithmetic sequence and then asked the students to make an *observation* and to ask a *question* related to it.

2, 9, 16, 23, 30, 37, 44, . . .

She demonstrated an example of her expectations for the task as follows:

> Observation: I see that there is a number in the sequence that ends in zero.
> Question: I wonder if there are other such numbers.

After a variety of observations and questions were generated, Laura asked students to answer some of their questions and try to prove their answers. Most observations related to the patterns in digits. For example, students noticed that even and odd numbers alternated. It was confirmed and proved—by considering sums of two odd and of even and odd numbers—that this pattern will continue indefinitely. Further, the repeating pattern of last digits was observed, confirmed, and proved.

Laura had used this task several times previously with similar groups of prospective elementary school teachers and was familiar with the "observations space" that they usually generate. Therefore, the following observation and question appeared somehow unexpected.

> There are two square numbers, 9 and 16, one after the other.
> Does this happen again, that is, are there other consecutive square numbers?

Laura was familiar with the fact that the sequence of differences of consecutive square numbers is the increasing arithmetic sequence of odd numbers. She directed students to the exploration of differences of consecutive square numbers, which suggested that the difference of 7—the constant difference in the given sequence—cannot reappear, and guided students toward the negative answer to the initial question. However, Laura felt that the student's observation could be rephrased and lead to a more interesting question. As such, "consecutive square numbers" can be seen as squares of consecutive integers, rather than consecutive terms in the sequence. So the question was modified to be "Are there another two terms in this sequence that are consecutive square numbers?" This question engaged students in purposeful exploration.

First, by extending the sequence, students found that 100 and 121, which are $10^2$ and $11^2$, were elements in this sequence. At this point, students were asked to predict what other square numbers may appear in the sequence. It was noted that $10 = 3 + 7$, and $11 = 4 + 7$. So, since $3^2$, $4^2$, $10^2$, $11^2$ were elements in the sequence, it was predicted that that $17^2$ and $18^2$ were also in the sequence. Laura asked whether it was possible to confirm this prediction without extending the sequence. It is significant to note that the formula for the general element in the arithmetic sequence was not in the active repertoire of this group of students. Laura's purpose was to guide students toward exploring patterns, rather than introducing formulas to the conversation. It was agreed that subtracting 2 from any number in the sequence leads to a multiple of 7, and this property should hold for all of the elements because they are generated by adding 7. Following this rule, or test, for sequence membership, it was confirmed that 289 ($17^2$) and 324 ($18^2$) were indeed elements in the sequence, because 287 and 322 are multiples of 7.

Laura invited students to check for additional "suspects"; that is, consecutive integers (like 10, 11) whose squares both appear in the original sequence 2, 9, 16, 23, 30, 37, 44, ... Then it was time to generalize these ideas. The suspects for consecutive-square membership were always of the form $(3 + 7n, 4 + 7n)$, for $n = 1, 2, 3, \ldots$ So the task was to show that subtracting 2 from $(3 + 7n)^2$ or from $(4 + 7n)^2$ results in a multiple of 7. The following manipulation confirmed the conjecture:

$$(3 + 7n)^2 = 9 + 42n + 49n^2 = 2 + (7 + 42n + 49n^2) = 2 + (\text{Multiple of } 7)$$

$$(4 + 7n)^2 = 16 + 56n + 49n^2 = 2 + (14 + 56n + 49n^2) = 2 + (\text{Multiple of } 7)$$



## DISHWASHERS' GAME: LAURA'S STORY 2

During some problem-solving activities, a class of elementary school teachers in a professional development setting was engaged in the following problem:

> Consider the following game that a wife, who doesn't like washing dishes, suggests to her husband, who also tries to avoid the chore. (Still, dishes have to be washed).
> "There are 2 green marbles and 1 red marble in a box. If you pick both green, I will wash the dishes, if you pick one green and one red, you do the dishes."
> Is the game fair?

Initially, the game was presented to students to provoke discussion of equiprobability; that is, the existence of two options (same color, different color) does not entail that they have the same chances of occurrences. The students found, first by trial and error and then by considering all possibilities, that the game proposed by the wife was unfair. If the marbles are labeled as G1, G2, and R, then the possible ways of picking two marbles are (G1, G2), (G1, R), and (G2, R). As such, the chances of picking different colored marbles are twice those of picking two of the same color.

Once this conclusion was reached, Laura asked, "What should be in the box to make the game fair, when picking two marbles?" After a short and humorous discussion of "what is fair" in the kitchen, it was agreed—but just for the purpose of this problem—that "fair" means that both partners will have equal chances of getting the chore; that is, the probability of picking two marbles of different colors must be equal to that of picking two of the same color (both red or both green).

The immediate first try was to work with two green and two red marbles, which, to the surprise of many, also results in unfair game. A composition of marbles that does result in a fair game—three green and one red—was found by trial and error. In this case, the possible combinations of two marbles are (G1, G2), (G1, G3), (G2, G3), (G1, R), (G2, R), (G3, R). As can be seen from this list, in three cases both marbles are green and in three cases there is one green and one red marble, so the probability of getting the same color is the same as the probability of getting different colors.

As Laura was about to conclude this activity, one student asked, “Are there other options for a fair game?” The teacher had not previously explored the question by herself, but she was able to evaluate the mathematics involved in the task and guide students toward a general solution. She scribbled something for herself on a small piece of paper and concluded that in order to solve the general problems the students had to be familiar with the “choose” function (that is, with a way to determine the number of combinations when choosing $k$ objects out of $n$, often read as *n-choose-k* and denoted $\binom{n}{k}$) and with triangular numbers. Luckily, the students were familiar with the choose function from their previous engagement with several combinatorial tasks. (We note in passing that reference to the choose function is not essential for the solution, but it is appropriate to use it given students’ prior experience with it.)

The next class activity engaged students in an exploration of square numbers and triangular numbers. Among a variety of patterns and relationships, the students noted that any square number is a sum of two consecutive triangular numbers, which can “proved without words,” shown in Figure 1.

It was further noted and emphasized that the “side” of the square (that is, the number squared) is the difference of the two consecutive triangular numbers. For example, considering the sequence



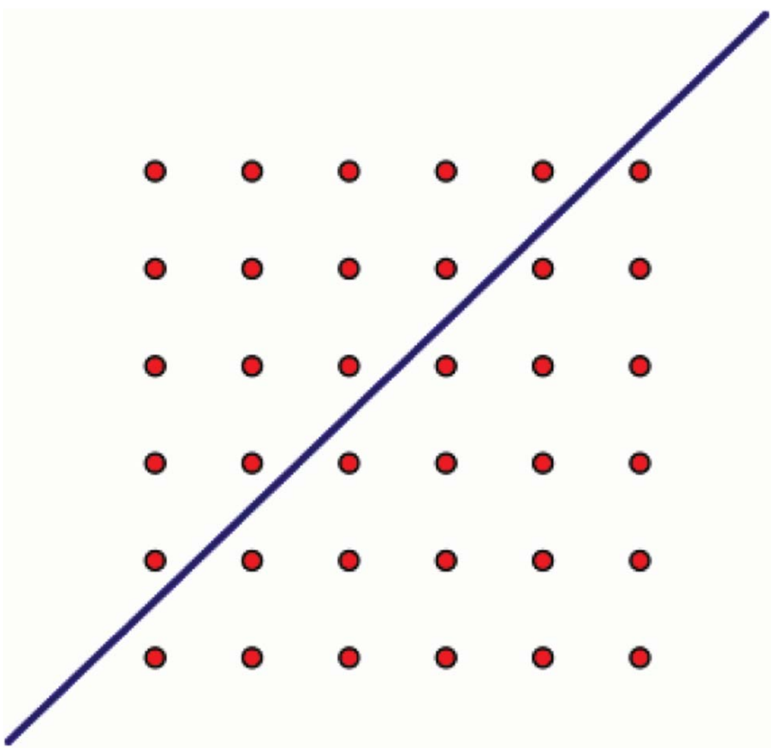

FIGURE 1 A Square Number is the Sum of Consecutive Triangular Numbers: Visual Illustration. (color figure available online)

of triangular numbers 1, 3, 6, 10, 15, 21, 28, ...

$$3 + 6 = 9 = (6 - 3)^2$$
$$6 + 10 = 16 = (10 - 6)^2$$
$$10 + 15 = 25 = (15 - 10)^2$$

etc.

After this seemingly unrelated activity, the class returned to the consideration of the generalized dishwashers game. After considerable prompting, it was concluded that if there are $g$ green and $r$ red marbles, the number of options for choosing two marbles of the same color is

$$\binom{r}{2} + \binom{g}{2}$$

and the number of options choosing two of different colors is $r \times g$.

For the game to be fair, these two expressions have to be equal:

$$\binom{r}{2} + \binom{g}{2} = r \times g$$

Therefore,

$$\frac{r!}{2!(r-2)!} + \frac{g!}{2!(g-2)!} = r \times g$$
$$r(r-1) + g(g-1) = 2rg$$
$$r^2 - r + g^2 - g = 2rg$$
$$r^2 - 2rg + g^2 = r + g$$
$$(r-g)^2 = r + g$$

This equation can be seen as a square number represented as a sum of two numbers. That is, the equality holds if $r$ and $g$ stand for consecutive triangular numbers, the sum of which is a square number.

## Fraction in Between Fractions: Alan Bishop's Story

Our third story, unlike the first two, is borrowed from the literature. On more than one occasion, Alan Bishop has recounted an event that clearly stayed with him for many years. The original account is in Bishop (1976). In one version (Bishop, Clarkson, Fitzsimons, & Seah, 2000), he writes as follows:

> This happened to me many years ago, and I remember it well. You are studying fractions with a lively class of 12 year old students, and you ask them to suggest a fraction that lies between one half and three-quarters. One particularly eager student offers the answer "two thirds". When you ask how she knows that it lies between the other two fractions, she answers: "Well you can see that on the top the numbers go 1, 2, 3 and on the bottom they go 2, 3, 4. On the top, the 2 is between the 1 and the 3, and on the bottom, the 3 lies between the 2 and the 4, so therefore two thirds must be between the other two fractions!"

The focus of the 1976 paper was on teachers' decision making, and Bishop only told us about various techniques that he used in "buying time" to think through some substantive and classroom management decisions. Bishop's purpose in recent versions of the story (e.g., Bishop, 2001) is to suggest that the teacher's response to the girl will be indicative of the values of that teacher. We have invited teachers and prospective teachers in several countries to consider and respond to the story, emphasizing mathematical rather than more generic features of the scenario. We shall consider their responses, and our own, later.

## RESPONDING TO CONTINGENCY MATHEMATICALLY

We have described three stories from mathematics classrooms—not invented stories but accounts of actual events that occurred in the course of the professional lives of the teachers involved in interaction with their students. Our first criterion for selecting these scenarios for scrutiny was their contingent nature: in each case, something occurred in the lesson that fell outside the teacher's "lesson image" (Schoenfeld, 1998) when it was planned. As we remarked earlier, this placed the teacher in the position of having to improvise, to act spontaneously, to have to think on their feet. Rowland, Thwaites, and Jared (2011) found that the great majority of contingent moments are triggered by unexpected contributions by students. Furthermore, the teacher's response to unexpected ideas and suggestions from students is one of three kinds: to ignore, to acknowledge but put aside, and to acknowledge and incorporate. There are sometimes good reasons for choosing the second of these options, and perhaps even the first, but we find it interesting, both mathematically and pedagogically, to explore possibilities of the third kind.

Again, we chose these three scenarios because of the realized or potential mathematical possibilities that we perceive in them ourselves. What we want to unfold here are the developments that a "mathematical disposition" in the teacher made (or could make) available that would not, or probably would not, come to light otherwise. Over and above the particularities of these three scenarios, an overtly mathematical (as opposed to generalist managerial) response by the teacher enables the students to witness what mathematicians do, how they operate and behave, how sometimes they get stuck, and what delight and frustration look like in a mathematician. There is the possibility that they will experience being drawn into the mathematician's world as a collaborator or a legitimate peripheral participant (Lave & Wenger, 1991).



## REFLECTING ON LAURA'S STORIES

Each of these stories exemplifies mathematical ways of being, or modes of enquiry, in different ways. The first story involving Laura—about the sequence of numbers—makes the point that this way of being, and acting, is not necessarily associated with advanced mathematical content (if, by *advanced* we mean university-level, or even postcompulsory). In fact, Laura was sensitive to the active repertoire of her students and directed the enquiry accordingly. This kind of sensitivity is not overtly mathematical and might be better related to Shulman's (1986) notion of curricular knowledge. In any case, she had extensive experience of working with groups of preservice elementary teachers. Our account of this story notes that Laura had used this task on previous occasions with similar groups and that she was familiar with their likely observations space. In that sense, she believed herself to be operating in familiar territory—she expected no surprises. In a description of his "theory of teaching-in-context," Alan Schoenfeld wrote:

> The teacher's lesson image includes knowledge of his or her students and how they may react to parts of the planned lesson; . . . I can tell you, before the class starts, how things are likely to unfold . . . there are many branch points and contingencies. However, I know what most of them are likely to be. And, there are few surprises. (1998, pp. 17–18)

In this case, Laura *was* surprised by an unexpected observation, and a related question, from a student. Laura's thinking about 2, 9, 16, 23, . . . was no doubt significantly framed by her mathematical recognition that this is an arithmetic sequence. The student, on the other hand, seemed less fettered by such conventional thinking and was open to "seeing" anything of interest in the numbers before her, such as 9 and 16 side by side. (Does this happen again? That is, are there other consecutive square numbers?)

Laura knew about differences between consecutive square numbers and that therefore a simple answer to the student's question could be "No: these are the only consecutive square numbers in this sequence." In fact, she responded by initiating an investigation of differences between consecutive square numbers, by which the students were enabled to come the same negative conclusion themselves. This response fell safely within the compass of Laura's substantive mathematics content knowledge (Schwab, 1978) and enriched the students' learning from the sequence activity. But, in the moment, Laura also opted to improvise, to engage in enquiry *with* the students, by means of a small modification to the question or perhaps an alternative interpretation. The implicit question, "Are there two consecutive terms of the sequence that are square numbers?" becomes "Are there two terms of the sequence that are consecutive square numbers?" In doing so, Laura calls upon her *syntactic* content knowledge (Schwab, 1978); that is, her knowledge and understanding of mathematical modes of enquiry. This is evidenced first in problem posing via reinterpreting the student's question (Brown & Walter, 1983)—stressing *consecutive squares* while ignoring *consecutive terms* in the given arithmetic sequence—and then in problem solving. How would one begin to answer such a question? Not by consulting a reference book, or the worldwide web, but by engaging with the evidence—the data. First, are there squares other than 9, 16, in the sequence? The students search the data and find 100 and also 121. In a sense, the alternative question is now answered in the affirmative, but Laura takes the enquiry to another level: generalization. The students could have searched for another pair of consecutive squares, though this could be laborious if 289, 324 fell outside the students' "square repertoire," as Laura may have suspected. She suggested that they stand back and *predict*, with a shift of attention from data to structure. This led to the prediction that $17^2$ and $18^2$ will be the next pair. The story does not record the students' affective response to the confirmation of their prediction, but in such moments students' perceptions of what mathematics is, and what it could be for them and—because these students are also teachers—for *their* students, can change radically.

Rather too often, this is where this enquiry would end: if we keep adding 7 to each member of the pair (so that $24^2$, $25^2$ is next), then these also appear as terms in the sequence. This inductive reasoning is vital to mathematical creativity, but the mathematics remains incomplete. In the U.K., mathematical proof effectively became a casualty of a well-intended mathematical investigation component to school mathematics, stressing induction but neglecting deduction, in the 1980s and 1990s (Hoyles, 1997). Laura knows that mathematics differs from science in its epistemological foundation and carefully supports the students' algebraic representation of the sequence and manipulation of the squares of $(3 + 7n)$, $(4 + 7n)$. Her substantive knowledge of modular arithmetic may well have assured her in advance that the squares of both would be congruent to 2 modulo 7; her knowledge of her students informs her that she must find another way to arrive at the same result.

The second story involving Laura, the dishwashers and a fair game, has some similarities with the first and some differences from it. The mathematical content was somewhat more advanced, in terms of school curricula, and probabilistic reasoning is known to pose significant obstacles to students and to bring to light a number of misconceptions (e.g., Kahneman, Slovic, & Tversky, 1982). Laura presented the problem, the game, in such a way as to bring one such misconception into the open. Arguably, her pedagogical content knowledge played a significant part in that decision. The students investigated the game empirically, and this served to provoke doubt in their earlier assumption of fairness. We are not told whether Laura suggested listing all the possible ways of choosing two marbles or whether one or more of the students proposed doing so. This listing is not without complication, of course, and underpinning it is the notion that a sample space with uniform probability distribution can be identified. Once this was achieved, the outcome space {same-color, different-color} was seen to be biased. In a sense, the dishwasher game is completed at this point, as a mathematics didactic task, and has hopefully achieved its purpose of challenging a common probabilistic reasoning heuristic. But Laura takes it one stage further by asking how the game could be made fair. As with the sequence activity, Laura is comfortable with mathematical modes of enquiry and committed to initiating her students into these modes. There was a slight risk that a student might propose a solution that she did not already know, but we see later that Laura has good command of the substantive content and would probably be happy to consider the correctness of any "fairness" solution proposed. The trial and error leads to the three green, one red solution by varying the numbers of green and red marbles and listing the sample space in each case to identify the relative likelihood of the two events (same/different color). The unexpectedness of the solution—certainly not their first choice—might be expected to provoke the positive affective response to which we referred earlier.

Jumping ahead to the general solution, we can attest to our own sense of delight in a result that was unknown to us previously and an unexpected application of a connection between triangular and square numbers that we knew previously only as a "pure" result. The story suggests that the impetus for this generalization came from the student who asked about other options. We infer that Laura was not expecting this contingent enquiry and that she did not have an answer to it ready-made. We do not know how she responded to the student in the first instance, but Laura finds a space in the lesson real-time to think the problem through for herself, as she "scribbled" on the piece of paper. Perhaps the students are busy with a task that can proceed unsupervised while Laura makes some time to consider both the mathematical and, presumably, teaching situation.

We recognize that contingent situations occur—especially with young children—when the teacher *cannot* suspend the class supervision role and the quality thinking time must be delayed to an opportunity in the staffroom, at home, with a friend/partner, perhaps in a sleepless night. In any case, the teacher's challenge is typically in two stages: knowing for oneself first and then making the finding available to the particular student class. The first of these draws upon the teacher's own content knowledge, perhaps advanced, perhaps not. The second is equally challenging and draws upon the teacher's pedagogical content knowledge and ingenuity, crucially on what these students will find accessible, meaningful, and helpful.

Laura's enquiry orientation directs her to the algebraic investigation of the generalized problem, leading to the equation $(r - g)^2 = r + g$. (We shall refer to this equation as *.) Her substantive content knowledge prompts the realization that consecutive triangular $g$, $r$, numbers would "fit" this equation. Our own interest was aroused sufficiently to want to know whether there could be other solutions; that is, is it the case that the equality (*) holds *if and only if* $g$ and $r$ are consecutive triangular numbers? Well, (*) states that a square number $m^2$, say, is the sum of two integers $a$, $b$, (say), where $m$ itself is the difference of $a$, $b$. Symbolically:

$$m^2 = a + b$$

$$m = b - a$$

The sum and difference of these two equations give $a = \frac{1}{2}\ (m^2 - m)$, $b = \frac{1}{2}\ (m^2 + m)$: two consecutive triangular numbers, and so these are indeed the only solutions.

Because the mathematical content in the dishwasher story is somewhat more advanced than that in the sequence story, it may be that Laura had to give the students a firmer steer toward these insights than she did in the search for consecutive squares earlier. But the significance of the story is that she was interested in the student's question ("Are there other solutions?"), was disposed to investigate it herself, and was able to recognize how the solutions connected with a rich and interesting topic—triangular numbers—that was part of her repertoire and very relevant to the prospective elementary teachers.



## REFLECTING ON ALAN BISHOP'S STORY

As we commented earlier, we have offered this "fraction in between fractions" story to a number of audiences, asking how they would respond to the girl. For example, some prospective secondary teachers agreed that the girl is correct, seemingly focusing more on her identification of $\frac{2}{3}$ as an answer and less on the reasoning by which she arrived at it. In contrast, some master's students at a different university were unimpressed by the correct answer, saying that they would tell the girl that this was not the correct way to solve this problem and would remind her about finding the arithmetic mean. Apart from the values issue that Bishop brings to more recent accounts (e.g., Bishop, 2001), either response would close the door on some interesting mathematical opportunities. When (as is more typically the case) audiences grasp these opportunities, they invariably observe that there is some ambiguity about the general principle underpinning the girl's reasoning. Because the principle is tacit, and because we cannot (now) ask her, we can only speculate on the possibilities, drawing on the suggestions offered to us.

She said, "On the top the numbers go 1, 2, 3 and on the bottom they go 2, 3, 4." One possibility is that her tacit conjecture C1 is this:

> (C1): Whenever the numerators of three fractions are *consecutive integers*, and the denominators likewise, the second fraction will be between the other two.

A more general version of this (though there is no evidence to suggest that she intended it) might be one of the following:

> (C2a): Whenever the numerators of three fractions are in *arithmetic progression*, and the denominators likewise, the second fraction will be between the other two.

(C2b): Whenever the second numerator is the *arithmetic mean* of the first and third, and the denominators likewise, the second fraction will be between the other two.

The two versions of C2 are equivalent mathematically but suggest different perceptions of what the girl "sees."

A third version, and perhaps the most neutral interpretation of the student's words, "The 2 is between the 1 and the 3, and on the bottom, the 3 lies between the 2 and the 4," can be stated:

(C3): Given three fractions, if the numerator of the second is *greater than that of the first and less than that of the third*, and the denominators likewise, the second fraction will be between the other two.



Our own response to the third of these three conjectures differs from the first two, in the following sense. Advanced mathematical knowledge (AMK; Zazkis & Mamolo, 2011) of continued fractions gives us assurance that C1 and C2 are true (the first is a special case of the second, of course). This substantive content is typically included in a second undergraduate course in the theory of numbers, in connection with continued fractions and Diophantine approximation; that is, rational approximation to irrational numbers. If $\frac{a}{b}$ and $\frac{c}{d}$ are consecutive convergents of a continued fraction (the $(n-1)$th and $n$th, say) and $k$ is the $(n+1)$th term of the same continued fraction, then the $(n+1)$th convergent is $\frac{ak+c}{bk+d}$ and lies between the previous two convergents. This "lies between" property can be proved algebraically, but an attractive geometric proof can be developed from the representation of $\frac{a}{b}$ by a line in the Cartesian plane with the same gradient, specifically the line through the origin and the point $(b, a)$ (e.g., Stark, 1970). However, this approach was *not* adopted in most of the number theory texts in an opportunity sample that we consulted:[1] it is perhaps significant that, according to the preface, Stark's (1970) book is aimed at prospective teachers.

The result C2 (and C1 as a special case) follows when $k = 1$. For an audience of teachers, or prospective teachers, this can be of particular interest because of the well-documented fraction addition error "adding numerators and adding denominators" (e.g., Lankford, 1972). This is readily seen to be equivalent to C2b, because the arithmetic mean of the numerators is half their sum, and likewise the denominators. It is salutary to realize that this common error produces a fraction between, and therefore not greater than, the addends.

There are also connections with Farey series: in fact, $\frac{a+c}{b+d}$ is the mediant, or "Farey mean," of $\frac{a}{b}$ and $\frac{c}{d}$. Our point here is that AMK gives access to mathematical results and connections that prepare the teacher for a response to conjectures C1, C2. This is not quite the same as Schoenfeld's (1998, p. 18) somewhat battle-weary remark that "there are few surprises." The girl's "in-between" suggestion *would* come as a surprise but would not unsettle the teacher who is able to evaluate it from the vantage point of his or her own mathematical content resources.

Conjecture C3 is somewhat different. We suspected that it is false: it significantly extends the boundaries of the "consecutive convergents" result, which itself is more general that C2. So we looked for a counterexample, a powerful and standard method of refutation, reasoning that if the numerators and/or denominators of $\frac{a}{b}$ and $\frac{c}{d}$ are not too close, this would give several options. Considering $\frac{1}{2}$ and $\frac{3}{10}$, for example, the numerator of the third fraction must be 2, but if we take the denominator much closer to 2 than to 10, the in-between property may fail. Indeed, $\frac{2}{3}$ is a suitable counterexample. Our curiosity having been further aroused, we used the geometric

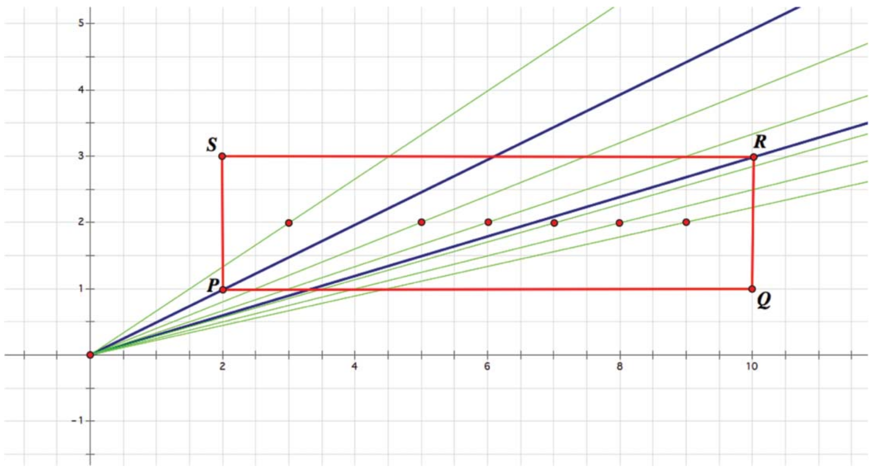


FIGURE 2 Representing the "Fraction in Between Fractions" Conjecture Geometrically. (color figure available online)

representation idea to investigate and, indeed, to explain this further. Figure 2 shows $\frac{1}{2}$ and $\frac{3}{10}$ represented as the slopes of the heavy lines joining the origin to P(2, 1) and R(10, 3), respectively.

The set of all permissible fractions $\frac{m}{n}$ with $1 < m < 3$ and $2 < n < 10$ is represented by lines joining the origin to each of the integer-lattice points strictly inside the rectangle PQRS shown. We see that there are six fractions available: two support conjecture C3, but four refute it. This might be a powerful representation to include in our imagined response to the girl's conjecture C3. It would need some preparation, of course, as an apparent digression from the discussion of fractions. But then Laura's introduction of triangular numbers also seemingly digressed from the original consideration of probability in the dishwasher problem.

Finally, in relation to Conjecture C2 (or the equivalent "adding numerators and adding denominators" procedure), we note that $\frac{a+c}{b+d}$ can be shown to lie between $\frac{a}{b}$ and $\frac{c}{d}$ by reference to an analogy of the following kind. Suppose, for example, that we have two jugs of orange juice diluted with water. Jug A contains 7 cL altogether, of which 4 cL is orange juice; jug B 3 cL altogether, of which 2 cL is orange juice. The drink in jug A will be weaker than that in jug B because $\frac{4}{7} < \frac{2}{3}$. If now we combine the contents of both jugs in a larger jug C and stir, the mixture cannot be stronger than that originally in jug B, nor can it be weaker than that originally in jug A. Because jug C contains 10 cL altogether, of which 6 cL is orange juice, it follows that $\frac{6}{10}$ lies between $\frac{4}{7}$ and $\frac{2}{3}$.

We are reliably informed that Chinese teachers refer to this kind of argument as "the sugar rule," where the analogy is with sweetness of sugared drinks. We note that it would be unacceptable among mathematicians as a formal proof, because pure mathematics does not include concepts such as *sweetness*. However, Shulman (1986) specifically included the use of analogies in his qualitative description of pedagogical content knowledge. G. J. Stylianides and A. J. Stylianides (2010) included a similar analogy in their discussion of a fractions task developed from a related scenario, specifically related to the mediant of two fractions and mathematically equivalent to our version C2b, which they used in a mathematics content course for prospective teachers.

Furthermore, Star and G. J. Stylianides (2013) return to the same fractions task, this time to discuss its potential to assess prospective teachers' procedural and/or conceptual knowledge.

## CONTINGENCY AND MATHEMATICAL MODES OF INQUIRY

In this article, we presented three contingent situations that arose in mathematics classrooms. In the first two cases, we saw how the teacher's mathematical knowledge informed and guided her response to situations that had not been anticipated in planning. In the third, we discussed possible directions for the lesson to proceed following the contingent event, drawing upon our own mathematical resources.

In the first two stories we exemplified how prospective and practicing teachers were guided toward particular mathematical modes of inquiry (Watson & Barton, 2011): working toward general structure, linking knowledge from different areas of mathematics, and examining generalized conjectures. We do not see in these examples any evidence of the teacher's advanced mathematical knowledge; that is, substantive knowledge acquired in studying mathematics beyond high school (Zazkis & Leikin, 2010) in the sense of using any particular theorems or concepts. The mathematics used in these stories is introductory combinatorics (that could have been avoided) and rather simple algebraic manipulations. Nevertheless, we believe that it is extended exposure to mathematics that serves as a support structure for the teacher's willingness to conjecture, to experiment, to take risks, and to take advantage of contingent opportunities as they arise. For example, it is likely that Laura's knowledge of modular arithmetic guided her investigation with the class of teachers. Indeed, knowledge of mathematics beyond the demands of the immediate curriculum offers some guidance to teachers in making an in-the-moment judgement of the mathematical potential of deviating from the intended instructional path. Using Schwab's (1978) distinctions, both syntactic and substantive knowledge can be activated in contingent teaching situations. The former refers to strategic knowledge about the nature of enquiry in the field and the mechanisms through which new knowledge is introduced and accepted in the community, rather than (substantive) knowledge of facts, concepts, and theorems. In the case of mathematics, syntactic knowledge is most likely to be acquired by *doing* mathematics, by posing and solving problems, by "signing up" to the world of mathematics. This kind of mathematical commitment is strongly related to, perhaps the same as, that described by Mason and Davis (2013) in their article.

The third story differs from the first two because it describes how opportunities arose in a mathematics classroom but stops short of describing whether the teacher took advantage of the opportunity. In fact, we do not know how the lesson proceeded mathematically. However, we have used it as a fruitful starting point in our own teaching with prospective and serving teachers as a scenario to provoke reflection, discussion, and learning. We have presented the story to different groups of teachers and invited their reactions to this scenario and hypothetical opportunity. In most cases the reaction was mathematically limiting, in the sense that responses were evaluative (correct/incorrect) rather than opening the door for exploration. As such, we speculated on how this opportunity could have been used had the teacher possessed the relevant knowledge and inclination. It took some prompting for teachers in our classes to formulate and explore generalized conjectures related to the presented "fraction-in-between-fractions" strategy. We argued that having relevant substantive knowledge could have resulted in an immediate ability to evaluate some of the conjectures and could have, putting to work one's syntactic knowledge, opened the gate for further exploration with students.

Of course, we do not intend to suggest that everything in our own investigation is suitable material for 12-year-olds, although some of that content could be made accessible to them. Nevertheless, we submit that some generalized conjectures, such as (C1) and (C3), can be explored and either supported or refuted even when children are just beginning to learn how to compare fractions. This may also lead students into additional mathematical modes of enquiry, enabled and supported by teachers' syntactic knowledge, such as examination of particular strategies, asking "what if," or noticing invariants. For example, having refuted a statement, one can ask under what conditions it would hold true. C2, of course, provides a particular example, in the case of the refuted C3, but what is the general case? We wonder under what conditions on the numerators and denominators of the two given fractions C3 holds true? This is in the spirit of "If not, what yes?" strategy suggested by Koichu (2008), who proposed an extension of the classical "what if not" of Brown and Walter (1983). Koichu's (2008) approach is based on inviting students to reconsider a mathematical claim that has been refuted by asking, "Since this statement is wrong, which one would be correct?" This approach results in progressive modifications of the presented claim, in the spirit of Lakatos (1976). In a similar spirit, we wonder which arithmetic sequences have multiple pairs of elements that are consecutive square numbers? Or, what composition of marbles of three (or more) colors will result in a fair game for dishwashers? We leave these questions as prompts for readers to engage in their own mathematical enquiry, in their chosen modes.

We endorse and emphasize an interesting and significant point made by Chick and Stacey (2013): that contingent situations in mathematics classrooms usually pose problems of two kinds for the teacher. First, there is a mathematical problem to be solved by the teacher in order that she or he achieve substantive clarity about the matter under consideration; secondly, the teacher must solve a pedagogical problem directed at facilitating the students' accessing and engaging with the solution of the mathematical problem. Though the mathematical content involved in the situations considered in this article is different from the scenarios exemplified by Chick and Stacey (2013), the overarching conclusion is similar: that a teacher's responses to problematic contingent moments that arise in teaching mathematics are fundamentally dependent on their mathematical knowledge, which prompts and guides pedagogical implementation.

## NOTE

1. Davenport (2008, p. 102) attributed the geometrical interpretation of convergents of continued fractions to Felix Klein (1907) but added that "the idea seems to be due [earlier] to H. J. S. Smith." The first edition of the Klein volume cited is earlier than 1907; in fact, Smith's dates are 1826–1883 (see Glaisher, 1894).

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