UBC-SFU-UVic-UNBC Calculus Examination

Formula Sheet for 5 June 2003

Exact Values of Trigonometric Functions

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π
$\sin\theta$	0	$1/2$	$\sqrt{2}/2$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0
$\cos\theta$	1	$\sqrt{3}/2$	$\sqrt{2}/2$	$1/2$	0	$-1/2$	$-\sqrt{2}/2$	$-\sqrt{3}/2$	-1

Trigonometric Definitions and Identities

$\sin(-\theta) = -\sin\theta$

$\cos(-\theta) = \cos\theta$

$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta$

$\sin 2\theta = 2\sin\theta\cos\theta$

$\cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi$

$\cos 2\theta = \cos^2\theta - \sin^2\theta$

$$\sin^2\theta = \frac{1-\cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

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[3] **1.** Given $f(x) = \dfrac{\sin x}{\ln(x^2 + e^{\cos x})}$, find $f'(x)$. (Algebraic simplification is not required.)

ANSWER:

[5] **2.** The curve $y = \ln(3x+1)$ has a tangent line of slope $\dfrac{1}{2}$ at some point. Find an equation for this line.

ANSWER:

Show your work:

Continued on page 4

[7] **3.** Find all real numbers A for which each function $y = \dfrac{A}{1 + ke^{-3t}}$ (where k is a constant) satisfies the differential equation $\dfrac{dy}{dt} = y(3 - y)$ at each point of its domain.

ANSWER:

Show your work:

[8] **4.** Find all real numbers k for which the curve $y = x^2 e^{-kx}$ has an inflection point at $x = 1$.

ANSWER:

Show your work:

Continued on page 5

[8] **5.** A stalagmite has the shape of a perfect circular cone. Its height is 200 mm and is increasing at a rate of 3 mm per century. Its base radius is 40 mm and is decreasing at a rate of 0.5 mm per century. Is its volume increasing or decreasing, and at what rate?

ANSWER:

Show your work:

Continued on page 6

[8] **6.** Find all points on the curve $2x^2 + y^2 + 2xy = 10$ where the tangent line is horizontal.

ANSWER:

Show your work:

Continued on page 7

[9] **7.** At what point on the parabola $y = 1 - x^2$ does the tangent line have the property that it cuts from the first quadrant a triangle of minimum area?

ANSWER:

Show your work:

Continued on page 8

[8] **8.** A particle is travelling along the curve $y = x^2 - 5$. At the instant that the particle reaches the point $(3, 4)$, the particle's distance from the origin is increasing at the rate of 2 units per second. How fast is the particle's x-coordinate increasing at this instant?

ANSWER:

Show your work:

Continued on page 9

[6] **9.** The following facts are known about the function $f(x)$:

(a) $f(2) = 4$ and (b) $f'(x) = (x^4 + 1)^{-1}$ for all x.

(i) Use a linear approximation to estimate $f(2.05)$. (Call your answer α.)

ANSWER:

$\alpha =$

(ii) Taking α from part (i), circle the correct statement below:

$\alpha < f(2.05)$ $\alpha = f(2.05)$ $\alpha > f(2.05)$

Clearly justify your selection, without using an antiderivative of $(x^4 + 1)^{-1}$.

Continued on page 10

[6] **10.** Given that $f''(t) = 6e^{-3t}$ for all t, and that $f(0) = 0$ and $f'(0) = 4$, find an explicit formula for $f(t)$.

ANSWER:

Show your work:

Continued on page 11

[8] **11.** Fred's body mass m satisfies the differential equation

$$\frac{dm}{dt} = \frac{C - 40m}{8000}.$$

Here m is Fred's mass in kilograms, t is the time in days, and C is Fred's energy intake, measured in Calories per day. If Fred's mass now is 100 kilograms, and he ingests 3000 Calories per day, how many days from now will his mass be 90 kilograms?

ANSWER:

Show your work:

Continued on page 12

[8] **12.** (i) Find numbers A and B such that the derivative of $Ae^{-3x}\cos x + Be^{-3x}\sin x$ with respect to x is $e^{-3x}\cos x$.

ANSWER:

Show your work:

(ii) Consider the triangular region T defined by $x \geq 0$, $y \geq 0$, and $x + y \leq \pi$. Find the area of the subset of T in which $y \leq e^{-3x}\cos x$.

ANSWER:

Show your work:

Continued on page 13

[8] **13.** Suppose that the function $f(x)$ is defined on an interval (a, b) where $a < 3 < b$.

(i) Express $f'(3)$ as a limit.

ANSWER:

(ii) Prove that if $f(3) = 9$ and $f'(3) = 4$ then $\lim_{x\to 3} \frac{\sqrt{f(x)} - 3}{x - 3} = \frac{2}{3}$.
Note: Assuming that $f'(x) \to 4$ as $x \to 3$ *is not allowed,* since this behaviour is not shared by all f with the given properties. Find a way to use part (i).

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[8] **14.** For each positive number a, let $V(a)$ denote the maximum value of $(a/x)^x$ on the interval where $x > 0$.

(i) Find an explicit formula for $V(a)$.

ANSWER:

Show your work:

(ii) Prove that $V'(a)/V(a)$ is independent of a and find its value.

ANSWER:

Show your work:

The End

This examination has 14 pages including this cover.

UBC-SFU-UVic-UNBC Calculus Examination

5 June 2003

Name: ______________________ **Signature:** ______________________

School: ______________________ **Candidate Number:** ______________________

Rules and Instructions

1. *Show all your work!* Full marks are given only when the answer is correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.
2. Calculators are optional, not required. Correct answers that are "calculator ready," like $3 + \ln 7$ or $e^{\sqrt{2}}$, are fully acceptable.
3. Any calculator acceptable for the Provincial Examination in Principles of Mathematics 12 may be used.
4. A basic formula sheet has been provided. No other notes, books, or aids are allowed. In particular, *all calculator memories must be empty when the exam begins.*
5. If you need more space to solve a problem on page n, work on the back of page $n - 1$.
6. CAUTION - Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.
7. Do not write in the grade box shown to the right.

1		3
2		5
3		7
4		8
5		8
6		8
7		9
8		8
9		6
10		6
11		8
12		8
13		8
14		8
Total		100