Key for 2004 Practise Exam #1

Note: There is no attempt here to describe all possible correct answers. In many cases other approaches to a question could garner full marks.

For the examiners, apart from the accuracy of the answers, the crucial test is whether the student has made clear the principles and/or method being used and whether those principles and/or method are sound.

1. For each of the following evaluate the limit if it exists and explain why it does not otherwise.

[2] (a) $\lim_{x \to 1} \dfrac{x^3 - 1}{x - 1}$

ANSWER: 3

JUSTIFICATION

For $x \neq 1$, $(x^3 - 1)/(x - 1) = x^2 + x + 1$. Thus the given limit is the same as

$$\lim_{x \to 1} \left(x^2 + x + 1\right) = \left(\lim_{x \to 1} x\right)^2 + \left(\lim_{x \to 1} x\right) + \lim_{x \to 1} 1 = 1 + 1 + 1 = 3.$$

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[3] (b) $\lim_{x \to 0} \dfrac{|x|}{x}$

ANSWER: does not exist

JUSTIFY YOUR ANSWER

For $x < 0$, $|x|/x = -x/x = -1$. Thus $\lim_{x \to 0-} \dfrac{|x|}{x} = \lim_{x \to 0-} -1 = -1$.

Similarly, $\lim_{x \to 0+} \dfrac{|x|}{x} = \lim_{x \to 0+} 1 = 1$. Since the one-sided limits are different, the given limit does not exist. ■

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[3] (c) $\lim_{x \to 1-} \dfrac{1 - x}{(\pi/2) - \sin^{-1} x}$

ANSWER: 0

JUSTIFY YOUR ANSWER

By L'Hospital's Rule the given limit is equal to $\lim_{x \to 1-} \dfrac{-1}{-1/\sqrt{1 - x^2}}$ provided the latter exists. But the latter limit is equal 0 since $\sqrt{1 - x^2} \to 0$ as $x \to 1-$. ■

2. Let $f(x) = \dfrac{\sec x - \tan x}{\sec x + \tan x}$.

[4] (a) Find an expression for $f'(x)$

ANSWER:

$$\frac{-2\sec x\,(\sec x - \tan x)}{\sec x + \tan x}$$

JUSTIFY YOUR ANSWER

The first step is to use the quotient rule for derivatives to get:

$$\frac{(\sec x + \tan x)\frac{d}{dx}(\sec x - \tan x) - (\sec x - \tan x)\frac{d}{dx}(\sec x + \tan x)}{(\sec x + \tan x)^2}$$

[2] (b) Simplify $f'(x)\,/\,f(x)$

ANSWER:
$-2\sec x$

JUSTIFY YOUR ANSWER

With the answer for (a) given above, this is by inspection. ■

3. Let l_t denote the tangent line to the parabola $y = -x^2$ at the point $(t, -t^2)$.

[4] (a) Find the equation of l_t.

ANSWER:

$y + 2tx = t^2$

EXPLANATION

The slope of l_t is $\dfrac{dy}{dx}_{x=t} = -2t$. Thus the point slope equation of l_t is $y - (-t^2) = (-2t)(x - t)$. ■

[2] (b) Assuming that l_t meets the hyperbola $xy = 1$ in two points find the x-coordinates of those points.

ANSWER:

$$\frac{t^2 \pm \sqrt{t^4 - 8t}}{4t}$$

EXPLANATION

Substituting $1/x$ for y in the equation for l_t we get $(2t)x^2 + (-t^2)x + 1 = 0$. The roots are

$$\frac{t^2 \pm \sqrt{t^4 - 8t}}{4t}.$$

[2] (c) Find a value of t such that the tangent line l_t is also tangent to $xy = 1$.

ANSWER:

$t = 2$

EXPLANATION

The slope of the tangent line is negative at every point of $xy = 1$. Therefore, if l_t is tangent to $xy = 1$, then $t > 0$. Also, by inspection of the roots from (b), when $t > 0$, the point(s) of intersection of l_t with the hyperbola lie in the first quadrant. The natural place to look for a tangent is where the two points of intersection coincide, i.e., where $t^4 - 8t = 0$. Since $t > 0$, this gives $t = 2$. The corresponding point of $xy = 1$ is $(1/2, 2)$. Since the slope of $xy = 1$ at $(1/2, 2)$ is -4, l_2 is indeed tangent to $xy = 1$. ■

[6] **4.** Consider the curve whose equation is $2(x^2 + y^2)^2 = 25xy$.

Find all points of the curve at which $\dfrac{dy}{dx} = 0$.

ANSWER:

$$\left(\frac{5 \cdot 3^{1/4}}{4\sqrt{2}}, \frac{5 \cdot 3^{3/4}}{4\sqrt{2}}\right), \left(-\frac{5 \cdot 3^{1/4}}{4\sqrt{2}}, -\frac{5 \cdot 3^{3/4}}{4\sqrt{2}}\right)$$

SHOW YOUR WORK

Implicit differentiation gives $4(x^2 + y^2)\left(2x + 2\dfrac{dy}{dx}\right) = 25\left(y + x\dfrac{dy}{dx}\right)$. Setting $\dfrac{dy}{dx} = 0$ we get $x^2 + y^2 = 25y/8x$. Subsituting for $x^2 + y^2$ in the original equation, we get $2\left(25y/8x\right)^2 = 25xy$ which gives $25y = 32x^3$. Substituting for $25y$ now gives $(x^2 + y^2)^2 = 16x^4$, which gives $y/x = \pm\sqrt{3}$. Clearly, $y/x > 0$. So $y = \sqrt{3}x$. The rest is straightforward. ■

[3] **5.** (a) Find the general antiderivative of $\dfrac{1}{1+4x^2}$.

ANSWER:
$(1/2)\arctan 2x \,+\, C$

SHOW YOUR WORK

The starting point is $\dfrac{d}{dx}\arctan x = \dfrac{1}{1+x^2}$. By the Chain Rule, $\dfrac{d}{dx}\arctan 2x = \dfrac{2}{1+(2x)^2}$. ■

[3] (b) Find a function y defined on $(0,\infty)$ such that

$$\frac{dy}{dx} = 2x + \frac{1}{x^2}, \;\; y(1) = 0.$$

ANSWER:
$x^2 - \dfrac{1}{x}$

SHOW YOUR WORK

Taking antiderivatives we have

$$y = \int 2x\,dx + \int \frac{1}{x^2}\,dx = x^2 - \frac{1}{x} + C\,.$$

To get $y(1) = 0$ we need $C = 0$. ■

[6] **6.** A spherical planet has radius R and a narrow hole bored along a diameter. An object falling towards the centre through the hole satisfies the differential equation

$$\frac{d^2x}{dt^2} = -c^2 x$$

where c is constant, x the distance from the centre, and t the elapsed time.

Initially,i.e. at $t = 0$, the object is at rest at the surface of the planet.

The motion of the falling object is given by an equation

$$x = A \cos ct + B \sin ct\,.$$

Given that $R = 2 \times 10^7$ and $c = 4 \times 10^{7/2}$, compute the values of the constants A, B from the initial data.

Show that the time taken for the object to fall to the centre is $\pi/2c$.

ANSWER:

$A = R,\ B = 0$

EXPLANATION

Differentiating with respect to t we get

$$\frac{dx}{dt} = -Ac \sin ct \ + \ Bc \cos ct.$$

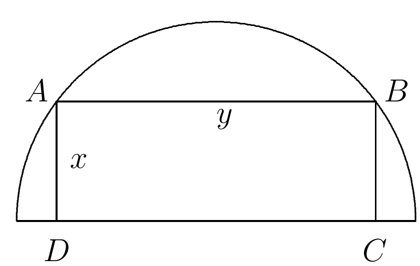
Since $(x)_{t=0} = R$ and $\left(\dfrac{dx}{dt}\right)_{t=0} = 0$, we have $A = R$ and $B = 0$. Thus $x = R \cos ct$.

The least $t > 0$ for which $x = 0$ is given by $ct = \pi/2$, which is the same as $t = \pi/2c$. ■

[8] **7.** A rectangle $ABCD$ with sides of length x, y is inscribed in a semicircle of radius 1 as shown in the figure.

The rectangle is chosen so that its perimeter $2x+2y$ is as large as possible.

Show that $xy = 4/5$.



EXPLANATION

Let $P = 2x + 2y$. Then $dP/dx = 2 + 2dy/dx$. From Pythagoras' theorem we have $x^2 + (y/2)^2 = 1$, which is $4x^2 + y^2 = 4$. Differentiating we get $8x + 2ydy/dx = 0$, so $dP/dx = 2 - (8x/y)$. Setting $dP/dx = 0$ we get $y = 4x$. Together with $4x^2 + y^2 = 4$ this gives $x = 1/\sqrt{5}$ and $y = 4/\sqrt{5}$. Hence $xy = 4/5$. As x increases through $1/\sqrt{5}$, dP/dx changes from negative to positive. So the critical point gives a maximum for P. ■

8. Let $f(x) = \sqrt[4]{10^{100} + x} - 10^{25}$.

[4] (a) Give the best possible linear estimate for the function $f(x)$ for small values of x, i.e., for x close to 0.

A *linear estimate* is a function $Ax + B$, where A and B are constants to be determined.

ANSWER:
$(10^{-75}/4)\, x$

SHOW YOUR WORK

Note that $f(0) = 0$, so $B = 0$. Also,

$$\lim_{x\to 0} \frac{f(x)}{x} = f'(0) = (1/4)\left(x + 10^{100}\right)^{-3/4}\Big|_{x=0} = 10^{-75}/4\,.$$

So $A = 10^{-75}/4$. ■

[4] (b) Is your linear estimate an understimate (meaning $Ax+B \le f(x)$ whenever $|x| < 1$), an overestimate (meaning $Ax + B \ge f(x)$ whenever $|x| < 1$), or neither?

Explain your answer using the Mean Value Theorem or any other theorem known to you which fits the context.

ANSWER

Let α denote $10^{-75}/4$. The key point is that $f'(x)$ is decreasing for $x > -10^{100}$.
Consider $x > 0$. By the MVT there exists c, $0 < c < x$, such that $f(x) = xf'(c)$. So $f(x) < xf'(0) = \alpha x$. Now consider x such that $-1 < x < 0$. By the MVT there exists c, $x < c < 0$, such that $f(x) = xf'(c)$. So again $f(x) < xf'(0) = \alpha x$. We conclude that αx is an overestimate of $f(x)$.
■

[8] **9.** Newton's law of cooling may be stated as:

$$\frac{dT}{dt} = kT, \tag{1}$$

where T is the *difference between* the temperature of a specified object at time t and the ambient temperature (usually a constant), t is the elapsed time, and k is a constant which depends on the particular case.

A kettle initially at 100° (Celsius) cools to 80° in one minute. The ambient temperature is constant at 20°.

Show that the time taken for the kettle to cool from 80° to 60° is:

$$\ln(2/3)\,/\ln(3/4) \approx 1.4 \text{ minutes}$$

EXPLANATION

The differential equation is the classical decay equation. It can be rewritten as $d(\ln T)/dt = k$. Taking antiderivatives we get $\ln T = kt + C$. Let u denote the time from 80° to 60° in minutes. Substituting the data for $t = 0$, $t = 1$, and $t = 1 + u$, we get

$$\ln 80 = C,\ \ln 60 = k + C,\ \ln 40 = k(1 + u) + C.$$

Subtracting the first equation from the second, and the second from the third, gives $\ln(3/4) = k$ and $\ln(2/3) = ku$. Hence the result. ■

10. Let $f(x) = x^3(x - 1)^2$.

[4] (a) Find the largest interval on which $f(x)$ is decreasing.

ANSWER:

SHOW YOUR WORK

Note that $f(x) = x^5 - 2x^4 + x^3$. Therefore

$$f'(x) = 5x^4 - 8x^3 + 3x^2 = x^2(5x - 3)(x - 1)\,.$$

By inspection, $f'(x) \geq 0$ for all $x \leq 3/5$ and all $x \geq 1$, while $f'(x) < 0$ on $(3/5, 1)$. So $[3/5, 1]$ is the largest interval on which $f(x)$ is decreasing. ■

[4] (b) Find all points of inflection of the function $f(x)$.

ANSWER:
$$0,\ \frac{6 \pm \sqrt{6}}{10}$$

SHOW YOUR WORK

Continuing our work above, we see that

$$f''(x) = 20x^3 - 24x^2 + 6x = x\left(x - \frac{6+\sqrt{6}}{10}\right)\left(x - \frac{6-\sqrt{6}}{10}\right).$$

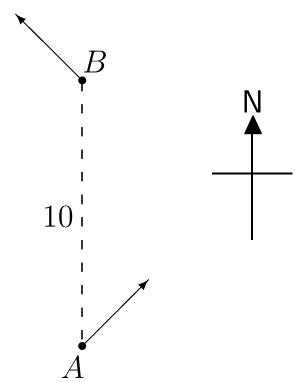
Note that $f''(x)$ changes sign at each of its three zeros. So each zero of $f''(x)$ is an inflection point of $f(x)$. ■

[6] **11.** At time $t = 0$ plane A is traveling NW at 500 mph while plane B, 10 miles north of plane A, is traveling NE at 500 mph.

The planes are travelling with constant speed and direction.

Let $f(t)$ denote the distance between the planes so that $f(0) = 10$.

Show that $f'(0) = 0$.



SHOW YOUR WORK

As origin O of coordinates take the initial position of plane A. Let the posiive x-axis point west and the positive y-axis point north. Then after t hours the respective positions of planes A, B are

$$(250\sqrt{2}t, 250\sqrt{2}t), \quad (-250\sqrt{2}t, 10 + 250\sqrt{2}t).$$

So

$$f(t)^2 = (2 \cdot 500^2)t^2 + 10^2.$$

Differentiating we get

$$2f(t)f'(t) = (4 \cdot 500^2)t.$$

Letting $t = 0$ we get $2f(0)f'(0) = 0$. Therefore $f'(0) = 0$. ■

[3] **12.** (a) Let $f(x)$ be a function whose domain includes $(0, \infty)$ and $c > 0$. Define $f'(c)$ as a limit.

DEFINITION

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

[5] (b) Using only the limit laws and the definition given in part (a) show that, if $f(x) = 1/x$, then

$$f'(c) = -1/c^2 \qquad (c > 0).$$

EXPLANATION

Observe that, when $x \neq c$,

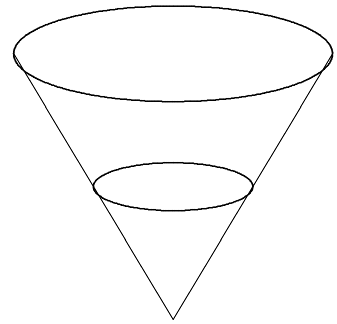
$$\frac{f(x) - f(c)}{x - c} = \frac{(1/x) - (1/c)}{x - c} = \frac{(c - x)/(xc)}{x - c} = -\frac{1}{xc}\,. \tag{2}$$

Suppose $c > 0$. Then, from (2),

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} -\frac{1}{xc} = \frac{-1}{c} \lim_{x \to c} \frac{1}{x} = \frac{-1}{c} \left(1 \Big/ \lim_{x \to c} x \right) = \frac{-1}{c^2}\,.$$

This is enough. ∎

13. A water tank is in the shape of an inverted right circular cone. It has diameter 6 feet at the top and height 4 feet. The tank is being filled with water at a rate of 10 cubic feet per minute . A leaf is floating in the centre of the tank and moving along the axis of the tank as the water level rises.



[4] (a) What is the velocity of the leaf when the water level is 2 feet?

ANSWER: $\frac{40}{9\pi}$ feet/minute

SHOW YOUR WORK

Let $V(x)$ denote the volume of water in the tank when the depth is x. By the formula for the volume of a cone, $V(4) = (\pi/3)4 \cdot 3^2 = 12\pi$ cubic feet. When the depth is x all the linear dimensions of the cone are reduced in the proportion $x : 4$. Therefore

$$V(x) = 12\pi \left(\frac{x}{4}\right)^3 = \frac{3\pi}{16}x^3.$$

Differentiating with respect to time t, measured in minutes,

$$10 = \frac{dV}{dt} = \frac{9\pi}{16}x^2\frac{dx}{dt}. \tag{3}$$

Letting $x = 2$ we get $dx/dt = \dfrac{40}{9\pi}$. ■

[4] (b) What is the acceleration of the leaf when the water level is 2 feet?

ANSWER: $-\left(\frac{40}{9\pi}\right)^2$ feet/minute^2

SHOW YOUR WORK

Differentiating (3) we get

$$0 = 2x\left(\frac{dx}{dt}\right)^2 + x^2\,\frac{d^2x}{dt^2}\,.$$

Substituting $x = 2$, we get

$$\left.\frac{d^2x}{dt^2}\right|_{x=2} = -\frac{2}{x}\left.\left(\frac{dx}{dt}\right)^2\right|_{x=2} = -\left(\frac{40}{9\pi}\right)^2.$$

[6] **14.** Compute the area enclosed between the parabolas $y = 1 - x^2$ and $y = x^2 - 1$.

Note that these parabolas are mirror images of each other in the x-axis.

ANSWER:

$8/3$

SHOW YOUR WORK

Note that the parabolas intersect at $(-1, 0)$ and $(1, 0)$. The area between them is bounded above by $y = 1 - x^2$ $(x \in [-1, 1])$, and below by $y = x^2 - 1$ $(x \in [-1, 1])$.

We need the principle that, if $F(x)$ is an antiderivative of $f(x)$, then the area under $y = f(x)$ between $x = a$ and $x = b$ is $F(b) - F(a)$. By symmetry the area between the parabolas is twice the area below $y = 1 - x^2$ between $x = -1$ and $x = 1$. By inspection, $F(x) = x - (1/3)x^3$ is an antiderivative of $1 - x^2$. So the desired area is

$$2\,(F(1) - F(-1)) = 2\,[(1 - (1/3)) - (-1 - (-1/3))] = 8/3\,.$$