Key for 2004 Practise Exam #2

Note: There is no attempt here to describe all possible correct answers. In many cases other approaches to a question could garner full marks.

For the examiners, apart from the accuracy of the answers, the crucial test is whether the student has made clear the principles and/or method being used and whether those principles and/or method are sound.

1. For each of the following evaluate the limit if it exists and explain why it does not otherwise.

[2] (a) $\lim_{x\to 1^+} \dfrac{1-\sqrt{x}}{1-x}$

ANSWER: $1/2$

JUSTIFY YOUR ANSWER

Note that for, $x < 1$, we have $\dfrac{1-\sqrt{x}}{1-x} = \dfrac{1}{1+\sqrt{x}}$. So, using the limit laws, we get

$$\lim_{x\to 1^+} \frac{1-\sqrt{x}}{1-x} = \lim_{x\to 1^+} \frac{1}{1+\sqrt{x}} = \frac{1}{1+\sqrt{\lim_{x\to 1^+} x}} = \frac{1}{1+1} = 1/2 \ .$$

[3] (b) $\lim_{x\to 0} \dfrac{f(x+2)-f(2)}{x}$

where $f'(x) = x^2 + \ln(x-1)$

ANSWER: 5

JUSTIFY YOUR ANSWER

Note that $f'(x) = 2x + (1/(x-1))$. Therefore $f'(2) = 5$. The given limit is the definition of $f'(2)$. ■

[3] (c) $\lim_{x\to 0} (\cos x)^{1/x^2}$

ANSWER: $1/\sqrt{e}$

JUSTIFY YOUR ANSWER

Since e^x is a continuous function, the given limit is $e^{\lim_{x\to 0} \frac{\ln\cos x}{x^2}}$. By l'Hospital's rule,

$$\lim_{x\to 0} \frac{\ln\cos x}{x^2} = \lim_{x\to 0} \frac{-\sin x}{2x\cos x} = \lim_{x\to 0} \frac{-\cos x}{2\cos x - 2x\sin x} = -1/2.$$

[3] **2.** (a) Let $f(x) = \dfrac{\sqrt{x}-1}{\sqrt{x}+1}$.

Find $f'(9)$.

ANSWER: $1/48$

JUSTIFY YOUR ANSWER

Using the quotient rule, we get

$$f'(x) = \frac{(\sqrt{x}+1)(1/\sqrt{x}) - (\sqrt{x}-1)(1/\sqrt{x})}{2\left(\sqrt{x}+1\right)^2} = \frac{1}{\sqrt{x}\left(\sqrt{x}+1\right)^2}\,.$$

[3] (b) Let $f(x) = \sqrt{1+\sqrt{x}}$.

Find $f'(9)$.

ANSWER: $1/24$

JUSTIFY YOUR ANSWER

Using the rule for a 'function of a function' we have

$$f'(x) = \frac{1/(2\sqrt{x})}{2\sqrt{1+\sqrt{x}}} = \frac{1}{4\sqrt{x}\,\sqrt{1+\sqrt{x}}}\,.$$

3. Let C be the curve in the xy-plane which is the graph of the equation $y = x^3 - x^2 + 1$, and P be $(1, 1)$. Let l be the line which is tangent to C at P.

[4] (a) Find the equation of l.

ANSWER: $y = x$

EXPLANATION: $dy/dx|_{x=1} = 1$. So the point slope form of the tangent line is $y - 1 = x - 1$. ■

[4] (b) Find another line tangent to C which is parallel to l.

ANSWER: $y - (23/27) = x + (1/3)$

EXPLANATION: For the tangent line at the point $(a, y(a))$ to be parallel to $y = x$ we need its slope to be 1, which is the same as $dy/dx|_{x=a} = 3a^2 - 2a = 1$. The roots of $3a^2 - 2a - 1$ are $-1/3$, 1. So the other tangent line with slope 1 is the one at $(-1/3,\ 23/27)$. ■

[4] **4.** (a) The number $2^{1/3}$ is being approximated by applying Newton's method to the function $y = x^3 - 2$ with initial estimate $x_0 = 1$.

What are the next two estimates? Give your answers as rational numbers.

ANSWERS:

$x_1 = 4/3$

$x_2 = 91/72$

SHOW YOUR WORK

Newton's method amounts to this: if a is an estimate for the zero r of $f(x) = 0$, then the next estimate is the x-coordinate of the point at which the tangent line at $(a, f(a))$ to $y = f(x)$ meets $y = 0$. Since the tangent line is $y - f(a) = f'(a)(x - a)$, the next estimate is $a - (f(a)/f'(a))$.

In this case $x - (f(x)/f'(x)) = x - \dfrac{x^3 - 2}{3x^2} = 2\left(\dfrac{x^3+1}{x^2}\right)$. Thus $x_1 = 4/3$ and $x_2 = 2\left(\dfrac{x_1^3+1}{x_1^2}\right) = 91/72$. ■

[4] (b) Suppose that Newton's method is being used to approximate the zero r of $f(x) = 0$ with initial estimate $x_0 < r$. By "zero" we mean that $f(r) = 0$.

It is given that $f(x_0) > 0$, and that $f'(x)$ is defined and increasing on $[x_0, r]$.

Using the Mean Value Theorem, or otherwise, explain carefully why the next estimate x_1 satisfies $x_0 < x_1 < r$.

EXPLANATION

It is given that $f(x_0) > 0$ and that $f'(x)$ is increasing on $[x_0, r]$. It follows that $f'(x_0) < 0$; otherwise, $f(x)$ is increasing on $[x_0, r]$, contradiction. It follows that

$$x_0 < x_0 - \frac{f(x_0)}{f'(x_0)} = x_1 .$$

By the Mean Value Theorem there exists c in (x_0, r) such that

$$\frac{f(r) - f(x_0)}{r - x_0} = f'(c).$$

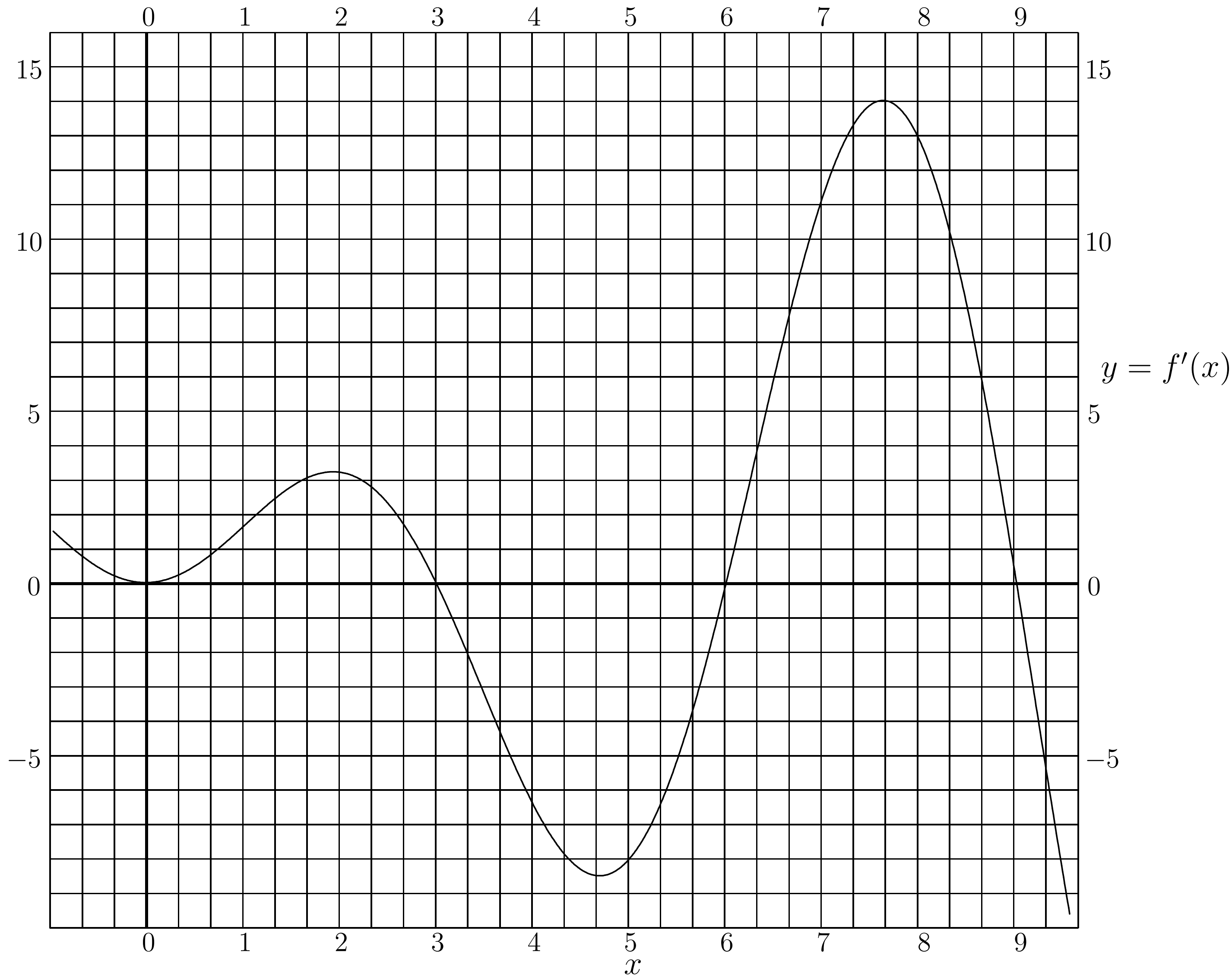
Since the derivative is increasing,

$$\frac{f(r) - f(x_0)}{r - x_0} = \frac{-f(x_0)}{r - x_0} > f'(x_0).$$

Cross-multiplying gives

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} < r$$

as required. ■



[6] **5.** Above is shown the graph of $y = f'(x)$ on the interval $I = [-1, 9.5]$. The graph touches the axis at $x = 0$.

By inspection of the graph estimate the values of x in I at which $f(x)$ attains local maxima/minima. Also, estimate the estimate the values of x in I at which $f(x)$ has points of inflection.

No explanation need be given.

x-coordinates of local minima of $f(x)$	x-coordinates of local maxima of $f(x)$	x-coordinates of points of inflection of $f(x)$
6	$3,\ 9$	$0,\ 2,\ 4.7,\ 7.6$

[6] **6.** One model of population growth is represented by the differential equation

$$\frac{dP}{dt} = kP(A - P) \tag{1}$$

where P is the population, t is time, and k and A are positive constants.

The equation (1) has some solutions of the form

$$P = \frac{Be^{Akt}}{1 + Ce^{Akt}}. \tag{2}$$

What relation between the constants A, B, C implies that the function P given by (2) satisfies (1).

ANSWER:

$ABC = B^2$

EXPLANATION

Substituting the value of P given by (2) into (1), we get

$$\frac{d}{dt}\frac{Be^{Akt}}{1 + Ce^{Akt}} = k\left(\frac{Be^{Akt}}{1 + Ce^{Akt}}\right)\left(A - \frac{Be^{Akt}}{1 + Ce^{Akt}}\right)$$

which is equivalent to

$$\frac{d}{dt}\frac{B}{e^{-Akt} + C} = k\left(\frac{B}{e^{-Akt} + C}\right)\left(A - \frac{B}{e^{-Akt} + C}\right)$$

which is equivalent to

$$\frac{kABe^{-Akt}}{(e^{-Akt} + C)^2} = \frac{kABe^{-Akt} + k(ABC - B^2)}{(e^{-Akt} + C)^2}.$$

7. A piece of wire of length 1 metre is cut into two pieces, where one of the pieces may have zero length. One piece is bent into a square, the other into a circle. A denotes the sum of the areas of the resulting square and circle.

[4] (a) How should the wire be cut so as to maximize A? Give the perimeter of the square as answer.

ANSWER:

0

[4] (b) How should the wire be cut so as to minimize A? Give the perimeter of the square as answer.

ANSWER:

$4/(\pi + 4)$

EXPLANATION: Let x be the length of the perimeter of the square in metres and A denote the total area. Then

$$\frac{dA}{dx} = \frac{d}{dx}\left(\frac{x^2}{4} + \pi\left(\frac{1-x}{2\pi}\right)^2\right) = \frac{1}{8\pi(\pi+4)}\left(x - \frac{4}{\pi+4}\right)$$

From this we see that as x increases from 0, A is decreasing until $x = 4/(\pi+4)$, and that A is increasing as x increases from $4/(\pi+4)$ to 1. Thus the maximum of A is attained at one of the end points $x = 0$, $x = 1$, while the minimum is attained at the critical point $x = 4/(\pi+4)$.
Finally, notice that

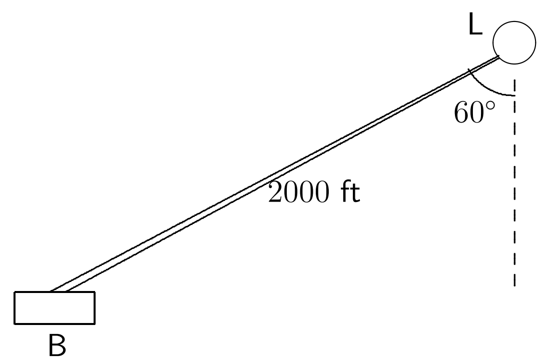
$$A(1) = 1/16 < \frac{1}{4\pi} = \pi(1/2\pi)^2 = A(0).$$

So $x = 0$ gives the maximum. ∎

[6] **8.** A beam of light from the lighthouse L strikes the north face of the building B 2000 feet away. From the lighthouse the building bears 60° west of south.

The beam rotates clockwise at a rate of 12° per second.

At how many feet per second does the beam travel across the north face of the building?

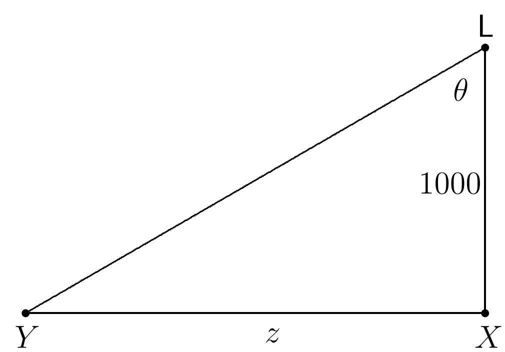


ANSWER:

$800\pi/3$

SHOW YOUR WORK

Imagine that the north face of B is extended to include the point X due south of L. Note that the length of LX is 1000 feet. Consider the point Y at which a ray from L hits the north face, and let z be the length of XY. The speed of the ray along the face is dz/dt.



Now, from the diagram,

$$z = 1000 \tan\theta$$

where for convenience θ is measured in radians.

It follows that

$$\frac{dz}{dt} = 1000 \sec^2\theta \, \frac{d\theta}{dt} = 1000 \sec^2\theta \, \frac{2\pi}{30}.$$

Letting $\theta = \pi/3$, we get the desired value. ∎

9. A ball is thrown vertically into the air with initial veloity v_0. The maximum height reached is h.

The equation governing the motion of the ball is

$$\frac{d^2x}{dt^2} = -g \tag{1}$$

where g is a constant and x is the height of the ball t seconds after release.

[4] (a) Find expressions for x and dx/dt in terms of t and v_0.

ANSWER:

$x = -(g/2)t^2 + v_0 t$

$dx/dt = -gt + v_0$

[4] (b) What initial velocity is required for the maximum height attained to be $4h$? Give your answer as a multiple of v_0.

ANSWER:

$2v_0$

EXPLANATION

Taking antiderivatives in (1), we get

$$\frac{dx}{dt} = -gt + C_1$$

and $C_1 = v_0$ since the initial velocity is v_0.

Taking antiderivatives again we get

$$x = -(g/2)t^2 + C_1 t + C_2 = -(g/2)t^2 + v_0 t + C_2$$

and $C_2 = 0$ since the initial height is 0. This completes part (a).

Note that maximum height is attained when $dx/dt = 0$, i.e., $C_1/g = v_0/g$ seconds after release. Substituting in the equation for x we see that $h = v_0^2/(2g)$. Since h varies with the square of the initial velocity, doubling v_0 gives rise to a fourfold increase in h. ∎

10. Let $f(x)$ be a function defined for all x in $(-\infty, \infty)$. Let a be a real number.

[2] (a) In terms of a limit define what it means for $f(x)$ to be *continuous at* $x = a$.

DEFINITION

$f(x)$ *is continuous at* $x = a$ if $\lim_{x \to a} f(x) = f(a)$.

[2] (b) In terms of a limit define what it means for $f(x)$ to be *differentiable at* $x = a$.

DEFINITION

$f(x)$ *is differentiable at* $x = a$ if $\displaystyle\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists.

[2] (c) Show that, if $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

EXPLANATION

Assume that $f(x)$ is differentiable at $x = a$. Note that for all $x \neq a$ we have

$$f(x) = f(a) + \left(\frac{f(x) - f(a)}{x - a}\right)(x - a).$$

Taking limits as x tends to a on both sides an using the limit laws, we get:

$$\begin{aligned}
\lim_{x \to a} f(x) &= \lim_{x \to a} f(a) + \lim_{x \to a}\left[\left(\frac{f(x) - f(a)}{x - a}\right)(x - a)\right] \\
&= f(a) + \lim_{x \to a}\left(\frac{f(x) - f(a)}{x - a}\right)\lim_{x \to a}(x - a) \\
&= f(a) + f'(a)\left(\lim_{x \to a} x - \lim_{x \to a} a\right) \\
&= f(a) + f'(a)\,(a - a) \\
&= f(a)
\end{aligned}$$

We conclude that $f(x)$ is continuous at $x = a$. ■

11. Evaluate the following antiderivatives:

[3] (a) $\displaystyle\int e^{2x+2}\,dx$

ANSWER:
$$\frac{e^{2x+2}}{2}+C$$

[3] (b) $\displaystyle\int \frac{1}{(1-2x)^3}\,dx$

ANSWER:
$$-\frac{1/2}{(1-2x)^3}+C$$

[3] (c) $\displaystyle\int \sec(\theta/2)\tan(\theta/2)\,d\theta$

ANSWER:
$$2\sec(\theta/2)+C$$

SHOW YOUR WORK

In each case we are exploiting one of the basic differentiation formulas. The general principle is that if $F(x)$ is an antiderivative of $f(x)$ and a, b are constants with $a \neq 0$, then

$$\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)+C\,.$$

For the rest we use,

(a) $\displaystyle\int e^x\,dx = e^x + C$

(b) $\displaystyle\int x^n\,dx = \frac{x^{n+1}}{n+1}+C$ when $n \neq 1$

(c) $\displaystyle\int \sec\theta\tan\theta\,d\theta = \sec\theta + C\,.$

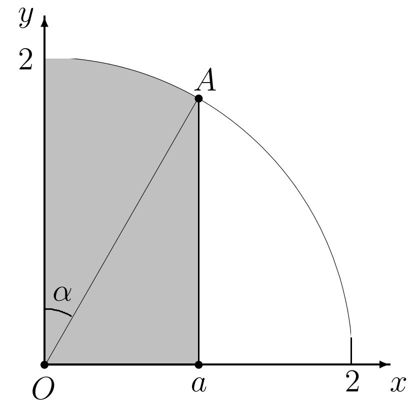
[5] **12.** The graph of $y = f(x)$ passes through $(1,-1)$, and for all $a > 0$ the slope of the graph at the point $(a, f(a))$ of the graph is $1/a$. Find $f(3)$.

ANSWER:

EXPLANATION: For $x > 0$, $f(x)$ is an antiderivative of $1/x$. So $f(x) = \ln(x)+C$. Since $f(1) = -1$, $C = -1$. Hence $f(3) = \ln(3) - 1$. ■

13. Consider the area under the arc of the circle $x^2 + y^2 = 4$, from $x = 0$ to $x = a$ in the first quadrant, shown in the figure.

A is the point $\left(a, \sqrt{4 - a^2}\right)$, and α is $\angle AOy$.



[2] (a) Write α as a function of a.
No explanation is required.

ANSWER:
$\arcsin(a/2)$

[2] (b) By considering two pieces, or otherwise, express the shaded area as function of a.

ANSWER:
$2\arcsin(a/2) + \dfrac{1}{2}a\sqrt{4 - a^2}$

SHOW YOUR WORK

The segment of the circle bounded by Oy, OA, and the arc of the circle from A to $(0, 2)$ has area $\dfrac{\alpha}{2\pi}(\pi 2^2) = 2\arcsin(a/2)$. The rest of the shaded area consists of the right triangle bounded by OA, $x = a$, and $0x$, which has area $\dfrac{1}{2}a\sqrt{4 - a^2}$. ■

[2] (c) From part (b) find an antiderivative for $\sqrt{x^2 - 4}$.

ANSWER:
$2\arcsin(x/2) + \dfrac{1}{2}x\sqrt{4 - x^2}$

SHOW YOUR WORK

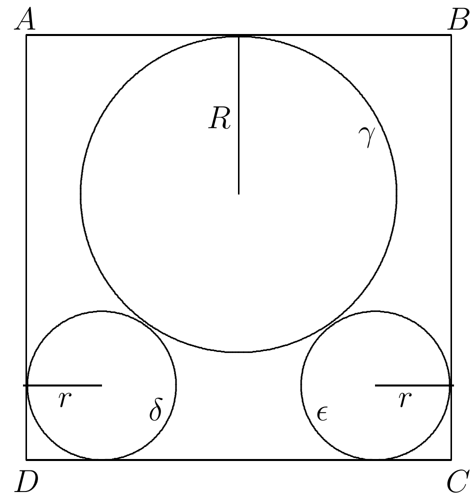
If $F(x)$ is an antiderivative of a continuous function $f(x)$ and $f(x) \geq 0$ on $[b, c]$, then the area under $y = f(x)$ from $x = b$ to $x = c$ is $F(c) - F(b)$. Conversely, if $F(a)$ gives the area under $y = f(x)$ from $x = b$ to $x = a$, where b is fixed, then $F(x)$ is an antiderivative of $f(x)$. ■

14. Three non-overlapping circles γ, δ, ϵ lie within a square $ABCD$ of side 2.

γ is constrained to touch the midpoint of AB.
δ is constrained to touch γ, AD, and CD.
ϵ is constrained to touch γ, BC, and CD.

Let R denote the radius of γ, and r the common radius of δ and ϵ. Let s denote $1 - r$.

From the geometrical constraints it follows that $R = s + \dfrac{s^2}{4}$. This relation may be assumed below.



Let y denote the sum of the area of the three circles.

[2] (a) Express y as a function of s.

ANSWER:
$$y = \pi\left[\left(s + \frac{s^2}{4}\right)^2 + 2(1-s)^2\right]$$

[2] (b) Show that $d^2y/ds^2 > 0$ for all s.

[3] (c) Find an algebraic equation for the value of s which gives the absolute minimum of y.

ANSWER:
$$s^3 + 6s^2 + 24s - 16 = 0$$

[3] (d) Find the value of s which gives the absolute maximum of y.

ANSWER:
$$s = 2(\sqrt{2} - 1)$$

JUSTIFICATION: The minimum and maximum values of s are $s_0 = 1/2$ and $s_1 = 2(\sqrt{2} - 1)$. The latter value corresponds to $R = 1$.

It is clear that $y = \pi R^2 + 2\pi r^2 = \pi\left[\left(s + \dfrac{s^2}{4}\right)^2 + 2(1-s)^2\right]$.

Also, $\dfrac{dy}{ds} = \dfrac{\pi}{4}\left(s^3 + 6s^2 + 24s - 16\right)$ and $\dfrac{d^2y}{ds^2} = \dfrac{3\pi}{4}\left(s^2 + 4s + 8\right)$.

Since the discriminant of the quadratic $s^2 + 4s + 8$ is negative, $\dfrac{d^2y}{ds^2} > 0$ for all s. Also, dy/ds is negative at s_0 and positive at s_1. So the cubic dy/ds has a unique real root which gives the absolute minimum of y. Since $d^2y/ds^2 > 0$, the absolute maximum of y occurs at one of the end points. ■