SFU - UBC - UNBC - UVic

Calculus Challenge Exam

Second Practice, May 10, 2004

Host: SIMON FRASER UNIVERSITY

Student signature

INSTRUCTIONS

1. Show all your work. Full marks are given only when correct, and is supported with a written derivation that is orderly, logical, and complete. Part marks are available in every question.
2. Calculators are optional, not required. Correct answers that are calculator ready, like $3 + \ln 7$ or e^2, are preferred.
3. Any calculator acceptable for the Provincial Examination in Principles of Mathematics 12 may be used.
4. A basic formula sheet has been provided. No other notes, books, or aids are allowed. In particular, all calculator memories must be empty when the exam begins.
5. If you need more space to solve a problem on page n, work on the back of page n - 1.
6. CAUTION - Candidates guilty of any of the following or similar practices shall be dismissed from the examination immediately and assigned a grade of 0:
   (a) Using any books, papers or memoranda.
   (b) Speaking or communicating with other candidates.
   (c) Exposing written papers to the view of other candidates.

Question	Maximum	Score
1	8	
2	6	
3	8	
4	8	
5	6	
6	6	
7	8	
8	6	
9	8	
10	6	
11	9	
12	5	
13	6	
14	10	
Total	100	

1. For each of the following evaluate the limit if it exists and explain why it does not otherwise.

[2] (a) $\lim_{x \to 1^+} \frac{1 - \sqrt{x}}{1 - x}$

ANSWER:

JUSTIFY YOUR ANSWER

[3] (b) $\lim_{x \to 0} \frac{f(x+2) - f(2)}{x}$
where $f'(x) = x^2 + \ln(x-1)$

ANSWER:

JUSTIFY YOUR ANSWER

[3] (c) $\lim_{x \to 0} (\cos x)^{1/x^2}$

ANSWER:

JUSTIFY YOUR ANSWER

[3] **2.** (a) Let $f(x) = \dfrac{\sqrt{x}-1}{\sqrt{x}+1}$.

Find $f'(9)$.

ANSWER:

JUSTIFY YOUR ANSWER

[3] (b) Let $f(x) = \sqrt{1+\sqrt{x}}$.

Find $f'(9)$.

ANSWER:

JUSTIFY YOUR ANSWER

3. Let C be the curve in the xy-plane which is the graph of the equation $y = x^3 - x^2 + 1$, and P be $(1, 1)$. Let l be the line which is tangent to C at P.

[4] (a) Find the equation of l.

ANSWER:

EXPLANATION

[4] (b) Find another line tangent to C which is parallel to l.

ANSWER:

EXPLANATION

[4] **4.** (a) The number $2^{1/3}$ is being approximated by applying Newton's method to the function $y = x^3 - 2$ with initial estimate $x_0 = 1$.

What are the next two estimates? Give your answers as rational numbers.

ANSWERS:

$x_1 =$

$x_2 =$

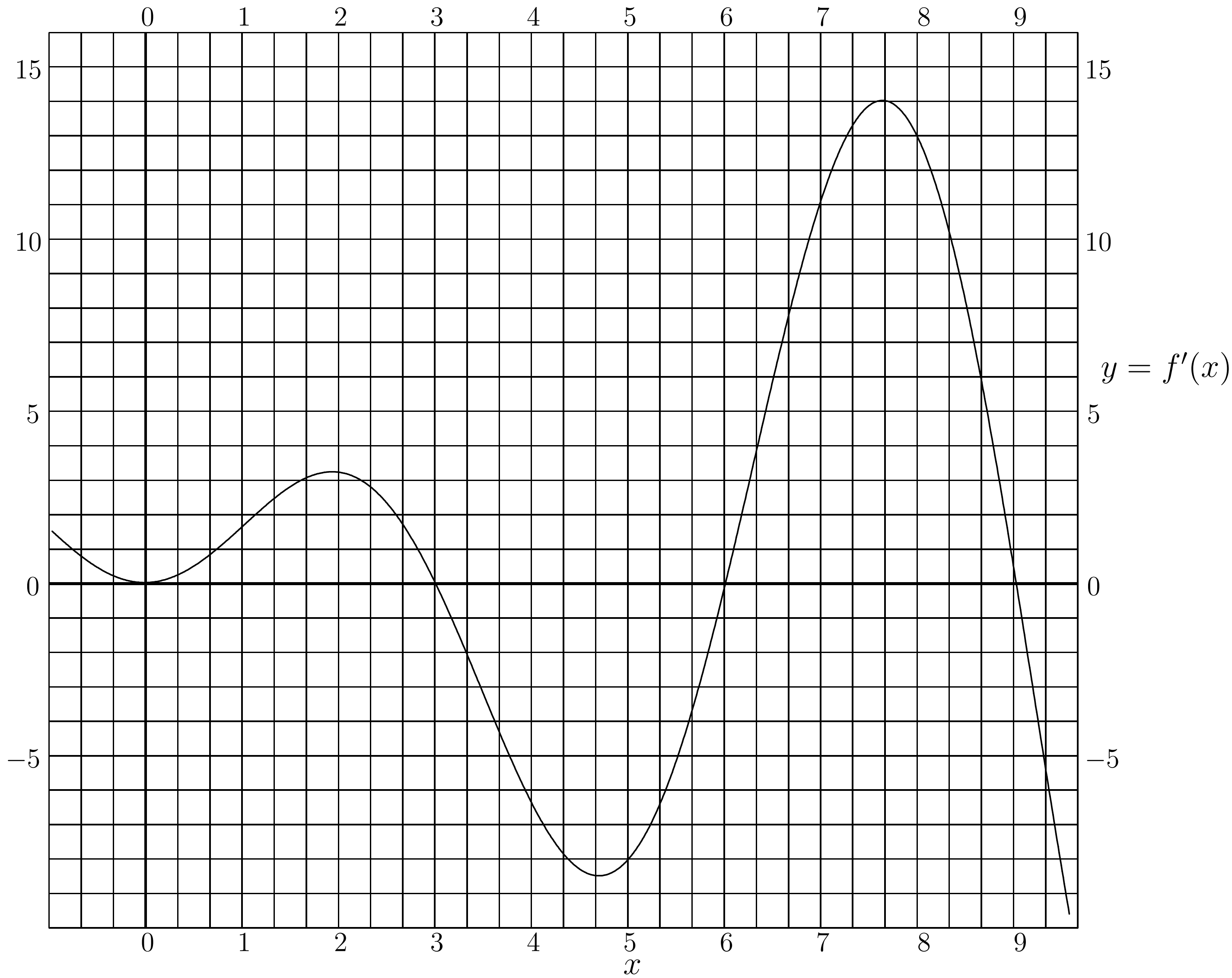
SHOW YOUR WORK

[4] (b) Suppose that Newton's method is being used to approximate the zero r of $f(x) = 0$ with initial estimate $x_0 < r$. By "zero" we mean that $f(r) = 0$.

It is given that $f(x_0) > 0$, and that $f'(x)$ is defined and increasing on $[x_0, r]$.

Using the Mean Value Theorem, or otherwise, explain carefully why the next estimate x_1 satisfies $x_0 < x_1 < r$.

EXPLANATION



[6] **5.** Above is shown the graph of $y = f'(x)$ on the interval $I = [-1, 9.5]$. The graph touches the axis at $x = 0$.

By inspection of the graph estimate the values of x in I at which $f(x)$ attains local maxima/minima. Also, estimate the estimate the values of x in I at which $f(x)$ has points of inflection.

No explanation need be given.

x-coordinates of local minima of $f(x)$	x-coordinates of local maxima of $f(x)$	x-coordinates of points of inflection of $f(x)$

[6] **6.** One model of population growth is represented by the differential equation

$$\frac{dP}{dt} = kP(A - P) \qquad (1)$$

where P is the population, t is time, and k and A are positive constants.

The equation (1) has some solutions of the form

$$P = \frac{Be^{Akt}}{1 + Ce^{Akt}}. \qquad (2)$$

What relation between the constants A, B, C implies that the function P given by (2) satisfies (1).

ANSWER:

EXPLANATION

7. A piece of wire of length 1 metre is cut into two pieces, where one of the pieces may have zero length. One piece is bent into a square, the other into a circle. A denotes the sum of the areas of the resulting square and circle.

[4] (a) How should the wire be cut so as to maximize A? Give the perimeter of the square as answer.

ANSWER:

EXPLANATION

[4] (b) How should the wire be cut so as to minimize A? Give the perimeter of the square as answer.

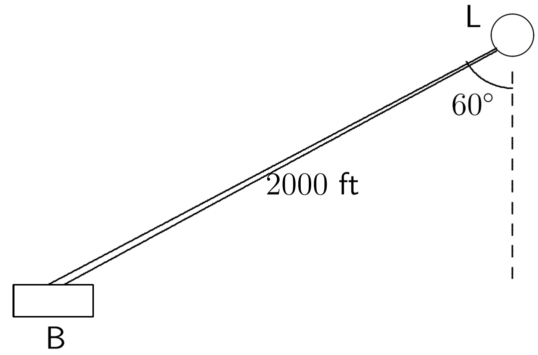
ANSWER:

EXPLANATION

[6] **8.** A beam of light from the lighthouse L strikes the north face of the building B 2000 feet away. From the lighthouse the building bears $60°$ west of south.

The beam rotates clockwise at a rate of $12°$ per second.

At how many feet per second does the beam travel across the north face of the building?



ANSWER:

SHOW YOUR WORK

9. A ball is thrown vertically into the air with initial veloity v_0. The maximum height reached is h.

The equation governing the motion of the ball is

$$\frac{d^2x}{dt^2} = -g \tag{1}$$

where g is a constant and x is the height of the ball t seconds after release.

[4] (a) Find expressions for x and dx/dt in terms of t and v_0.

ANSWER:

EXPLANATION

[4] (b) What initial velocity is required for the maximum height attained to be $4h$? Give your answer as a multiple of v_0.

ANSWER:

EXPLANATION

10. Let $f(x)$ be a function defined for all x in $(-\infty, \infty)$. Let a be a real number.

[2] (a) In terms of a limit define what it means for $f(x)$ to be *continuous at* $x = a$.

DEFINITION

[2] (b) In terms of a limit define what it means for $f(x)$ to be *differentiable at* $x = a$.

DEFINITION

[2] (c) Show that, if $f(x)$ is differentiable at $x = a$, then $f(x)$ is contiuous at $x = a$.

EXPLANATION

11. Evaluate the following antiderivatives:

[3] (a) $\int e^{2x+2}\,dx$

ANSWER:

SHOW YOUR WORK

[3] (b) $\int \frac{1}{(1-2x)^3}\,dx$

ANSWER:

SHOW YOUR WORK

[3] (c) $\int \sec(\theta/2)\tan(\theta/2)\,d\theta$

ANSWER:

SHOW YOUR WORK

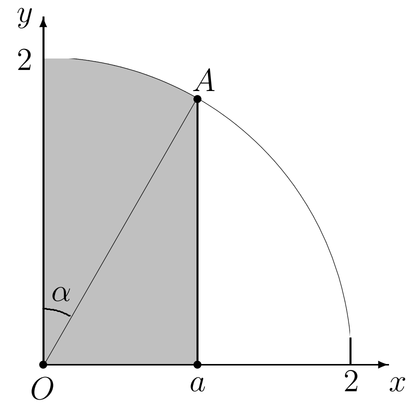
[5] **12.** The graph of $y = f(x)$ passes through $(1, -1)$, and for all $a > 0$ the slope of the graph at the point $(a, f(a))$ of the graph is $1/a$. Find $f(3)$.

ANSWER:

EXPLANATION

13. Consider the area under the arc of the circle $x^2 + y^2 = 4$, from $x = 0$ to $x = a$ in the first quadrant, shown in the figure.

A is the point $\left(a, \sqrt{4 - a^2}\right)$, and α is $\angle AOy$.



[2] (a) Write α as a function of a.
No explanation is required.

ANSWER:

[2] (b) By considering two pieces, or otherwise, express the shaded area as function of a.

ANSWER:

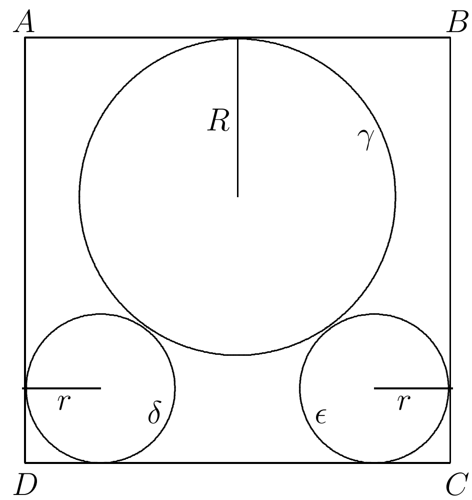
SHOW YOUR WORK

[2] (c) From part (b) find an antiderivative for $\sqrt{4 - x^2}$.

ANSWER:

SHOW YOUR WORK

14. Three non-overlapping circles γ, δ, ϵ lie within a square $ABCD$ of side 2.



γ is constrained to touch the midpoint of AB.
δ is constrained to touch γ, AD, and CD.
ϵ is constrained to touch γ, BC, and CD.

Let R denote the radius of γ, and r the common radius of δ and ϵ. Let s denote $1 - r$.

From the geometrical constraints it follows that $R = s + \dfrac{s^2}{4}$. This relation may be assumed below.

Let y denote the sum of the areas of the three circles.

[2] (a) Express y as a function of s.

ANSWER:

$y =$

SHOW YOUR WORK

[2] (b) Show that $d^2y/ds^2 > 0$ for all s.

EXPLANATION

[3] (c) Find an algebraic equation for the value of s which gives the absolute minimum of y.

ANSWER:

JUSTIFICATION

[3] (d) Find the value of s which gives the absolute maximum of y.

ANSWER:

$s =$

JUSTIFICATION