Approximation Algorithms (Load Balancing)

July 16, 2014

Problem Definition :

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Each job J_i has a processing time $t_i \geq 0$.

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Goal :

We want to assign (load) the jobs to machines such that the maximum load is minimized.
In other words, we would like to balance the loads.

Let $A(i)$ be the set of jobs that are assigned to M_i. Then the load of M_i denoted by $T_i = \sum_{j \in A(i)} t_i$.

We wish to minimize $T = \max_i T_i$.

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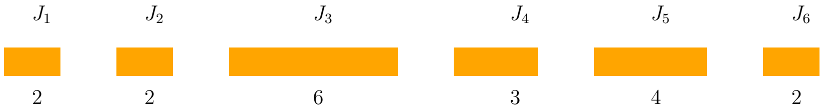
The load balancing problem in *NP*-complete. Even when there are two machines.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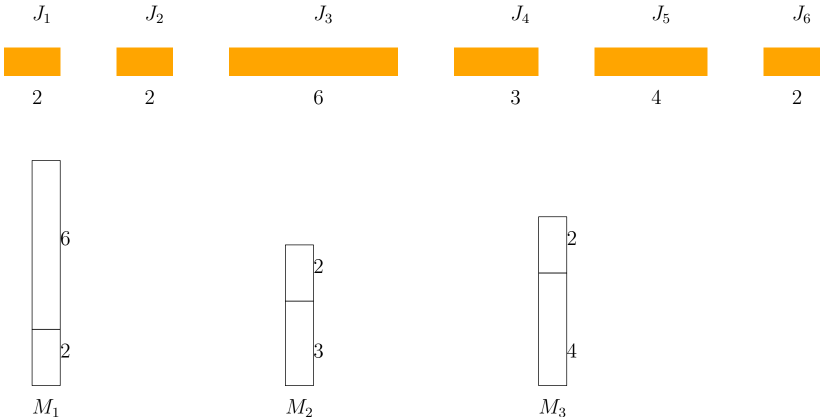
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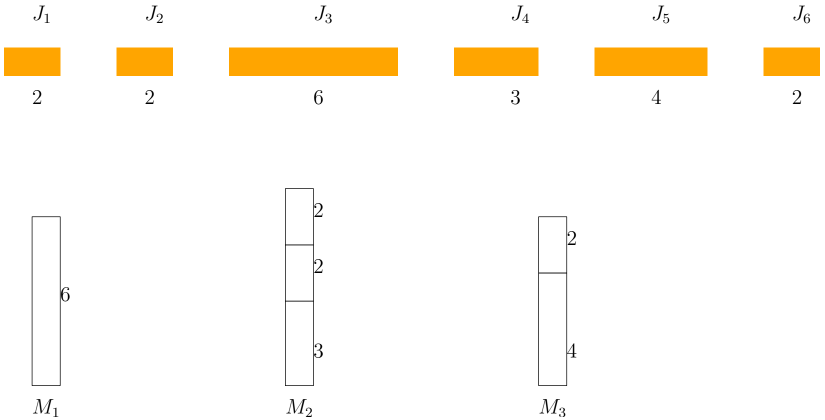
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M_1
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Greedy-Balance

1. Set $T_i = 0$ and $A(i) = \emptyset$ for all machines M_i.
2. **for** $j = 1$ to n
3. Let M_i be a machine with minimum load $(\min_k T_k)$.
4. Assign job j to machine M_i.
5. Set $A(i) \leftarrow A(i) \cup \{J_j\}$
6. Set $T_i \leftarrow T_i + t_j$

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Consider the time we add job j into machine M_i. The load of machine M_i was $T_i - t_j$ before adding J_j to M_i. Also $T_i - t_j$ was the smallest load. Every other machine has load at least $T_i - t_j$. Therefore :

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$$m(T_i - t_i) \leq \sum_k T_k$$

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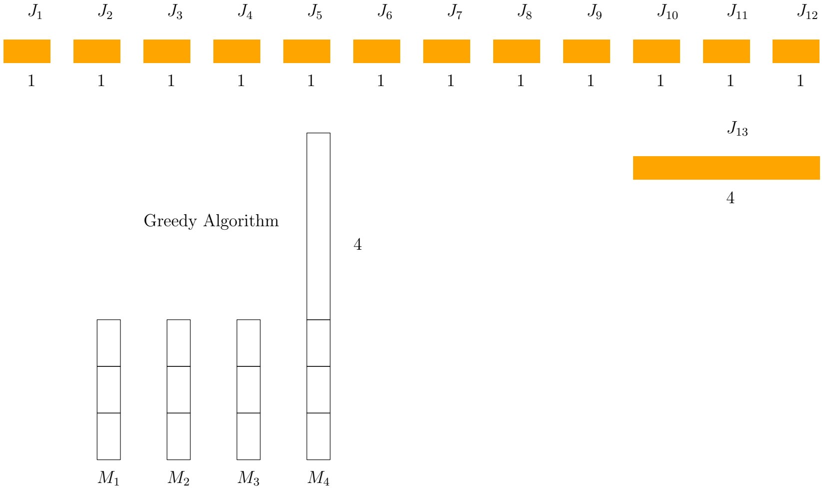
$T_i - t_j \leq \frac{1}{m} \sum_j t_j \leq T^*$. Also we know that $t_j \leq T^*$.

Therefore load of M_i after adding J_j is $T_i = (T_i - t_j) + t_j \leq 2T^*$. $\square$

The Greedy-Balance could actually be as close as possible to $2T^*$.



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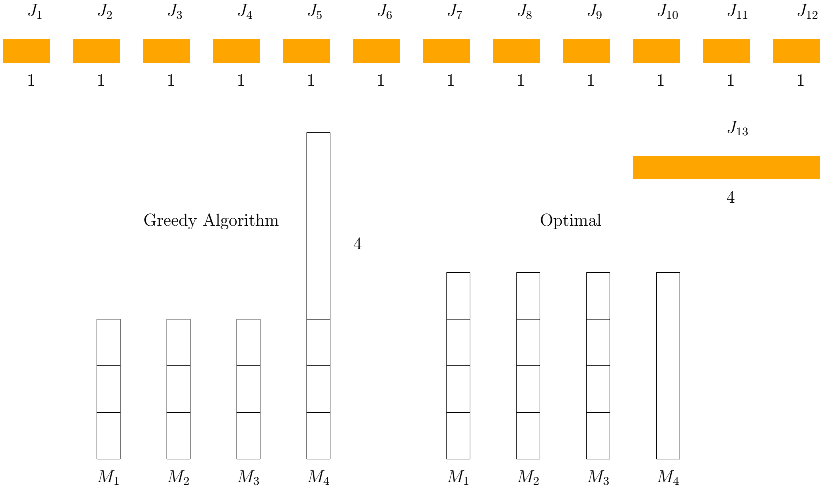

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The first $m(m-1)$ jobs each with time $t_j = 1$ and the last job $n = m(m-1)+1$ has time $t_n = m$.

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The optimal has $T^* = m$ while the Greedy algorithms has max load $2m-1$.

An Improved Approximation Algorithm

Sort-Balance

1. Set $T_0 = 0$ and $A(i) = \emptyset$ for all machines M_i.
2. Sort the jobs in decreasing order of processing times t_j.
3. Assume $t_1 \geq t_2 \geq ... \geq t_n$.
4. **for** $j = 1$ to n
5. Let M_i be a machine with minimum load ($\min_k T_k$).
6. Assign job j to machine M_i.
7. Set $A(i) \leftarrow A(i) \cup \{J_j\}$
8. Set $T_i \leftarrow T_i + t_j$

If there are more than m jobs, then $T^* \geq 2t_{m+1}$.

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Lemma

Algorithm Sort-Balance produces an assignment of jobs to machines with max load $T \leq \frac{3}{2}T^*$.

Using similar analysis as in the previous lemma (leave it as exercise)

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But the running time of the algorithm is

$$\mathcal{O}(n^{(\frac{1}{\epsilon})^{1.5}})$$

where n is the number of jobs.