Practice and Exercises #1

Part 1: Logic, Proofs, and Set Theory

- **Exercise 1:**

Let $\mathbb{Q}$ be the set of all rational numbers. If $x \in \mathbb{Q}$, which of the following statement imply which of the others? Justify your answer.

P $x \geq 0$

Q $\exists w \in \mathbb{Q}$ such that $x = w^2$

R $nx > -1 \ \forall n \in \mathbb{N}$

S $\exists n \in \mathbb{N}$ such that $nx > -1$

- **Exercise 2:**

Prove by induction that

$$(1+a)^n \geq 1 + na + \frac{1}{2}n(n-1)a^2$$

whenever $a > 0$ and $n \in \mathbb{N}$.

- **Exercise 3:**

Let $I = \{x \in \mathbb{R} \ : \ 0 \leq x \leq 2\}$ and $J = \{x \in \mathbb{R} \ : \ 1 \leq x \leq 3\}$. Illustrate in a single diagram the sets $I \times J$ and $J \times I$.

- **Exercise 4:**

Consider 3 statements A, B, C. Prove the following:

$$(A \vee (B \wedge C)) \Leftrightarrow ((A \vee B) \wedge (A \vee C))$$

- **Exercise 5:**

Let

$$A = \{x \in \mathbb{R} : 1 \leq x^2 \leq 4\} \qquad B = \{x \in \mathbb{R} : 1 < x^2 \leq 4\}$$

Is A bounded above? if so find $\sup A$. Is A bounded below? if so find $\inf A$. Does A have a greatest element? if so find it. Does A have a least element? if so find it. Repeat for set B.

- **Exercise 6:**

Let $A = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 < 2\}$, $B = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 > 2\}$. Prove that if $a \in A$ and $b \in B$ then $a < b$. All numbers mentioned in your proof should be rational.

- **Exercise 7:**

Let $A = \{x \in \mathbb{Q} : x > 0 \text{ and } x^2 < 2\}$.

1. Show that A is nonempty and bounded above. Let $s \in \mathbb{R}$ such that $s = \sup A$.
2. Let v be a positive real number such that $v^2 > 2$. Prove that $\exists y \in \mathbb{R}$ such that $0 < y < v$ and $y^2 > 2$.
3. Let v and y be as above. Show that y is an upper bound for A. Deduce that $v > s$.
4. Let u be a positive real number such that $u^2 < 2$. Prove that $\exists w \in \mathbb{R}$ such that $u < w$ and $w^2 < 2$.
5. Let u and w be as above. Using the axiom of Archimedes[1], show that there is a rational number x such that $u < x < w$. Deduce that u is not an upper bound of A.
6. Using 3 and 5 above, show that $s^2 = 2$.
7. Show how your answers may be modified to prove that every positive real number has a square root.

- **Exercise 8:**

Let X be a nonempty set and let d be a function from $X \times X$ to $\mathbb{R}$ with the following 2 properties:

P1 $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$;

P2 $d(x, y) = 0$ if and only if $x = y$.

Prove that d is a metric on X.

- **Exercise 9:**

1. Given a nonempty set X, prove that the following is a metric on X:

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

NB: this is the discrete metric.

[1]For every $x \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that $N > x$.

2. Let the set X consist of the object w and at least one other member. Prove that the following is a metric on X:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x = w \text{ and } y \neq w \\ 1 & \text{if } x \neq w \text{ and } y = w \\ 2 & \text{otherwise} \end{cases}$$

- **Exercise 10:**

All questions are independent.

1. Let X be a metric space. Prove that the set $\varnothing \subset X$ is open, closed, bounded and compact.
2. Why does every set in $\mathbb{R}$ that is nonempty, closed, and bounded above have a greatest member?
3. Let P be the set of all positive real numbers, with the obvious metric. Prove that the "half-open" interval $(0,1] = \{x \in \mathbb{R} : 0 < x \leq 1\}$ considered as a subset of the metric space P is closed, bounded but not compact.
4. Let X and Y be sets of points in $\mathbb{R}^l$ and $\mathbb{R}^m$ respectively; then their cartesian product $X \times Y$ may b regarded as a subset of $\mathbb{R}^{l+m}$. prove the following:
   (a) If X and Y are bounded sets, so is $X \times Y$.
   (b) If X and Y are closed sets, so is $X \times Y$.
   (c) If X and Y are compact sets, so is $X \times Y$.