Practice and Exercises #2

Part 2: Elements of Analysis

- **Exercise 1:**

Let the functions $f(.)$ and $g(.)$ be defined for all $x \in \mathbb{R}$ by

$$f(x) = \begin{cases} |x| & \text{if } x < 1, \\ 1 & \text{if } x \geq 1, \end{cases} \qquad g(x) = \begin{cases} x^2 & \text{if } x < 2, \\ 4 & \text{if } x \geq 2, \end{cases}$$

1. Determine the domain and range of $y = f(x)$, $y = g(x)$, $y = f(g(x))$, and $y = g(f(x))$.

2. Sketch the graphs of the above 4 functions.

3. Determine the inverse of each function (when possible).

- **Exercise 2:**

All questions are independent.

1. Assume that $x_n \to x$ and u is a real number such that $x_n < u$ for all n. Can we be sure that $x \leq u$? Can we be sure that $x < u$? Justify.

2. Show that if $a_n \leq x_n \leq b_n$ for all n with $a_n \to x$ and $b_n \to x$ then $x_n \to x$.

   Note: this result is often referred to as the "sandwich theorem".

3. Let $\{x_n\}$ be any sequence of real numbers. Which of the following propositions are true?

   (a) If $\lim_n x_n = x$ then $\lim_n x_{n+1} = x$.

   (b) If $\lim_n x_n = x$ then $\lim_n x_{2n} = x$.

   (c) The converse of (a).

   (d) The converse of (b).

4. Show that if $x_n \to x$ then $|x_n| \to |x|$. Hence, show that if $f : I \to \mathbb{R}$ is a continuous function, then the function $|f|$ defined by $|f|(x) = |f(x)|$ is also continuous.

- **Exercise 3:**

Let I be an interval and let f be a function from I to $\mathbb{R}$. Recall that f is continuous if and only if, for any $x_0 \in I$. there exists a real number $\delta > 0$ such that

$$|f(x) - f(x_0)| < \epsilon \text{ whenever } x \in I \text{ and } |x - x_0| < \delta$$

In general, δ depends both on ϵ and x_0. We say that f is **uniformly continuous** if, for each $\epsilon > 0$ δ can be chosen so that it depends only on ϵ and not on x_0. The precise definition is as follows: the function $f : I \to \mathbb{R}$ is uniformly continuous if, for any $\epsilon > 0$, there exists a real number $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \epsilon \text{ whenever } x_1, x_2 \in I \text{ and } |x_1 - x_2| < \delta$$

1. Using the precise definition, prove that any uniformly continuous function $f : I \to \mathbb{R}$ is also continuous.

2. Prove that if I is the closed interval $[a, b]$, any continuous function $f : I \to \mathbb{R}$ is uniformly continuous. *Hint: use contraposition.*

3. Let I be the open interval $(0, 1)$ and let $f(x) = 1/x$ $\forall x \in I$. Prove that f is continuous but not uniformly continuous.