Practice and Exercises #3

Part 3: Linear Algebra and Matrices

- **Exercise 1:**

Consider 2 conformable matrices $A = [a_{ij}]$ and $B = [b_{ij}]$. Prove that $(AB)' = B'A'$.

- **Exercise 2:**

Find examples illustrating the facts stated in Lemma 1.3 p4.

- **Exercise 3:**

Characterize the set of all bases in $\mathbb{R}$. Do the same for $\mathbb{R}^n$. Persuade yourself that this is the set of all nonsingular $n \times n$ matrices.

- **Exercise 4:**

Use the properties of transpose and inverse to prove that:

1. $A^{-k} = (A^k)^{-1}$.
2. Consider the matrix $Z = X(X'X)^{-1}X'$ where X is an arbitrary $m \times n$ matrix.
   (a) Can you simplify the expression for Z?
   (b) Show that Z is symmetric.
   (c) Show that Z is idempotent.

- **Exercise 5:**

Let A be a $n \times n$ matrix and λ a real number. Find the relationship between $\det(A)$ and $\det(\lambda A)$. Formal proof required.

- **Exercise 6:**

Let A be $m \times n$ matrix. We are using the notations introduced in section 3.3.

1. Let $m = n$, and consider $Ax = b$. Suppose that $|A| = 0$, so that the system is either underdetermined or overdetermined. What of the two cases arises if $b = 0$? And what happens if $b \neq 0$?
2. Consider now $m > n = rank(A)$ and characterize the appropriate partition that gives $X^* = \{\tilde{A}^{-1}\tilde{b}\}$.

- **Exercise 7:**

All questions are independent. Let X be an arbitrary $m \times n$ real matrix.

1. Show that $X'X$ is positive semidefinite.
2. Suppose $m \geq n$ and $rank(X) = n$. Show that $X'X$ is positive definite.
3. Show that a positive definite matrix is nonsingular.
4. Show that if X is symmetric and idempotent, then X is also positive semi-definite.