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Econ 798, Introduction to Mathematical Economics
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(based on the notes of Prof. A. Karaivanov)

Part 1: Logic, Proofs and Set theory

1 Logic

- Economics uses **mathematics and formal logic** as its main methodology nowadays.
- **Logic** is the set of accepted rules to "sensibly" connect statements.
- **Mathematics** is simply a language used to make statements - that is, a shorthand notation that we use to connect a set of accepted axioms and assumptions (e.g. 1+1=2) to a result that we want to prove. If we are careful, we could do this whole process in any other language (e.g. English) although spoken languages may lack some special "words" that mathematics has.
- Think of mathematics as a **special language based on logic**.
- All theorems or results we prove in economics (or math) are logically equivalent to the axioms we start with and the assumptions we make. The results are nothing but a transformation of the assumptions following the rules of logic. Thus, (assuming the math is right) one cannot criticize a result one does not agree with - one has to find which assumption/axiom one does not like or agree with.
- Logic: "playing" with statements:
  - example: (A) all living individuals have heads and (B) John is a living individual **imply** (C) John has a head
  - BUT: (A) it is always cloudy when it rains and (B) it is not raining **do not imply** (C) there are no clouds.

1.1 Logical operators

- **AND** ("conjunction"): "A and B" means both A and B are (assumed) true.
  Written: $A \wedge B$
- **inclusive OR** ("inclusive disjunction"): either A or B or both are true.
  Written: $A \vee B$

- **exclusive OR** ("exclusive disjunction"): either A or B but not both are true.
  Written $A \oplus B$

- **negation**: the logical opposite of A.
  Written: $\sim A$

1.2 Truth tables

- a simple way to remember how logical operators work. Below T denotes "true", F denotes "false".

A	B	$A \wedge B$	$A \vee B$	$A \oplus B$	$\sim A$
T	T	T	T	F	F
T	F	F	T	T	F
F	T	F	T	T	T
F	F	F	F	F	T

- **Example:** if $x > 5$ and $x < 7$ then $8 > x > 4$ is true.

- **Exercise:** show that $A \oplus B = (A \wedge \sim B) \vee (\sim A \wedge B)$
  hint: add columns for each step to the table above.

- The basic Aristotelian principle is that a statement is either true or false - no in-between!

- **De Morgan Laws** (prove using truth tables)
  - L1: $\sim (A \wedge B) = (\sim A) \vee (\sim B)$
  - L2: $\sim (A \vee B) = (\sim A) \wedge (\sim B)$

1.3 More logical operators

- **Implication:** "if A then B".
  Written: $A \Rightarrow B$
  - means that whenever A is true then B is also true.
  - says nothing about whether B is true when A is false!
  - $A \Rightarrow B$ also means that A is a *sufficient condition* for B or that B is a necessary condition for A.
  - A being *sufficient condition* for B means that to prove B is true, it is enough (sufficient) to show that A is true.

- B being *necessary condition* for A means that if A is true then B must be necessarily true; however B being true may not be sufficient for A to be true (other conditions may be needed).
- **Example:** if $x < 0$ then $x < 2$

- **Converse:** $B \Rightarrow A$ is the converse of $A \Rightarrow B$. It is possible one of these statements to be true but not the other.
  - **Example:** "if it rains then it is cloudy"; but "it is cloudy" does not imply that "it rains".

- **Equivalence:** "A if and only if B"
  Written $A \Leftrightarrow B$ (two-way implication)
  - means $A \Rightarrow B$ and $B \Rightarrow A$
  - **Example:** $x > 0 \Leftrightarrow 2x > 0$ is true, but $x > 0 \Leftrightarrow x^2 > 0$ is false.
  - Exercise: show that $A \Rightarrow B$ is equivalent to $(\sim B) \Rightarrow (\sim A)$

- **Contrapositive:** $(\sim B) \Rightarrow (\sim A)$ is called the contrapositive of $A \Rightarrow B$. It is a very useful tool for proofs.
  - **Example:** "If it is raining then it is cloudy" is logically equivalent to "If it is not cloudy then it is not raining"

- **Logical quantifiers** (additional useful notation)
  - "for all/any", denoted $\forall$
  - "there exists", denoted $\exists$
  - "such that", denoted by $:$ (or "s.t.")
  - "defined as", denoted by $\equiv$
  - **Examples:**
    * "all living individuals have heads"; the statement is true for each/any living person
    * as opposed to: "there exists a person who is tall" - need to find only one such person to be true.
    * "the square of any real number is positive" is FALSE; the square of any real number is non-negative" is TRUE.
    * $\forall$ and $\exists$ are related: $\forall x$ real number, $x^2 \geq 0$ is equivalent to $\sim \exists x$ (does not exist x real number) such that $x^2 < 0$

      NB: if some statement is true for all x, then there does not exist an x such that this same statement is not true.

2 Methods of proof

- We use the logic rules and notation described above to construct "proofs" - chains of statements connected by logical operators. We start by some assumptions (plus a set of accepted axioms) and transform them along the way to the result we want to show. Remember: if the proof is correct, the end result should be logically equivalent to the set of assumptions made in the beginning!

- Suppose we want to prove that $A \Rightarrow B$. What options do we have available?

1. **Direct proof (proof by deduction)**
   - find a sequence of statements $A_1, A_2, \cdots$ such that $A \Rightarrow A_1 \Rightarrow A_2 \Rightarrow \cdots \Rightarrow B$

2. **Proof by contradiction (indirect proof)**
   - use the contrapositive: to prove that $A \Rightarrow B$, prove instead the equivalent statement $\sim B \Rightarrow \sim A$
   - **Example:** if (A) n is a prime number different from 2, then (B) n is odd. The contrapositive is: if "n is not odd", then "n is not a prime or n is not equal to 2".
   - indirect proofs are often much easier than direct ones. Why? b/c you just need to find a contradiction. It is like replacing "for all" with "there exists".
   - *Illustration:*

     (a) if x and y are real numbers, then $(x + y)^2 = x^2 + y^2 + 2xy$

     (b) if x and y are real numbers, then $(x + y)^2 = x^2 + y^2$

     Clearly, (a) is true and (b) is not! It is important to realize that proving (a) needs to be done algebraically with x and y interpreted as **arbitrary real numbers**: e.g. verifying that the numerical conclusion holds for *some* numbers is not enough.
     By contrast, to prove (b) is false, it suffices to find **one** counterexample - ie one pair of real numbers for which the statement does not hold.
   - **Example:** show that $\sqrt{2}$ is an irrational number.

3. **Proof by induction**
   - useful for statements that can be indexed by integers
   - a tricky method but very powerful tool sometimes
   - the so-called induction principle: consider a sequence of statements, P(1), P(2) indexed by the natural numbers
   - suppose we can verify that:

    * statement P(1) is TRUE
    * whenever statement $P(k)$ is true for any k integer then $P(k+1)$ is also true.
  - the induction principle then says ("leap of logic") that all P are true (until k infinity)
  - **Example:** show that for any integer n, $1+2+\cdots+n = n(n+1)/2$.

3 Set theory basics

3.1 What is a set?

- **a set** is simply a well-defined collection of objects - an abstract tool:
  - the "set of students in our class";
  - the "set of the days of the week";
  - the "set of prime numbers", ...
- a set can have finitely or infinitely many elements

3.2 Set notation

- $a \in A$ means "a is an element of the set A".
  - $a \notin A$ means "a is not an element of the set A".
  - $A = \{a, b, c\}$: the set consists of three elements, a, b, c.
  - $A = \{x \in \mathbb{R}\ s.t.\ x+5 > 0\}$: the set A consists of all the real numbers x s.t. $x+5 > 0$.
  - **the empty set** (set with no element) is denoted $\emptyset$
    * **Example:** the set $A = \{x \in \mathbb{R}\ s.t.\ x^2 < 0\}$ is empty.
  - $B \subseteq A$ means "B is a subset of A: this means that all elements of set B are also elements of set A; but some elements of A may not be in B.
    * **Example:** the set $B = \{x \in \mathbb{R}\ s.t.\ x > 5\}$ is a subset of $\mathbb{R}$.
    * **Exercise:** show that if $A \subseteq B$ and $B \subseteq A$ then $A = B$.
  - $A \nsubseteq B$ means A is not a subset of B: this means that at least one element of A is not in B.

3.3 Some commonly used sets

- the set of real numbers (reals) $\mathbb{R}$
  - the set of integers $\mathbb{Z}$

- the set of natural numbers $\mathbb{N}$
- the set of rational numbers (numbers that can be expressed as fractions) $\mathbb{Q}$
- the set of non-negative real numbers $\mathbb{R}_+$ (or $\{x \in \mathbb{R}\ s.t.\ x \geq 0\}$).
- the set of positive real numbers $\mathbb{R}_+^*$ (or $\mathbb{R}_{++}$ or $\{x \in \mathbb{R}\ s.t.\ x > 0\}$).
- the set of n-dimensional vectors $\mathbb{R}^n$ (think about them as an ordered collection of n real numbers).

3.4 Set operations

1. **Union**: $A \cup B$ is the set all elements that are either in A or in B or in both: $A \cup B = \{x : x \in A \vee x \in B\}$.
   - if applied to many sets, we write: $C = \bigcup_{i=1}^{n} A_i$ (and n can be finite or infinite).
2. **Intersection**: $A \cap B$ is the set of all elements common to (in) both A and B (this can be the empty set!): $A \cap B = \{x : x \in A \wedge x \in B\}$.
   - if applied to many sets, we write: $C = \bigcap_{i=1}^{n} A_i$ (and n can be finite or infinite).
3. **Subtraction**: $A \setminus B$ is the set of all elements of A that are not in B: $A \setminus B = \{x : x \in A \wedge x \notin B\}$.
4. **Complement**: in many cases, all sets we study are subsets of some "universal set", say U (e.g. $U = \mathbb{R}$, the set of all real numbers). Then we write: $A^c \equiv U \setminus A$
   - we have: $A \cup A^c = U$ (the Aristotelian principle)

- **Examples**: let $A = \{x \in \mathbb{R} : -4 < x \leq 3\}$ and $B = \{x \in \mathbb{R} : -1 \leq x < 7\}$. Then,
  - $A \cup B = \{x \in \mathbb{R} : -4 < x < 7\}$;
  - $A \cap B = \{x \in \mathbb{R} : -1 \leq x \leq 3\}$;
  - $A \setminus B = \{x \in \mathbb{R} : -4 < x < -1\}$;
  - $A^c = \{x \in \mathbb{R} : x \leq -4\ or\ x > 3\}$;
- **Venn diagrams**: a useful way to represent sets and perform set operations visually.
  - draw a set, union, intersecion, subtraction, complement.
- **Exercise**:
  - show that $(A \cap B) \cap C = A \cap (B \cap C)$ and $(A \cup B) \cup C = A \cup (B \cup C)$.

– show that $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$.

– show that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.

– show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

– show that $(A \cup B)^c = A^c \cap B^c$

3.5 Ordered sets

Let S be a set. An **order** on S is a relation (denoted "$<$") with the following properties:

(i) **Completeness:** if $x \in S$ and $y \in S$ then one and only one of these 3 statements is true, $x < y$, or $x = y$, or $y < x$.

(ii) **Transitivity:** if $x, y, z \in S$ then if $x < y$ and $y < z$ then $x < z$.

- **Definition:** an **ordered set** is a set on which an order has been defined.

- **Example**: $\mathbb{R}$ is an ordered set if $x < y$ is defined to mean $y - x$ is a positive real number.

3.6 Metric spaces

Metric spaces are among the most important types of sets we use in economics.

- **Definition:** a set X whose elements we will call *points* is a **metric space** if, for any 2 points p and q in X, there is an associated real number $d(p, q)$ called *distance* (or *metric*) from p to q defined such that,

  (a) $d(p, q) > 0$ if $p \neq q$ and $d(p, p) = 0$

  (b) $d(p, q) = d(q, p)$

  (c) $d(p, q) \leq d(p, r) + d(r, q)$ for any $r \in X$

- **Example:** the set of real numbers $\mathbb{R}$ with the metric $d(x, y) = |x - y|$ (also called the Euclidian distance) where $|z|$ denotes the absolute value of z is a metric space. The set of real n-dimensional vectors $\mathbb{R}^n$ is a metric space too.

- **Open (closed) ball:** if $x \in \mathbb{R}^n$ and r positive real number, the open (closed) ball B with center at x and radius r is defined as the set $\{y \in \mathbb{R}^n : d(x, y) < r\}$ where d is the Euclidian distance in $\mathbb{R}^n$.

3.7 Convex sets

- **Definition:** a set $E \subseteq \mathbb{R}^n$ is convex if:

$$\lambda x + (1-\lambda)y \in E$$

whenever $x \in E$, $y \in E$, and $0 < \lambda < 1$.

- **Examples:** $\mathbb{R}^n$ is convex; $\mathbb{R}^n_+$ is convex.

- Exercise:
  - show that all (open or closed) balls are convex.
  - Exercise: show that the intersection of convex sets is a convex set

3.8 Closed, Open, Bounded, and Compact sets

- **Definitions:** let X be a metric space. All points and sets mentioned below are elements or subsets of X.

  (a) a **neighborhood** of point $p \in X$ is a set $N_r(p)$ consisting of all q s.t. $d(p,q) < r$ for some $r > 0$ (note that this is more general than the definition of a ball! Why?)

  - Neighborhoods are: segments in $\mathbb{R}^1$, circles in $\mathbb{R}^2$, and spheres in $\mathbb{R}^3$.

  (b) a point $p \in X$ is a **boundary** point of the set E if every neighborhood $N_r(p)$ of p contains a point $q \neq p$ such that $q \notin E$ and a point $z \neq p$ s.t. $z \in E$.

  (c) a set E is **closed** if all its boundary points are in E (i.e., the set "contains its boundary").

  (d) a point p is an **interior** point of E if there exists a neighborhood $N_r(p)$ of p such that $N_r(p) \subseteq E$.

  (e) a set E is **open** if every point of E is an interior point of E.

  (f) a set E is **bounded** if $\exists M \in \mathbb{R}$ and a point $q \in X$ s.t. $d(p,q) < M$ for all $p \in E$.

- **Put simply:**
  - A set is closed if it contains its boundary.
  - A set is open if it does not (contain its boundary).

- **Theorem:** *every neighborhood is an open set.* (Rudin, Thm 2.19)

– Proof: take a neighborhood[1] $E = N_r(p)$ and let q be any point in E s.t. $q \neq p$. Then, there is a positive real number h s.t. $d(p,q) = r - h$. For all points s s.t. $d(q,s) < h$, we have $d(p,s) \leq d(p,q) + d(q,s) < r - h + h = r$. This implies that $s \in E$. Thus, all points s in the neighborhood with radius $h > 0$ around q belong to E. This implies that q is an interior point of E.

- **Example:** exercise 2.21 in Rudin p33.
- **Theorem:** *a set is open if and only if its complement is closed.*
  - Proof: 2.23 in Rudin p34.
- **Theorem:**
  - for any collection $\{G_i\}$ of open sets, the set $\bigcup_i G_i$ is open.
  - for any collection $\{G_i\}$ of closed sets, the set $\bigcap_i G_i$ is closed.
  - for any finite collection $\{G_1, G_2, \cdots, G_n\}$ of open sets, the set $\bigcap_{i=1}^n G_i$ is open.
  - for any finite collection $\{G_1, G_2, \cdots, G_n\}$ of closed sets, the set $\bigcup_{i=1}^n G_i$ is closed.
- **Definition (compact set):** a set $S \subseteq \mathbb{R}^n$ is compact if it is closed and bounded.
  - Examples: $[0,1]$ is compact in $\mathbb{R}$.

    $[0,1)$ is bounded but not closed (why?) so it is not compact.

    $\mathbb{R}^n$ is closed but not bounded so not compact.

3.9 More on the real number system: maximum and supremum

- Given 2 real numbers x and y, we define $\max(x,y)$ and $\min(x,y)$ to be respectively the greater and the lesser of x and y. Thus,

$$\max(x,y) = \begin{cases} x & \text{if } x \geq y, \\ y & \text{if } x < y, \end{cases} \qquad \min(x,y) = \begin{cases} y & \text{if } x \geq y, \\ x & \text{if } x < y. \end{cases}$$

- More generally, for any subset S of $\mathbb{R}$ which has a greatest member, we define $\max S$ to be that member; note however that an infinite subset of $\mathbb{R}$ may not have a greatest member. For example, if

$$S_1 = \{x \in \mathbb{R} \ : \ 0 < x \leq 1\} \qquad S_2 = \{x \in \mathbb{R} \ : \ 0 < x < 1\}$$

then $\max S_1 = 1$ but S_2 has no greatest member; for if x is any member of S_2, $(x+1)/2$ is a member of S_2 that exceeds x.

[1]"Take a neighborhood" requires a point p (the center) and a positive real r (the radius).

- Similarly, if the subset S of $\mathbb{R}$ has a least element, we denote it by $\min S$.

- **Completeness axiom of** $\mathbb{R}$: every nonempty subset of $\mathbb{R}$ that is bounded above has a least upper bound. If S is such a set, its least upper bound is called the supremum of S and is denoted $\sup S$.

- Consider again the above sets S_1 and S_2 for illustration. The set of all upper bounds for S_1 is $T = \{y \in \mathbb{R} \ : \ y \geq 1\}$, so $\sup S_1 = 1$. Further, y is an upper bound of S_2 if and only if $y \geq 1 - \beta$ for every positive number β, and this happens if and only if $y \geq 1$. Hence, the set of all upper bounds of S_2 is also T and $\sup S_2 = 1$. Note also that $\max S_1 = \sup S_1$.

- More generally:

  Proposition: If S is a non-empty subset of $\mathbb{R}$, the following statements are equivalent:

  (a) S has a greatest element;

  (b) S is bounded above and $\sup S \in S$.

  Further, if (a) holds, then $\max S = \sup S$.

  Proof as an exercise.

4 Vector spaces

- **Definition (vector):** Let $\mathbb{R}^k$ be the set of all ordered k-tuples, $x = (x_1, x_2, \cdots, x_k)$ where x_i are real numbers, called the coordinates of x. The elements, x of $\mathbb{R}^k$ are called **vectors**.

- **Definition (vector space):** a non-empty set $X \subseteq \mathbb{R}^k$ is a vector space if, for any $x, y \in X$, $x + y \in X$ and $cx \in X$ for all real numbers c.

  - $x + y$ is the vector with coordinates $x_i + y_i$ for all $i = 1, \cdots, k$.
  - cx is the vector with coordinates $c \times x_i$ for all $i = 1, \cdots, k$.

- **Definition (inner product):** the **inner product** of two vectors, x and y, denoted $x.y$ is the real number defined as:

$$x.y = \sum_{i=1}^{k} x_i y_i$$

- we say that two vectors are **orthogonal** if $x.y = 0$

  e.g., $x = [1, 2, 5]$ and $y = [2, 4, -2]$ are orthogonal. (check to verify!)

- **Definition (Euclidean norm):** the **Euclidean norm** of vector x denoted $\|x\|$ is:

$$\|x\| = \left(\sum_{i=1}^{k} x_i^2 \right)^{1/2}$$

- **Theorem:** suppose x, y, $z \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$. Then:

  (a) $\|x\| \geq 0$

  (b) $\|x\| = 0$ if and only if $x = 0$ (the zero vector, consisting of all k components equal to zero).

  (c) $\|\alpha x\| = |\alpha| \times \|x\|$

  (d) $|x.y| \leq \|x\| \times \|y\|$

  (e) $\|x + y\| \leq \|x\| + \|y\|$

  (f) $\|x - z\| \leq \|x - y\| + \|y - z\|$

- **Definition (Hyperplane)**: let $p \neq 0$ be a vector in $\mathbb{R}^n$ and let $a \in \mathbb{R}$. The set $H(p, a) \equiv \{x \in \mathbb{R}^n \ : \ p.x = a\}$ is called a **hyperplane**.

  – hyperplanes in $\mathbb{R}^2$ are straight lines, in $\mathbb{R}^3$ are planes, etc.

- **Separating hyperplane theorem:** let C be a non-empty convex set in $\mathbb{R}^n$ and let $x^* \in \mathbb{R}^n$ such that $x^* \notin C$. Then there exists a hyperplane $H(p, a)$ that "separates" C and x^*, i.e. such that $p.y \leq a$ for all $y \in C$ and $p.x^* \geq a$.

  – Note 1: non-strict inequalities in the above result are essential! This theorem is crucial for many general equilibrium results in economics.

  – Note 2: the point x^* can be replaced by another convex set (say B disjoint from C).