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Econ 798, Introduction to Mathematical Economics
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Part 6: Introduction to Dynamic Programming

1 Mathematical preliminaries

In this section I introduce some mathematical concepts and results which will be used in the applications of the theory in section 2.

1.1 Fixed point theorems

In economics we often have the problem of finding an equilibrium (steady state). Mathematically, this is frequently translated into solving a functional equation of the form $Tx = x$ where T is some "mapping": that is, the "image" of x (after applying T) is still x; here, T could be a function or something more general like a correspondence from some set X into itself.
If such an x exists, we call it a **fixed point of the mapping** T.

- **Theorem 1: Brouwer's fixed point theorem**

  Let A be a non-empty, compact, convex subset of $\mathbb{R}^n$ and f be a continuous function from A to A. Then f has a fixed point - that is, $\exists x \in A$ s.t. $f(x) = x$.

  "Proof" for the case $n = 1$: Show first that A must be a closed interval, $[a, b]$. Define $g(x) \equiv f(x) - x$. Then, by definition, g is continuous and $g(a) \geq 0$ and $g(b) \leq 0$ (since $f(x) \in [a, b]\ \forall x \in [a, b]$). Then, necessarily, there must exist a point $c \in [a, b]$ at which $g(c) = 0$ [this is the Theorem of intermediate values]. It is a fixed point of $f(x)$.

- **Remarks:**

  - closed is important - e.g. take $f(x) = (x + 1)/2$ on $(-1, 1)$;
  - convex is important - e.g. think of rotating a donut;
  - continuous is important - e.g. think of a line not crossing the 45-degree line on [0, 1];
  - from A to A is important - e.g. $f(x) = x + 2$ on [0, 1];
  - bounded is important - take $A = \mathbb{R}$ and $f(x) = x + 1$.

The above theorem is only applicable to functions (or single-valued mappings). In order to generalize it to more general mappings (e.g. multi-valued mappings, we first need to define some mathematical concepts. *Correspondence* refers to a mapping from a set X to a set Y which returns, for any single argument in X, a set in Y as output: in other words, a correspondence maps from a real number into a set (such as an interval).

Why do we need correspondence in economics? Think about an individual allocating its budget (say x expressed in \$) to consume bundles of goods. The associated feasible set of bundles (or set of affordable bundles that the individual can choose from) is described using a correspondence related to the cost of the bundle, and specifically contains all bundles with a cost between 0 and x.

- **Definition: continuity of a correspondence**

    (a) A correspondence $F : X \to Y$ is upper hemicontinuous (uhc) at $x \in X$ if $F(x)$ is non-empty and if for every sequence $\{x_n\} \in X$ converging to x, and for every converging sequence $\{y_n\}$ with $y_n \in F(x_n)\ \forall n$, we have that the limit point y of the sequence $\{y_n\}$ (here, $y_n \to y$) is s.t. $y \in F(x)$.

    (b) A correspondence $F : X \to Y$ is lower hemicontinuous (lhc) at x if $F(x)$ is non-empty and if for every $y \in F(x)$ and every sequence $\{x_n\} \in X$ converging to x, there exists a subsequence of $\{x_n\}$ denoted $\{x_{n_k}\}$, and an associated sequence $\{y_k\}$ s.t. $y_k \in F(x_{n_k})$ and $y_k \to y$.

    (c) A correspondence is continuous if it is both lower and upper hemicontinuous.

Recall that the continuity of functions can be characterized using either sequences or open sets. The above characterization of continuity of correspondence is based on sequences. There is an alternative characterization which is based on open sets. We will practice both and you can choose to use the characterization that you find most appealing.

- **Theorem 2: Kakutani's fixed point theorem**

    Let S be non-empty, compact, convex subset of $\mathbb{R}^n$. Let F be an upper hemicontinuous correspondence from S into[1] 2^S s.t. $\forall x \in S$ the set $F(x)$ is non-empty, closed and convex. Then, there exists $x^* \in S$ s.t. $x^* \in F(x^*)$ - a fixed point of F.

Counterexample: the following correspondence satisfies all conditions apart from $F(x)$ being convex at $x = 0.5$, and it is obviously not continuous.

$$F(x) = \begin{cases} 0.6 & \text{if } 0 \leq x < 0.5 \\ \{0.6, 0.4\} & \text{if } x = 0.5 \\ 0.4 & \text{if } 0.5 < x \leq 1 \end{cases}$$

[1] 2^S denotes the power set of S which represents the set of all subsets of S.

1.2 The contraction mapping theorem

The main result derived in this section is a very general fixed point theorem which can be applied to basically any type of mappings (operators) satisfying some regularity conditions. Before we state the result we need to define some concepts.

- **Definition: Contraction mapping, CM**

  Let (S, ρ) be a metric space with metric ρ and $T : S \to S$ maps S (some set with elements numbers or functions, etc.) into itself.
  T is called a contraction mapping (with modulus β) if for some $\beta \in (0, 1)$

$$\rho(Tx, Ty) \leq \beta\rho(x, y) \qquad \text{for all } x, y \in S \text{ and } x \neq y$$

In the above definition, Tx denotes the image of x after applying the mapping T.

- **Example:**

  Take $S = [a, b] \subseteq \mathbb{R}$ with $\rho(x, y) = |x - y|$. Then $T : S \to S$ is a CM if for some $\beta \in (0, 1)$

$$\frac{|Tx - Ty|}{|x - y|} < \beta < 1 \qquad \text{for all } x, y \in S, \text{ and } x \neq y$$

  For example, if T is a function with slope uniformly less than 1, it is a CM.
  A point x satisfying $Tx = x$ is called a *fixed point of* T.

- **Theorem 3: Contraction mapping theorem**

  If (S, ρ) is a complete metric space (such that every convergent sequence in it converges to a point in it) and $T : S \to S$ is a contraction mapping with modulus β then:

  (a) T has a unique fixed point in S denoted v;

  (b) For any $v_0 \in S$, $\rho(T^n v_0, v) \leq \beta^n \rho(v_0, v)$, $n = 1, 2, \cdots$ where $T^n x$ means applying T n times on x - i.e. $T^{n+1}x = T(T^n x)$.

Note that (b) bounds the distance between the n-th iteration and the fixed point. However, if v is not known, this bound is unknown too and thus useless. Instead, we can show that for any $v_0 \in S$

$$\rho(T^n v_0, v) \leq \frac{1}{1 - \beta}\rho(T^n v_0, T^{n+1} v_0)$$

- **Corollary 1**

  Let (S, ρ) be a complete metric space and $T : S \to S$ be a CM with a fixed point $v \in S$. If S' is a closed subset of S and $T(S') \subseteq S'$ then $v \in S'$.

The above statement tells us that if T shrinks the set S when applied repeatedly, then the fixed point stays inside the resulting set. This result is very useful for some game theoretical applications.

- **Corollary 2: N-stage contraction theorem**

  Let (S, ρ) be a complete metric space, $T : S \to S$ and suppose that for some integer $N \geq 1$, $T^N : S \to S$ is a CM with modulus β. Then:

  (a) T has exactly one fixed point in S denoted v;

  (b) For any $v_0 \in S$, $\rho(T^{kN} v_0, v) \leq \beta^k \rho(v_0, v)$, $k = 0, 1, \cdots$.

The above theorems do not tell us however how we can find those contraction mappings, i.e. how to check if an operator T is a CM. The following result provides sufficient conditions.

- **Theorem 4: Blackwell's conditions**

  Let $X \subseteq \mathbb{R}^l$ and $B(X)$ be a space of bounded functions $f : X \to \mathbb{R}$ with the sup norm (i.e. $||f|| = \sup_{t \in X} |f(t)|$). Let $T : B(X) \to B(X)$ be an operator satisfying:

  (i) monotonicity: if $f, g \in B(X)$ and $f(x) \leq g(x)$ $\forall x \in X$ then $(Tf)(x) \leq (Tg)(x)$, $\forall x \in X$;

  (ii) discounting: $\exists \beta \in (0,1)$ s.t. $(T(f + a))(x) \leq (Tf)(x) + \beta a$, $\forall f \in B(X)$, $\forall a \geq 0$, $\forall x \in X$, where $(f + a)(x)$ is the function defined by $(f + a)(x) = f(x) + a$.

  Then T is a contraction mapping with modulus β.

2 Dynamic programming

2.1 Preliminaries

We are interested in problems of the type (many examples in macroeconomics):

$$\sup_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1}) \quad s.t. \quad x_{t+1} \in \Gamma(x_t), \; t = 0, 1, \cdots \quad , \text{ and } x_0 \in X \text{ given} \qquad (SP)$$

We will call the above problem the Sequence Problem (SP) as it involves finding a sequence $\{x_{t+1}\}$ to maximize the discounted sum $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$ subject to the constraint that x_{t+1} is in some set depending on x_t. t is usually interpreted as time; $\sum_{t=0}^{\infty} \beta^t F(x_t, x_{t+1})$ is called the objective function, and $\Gamma(x_t)$ the set of constraints.

Notice that we have an infinite number of variables so we cannot simply set up the system of first order conditions and solve for the optimal x_t. It turns out that a possible way to solve the above problem is to transform it into a different one which will allow us to apply the theory about contraction mappings.

Denote by $v(x)$ the value corresponding to the supremum (least upper bound) of the objective from time t onwards,

$$\begin{aligned} v(x_0) &= \sup_{\{x_{t+1}\}_{t=0}^{\infty}, x_{t+1}\in\Gamma(x_t)} F(x_0,x_1)+\beta F(x_1,x_2)+\beta^2 F(x_2,x_3)+\cdots \\ v(x_1) &= \sup_{\{x_{t+1}\}_{t=1}^{\infty}, x_{t+1}\in\Gamma(x_t)} F(x_1,x_2)+\beta F(x_2,x_3)+\beta^2 F(x_3,x_4)+\cdots \\ \cdots \end{aligned}$$

Note that continuing in this recursive way the subscripts become irrelevant as all we care about is what the current-period value x_t is. That is, the maximization problem is exactly the same at any period (see Example below: a cake has to be eaten in infinitely many remaining periods by the exact same consumer) - the only thing that changes is the initial value. Hence, intuitively we can re-write the original (SP) problem as a "generic" problem with current value x in which we search for the function v that satisfies:

$$v(x) = \sup_{y\in\Gamma(x)} F(x,y)+\beta v(y)\ , \forall x\in X \qquad (FE)$$

The above is an equation in the unknown function v and is called the Functional Equation (FE) while v; the function v that satisfies it is called the value function. Dynamic programming deals with solving dynamic optimization problems using FEs.

We now study the relationship between the solutions to the (SP) and the (FE), and develop methods to analyze the latter. We start with some important definitions.

- **Definition: Graph**

  Let $\Gamma : X \to Y$ be a correspondence and define the set $A = \{f(x,y) : y \in \Gamma(x)\}$.
  A is called the graph of Γ.

  Example: take $x \in [0,1]$ and let $\Gamma(x) = [0,x]$. Then the graph of Γ is the area below the 45-degree line between $x = 0$ and $x = 1$ in a two-dimensional graph with x on the horizontal axis and values from $\Gamma(x)$ on the vertical axis.

- **Theorem 5: Theorem of the maximum**

  Let $X \subseteq \mathbb{R}^l$, $Y \subseteq \mathbb{R}^m$, $f : X \times Y \to \mathbb{R}$ be a continuous function and let $\Gamma : X \to Y$ be a compact-valued and continuous correspondence. Then the function $h : X \to \mathbb{R}$ defined as

  $$h(x) = \max_{y\in\Gamma(x)} f(x,y)$$

  is continuous and the correspondence $G : X \to Y$ defined as $G(x) = \{y \in \Gamma(x) : h(x) = \max_{y\in\Gamma(x)} f(x,y)\}$ is non-empty, compact-valued and uhc.

The above theorem provides conditions under which (i) a value function is continuous; and (ii) under which the set $G(x)$ of maximizers of f is non-empty and compact-valued.
The following result is related to the theorem of the maximum.

- **Corollary**

  Suppose that, in addition to the conditions in Theorem 5, we have Γ convex-valued and f strictly concave in y. Then G is a single-valued and continuous function, $g(x)$.

2.2 The Principle of Optimality

Armed with the theory from the previous section we are ready to go back to the question: what is the relationship between the solutions to the problems (SP) and (FE)?
The general idea is that:
(i) the solution to the (FE) evaluated at x_0 gives the value of the supremum of (SP);
(ii) a sequence $\{x_{t+1}\}_{t=0}^{\infty}$ attains the supremum in (SP) if and only if it satisfies:

$$v(x_t) = F(x_t, x_{t+1}) + \beta v(x_{t+1}) \text{ , for } t = 0, 1, \cdots$$

These ideas, first stated by Bellman, are known as the **Principle of Optimality**. In the discussion below we study the conditions under which the Principle of Optimality holds.
We start with some notation and terminology.

- **Notation:**

- X is the set of all possible values for the so-called state variable x and is called the state space.

- $\Gamma : X \to X$ is a correspondence describing the feasibility constraints or the feasible set.

- $A = \{(x, y) \in X \times X : y \in \Gamma(x)\}$ is the graph of Γ.

- $F : A \to \mathbb{R}$ is called the return function.

- $\beta \geq 0$ is called the discount factor.

Thus X, Γ, F and β are given in our problem. First, we need to establish conditions under which (SP) is well-defined, i.e. the feasible set is non-empty and the objective function is well defined for all points in the feasible set.
Call a sequence $\{x_t\}_{t=0}^{\infty}$ in X a plan. Given $x_0 \in X$, let

$$\Pi(x_0) = \{\{x_t\}_{t=0}^{\infty} : x_{t+1} \in \Gamma(x_t),\ t = 0, 1, \cdots\}$$

be the set of feasible plans from x_0. Let $\tilde{x} = (\tilde{x}_0, \tilde{x}_1, \cdots)$ denote a typical element in $\Pi(x_0)$.

The following assumption ensures that $\Pi(x_0)$ is non-empty, $\forall x_0 \in X$.

- **Assumption A1:** $\Pi(x)$ is non-empty for all $x \in X$.

Next, we need to ensure that the objective function is well-defined. We need the following.

- **Assumption A2:** For all $x_0 \in X$ and all $\tilde{x} \in \Pi(x_0)$, $\lim_{n\to\infty} \sum_{t=0}^{n} \beta^t F(\tilde{x}_t, \tilde{x}_{t+1})$ exists.

There are many ways to satisfy A2; the simplest is to assume that F is bounded (for this, it may help to assume or know that X is bounded) and $\beta \in (0,1)$.
For each $n = 0, 1, \cdots$, define the partial sum u_n from a feasible plan $\tilde{x}$ as:

$$u_n : \Pi(x_0) \to \mathbb{R} \quad \text{with} \quad u_n(\tilde{x}) \equiv \sum_{t=0}^{n} \beta^t F(\tilde{x}_t, \tilde{x}_{t+1})$$

Using A2, we can then define its limit as:

$$u(\tilde{x}) \equiv \lim_{n\to\infty} u_n(\tilde{x})$$

Since under assumptions A1-A2 the objective function is well-defined and the set of feasible plans is non-empty, we can then define the supremum function, $v^* : X \to \mathbb{R}$ by:

$$v^*(x_0) \equiv \sup_{\tilde{x}\in\Pi(x_0)} u(\tilde{x})$$

that is, $v^*(x_0)$ is the value of the supremum in (SP).

We are interested in the connection between the supremum function v^* and the solutions (called value functions) v to the problem (FE). Note that under assumptions A1-A2 the function $v^*(x_0)$ is uniquely defined for any x_0 (think why!) whereas v may not be.

- **Definition: "satisfies the FE"**

  Suppose $|v^*(x_0)| < \infty$. We say that v^* "satisfies the (FE)" if the following conditions hold:

  (i) $\forall y \in \Gamma(x_0)$: $v^*(x_0) \geq F(x_0, y) + \beta v^*(y)$.

  (ii) $\forall \epsilon > 0, \exists y \in \Gamma(x_0)$ s.t $v^*(x_0) \leq F(x_0, y) + \beta v^*(y) + \epsilon$.

We are now ready for our first result on the relationship between v^* in (SP) and the function v in (FE).

- **Theorem 7**

  Let X, Γ, F and β satisfy A1-A2.
  Then the supremum function v^* defined above satisfies the (FE).

The above theorem states that under A1 and A2 the solutions to the (SP) solve the (FE). We will call the solution to the (FE) v, the value function. Next, we establish a partial inverse.

- **Theorem 8**

  Let X, Γ, F and β satisfy A1-A2. If v is a solution to the (FE) and satisfies

  $$\lim_{n\to\infty} \beta^n v(x_n) = 0 \quad \text{for all } \{x_0, x_1, \cdots\} \in \Pi(x_0) \text{ and all } x_0 \in X \qquad (*)$$

  Then $v = v^*$.

The above result implies that v^* is the only solution to the (FE) which satisfies the boundedness condition (*). This means that (FE) has at most one solution satisfying (*), but it may have other.

Next, we characterize the feasible plans that attain the optimum in (SP). Call a feasible plan $\tilde{x}^* \in \Pi(x_0)$ an optimal plan from x_0 if it attains the supremum in (SP) - i.e. if $u(\tilde{x}^*) = v^*(x_0)$. The next two theorems describe the connection between optimal plans and those that satisfy the equation:

$$v(x_t) = F(x_t, x_{t+1}) + \beta v(x_{t+1}) \qquad \text{for } t = 0, 1 \qquad (**)$$

- **Theorem 9**

  Let X, Γ, F and β satisfy A1-A2. Let also $\tilde{x}^* = \{x_0^*, x_1^*, \cdots\} \in \Pi(x_0)$ be an optimal plan from x_0 (with $x_0^* = x_0$). Then:

  $$v^*(x_t^*) = F(x_t^*, x_{t+1}^*) + \beta v^*(x_{t+1}^*) \quad \text{for all } t = 0, 1, \cdots$$

  i.e. the optimal plan satisfies (**) for the function v^*.

- **Theorem 10**

  Let X, Γ, F and β satisfy A1-A2 and $\tilde{x}^* \in \Pi(x_0)$ be a feasible plan satisfying (**) and

  $$\lim_{t\to\infty} \sup \beta^t v^*(x_t^*) \le 0$$

  Then $\tilde{x}^*$ attains the supremum in (SP) - i.e. it solves the (SP).

The above theorems show under what conditions we may use the (FE) in (**) to solve the problem (SP). Again, notice that we need a boundedness condition to be satisfied to go in the opposite direction, that is from (FE) to (SP).

- **Some more terminology:**

  We will call a non-empty correspondence $G : X \to X$ with $G(x) \subseteq \Gamma(x)$, $\forall x \in X$ a policy correspondence: notice that $G(x)$ is a feasible set of "actions" if the state is x. If $G(x)$ is single-valued, we call it a policy function since it can be interpreted as a feasible action (policy) to be taken given x. If the elements of the feasible plan $\tilde{x} \in \Pi(x_0)$ satisfy $x_{t+1} \in G(x_t)$ for all t, we say that $\tilde{x}$ is generated from G.

We can also define the optimal policy correspondence as

$$G^*(x) = \{y \in \Gamma(x) : v^*(x) = F(x,y) + \beta v^*(y)\} \qquad (***)$$

Theorem 9 states that every optimal plan $\tilde{x}^*$ is generated from G^* and Theorem 10 states that any plan $\{x_t^*\}$ generated from G^* is an optimal plan if a boundedness condition is satisfied.

2.3 Bounded returns

In this section we provide a sharper characterization of the solution to the (FE) under more assumptions on Γ, F, X and β. We will study functional equations of the form:

$$v(x) = \max_{y \in \Gamma(x)} [F(x,y) + \beta v(y)] \qquad (E1)$$

Formally, we make two new (stronger) assumptions in place of A1 and A2:

- **Assumption A3:** X is a convex set in $\mathbb{R}^l$ and $\Gamma : X \to X$ is non-empty, compact-valued and continuous.

- **Assumption A4:** $F : A \to \mathbb{R}$ is bounded and continuous and $\beta \in (0,1)$.

  Clearly, A3-A4 imply A1-A2 (verify why!). Also notice that by the theorems above, under A3 and A4 the solutions to (FE) and (SP) coincide.

  Define the operator T on $\mathcal{C}(X)$ - the space of bounded, continuous functions $f : X \to \mathbb{R}$ with the sup norm ($||f|| = \sup_{x \in X} |f(x)|$) as:

$$(Tf)(x) = \max_{y \in \Gamma(x)} [F(x,y) + \beta f(y)]$$

  i.e. (FE) is equivalent to $Tv = v$.

- **Theorem 11**

  Let X, Γ, F and β satisfy A3-A4. Then:

  (i) the operator T maps the space $\mathcal{C}(X)$ into itself, i.e., $T : \mathcal{C}(X) \to \mathcal{C}(X)$.

  (ii) T has a unique fixed point, $v \in \mathcal{C}(X)$.

  (iii) For any $v_0 \in \mathcal{C}(X)$ and given v from (ii), we have:

$$||T^n v_0 - v|| \leq \beta^n ||v_0 - v|| \quad \text{for } n = 0, 1, \cdots$$

(iv) Given v from (ii), the optimal policy correspondence $G : X \to X$ defined as

$$G(x) = \{y \in \Gamma(x) : v(x) = F(x, y) + \beta v(y)\} \qquad (E2)$$

is compact-valued and uhc.

Theorems 8-11 establish that, under A3-A4, the supremum (value) function v is bounded and continuous. Theorems 10 and 11 imply that there exists at least one optimal plan since every plan generated by the non-empty correspondence G is optimal.
To characterize the value function v and the optimal policy G more sharply we need more assumptions on F and Γ. Let us add two more.

- **Assumption A5:** For any y, $F(., y)$ is strictly increasing in each of its first l arguments[2].

- **Assumption A6:** Γ is monotone, i.e. $x \leq x_0$ implies $\Gamma(x) \subseteq \Gamma(x_0)$.

- **Theorem 12**

  Let X, Γ, F and β satisfy A3-A6 and let v be the unique solution to (E1).
  Then v is strictly increasing.

Alternatively, we can make the following assumptions:

- **Assumption A7:** $F(x, y)$ is strictly concave. [Note: must be true in all arguments]

- **Assumption A8:** $\Gamma(x)$ is convex, i.e. for all $\theta \in (0, 1), x, x' \in X, y \in \Gamma(x)$ and $y' \in \Gamma(x')$, we have

$$\theta y + (1 - \theta)y' \in \Gamma(\theta x + (1 - \theta)x')$$

- **Theorem 13**

  Let X, Γ, F and β satisfy A3-A4 and A7-A8, v satisfies (E1) and G satisfy (E2).
  Then the value function, v is strictly concave and the optimal policy G is a continuous, single-valued function.

Notice that what is interesting in Theorems 12 and 13 is that T preserves certain properties of F - i.e. v inherits them.

[2]Recall that $x \in X \subseteq \mathbb{R}^l$, so these first l arguments are simply the (vector of) state variables x.

- **Theorem 14: Convergence of the value and policy functions**

  Let X, Γ, F and β satisfy A3-A4 and A7-A8, v satisfy (E1) and g satisfy (E2). Let $\mathcal{C}'(X)$ be the space of bounded, continuous, strictly concave functions $f : X \to \mathbb{R}$ and $v_0 \in \mathcal{C}'(X)$. Let the sequence $\{v_n, g_n\}_{n=0}^{\infty}$ be defined as:

$$\begin{aligned} v_{n+1} &= Tv_n \\ g_n(x) &= \arg\max_{y \in \Gamma(x)} [F(x,y) + \beta v_n(y)] \quad \text{for } n = 0, 1, \cdots \end{aligned}$$

  Then g_n converges to g pointwise. If X is compact, the convergence is uniform.

The above theorem is very useful - it shows that we can solve the (FE) by iterating on it, which is how it is actually done in most cases when numerical techniques are employed. Since it is rare that one can solve the (FE) for v explicitly the above result is valuable.

Now that we have established the conditions for the existence of a unique solution $v \in \mathcal{C}(X)$ of (E1) we may like to treat this equation as an ordinary optimization problem and use standard calculus (first-order conditions) to further characterize the optimal policy g. In order to do this, however, we need to know if v is differentiable.
Imagine for the moment that the latter was true. Then we will have the first-order condition:

$$\frac{\partial F(x, g(x))}{\partial y} + \beta v'(g(x)) = 0$$

which can be used to study the properties of the optimal policy function $g(x)$ using the Implicit Function Theorem.
To ensure differentiability of the value function v, we need an additional assumption:

- **Assumption A9:** F is continuously differentiable on the interior of A.

- **Theorem 15: Benveniste/Scheinkman**

  Let X, Γ, F and β satisfy A3-A4 and A7-A9, v satisfies (E1) and g satisfies (E2).
  If $x_0 \in int(X)$ and $g(x_0) \in int[\Gamma(x_0)]$ then v is continuously differentiable at x_0 with (partial) derivatives given by:

$$\frac{\partial v(x_0)}{\partial x_i} = \frac{\partial F(x_0, g(x_0))}{\partial x_i} \quad \text{for } i = 1, \cdots, l$$

  Notice that the Envelope Theorem is used to derive the result above.

2.4 Example: Cake eating problem

Suppose we have a cake of size k_0. A consumer with log utility, $u(c_t) = \ln c_t$ wants to decide c_t - that is, how much of the cake to eat at each period t (with $t = 0, 1, 2, \cdots$).
One easy way to write this problem is:

$$\max_{\{c_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t) \quad s.t. \quad \sum_{t=0}^{\infty} c_t = k_0 \, , c_t \geq 0 \, , k_0 > 0 \text{ given}$$

We can think of the same problem in an equivalent way: instead of deciding how much to consume c_t at each period t, we can decide how much cake to leave for the following period k_{t+1} - with the cake size at period t denoted as k_t. The current cake size k_t is the natural state variable (x_t in our general framework) for this problem (think why! the intuition is that everything else is the same once you know k_t - maximizing the same utility, infinite number of periods, etc.).
That is, the above problem can be re-written as a valid sequence problem (SP) as follows:

$$\begin{aligned} &\max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(k_{t+1} - k_t) \qquad (SP) \\ s.t. \qquad &0 \leq k_{t+1} \leq k_t \quad \text{for all } t \text{ and } k_0 > 0 \text{ given} \end{aligned}$$

Start by convincing yourself that the proper state variable (summarizing all information that you need in order to make a decision of the future k's) is the current amount of cake, k_t. **Dropping the time subscript** [do not forget!!!], call k the beginning-of-period cake size and k' the end-of-period (or, beginning of next period) cake size. Then, the (FE) problem associated with (SP) is:

$$v(k) = \max_{k' \in [0,k]} \ln(k - k') + \beta v(k') \quad \text{for all } k \in [0, k_0] \qquad (FE)$$

In this problem, we have [Think why this is the case!!]:

- the state space $X = [0, k_0]$;
- the return function $F(k, k') = \ln(k - k')$;
- the feasibility correspondence $\Gamma(k) = [0, k]$;
- the discount factor β.

How to solve the (FE) problem? Use the **guess and verify** approach[3]. Guess $v(k) = A + B \ln k$. Take the FOC of the right hand side (here I just assume this is necessary and sufficient, which it is for this problem, but in general need to verify usual conditions):

$$\frac{1}{k - k'} = \frac{\beta B}{k'}$$

[3]This only works with some [simple] functional forms; in general, a computer is needed to solve it numerically.

In other words, the optimal choice must satisfy $k' = \frac{\beta B}{(1+\beta B)}k$. This gives us the optimal policy $g(k) = \frac{\beta B}{(1+\beta B)}k$. This implies $c = k - k' = \frac{k}{(1+\beta B)}$ (where c denotes current period consumption).

If our guess is correct, after substituting

$$k' = g(k) \quad \text{(the maximizer) and} \quad v(k') = v(g(k)) = A + B\ln[g(k)]$$

into the right hand side of (FE), we must get an expression of the form $A + B\ln k$! Why? since the LHS must equal $v(k)$ when we plug in the maximizer $k' = g(k)$ in the RHS!
Note that the same function v sits on both sides of FE.

Let's see... we have that $\ln(k - g(k)) + \beta v(g(k))$ equals:

$$\ln\left(\frac{k}{1+\beta B}\right) + \beta\left[A + B\ln\left(\frac{\beta B}{1+\beta B}k\right)\right]$$

Notice that this expression is indeed of the form $A + B\ln k$ [after a little algebra]. Thus, our guess is verified (correct). We also know then that $v(k) = A + B\ln k$ for all k. By comparing terms we see that we must have $B = 1 + \beta B$ i.e. $B = 1/(1-\beta)$ and so the optimal policy is $k' = g(k) = \beta k$ and hence $c = (1-\beta)k$. Overall, the optimal policy entails that a fraction $(1-\beta)$ of the cake is consumed each period.

Using the equivalence of (SP) and (FE), we construct the sequence $\{c_t\}_{t=0}^{\infty}$ solving the original problem (or, the corresponding sequence $\{k_{t+1}\}_{t=0}^{\infty}$), by iterating over the following equations for all t:

$$\begin{aligned} c_t &= (1-\beta)k_t \\ k_{t+1} &= \beta k_t \\ k_0 & \quad \textit{given} \end{aligned}$$

How about the constant A? We don't really need it, but comparing terms, we must have

$$A = \beta A - \ln(1+\beta B) + \beta B\ln\beta B - \beta B\ln(1+\beta B)$$

and so

$$A = \frac{\ln(1-\beta) + \frac{\beta}{1-\beta}\ln(\beta)}{1-\beta}$$