Regularizations of the Dirac delta distribution

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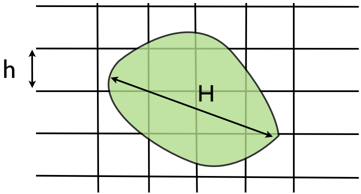
Introduction

- Solve a PDE
$$\mathcal{L}(u) = \mathcal{S}, \quad \text{in } \Omega.$$
- $\mathcal{L}$–differential operator, $\mathcal{S}$– source term with δ singularity.
- Approximate problem
$$\mathcal{L}(u_H) = \mathcal{S}_H, \quad \text{in } \Omega$$
so that $\mathcal{S}_H \to \mathcal{S}$ as $H \to 0$.
- In practice we discretize the problems (Finite Differences, Finite Volumes, Spectral, etc)
$$\mathcal{L}(u_h) = \mathcal{S}_h$$
$$\mathcal{L}(u_{H,h}) = \mathcal{S}_{H,h}$$
- We want $u_{H,h} \to u$ in some sense as $H, h \to 0$.



Introduction

Questions

Q1: How do we construct 'good' approximations $\mathcal{S}_H$ to $\mathcal{S}$?

Q2: How does the choice of $\mathcal{S}_H$ affect the convergence $u_{H,h} \to u$?

Q3: What form of convergence should be used to examine $\mathcal{S}_H \to \mathcal{S}$?

Q4: What form of convergence should be used to examine $u_{H,h} \to u$?

$$\|u - u_{H,h}\|_X \leq \underbrace{\|u - u_H\|_X}_{\text{regularization error}} + \underbrace{\|u_H - u_{H,h}\|_X}_{\text{discretization error}} .$$

- Discretized errors are well studied[1,2,3], so we focus on the regularization error.

[1] Anna-Karin Tornberg and Björn Engquist. "Regularization techniques for numerical approximation of PDEs with singularities". In: *Journal of Scientific Computing* 19.1–3 (2003), pp. 527–552.

[2] Johan Waldén. "On the approximation of singular source terms in differential equations". In: *Numerical Methods for Partial Differential Equations* 15.4 (1999), pp. 503–520.

[3] Yang Liu and Yoichiro Mori. "Properties of discrete delta functions and local convergence of the immersed boundary method". In: *SIAM Journal on Numerical Analysis* 50.6 (2012), pp. 2986–3015.

Weak-$*$ convergence of $\tilde{\delta}_H \to \delta$

- Assume $\mathcal{S} = \delta$, we want a sequence of elements $\delta_H \in \mathcal{H}_0^s(\Omega)$ with a compact support of diameter $\mathcal{O}(H)$.
- Take corresponding elements $\tilde{\delta}_H \in \mathcal{H}^{-s}(\Omega)$

$$\tilde{\delta}_H(\phi) := (\delta_H, \phi)_{L^2(\Omega)}, \qquad del\tilde{t}a_H(\phi) \to \delta(\phi\rangle), \qquad \forall \phi \in C_c^\infty(\Omega).$$

- Use Taylor's theorem

$$\left|\delta(\phi) - \tilde{\delta}_H(\phi)\right| \leq \left|\phi(0)\left(1 - (\delta_H, \mathbf{1}_\Omega)_\Omega\right) + \sum_{1\leq|\alpha|\leq m} \frac{\partial^\alpha \phi(0)}{\alpha!} \left(\delta_H, \mathbf{x}^\alpha\right)_\Omega\right| + \left|\sum_{|\beta|=m+1} \left(\delta_H, R_\beta(\mathbf{y})(\mathbf{x})^\beta\right)_\Omega\right|.$$

- Moment conditions

$$\begin{aligned} &(\delta_H, \mathbf{1}_\Omega)_\Omega = 1 \quad \text{and} \quad (\delta_H, \mathbf{x}^\alpha)_\Omega = 0 \quad \text{for } 1 \leq |\alpha| \leq m, \\ &|\tilde{\delta}_H(\phi) - \delta(\phi)| \leq C(\phi, m) H^{m+1}. \end{aligned} \tag{1}$$

$(W_{-\alpha})^*$ convergence of $\tilde{\delta}_H \to \delta$

- (Weighted-L^2 space) For $\Omega \subset \mathbb{R}^n$, constant $\beta \in (-\frac{n}{2}, \frac{n}{2})$, measurable $u \in L^2_\beta(\Omega)$ if

$$\|u\|_{L^2_\beta(\Omega)} := \left(\int_\Omega |u(x)|^2 \, |x|^{2\beta} \, dx \right)^{\frac{1}{2}} < \infty. \tag{2}$$

- (Weighted Sobolev space)

$$W_\beta := \{u : u|_{\partial\Omega} = 0 \text{ and } \|u\|_{L^2_\beta(\Omega)} + \|\nabla u\|_{L^2_\beta(\Omega)} < \infty\}. \tag{3}$$

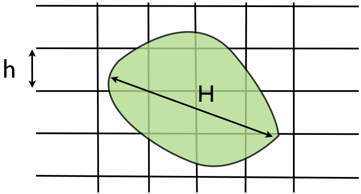
$(W_{-\alpha})^*$ convergence of $\tilde{\delta}_H \to \delta$

- Fix $\frac{n}{2} - 1 < \alpha < \frac{n}{2}$, then $\delta \in (W_{-\alpha})^*$ (Theorem 4.7 of Agnelli et al.[4]).
- Let h be the mesh size

Proposition

Let δ_H satisfy m moment conditions and assume that $H = \mathcal{O}(h^\beta)$ with $\beta > 0$, then

$$\|\tilde{\delta}_H - \delta\|_{(W_{-\alpha})^*} = \mathcal{O}\left(H^{\frac{1}{\beta}(\alpha+(\beta-1)(m+1))}\right).$$



[4] Juan Pablo Agnelli, Eduardo M. Garau, and Pedro Morin. "A posteriori error estimates for elliptic problems with Dirac measure terms in weighted spaces". In: *ESAIM: Mathematical Modelling and Numerical Analysis* 48 (2014), pp. 1557–1581.

Constructing δ_H

- Moment conditions

$$(\delta_H, \mathbf{1}_\Omega)_\Omega = 1 \quad \text{and} \quad (\delta_H, \mathbf{x}^\alpha)_\Omega = 0 \quad \text{for } 1 \leq |\alpha| < m.$$

Finite dimensional moment problem

Let $\mathcal{H}$ be an infinite dimensional Hilbert space and fix $m \in \mathbb{N}$. Given linearly independent $\varphi_i \in \mathcal{H}$ and scalars c_i for $i = 0, \ldots, m$, find $q \in \mathcal{H}$ such that

$$(q, \varphi_i)_\mathcal{H} = c_i \qquad \text{for } i = 0, \ldots, m. \tag{4}$$

Let $\{\psi_k\}_{k=0}^\infty$ form a Riesz basis in $\mathcal{H}$ so that **span** $\{\varphi_k\}_{k=1}^m =$ **span** $\{\psi_k\}_{k=1}^m$. Then any $\bar{q} = \sum_{j=0}^m \beta_j \psi_j \in \mathcal{H}$ is a solution if β_js solve

$$\sum_{i=0}^m (\varphi_i, \psi_j)_\mathcal{H} \, \beta_j = c_i \qquad \text{for } i = 0, \ldots, m. \tag{5}$$

Furthermore, any $\tilde{q} = \bar{q} + \beta_k \psi_k$ is a solution if ψ_k is orthogonal to **span** $\{\psi_i\}_{i=0}^m$.

Constructing δ_H

- Choose number of moments $m \in \mathbb{N}$.
- Pick a domain $\Omega_H \subset \Omega$ with diameter H containing the origin.
- Pick a convenient orthonormal basis $\{\psi_k\}_k \in L^2(\Omega_H)$ (Polynomials, Trig functions).
- Set $\delta_H = \sum_{j=0}^{m} \beta_j \psi_j$.
- Solve the linear system

$$\sum \beta_k \left(\psi_k, \mathbf{1}_\Omega\right)_\Omega = 1 \quad \text{and} \quad \sum \beta_k \left(\psi_k, \mathbf{x}^\alpha\right)_\Omega = 0, \quad \text{for } 1 \leq |\alpha| < m. \tag{6}$$

- We are free to add more conditions!

Constructing δ_H

Example: Radial regularizations in 2D

Choose $\delta_H(\mathbf{x}) = H^{-2}\eta_{m,p}(\mathbf{x}/H)$ such that $\eta_{m,p}$ is radially symmetric and supported on $B(0,1)$. Rewrite moment problem in radial coordinates and choose Legendre polynomials as the basis.

$$\eta_{1,1}(r) = \frac{6}{\pi}(3-4r), \quad \eta_{2,2} = \frac{12}{\pi}(15r^2 - 20r + 6).$$

Example: Tensor product regularizations in 2D

Choose $\delta_H = (H^{-1}\eta_{m,p}(x/H)) \otimes (H^{-1}\eta_{m,p}(y/H))$ for $\eta_{m,p}(x)$ supported on $[-1,1]$. Write moment conditions in 1D with Legendre polynomials as the basis.

$$\eta_{1,0}(x) = \frac{1}{2}, \quad \eta_{2,2}(x) = \frac{9}{2}18x + 15x^2.$$

Constructing δ_H

Symbol	Dim	Moment	Expression	Reference
$\eta_{1,0}$	1D	1	$\frac{1}{2}$	
$\eta_{1,1}$	1D	1	$1 - r$	[1,2,3]
$\eta_{2,2}$	1D	2	$\frac{9}{2} - 18r + 15r^2$	
$\eta_{2,3}$	1D	2	$-30r^3 + 60r^2 - 36r + 6$	
$\eta_{2,5}$	1D	2	$168r^5 - \frac{945}{2}r^4 + 450r^3 - 150r^2 + \frac{9}{2}$	
$\eta_{1,cos}$	1D	1	$\frac{1}{2}(1 - \cos(\pi r))$	[1,2,3,4]
$\eta_{1,1}$	2D	1	$\frac{6}{\pi}(3 - 4r)$	
$\eta_{1,2}$	2D	1	$\frac{12}{\pi}(5r^2 - 8r + 3)$	
$\eta_{2,2}$	2D	2	$\frac{12}{\pi}(15r^2 - 20r + 6)$	
$\eta_{2,3}$	2D	2	$\frac{-60}{\pi}(7r^3 - 15r^2 + 10r - 2)$	
$\eta_{2,5}$	2D	2	$\frac{84}{\pi}(24r^5 - 70r^4 + 70r^3 - 25r^2 + 1)$	

[1] Anna-Karin Tornberg and Björn Engquist. "Numerical approximations of singular source terms in differential equations". In: *Journal of Computational Physics* 200.2 (2004), pp. 462–488

[2] Richard P. Beyer and Randall J. LeVeque. "Analysis of a one-dimensional model for the immersed boundary method". In: *SIAM Journal on Numerical Analysis* 29.2 (1992), pp. 332–364

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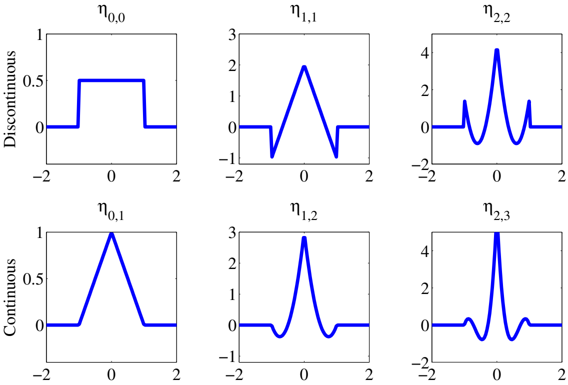
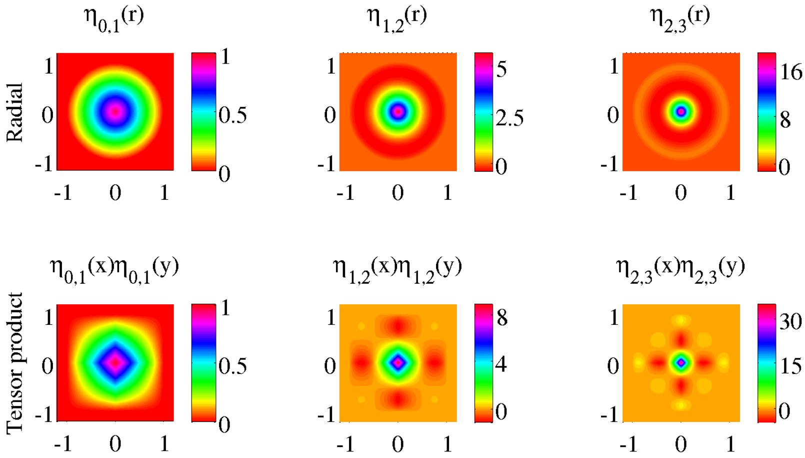
[4] Anna-Karin Tornberg. "Multi-dimensional quadrature of singular and discontinuous functions". In: *BIT* 42.3 (2002), pp. 644–6695

Constructing δ_H: Examples in 1D



Constructing δ_H: Examples in 2D



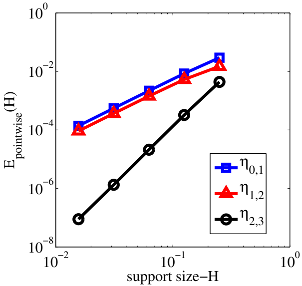
Convergence of $u_H \to u$: Elliptic PDEs

$$Lu = \delta, \quad Lu_H = \tilde{\delta}_H \quad \text{for } x \in \Omega, \quad \text{and} \quad u = u_H = 0 \quad \text{on} \quad \partial\Omega.$$

- L–constant coefficient elliptic operator, $\tilde{\delta}_H$–m-moment regularization.
- Pointwise convergence away from the support

$$|u(x) - u_H(x)| \leq C_m H^{m+1}, \quad \text{for} \quad x \in \Omega \setminus \text{supp } \tilde{\delta}_H. \tag{7}$$

- Helmholtz equation on a disk.

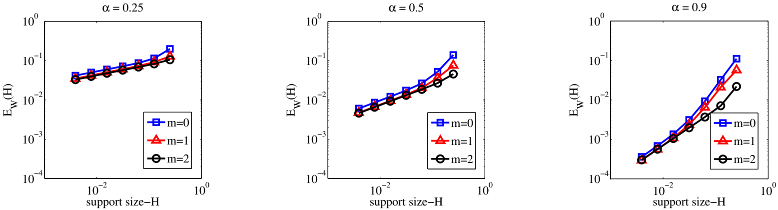


Convergence of $u_H \to u$: Elliptic PDEs

- L– constant coefficient second order elliptic operator.
- Weighted Sobolev norm (Section 2.1 of Agnelli et al.[4])

$$\|u - u_H\|_{W_\alpha} \leq C_* \left\|\delta - \tilde{\delta}_H\right\|_{(W_{-\alpha})^*} \leq \tilde{C}_* H^{\frac{1}{\beta}(\alpha+(\beta-1)(m+1))}. \qquad (8)$$

- Recall $H = \mathcal{O}(h^\beta)$.
- In the limit as $\beta \to 1$, convergence is independent of m.
- Helmholtz equation on a disk.



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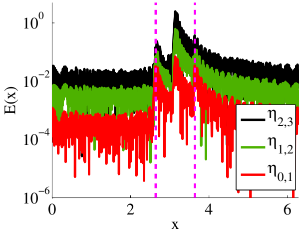
First order wave equation in 1D

$$\begin{cases} u_t + u_x = 0 & \text{in } \mathbb{T}(0,2\pi) \times (0,T), \\ u(x,0) = \delta_H(x) & \text{on } \mathbb{T}(0,2\pi) \times \{t=0\}. \end{cases} \quad (9)$$

- Analytic solution $u(x,y) = \delta_H(x-t)$.
- Pointwise error at $t = 36\pi$ (13 periods).

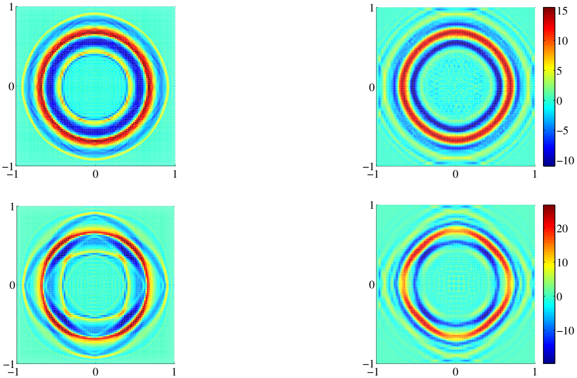


- Larger error with higher moments.
- Dissipative error.

Second order wave equation in 2D

$$\begin{cases} u_{tt} = u_{xx}, & \text{in } (-1.1)^2 \times (0, 0.7], \\ u(x,0) = \delta_H, & u_x(x,0) = u(x,t)|_{\partial[-1,1]} = 0. \end{cases} \tag{10}$$

- Similar to free space solution for short time.
- Should be radially symmetric.

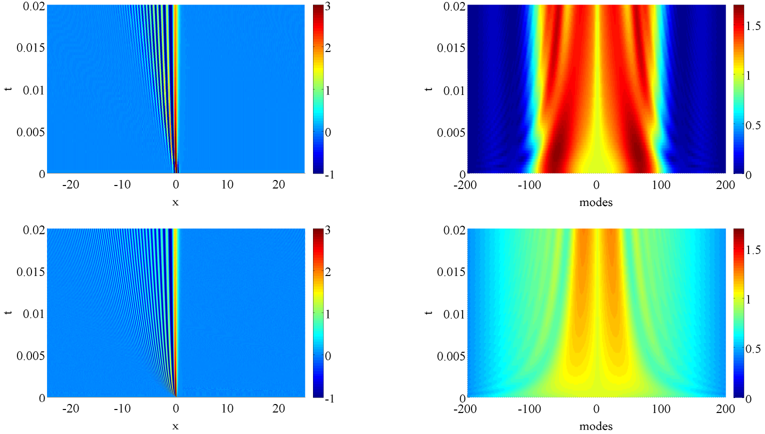


Korteweg-de Vries (KdV)

$$u_t + 6uu_x + u_{xxx} = 0 \quad \text{in} \quad \mathbb{T}([-8\pi, 8\pi]) \times (0, 0.05] \quad \text{and} \quad u(x,0) = \delta_H. \tag{11}$$

- Analytic solution

$$u(x,t) \approx \frac{1}{2}\text{sech}^2\left(\frac{1}{2}(x-t)\right) + \text{radiative waves.} \tag{12}$$



Closing remarks

Questions

Q1: How do we construct 'good' approximations $\mathcal{S}_H$ to $\mathcal{S}$?
A: Moment problem.

Q2: What form of convergence should be used to examine $\mathcal{S}_H \to \mathcal{S}$?
A: Weak-$*$ topology, Weighted Sobolev norm.

Q3: How does the choice of $\mathcal{S}_H$ affect the convergence $u_{H,h} \to u$?
A: Depends on the PDE operator and the choice of the norm.

Q4: What form of convergence should be used to examine $u_{H,h} \to u$?
A: Pointwise away from a compact set, Weighted Sobolev norm.

- A flexible framework to construct $\tilde{\delta}_H$.
- Compare existing regularizations in the literature (Tensor product vs Radial).

Thank you!

Bamdad Hosseini, Nilima Nigam, and John M. Stockie. "On regularizations of the Dirac delta distribution". In: (2014). URL: http://arxiv.org/abs/1412.4139

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