Chapter 1

Neoclassical growth theory

1.1 The Solow growth model

The general questions of growth:

- What are the determinants of long-run economic growth?
- How can we explain the vast differences in both output levels and growth rates across countries/time?

Solow's specific question: What do simple neoclassical assumptions imply about growth? His key assumptions include:

- Constant returns to scale.
- Perfect competition.
- Complete information.
- No externalities.

1.1.1 The basic model

Time and demography

Time is discrete. Our notation is going to use subscripts; X_t is the value of variable X at time t.

An aside: in Romer, most of the models are in continuous time, while I will generally use discrete time. Notation differs between continuous time and discrete time models, but *almost* any macro model can be written in either - the difference is usually a matter of taste and convenience. You will be responsible for learning to use both methods.

The economy has one consumer with infinite lifetime, and one firm.

Consumer

The consumer supplies labor L_t to the market, at market wage w_t. The consumer also owns all of the capital K_t and rents to the market at rental rate r_t. The consumer also owns the firm and receives its total profit π_t. Income is thus:

$$Y_t = r_t K_t + w_t L_t + \pi_t \tag{1.1}$$

Labor supply grows at exogenous rate g_L, i.e.:

$$L_{t+1} = (1 + g_L)L_t \tag{1.2}$$

Even though we have only one consumer, the fact that his labor supply is growing over time captures the idea of an expanding population.

Capital is accumulated by the consumer and depreciates at rate δ.

$$K_{t+1} = (1 - \delta)K_t + I_t \tag{1.3}$$

where I_t is gross investment. Notice that the price of capital is one. We can always set the price of one good in the economy to be one. Such a good is called the "numeraire" good.

If we were doing research right now, we would write down a general equilibrium model, with a utility function for the consumer which determines how labor supply responds to wages, and how investment responds to the return on capital. Soon we will do just that. But for now, we'll take the same shortcut that Solow took. We'll write down the consumer's decision rules directly.

1. The consumer saves some exogenous fraction s of income.

2. The consumer supplies all of his labor and capital, regardless of prices (inelastically).

so we can rewrite equation (1.3) as:

$$K_{t+1} = (1-\delta)K_t + sY_t \tag{1.4}$$

Firm

The firm can take capital and labor and convert it into output (consumption and new capital) which is then sold back to the consumer. The firm's technology is described by the production function $Y_t = F(K_t, A_tL_t)$.

A_t is the level of "technology" at time t. It grows at exogenous rate g_A:

$$A_{t+1} = (1+g_A)A_t \tag{1.5}$$

In this formulation, technological progress is "labor augmenting", meaning that it increases the effective amount of labor. We could also in principle have "capital augmenting" technical progress or "neutral" technological progress $Y_t = A_tF(K_t, L_t)$. It will turn out that a lot of the things we will do are a lot easier if we assume labor augmenting technical progress.

We will also assume that F is a "neoclassical" production function.

Definition 1.1.1 (Neoclassical production function) *The neoclassical production function $F(K, L)$ has the following properties:*

1. *F is homogeneous of degree 1. Formally, for any $c \geq 0$, $F(cK, cL) = cF(K, L)$. In other words, production exhibits constant returns to scale.*
2. *Both factors are necessary, i.e., $F(0, L) = F(K, 0) = 0$, for any K, L.*
3. *Both factors contribute to output:*

$$\frac{\partial F(K,L)}{\partial K} > 0 \tag{1.6}$$

$$\frac{\partial F(K,L)}{\partial L} > 0 \tag{1.7}$$

4. *The firm has decreasing returns in each product, or F is concave in both arguments:*

$$\frac{\partial^2 F(K,L)}{\partial K^2} < 0 \tag{1.8}$$

$$\frac{\partial^2 F(K,L)}{\partial L^2} < 0 \tag{1.9}$$

5. *The "Inada conditions" hold:*

$$\lim_{K\to 0} \frac{\partial F(K,L)}{\partial K} = \infty \tag{1.10}$$

$$\lim_{K\to \infty} \frac{\partial F(K,L)}{\partial K} = 0 \tag{1.11}$$

The firm's profits are

$$\pi_t = F(K_t, A_t L_t) - r_t K_t - w_t L_t \tag{1.12}$$

Because firm profits go to the consumer, the consumer's income is equal to the firm's total output:

$$Y_t = F(K_t, A_t L_t) \tag{1.13}$$

Markets

All goods are traded on a competitive market. Whenever we write down a macroeconomic model, we usually have to define an "equilibrium" in the model is.

An equilibrium for this model is a sequence of factor prices $\{w_t\}, \{r_t\}$ and allocations $\{K_t, L_t\}$ such that

1. The capital stock K_t, labor supply L_t, and technology level A_t are determined by equations (1.4), (1.2), and (1.5), and initial conditions K_0, L_0, and A_0, respectively.

2. Taking prices as given, the firm purchases capital K_t and labor L_t to maximize its profits (1.12).

3. Markets clear, that is, the capital and labor demand of the firm at prices w_t and r_t are equal to the supply.

In general an equilibrium has the same elements. The equilibrium itself is a vector or sequence of prices and allocations. We impose two types of conditions on them - first that the various agents in the model choose allocations to maximize utility, taking prices as given. Then we impose the condition on prices that markets clear. Usually in the applications we will be looking at, the market-clearing condtion pins everything down.

1.1.2 So what do we think?

This model would be unpublishable today, but managed to win Solow a Nobel Prize. The main methodological failing of the model from a current perspective is that the actions of the consumer are simply assumed. In a modern macro model you are expected to write down a utility function and set of budget constraints for the representative consumer, and derive the consumer's optimal actions. In a week or two we will do just this.

Notice that this model will be useless as a means of analyzing

- Business cycles (we need shocks to AD or AS)
- Monetary policy (we need money)
- Unemployment (we need either a labor/leisure tradeoff, or some reason why labor markets don't clear)
- Welfare effects of policy (we need a utility function and policy instruments)

However, it will prove very useful for analyzing the relative contribution of capital and technical progress to economic growth.

1.1.3 Wages and interest rates

Because the factor markets are competitive, factors are rented by firms at marginal revenue product, and firm profits are zero. The wage, therefore is

$$w_t = \frac{\partial F(K_t, A_t L_t)}{\partial L_t} \tag{1.14}$$

and the rental rate of capital is

$$r_t = \frac{\partial F(K_t, A_t L_t)}{\partial K_t} \tag{1.15}$$

Suppose that you invested in one unit of capital today. Tomorrow you would get r_{t+1} in rents on that capital, and $(1 - \delta)$ units of undepreciated capital. The net return on capital (also called the interest rate) is $r_{t+1} - \delta$.

Factor shares and output elasticity

Suppose we could estimate "Labor's share of output":

$$\alpha_L = \frac{\text{Total wages paid}}{\text{GDP}} \tag{1.16}$$

Generally, estimates of α_L run around 0.64. Capital's share, α_L is thus about 0.36.

Can we derive the implied value of labor's share from the model? Yes:

$$\begin{aligned} \alpha_L &= \frac{w_t L_t}{Y_t} \\ &= \frac{\partial Y_t}{\partial L_t}\frac{L_t}{Y_t} \end{aligned} \tag{1.17}$$

Notice that this is the output elasticity of labor.

By the same reasoning, capital's share is the output elasticity of capital. In other words, Solow's model and the data together imply that a one percent growth in the labor force leads to a 0.64 percent increase in output. A one percent increase in the capital stock increases output by 0.36 percent.

1.1.4 Growth accounting

How much of a country's growth can be explained by:

- Labor force growth
- Capital accumulation
- Technical progress

Solow's model allows us to decompose growth into these three components. First we make a first-order Taylor series approximation:

$$\begin{aligned} Y_{t+1} - Y_t &= \left.\frac{\partial Y}{\partial K}\right|_{K=K_t} * (K_{t+1} - K_t) \\ &+ \left.\frac{\partial Y}{\partial L}\right|_{L=L_t} * (L_{t+1} - L_t) \end{aligned} \tag{1.18}$$

$$\begin{aligned} &+ \left.\frac{\partial Y}{\partial A}\right|_{A=A_t} * (A_{t+1} - A_t) \\ &+ \text{approximation error} \end{aligned}$$

Divide by Y_t on both sides.

$$\frac{\Delta Y_t}{Y_t} = \frac{\partial Y}{\partial K}\frac{\Delta K_t}{Y_t} + \frac{\partial Y}{\partial L}\frac{\Delta L_t}{Y_t} + \frac{\partial Y}{\partial A}\frac{\Delta A_t}{Y_t} \tag{1.19}$$

The equation above decomposes GDP growth into portions that can be attributed to growth in the capital stock, the labor force, and the technology level. Now,

$$\begin{aligned} \frac{\partial Y}{\partial K}\frac{\Delta K_t}{Y_t} &= \frac{\partial Y}{\partial K}\frac{\Delta K_t}{K_t}\frac{K_t}{Y_t} \\ &= \alpha_K \frac{\Delta K_t}{K_t} \\ &= \alpha_K g_K \end{aligned} \tag{1.20}$$

We can do this for labor as well:

$$\begin{aligned} \frac{\partial Y}{\partial L}\frac{\Delta L_t}{Y_t} &= \frac{\partial Y}{\partial L}\frac{\Delta L_t}{L_t}\frac{L_t}{Y_t} \\ &= \alpha_L \frac{\Delta L_t}{L_t} \\ &= \alpha_L g_L \\ &= (1-\alpha_K) g_L \end{aligned} \tag{1.21}$$

and for productivity:

$$\begin{aligned} \frac{\partial Y}{\partial A}\frac{\Delta A_t}{Y_t} &= \frac{\partial Y}{\partial A}\frac{\Delta A_t}{A_t}\frac{A_t}{Y_t} \\ &= \frac{\partial Y}{\partial A}\frac{A_t}{Y_t}\frac{\Delta A_t}{A_t} \\ &= \alpha_A g_A \end{aligned} \tag{1.22}$$

Substituting back in:

$$g_y = \alpha_K g_K + (1-\alpha_K) g_L + \alpha_A g_A \tag{1.23}$$

or

$$\alpha_A g_a = g_y - \alpha_K g_K - (1-\alpha_K) g_L \tag{1.24}$$

So if we have observations on the growth rate of output, the labor force, and the capital stock, we can have an estimate on the growth rate of total factor productivity. Equation (1.24) defines the "Solow residual."

Sometimes people use the term Solow residual to refer to what I've called total factor productivity, so they call equation (1.24) the growth rate of the Solow residual.

So what is the Solow residual?

- Conservative version - the part of growth which is not explained by capital accumulation and labor force expansion. "A measure of our ignorance," as Solow himself put it.
- Ambitious version - the part of growth which is explained by technical progress.

So what are the critical assumptions here? Perfect competition and constant returns to scale. Without those factor shares and output elasticities are not identical. Note that we have yet to use the savings rate in any of this. As a result, the growth accounting procedure used above will still work in a more complex model which preserves these two features.

Results

Since we have estimates of capital's share of output, as well as the growth rates of output, capital, and labor, we can estimate the Solow residual for many industrialized countries. Here's a table that describes Solow residuals for some countries from 1947-1973:

Country	Capital's Share	GDP growth	from capital	from labor	TFP growth	% of growth from TFP
Canada	44	5.17	2.54	0.88	1.75	33.9
France	40	5.42	2.25	0.21	2.96	54.5
Germany	39	6.61	2.69	0.18	3.74	56.6
Italy	39	5.27	1.80	0.11	3.37	63.5
Japan	39	9.51	3.28	2.21	4.02	42.3
Netherlands	45	5.36	2.47	0.42	2.48	46.2
UK	38	3.73	1.76	0.03	1.93	51.9
US	40	4.02	1.71	0.95	1.35	33.6

These results, and Solow's results at the time, showed that growth in TFP accounted for a large fraction of total output growth. The conclusion drawn from these results is that technological advance is very important in explaining growth.

Let's take a look at some more recent figures. These are from a famous paper by Alwyn Young (1994 QJE) called "The tyranny of numbers: confronting the statistical realities of the East Asian growth experience".

Country	Capital's Share	GDP growth	from capital	from labor	TFP growth	% of growth from TFP
Hong Kong	37	7.3	3.09	2.00	2.20	30.1
Singapore	53	8.50	6.20	2.86	-0.40	–
South Korea	32	10.32	4.77	4.35	1.20	11.6
Taiwan	29	9.10	3.68	3.62	1.80	19.8

The results in this table ran counter to the very popular idea at the time that these countries experienced a "miracle" increase in productivity. Instead, the table points out that the high growth rates in these four countries can be traced primarily to rapid capital accumulation and labor force growth.

Extensions

This approach can be easily extended to more than two factors of production. For example, one can include energy usage, or differentiate between labor of workers with or without a college degree. As long as you have growth rates and factor shares, you can do this.

$$SRSD = g_y - \sum_{i=1}^{N} \alpha_i g_i \tag{1.25}$$

Another application is to estimate the returns to scale. Suppose you had time series of inputs and factor shares for a number of industries, and ran the regression

$$g_y = \beta_0 + \beta_1 \left(\sum_{i=1}^{N} \alpha_i g_i \right) + \epsilon \tag{1.26}$$

Under some restrictive assumptions, β_0 is the Solow residual, and β_1 is an estimate of the returns to scale. Early work in the late 1980's by Robert Hall found substantial increasing returns to scale ($\beta_1 > 1$) in the US economy, though subsequent research has called Hall's estimates into question.

1.1.5 Dynamics of the Solow model

Let's return to the Solow model. We have a system of difference equations:

$$K_{t+1} = (1-\delta)K_t + sF(K_t, A_t L_t) \tag{1.27}$$
$$A_{t+1} = (1+g_A)A_t \tag{1.28}$$
$$L_{t+1} = (1+g_L)L_t \tag{1.29}$$

and initial conditions K_0, A_0, L_0. In continuous time models, you get a system of differential equations. In order to analyze this system, we need to invest in some tools.

1.1.6 Dynamic tools I: Difference equations

In general, a difference equation is a function:

$$X_{t+1} = D(X_t) \tag{1.30}$$

X_t is called the state variable. It may be a vector, in which case the equation is of order k, where k is the number of rows. A solution is a sequence of X_t's which satisfy the equation. In order for the solution to be unique, k boundary conditions must also be imposed. These usually take the form of an initial value X_0, but in many economically interesting cases, take the form of $\lim_{t\to\infty} f(X_t) = 0$ for some function f. In our initial example

$$X_t = \left\{ \begin{array}{c} K_t \\ L_t \\ A_t \end{array} \right\}$$

and our boundary condition was X_0, the initial capital stock, population, and technology level. Later on, we will run into stochastic difference equations:

$$X_{t+1} = D(X_t) + \epsilon_{t+1} \tag{1.31}$$

where ϵ_t is some stochastic (random) process.

A high-order difference equation is easier to solve if you "uncouple" it into a set of independent first-order equations. In an ideal world, this can be done with simple substitution.

Suppose we have managed this and now we have a first-order system:

$$x_{t+1} = d(x_t) \tag{1.32}$$

where x_t is a scalar variable. In your homework (due tomorrow) you will use computer simulation to experiment with linear first-order difference equations.

To characterize the solution, we have four basic tools at our disposal:

1. Graphical analysis (phase diagrams)
2. Steady state analysis
3. Local analysis
4. Global analysis

Graphical analysis

Here is a phase diagram for an arbitrarily drawn difference equation. I'll also draw the line $x_{t+1} = x_t$, which is also called the 45 degree line. (Of course there's no diagram in these notes) One way to figure out how x_t evolves is to pick a point and go like so. (This part refers to a graph I draw on the board, so I can't describe it very well here).

Steady states

Each point where these two lines intersect is a fixed point of g, or a steady state of the system. If we start there, we stay there forever. Finding steady states is usually easy. Simply solve

$$x_\infty = d(x_\infty) \tag{1.33}$$

As you can see, there may be zero, one, or many steady states.

Local analysis

For a given steady state, we might ask if all of the starting points nearby will lead to the steady state eventually. If they will, the steady state is "locally stable". If not, the steady state is locally unstable. To find the local characteristics of a steady state, take a linear approximation.

$$(x_{t+1} - x_\infty) = d'(x_\infty)(x_t - x_\infty) + \epsilon \tag{1.34}$$

Fortunately, there are some easy ways to characterize the fixed points of a linear difference equation:

- If $-1 < d'(x_\infty) < 1$, then x_∞ is locally stable (convergent). If x_0 starts near x_∞, x_t will approach x_∞ as t approaches infinity. Otherwise it is locally unstable (divergent).
- $d'(x_\infty) < 0$, then x_t oscillates.
- $d'(x_\infty) > 0$, then x_t converges or diverges monotonically.

Global analysis

Global analysis, or nonlinear dynamics, asks the question, "for each possible starting point, what behavior does the system eventually settle down into?" For linear difference equations, global analysis and local analysis are the same thing. For nonlinear systems, some new behavior may appear:

- Periodic points: A point x of a difference equation $x_{t+1} = d(x_t)$ is a periodic point with period p if $x = d^p(x)$ and p is the smallest integer for which this is true.
- Chaotic points: No time for a definition. In essence, a point is chaotic if it never converges to a point or a periodic cycle. Chaotic sequences "look random" - they bounce around an uncountably infinite set of points, never going to the same point twice. If a difference equation has chaotic points, it has points of every finite periodicity.

Some macro models actually exhibit periodicity and chaos. Some economists have used that fact to make the following point. Most macroeconomists think business cycles are caused by the economy's response to shocks. It is theoretically possible that they are actually due to chaotic/periodic dynamics in the nonlinear difference equations describing the economy (sometimes called "endogenous business cycles")? This idea is very similar to a longstanding Marxist claim that cycles of boom and bust are an inherent feature of capitalism. Not surprisingly, this research agenda is more popular in Europe than in North America, and among those with more left-wing political views.

1.1.7 Dynamics of the Solow model, continued

So let's apply these things we've learned about dynamics to the Solow model. Our first step is to uncouple equation (1.27) into a set of simple first-order difference equations. To do this, define:

$$k_t \equiv \frac{K_t}{A_t L_t} \tag{1.35}$$

$$y_t \equiv \frac{Y_t}{A_t L_t} \tag{1.36}$$

If productivity is constant, then these new variables can be thought of as output per worker and capital per worker. With growing productivity, they are something a little harder to get your head around: capital per "effective worker".

Now divide (1.27) by $A_t L_t$ on both sides, and we get

$$\frac{K_{t+1}}{A_t L_t} = (1-\delta)\frac{K_t}{A_t L_t} + s\frac{F(K_t, A_t L_t)}{A_t L_t} \tag{1.37}$$

Now, remember that F is homogeneous of degree 1, so

$$\frac{F(K, AL)}{AL} = F(\frac{K}{AL}, 1) \tag{1.38}$$

This gives us:

$$\frac{K_{t+1}}{A_t L_t} = (1-\delta)\frac{K_t}{A_t L_t} + sF(\frac{K_t}{A_t L_t}, 1) \tag{1.39}$$

Multiply the left side by $\frac{(1+g_A)A_t*(1+g_L)L_t}{A_{t+1}L_{t+1}}$, we get:

$$\frac{(1+g_A)(1+g_L)K_{t+1}}{A_{t+1}L_{t+1}} = (1-\delta)\frac{K_t}{A_t L_t} + sF(\frac{K_t}{A_t L_t}, 1) \tag{1.40}$$

Substituting back in, we get

$$k_{t+1} = \frac{(1-\delta)k_t + sF(k_t, 1)}{(1+g_A)(1+g_L)} \tag{1.41}$$

Finally, we define a new function $f(k) = F(k, 1)$. Notice that $y_t = f(k_t)$, and that f inherits every feature of F except that it is not homogeneous of degree one. Rewriting (1.41) in terms of our new variables:

$$k_{t+1} = \frac{(1-\delta)k_t + sf(k_t)}{(1+g_A)(1+g_L)} \tag{1.42}$$

This is a first-order difference equation describing the evolution of "capital per effective worker" over time. We can write it as:

$$k_{t+1} = d(k_t) \tag{1.43}$$

where:

$$d(k) = \frac{sf(k) + (1-\delta)k}{(1+g_A)(1+g_L)} \tag{1.44}$$

What do we know about d? In this case:

- It passes through the origin: $d(0) = 0$.
- It's strictly increasing: $d'(k) > 0$.
- It's strictly concave: $d''(k) < 0$.
- A variant of the Inada conditions hold:

$$\begin{aligned} \lim_{k\to 0} d'(k) &= \infty \qquad (1.45) \\ \lim_{k\to\infty} d'(k) &= \frac{1-\delta}{(1+g_A)(1+g_L)} \in [0, 1] \end{aligned}$$

First we do the graphical analysis. We get an equation that looks like this. As you can see, we seem to have two fixed points. We know that zero is a fixed point. What about the other? Let k_∞ be the nonzero solution to:

$$k = \frac{sf(k) + (1-\delta)k}{(1+g_A)(1+g_L)} \tag{1.46}$$

Suppose momentarily that $g_A = g_L = 0$, i.e., no population or technology growth. Then k_∞ solves $sf(k_\infty) = \delta k_\infty$, i.e., total savings equals total depreciation.

Local analysis: We know that $d'(0) = \infty$, so it is an unstable steady state. We also know that $d'(k*) < 1$, so it is locally stable.

Global analysis: Since d is a strictly increasing function, it has no periodic points. Therefore it also has no chaotic points. It can be also proven that for all initial conditions $k_0 > 0$:

$$\lim_{t\to\infty} d^t(k_0) = k_\infty \tag{1.47}$$

We call such a k_∞ "globally asymptotically stable"

To sum up, from any initial positive level of the capital stock, k_t will converge monotonically to a steady state k_∞. What does this imply about the aggregate variables we are interested in? If we are at the steady state (k_t is constant), then:

- Since $\frac{K_t}{A_t L_t}$ is constant, K_t and $A_t L_t$ are growing at the same rate.
- Since the growth rate of A_t is g_A and the growth rate of L_t is g_L, the growth rate of K_t is $g_A + g_L$.
- Since F is homogeneous of degree 1, Y_t is also growing at rate $g_A + g_L$.
- Per capita output $\frac{Y_t}{L_t}$ grows at rate g_A.

The steady state for k_t corresponds to a "balanced growth path" for the original variables. A balanced growth path is a sequence in which all of the variables grow at a constant rate.

Notice that savings rates do not enter into long run growth rates. The only thing that matters for long-run growth in per capita output is the growth rate of technology. Increased savings only affects the *level* of per capita output. Why? Because of decreasing returns to capital. Eventually the productivity of one more unit of capital falls below the amount needed to cover its own depreciation.

1.1.8 Convergence

How does the initial level of capital affect growth rates?

- Convergence - poor countries grow faster than rich countries.
- Divergence - rich countries grow faster than poor countries.

Intuitively, the Solow model has decreasing returns to capital, so we would expect countries with very little capital to grow faster. Can we prove this?

Our difference equation for capital is:

$$k_{t+1} = \frac{sf(k_t) + (1-\delta)k_t}{(1+g_A)(1+g_L)} \tag{1.48}$$

We want an equation relating the growth rate to the current level of capital.

$$k_{t+1} - k_t = \frac{sf(k_t) + (1-\delta-(1+g_L)(1+g_A))k_t}{(1+g_A)(1+g_L)} \tag{1.49}$$

Dividing by k_t:

$$\text{growth rate} = \text{positive constant}\frac{f(k_t)}{k_t} + \text{constant} \tag{1.50}$$

So is $\frac{f(k)}{k}$ increasing or decreasing as k goes up? It's decreasing whenever f is concave, so growth is decreasing in capital.[1]

There are different ways of interpreting the convergence implications of the Solow model, depending on what you think the underlying parameters look like:

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[1]You don't have to be able to prove this, but here's how you would:

$$\frac{\partial}{\partial k}\frac{f(k)}{k} = \frac{kf'(k) - f(k)}{k^2} \tag{1.51}$$

If $kf'(k) > f(k)$, growth is increasing in k, otherwise it's decreasing.

Graphical method: $f'(k)$ is the slope of this line, $f(k)/k$ is the slope of this line. Obviously $f'(k) < f(k)/k$, which implies that $kf'(k) < f(k)$, which implies that growth is decreasing in capital.

Another way:

$$kf'(k) - f(k) = kf'(k) - \left(f(0) + \int_0^k f'(x)dx\right) \tag{1.52}$$

- Assume that all countries have the same savings rates and access to the same technology. Then the model implies that all countries are moving towards the same long-run income level. Sometimes called σ-convergence

- Assume that all countries have the same savings rates and rates of technological advance (though maybe different initial levels of technology) This implies that poor countries grow faster than rich countries. Sometimes called absolute convergence.

- Countries may have different savings rates, and as a result be moving towards different long-run income levels. However, if you control for the various determinants of long-run income (in particular savings rates), initial income will have a negative correlation with growth rates. Sometimes called β convergence.

There's a huge empirical literature on convergence. Stylized facts:

- Absolute convergence among U.S. states, Canadian provinces, Japanese perfectures, European reigions, etc. (Barro)

- Absolute convergence among "open" economies (Sachs and Warner 1995).

- No evidence for absolute convergence among the countries of the world. (Romer 1986)

- Strong evidence for β convergence. (Mankiw, Romer and Weil 1992, Barro (many papers), many others)

- Current distribution of income implies strong historical tendency for σ-divergence. In other words, if two countries are converging, the ratio

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$$= f'(k)\int_0^k dx - \left(\int_0^k f'(x)dx\right) \tag{1.53}$$

$$= \int_0^k (f'(k) - f'(x))dx \tag{1.54}$$

Since $f''(k) < 0$, we know that $f'(k) - f'(x)$ is negative if $x < k$. Growth rates are decreasing in capital.

of rich country income to poor country income must by shrinking. Current ratio of, say, U.S. income to India income, plus slow convergence, implies implausibly low Indian income 100 years ago. (Pritchett 1998)

Consensus developing: suitably similar economies, with economic ties between them, have a strong tendency to converge. For example, one would expect to find income gaps between EC countries to narrow. Countries with vast differences in institutions, infrastructure, etc., may not converge. Longstanding idea of "convergence clubs".

1.1.9 Assessing the Solow model

- Big successes:
  - A simple, transparent neoclassical model with clear implications. Has been critical to empirical work.
  - Use of Solow residual to estimate relative impact of technology and capital accumulation on income growth.
  - Raises convergence issue and allows for clearer discussion.
- Failures:
  - Convergence picture more complex than implied by the model. Model is only able to explain non-convergence, and vast differences in income accross countries by "technology". But technology moves easily across borders - someone in Chad owns a laptop computer, a technology which was unavailable anywhere 20 years ago.
  - Technology is completely exogenous, yet it is the key determinant of growth. Surely the technology level is the outcome of the purposeful actions of firms and individuals. If we wish to understand growth, we must model the development of new technology.
  - Technology is really just a residual. All we've really found is that, under neoclassical assumptions, accumulation of physical capital cannot explain long-run growth. It could be that human capital accumulation, learning by doing, public investment in infrastructure, etc. is driving growth. It could also be that the assumptions

are wrong - maybe we have increasing returns and imperfect competition.

Appendix A

Continuous-time version of the Solow model

Romer writes down the Solow model in continuous time. I'll show you how to translate.

Translating a discrete-time system into a continuous time system is quite easy. Two steps:

1. For every x_{t+1} you see, use algebra to convert it into an $x_{t+1} - x_t$
2. Replace $x_{t+1} - x_t$ with $\frac{\partial x_t}{\partial t}$ (usually denoted $\dot{x}_t$)
3. Replace $(1+g)^t$ (where g is a growth rate) with e^{gt}

That's it. So for the Solow model we have

$$K_{t+1} = (1-\delta)K_t + sF(K_t, L_t) \tag{A.1}$$
$$A_{t+1} = (1+g_A)A_t \tag{A.2}$$
$$L_{t+1} = (1+g_L)L_t \tag{A.3}$$

which becomes:

$$\dot{K}_t = sF(K_t, L_t) - \delta K_t \tag{A.4}$$
$$A_t = A_0 e^{g_A t} \tag{A.5}$$
$$L_t = L_0 e^{g_L t} \tag{A.6}$$