Lecture 12 - Centre of mass and collisions

What's important:
- centre of mass motion
- conservation of momentum
- collisions in one dimension

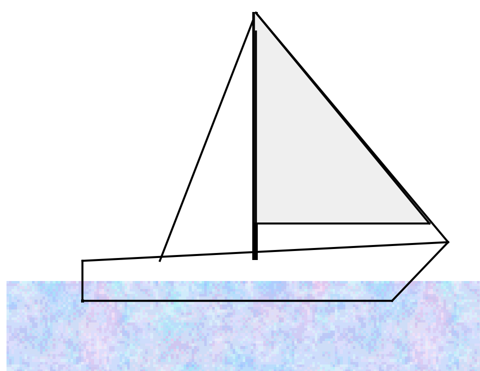
Demonstrations:
- collisions on an air track

Text: Walker, Secs. 9.4, 9.5, 9.6, 9.7
Problems:

Centre of mass

Most of the physical objects that we deal with have some spatial extension - they are not point-like objects. But when we discuss the motion of a ball, for example, we talk about the "position of the ball" as if it were a point. It seems natural enough to do so, but why does it work?

Consider the motion of a sailboat, complete with sails and flags. When we talk about



the position of the boat, we tend to ignore the sails and concentrate on the hull. Unquestionably, the sail provides the force to propel the boat, but the motion of the boat is dominated by the hull. In other words, we emphasize the (dynamic) importance of the different elements of an object according to their mass.

We can define a **weighted average** of a collection of objects (or of the components of an object) by the vector $\mathbf{R}_{cm}$

$$\mathbf{R}_{cm} = M_{tot}^{-1} \sum_{i=1}^{N} m_i \mathbf{r}_i$$

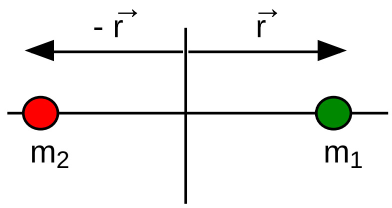
where M_{tot} is the total mass

$$M_{tot} = \sum_{i=1}^{N} m_i$$

As you can verify by inspection, the units of the masses on the right-hand side of this equation cancel, leaving $\mathbf{R}_{cm}$ with units of length.

Example

Consider two unequal mass objects, m_1 and m_2, at positions +**r** and -**r**:



$$\vec{R}_{cm} = \frac{1}{m_1 + m_2}(m_1\vec{r} - m_2\vec{r}) = \frac{m_1 - m_2}{m_1 + m_2}\vec{r}$$

Two special cases:

- Suppose $m_1 = m_2$, then $\mathbf{R}_{cm} = 0$ (i.e., the cm sits at the coordinate origin).
- Suppose $m_1 >> m_2$, then $m_1 + m_2 \sim m_1$ and $\mathbf{R}_{cm} \sim (m_1 / m_1)\, \mathbf{r} = \mathbf{r}$, as expected.

Motion of the cm

Now, $\Delta\mathbf{R}_{cm}/\Delta t$ is the velocity of the centre-of-mass, $\mathbf{v}_{cm}$; that is, $\mathbf{v}_{cm}$ is the slope of the $\mathbf{R}_{cm}$ *vs*. time graph. What is $\mathbf{v}_{cm}$ in terms of the individual velocities? It takes some mathematics (which you may want to skip)

$$\Delta\mathbf{R}_{cm}/\Delta t = \Delta(M_{tot}^{-1}\,\Sigma_{i=1}^{N}\, m_i\mathbf{r}_i)/\Delta t = M_{tot}^{-1}\,\Delta(\Sigma_{i=1}^{N}\, m_i\mathbf{r}_i)/\Delta t = M_{tot}^{-1}\,\Sigma_{i=1}^{N}\, m_i\,(\Delta\mathbf{r}_i/\Delta t)$$

to show that

$$\mathbf{v}_{cm} = M_{tot}^{-1}\,\Sigma_{i=1}^{N}\, m_i\mathbf{v}_i$$

In other words, the cm velocity is the weighted sum of the individual velocities. The same applies for the cm acceleration:

$$\mathbf{a}_{cm} = \Delta\mathbf{v}_{cm}/\Delta t = M_{tot}^{-1}\,\Sigma_{i=1}^{N}\, m_i\mathbf{a}_i$$

What determines the motion of the centre of mass? Suppose that individual particles in the system are subject to N forces (one for each particle):

$$\mathbf{F}_{net} = \Sigma_{i=1}^{N}\,\mathbf{F}_i.$$

Then by Newton's Second Law,

$$\mathbf{F}_{net} = \Sigma_{i=1}^{N}\,\mathbf{F}_i = \Sigma_{i=1}^{N}\, m_i\mathbf{a}_i = M_{tot}\,\{M_{tot}^{-1}\,\Sigma_{i=1}^{N}\, m_i\mathbf{a}_i\} = M_{tot}\,\mathbf{a}_{cm}$$

This expression says that $\mathbf{a}_{cm}$ obeys a dynamical equation of the same form as Newton's second law. We can also substitute $\mathbf{a}_{cm} = \Delta\mathbf{v}_{cm}/\Delta t$ to obtain

$$\mathbf{F}_{net} = M_{tot}\, \Delta\mathbf{v}_{cm}/\Delta t = \Delta\{M_{tot}\mathbf{v}_{cm}\}/\Delta t = \Delta\mathbf{P}_{tot}/\Delta t.$$

Summary

$$\mathbf{R}_{cm} = M_{tot}^{-1} \textstyle\sum_{i=1}^{N} m_i\mathbf{r}_i$$

$$\mathbf{v}_{cm} = \Delta\mathbf{R}_{cm} / \Delta t \qquad \mathbf{v}_{cm} = M_{tot}^{-1} \textstyle\sum_{i=1}^{N} m_i\mathbf{v}_i \qquad \text{as } \Delta t \text{ --> } 0$$

$$\mathbf{a}_{cm} = \Delta\mathbf{v}_{cm} / \Delta t \qquad \mathbf{a}_{cm} = M_{tot}^{-1} \textstyle\sum_{i=1}^{N} m_i\mathbf{a}_i \qquad \text{as } \Delta t \text{ --> } 0$$

and

$$\mathbf{F}_{net} = M_{tot}\,\mathbf{a}_{cm} = \Delta\mathbf{P}_{tot} / \Delta t.$$

Thus, $\mathbf{R}_{cm}$, $\mathbf{v}_{cm}$ and $\mathbf{a}_{cm}$ behave just like any kinematic set $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$, except that the dynamics is governed by $\mathbf{F}_{net}$. This is why we don't need to worry about the dynamics of atoms when we describe the motion of a car. In our study of dynamics, gravity acts through the centre of mass of an object.

Conservation of Momentum

In the previous lecture, we dealt with a fundamental conservation law of Nature: conservation of energy. There is another equally important conservation law - conservation of momentum. This law says that if the net external force on a system of N particles vanishes, then the total momentum $\mathbf{P}_{tot}$

$$\mathbf{P}_{tot} = \textstyle\sum_{i=1}^{N} \mathbf{p}_i$$

does not change with time

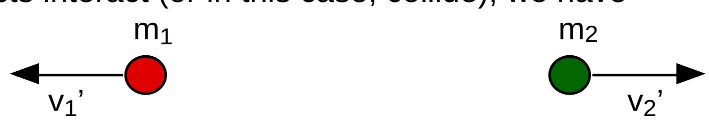
$$\Delta\mathbf{P}_{tot} / \Delta t = 0.$$

Note that conservation of momentum is a vector equation and applies component by component to the momentum vector $\mathbf{p}$. We now wish to apply the conservation of energy and momentum to the interaction of objects.

Collisions in One Dimension

Consider two objects whose initial velocities and masses are known:

After the objects interact (or in this case, collide), we have



Can we determine v_1' and v_2'? We know that momentum is conserved, so

$$m_1v_1 + m_2\underline{\underline{v_2}} = m_1v_1' + m_2v_2'$$

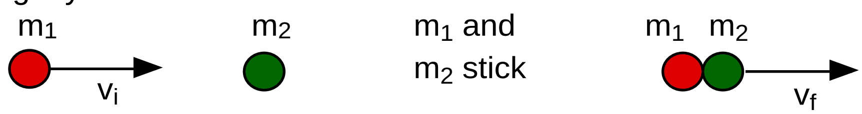
take sign into account

If the collision involves no dissipative forces, then we also have conservation of kinetic energy

$$\frac{1}{2}\ m_1 v_1^2 + \frac{1}{2}\ m_2 v_2^2 = \frac{1}{2}\ m_1 (v_1')^2 + \frac{1}{2}\ m_2 (v_2')^2$$

So, in this situation we have two equations and two unknowns, and we can solve for both v_1' and v_2' .

Consider a slightly different situation:



We can solve this by conservation of momentum alone:

$$m_1 v_i + 0 = (m_1 + m_2) v_f \qquad \text{or} \qquad v_f = [m_1/(m_1 + m_2)]\ v_i.$$

But conservation of kinetic energy also gives an equation relating v_i and v_f. Is this equation consistent with the results from conservation of momentum? To answer this question, we evaluate the kinetic energy before and after the collision as determined by the conservation of momentum equation:

$$K_i = \frac{1}{2}\ m_1 v_i^2 \qquad\qquad K_f = \frac{1}{2}\ (m_1 + m_2)\ v_f^2$$

$$= \frac{1}{2}\ (m_1 + m_2) \left(\frac{m_1}{m_1 + m_2} \right)^2 v_i^2$$

$$= \frac{1}{2} \left(\frac{m_1}{m_1 + m_2} \right) m_1 v_i^2$$

$$= \frac{m_1}{m_1 + m_2}\ K_i$$

$K_f < K_i$ and kinetic energy is not conserved. The difference in kinetic energy between the initial and final states has gone into heat or sound or whatever.

Rules:

- first apply conservation of momentum (**vector**, results in 1 to 3 equations).
- then evaluate the kinetic energies (1 equation)

We say the collision is **elastic** if kinetic energy is conserved. If kinetic energy is not conserved, the collision is **inelastic** and $K_f < K_i$. Thus, kinetic energy may not provide a constraint on the values of the momenta after the collision. Of course, even if kinetic energy is not conserved, the **total energy**, including heat *etc.*, must be conserved.