Lecture 13 - Rotational kinematics

What's important:

- definitions of angular variables
- angular kinematics

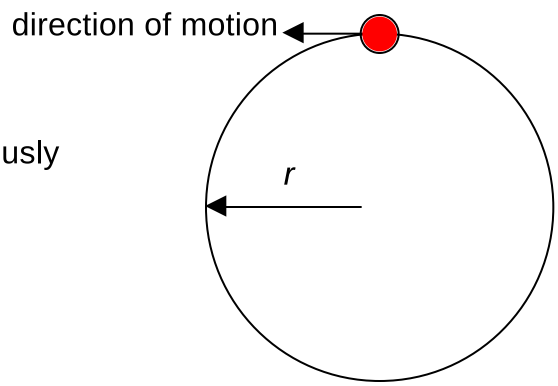
Demonstrations:

- ball on a string, wheel

Text: Walker, Secs. 10.1, 10.2, 10.3

Problems:

We now consider a type of motion in which the Cartesian description is fairly cumbersome. For example, if an object at the end of a string moves in a circular path at constant speed,



Then is has

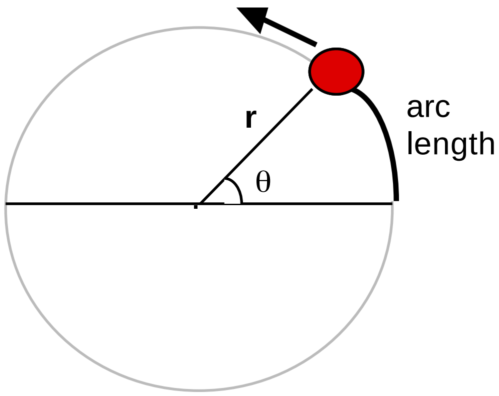
s = speed = constant
r = radius = constant
x, y and v_x, v_y vary continuously

Since s and r are constant, independent of the object's position, they do not specify the position or the velocity the object. While x, y and v_x and v_y do describe the object's motion, they are time-dependent and cumbersome.

We seek a description that makes use of the constancy of s and r, yet contains the time dependence in a functionally simple form. We start with uniform circular motion, and work towards general expressions for motion using angular variables.

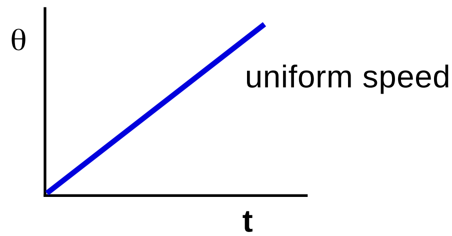
Uniform Circular Motion

In two dimensions, the position of an object can be described using the polar coordinates r and θ. If the object is moving in uniform circular motion, then r is constant, and the time dependence of the motion is contained in θ. As a kinematic variable, θ is the angular analogue of position.



$$\theta = \frac{\text{arc length}}{\text{radius}} \qquad \text{(in radians)}$$

The sign convention is that θ increases in a **counter-clockwise** direction. As a function of time, θ looks like (for uniform circular motion):

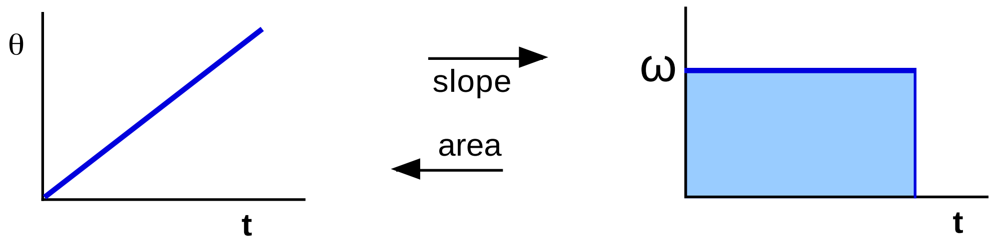


Now, if θ is the angular analogue of position, then the slope of the θ vs. t graph is the angular analogue of velocity. We define

angular speed
or
angular frequency of rotation

$$\omega \equiv \Delta\theta / \Delta t \quad \text{as } \Delta t \to 0$$

(in other words, ω is the rate of change of θ with time)



We can go back and forth from θ to ω by slopes and areas, just like with x and v. For example, from the area under the ω vs. t curve, we find

$$\theta = \theta_o + \omega t \qquad \text{(at constant } \omega)$$

Two other quantities used to describe uniform circular motion are the period T and the frequency f. During the period T, the object sweeps through one complete revolution, or 2π radians. Hence:

$\omega \equiv \Delta\theta / \Delta t = 2\pi / T$ using $\Delta\theta = 2\pi$ radians during the period T

The frequency f is T^{-1}, so we also have

$$\omega = 2\pi f$$

Note: $\omega = 1$ 1 radian / second

$f = 1$ 1 revolution / second

Angular and linear links

So far, we have the following equations for linear and angular kinematics

$\mathbf{x} = \mathbf{x}_o + \mathbf{v}t$ $\theta = \theta_o + \omega t$

$\mathbf{v} = \Delta\mathbf{x} / \Delta t$ $\omega = \Delta\theta / \Delta t$ as $\Delta t \to 0$

We have a relation between x and θ already. To obtain a relation between $\mathbf{v}$ and ω, we return to the definition of speed:

$$\text{speed} = |\mathbf{v}| = \frac{\Delta\ \text{distance}}{\Delta\ \text{time}} = \frac{2\pi\mathbf{r}}{T} \quad \leftarrow \text{circumference}$$

$$= \left(\frac{2\pi}{T}\right)\mathbf{r}$$

$$= \omega\mathbf{r}$$

$$\boxed{\mathbf{v} = \omega\mathbf{r}}$$ links linear and angular speeds

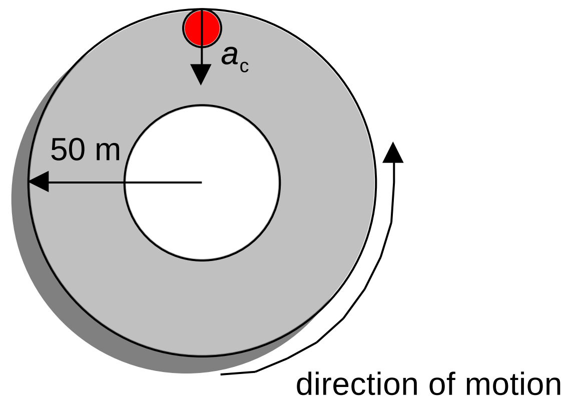
Another linking relation is through the centripetal acceleration a_c. From previous work, there is a centripetal acceleration even at constant speed:

$$\mathbf{a}_c = \frac{\mathbf{v}^2}{\mathbf{r}} = \frac{(\omega\mathbf{r})^2}{\mathbf{r}}$$

$$\boxed{\mathbf{a}_c = \omega^2\mathbf{r}} \qquad (\text{or } \mathbf{a}_c = \omega\mathbf{v})$$

Example

We have a doughnut-shaped space station on which we wish to produce artificial gravity through rotation. What must ω be so that $a_c = g$ at 50 m from the centre of the station?



Using $a_c = \omega^2 r$, we have

$$\omega^2 = 9.8 / 50$$

or

$$\omega = (9.8 / 50)^{1/2} = 0.44_3 \text{ rad/s}$$

(note that the units are OK)

Other quantities which we can calculate in this example are:

$$T = 2\pi/\omega = 2 \bullet 3.142 / 0.443 = 14.2 \text{ s.}$$

which is moderately fast and would not encourage the occupants of the space station to look out the windows. The corresponding speed at the perimeter of the station is:

$v = \omega r$ ---> $v = 0.44_3 \bullet 50 = 22.2$ m/s = 80 km/hr.