Lecture 15 - Angular momentum and torque

What's important:
- angular momentum and torque
- summary of angular kinematics and dynamics
- links between angular and linear equations

Demonstrations: none

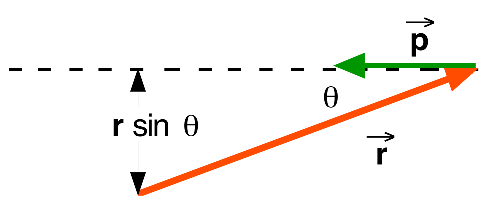
Vectors for ω and α

In the previous lectures, we introduced angular velocity and acceleration as scalar quantities. In fact, they are vectors, as are their dynamic counterparts torque and angular momentum. Here, we will consider a selection of problems where the vector nature of angular quantities doesn't matter. However, it should be kept in mind, particularly by students heading into biophysics, that the situations we deal with are not general, and a somewhat more complicated set of equations are needed to describe angular motion in general.

Angular Momentum

There are angular analogues of momentum and force. The *angular* momentum is

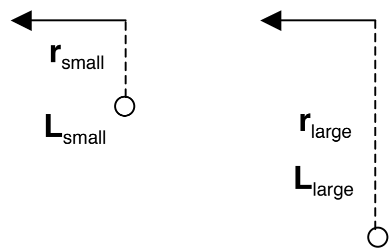
$$L = r\,p\,\sin\theta = (r\sin\theta)\;p = r_\perp\,p$$



Notes:

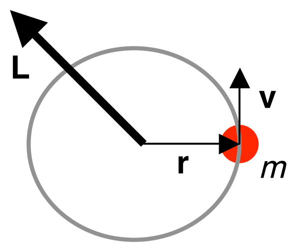
(i) If **r** is parallel to **p**, the angular momentum vanishes since $\theta = 0$.

(ii) L is defined with respect to some point in space.



Moment of inertia

The angular momentum L is related to the angular velocity ω through the moment of inertia. Consider the motion of a single mass m around a point:



$$\begin{aligned} L &= rp \\ &= r\,mv \\ &= r\,m\,\omega r \\ &= (mr^2)\,\omega. \end{aligned}$$

$I \equiv mr^2$ is the moment of inertia

$L = I\,\omega$ just like $\mathbf{p} = m\mathbf{v}$.

Torque

In linear kinematics, force is the rate of change of momentum:

$$\mathbf{F} = m\mathbf{a} = d\mathbf{p} / dt.$$

In angular motion, **torque** is the rate of change of **angular momentum**:

$$\tau = dL / dt.$$

The link between τ and $\mathbf{F}$ can be obtained from:

$$\tau = d(r_\perp\, p) / dt$$

Now,

$$\begin{aligned} \Delta(ab) &= (a + \Delta a)\cdot(b + \Delta b) - ab = ab + a(\Delta b) + (\Delta a)b + \Delta a\cdot\Delta b - ab \\ &= a(\Delta b) + (\Delta a)b \qquad \text{(dropping the very small term } \Delta a\cdot\Delta b\text{)} \end{aligned}$$

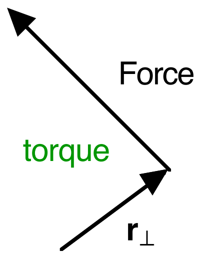
Thus

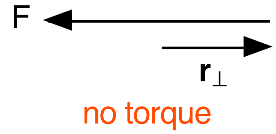
$$d(r_\perp\, p) / dt = p \cdot (dr_\perp / dt) + r_\perp \cdot (dp / dt)$$

But $r_\perp$ is constant here, so its time derivative vanishes and we are left with

$$\tau = r_\perp \cdot (dp / dt) = r_\perp\, F$$

The maximum torque occurs when **r** is perpendicular to **F**. If **r** is parallel to **F**, then there is no torque (the force still acts, but does not change the angular momentum):





Examples

- Torque from a car engine: 100 - 300 N•m
- Torque from a flagellum: 2-6 x 10^{-18} N•m

Summary

Linear	Links	Angular
$\mathbf{v} = d\mathbf{x} / dt$	$v = \omega r$	$\omega = d\theta / dt$
$\mathbf{a} = d\mathbf{v} / dt$	$a_{tan} = \alpha r$	$\alpha = d\omega / dt$
$\mathbf{p} = m\mathbf{v}$	$L = r_\perp p$	$L = I \omega$
$\mathbf{F} = d\mathbf{p} / dt$	$\tau = r_\perp F$	$\tau = dL / dt$

All changes occur in the $\Delta t \to 0$ limit. I is the moment of inertia.