Lecture 16 - Moments of inertia

What's important:
- conservation of angular momentum
- gyroscope
- moments of inertia

Demonstrations: bike wheel, rotating stool, weights

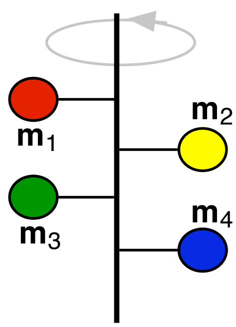
Conservation of angular momentum

We saw previously that linear momentum **p** is conserved unless there is a net external force: the applied force is the rate of change of linear momentum. So too with angular momentum: **Angular momentum L is conserved unless there is a net external torque**.

Consider how conservation of (scalar) angular momentum applies to a situation in which the moment of inertia changes. The moment of inertia I for a single mass m executing a circle of radius r about an axis is

$$I = mr^2.$$

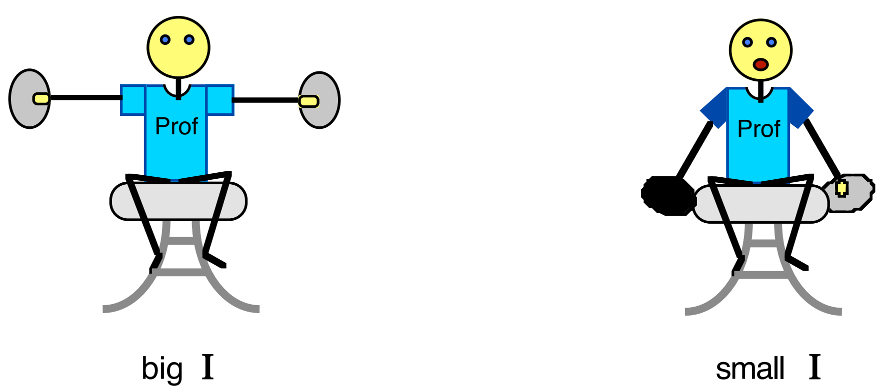
For a group of masses, all rotating with the same ω:



$$\mathbf{I} = \mathbf{m_1 r_1^2} + \mathbf{m_2 r_2^2} + \mathbf{m_3 r_3^2} + \mathbf{m_4 r_4^2}$$

$$= \sum_i \mathbf{m_i r_i^2}$$

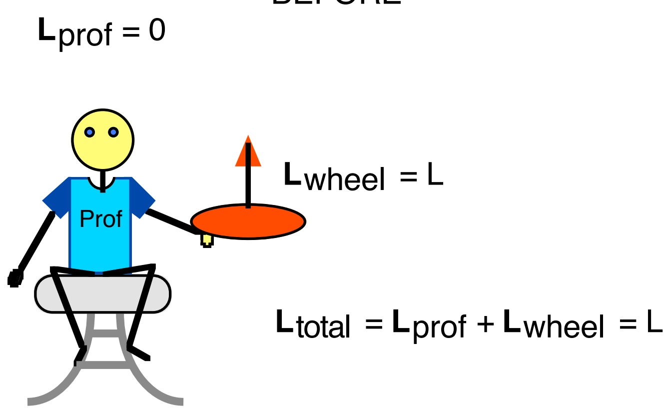
Demo: A slowly rotating prof has a big moment of inertia by holding weights out at the end of his arms. Dropping his arms into the vertical position reduces his moment of inertia. Because angular momentum is conserved, then the prof's angular velocity must increase as his moment of inertia decreases:



L is conserved. $\therefore$ **L** = $I_{big}\ \omega_{small}$ = $I_{small}\ \omega_{big}$

Demo: Prof sits on a stool which is free to rotate, and holds a rotating object in his hands: in the diagram, a bike wheel.

BEFORE



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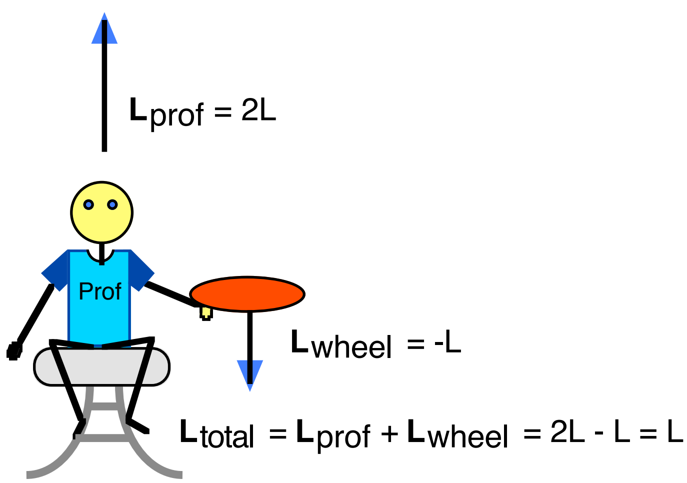


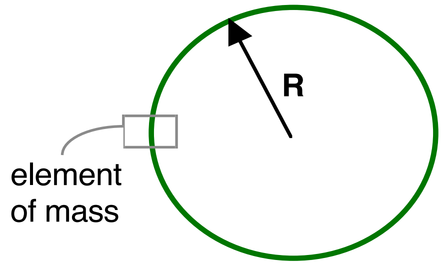
Although we have not discussed the vector nature of **L**, the rotating object has an angular momentum, pointing in a particular direction. Prof changes the direction of the angular momentum of the object by changing its orientation. But the total angular momentum of the system is conserved, with the result that the angular momentum of the prof must change in an equal and opposite way to the change of the rotating object. There are many other examples of conservation of angular momentum that don't refer to its vector nature, such as motion of comets.

Moments of Inertia in Detail

Moment of inertia for point-like masses: $I = \Sigma_i \, m_i \, r_i^2$.

If the mass distribution is continuous: $I = \Sigma_i \, r_i^2 \, \Delta m$

i) Ring of radius **R**,
mass **M**.

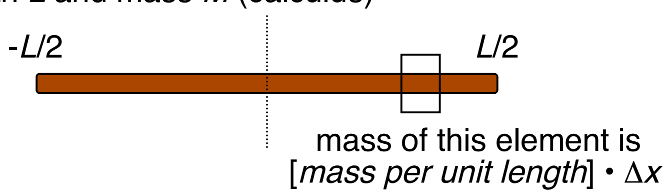


Break the mass of the ring up into N small segments each of mass $\Delta m = M / N$. Then I is just a sum over all elements Δm, each a distance R from the center:

$$I = \Sigma_i \, r_i^2 \, \Delta m = R^2 \, \Sigma_i \, \Delta m = MR^2.$$

If the axis of rotation were in the plane of the ring, the moment would drop by a factor of 2.

ii) Thin rod of length L and mass M (calculus)

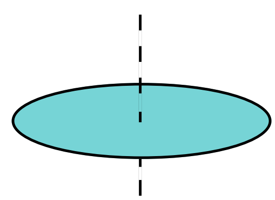


Axis through the centre: $I = M\,L^2/12$; axis through the end: $I = M\,L^2/3$.
Demo: pencil oscillating about an axis through its centre and through an end.

iii) Disk (more calculus!)

$$\mathbf{I} = \frac{1}{2} \, \mathbf{M} \, \mathbf{R}^2$$

(axis perpendicular to plane)

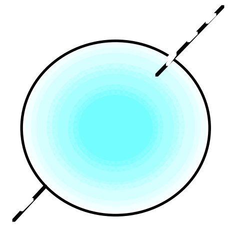


If the axis were in the plane of the disk, then the moment would drop by a factor of 2.

iv) Solid Sphere (still calculus!)

$$\mathbf{I} = \frac{2}{5}\,\mathbf{M}\,\mathbf{R}^2$$

(axis through centre)



v) Ellipsoid

This is a 3-dimensional shape is that swept out by a 2-dimensional ellipse when it is rotated about one of its symmetry axes (could be prolate or oblate). The axis of rotation of the ellipsoid itself could be around any axis. The shape is characterized by two lengths:

length $2a$ and width $2b$ (major and minor axes)

For rotation along the major axis:

$I = (2/5)\ Mb^2$

which looks just like the result for a sphere.

Example:

Let's approximate a bacterium as being an ellipsoid of length $2a = 6\ \mu$m and width $2b = 2\ \mu$m (in most cases, a better approximation for a cylindrical bacterium is a straight cylinder capped at each end by hemispheres). For a torque of 10^{-18} N•m (typical, if not a little low, for a flagellum) what angular acceleration would be experienced by the bacterium?

We need both the mass and moment of inertia of the bacterium. For the mass, use the density of water 10^3 kg/m^3 as being the average density for the cell:

$M = [density] \cdot [volume]$

$= [density] \cdot (4\pi/3)\ ab^2$ (ellipsoid)

$= 10^3\ (4\pi/3)\ (3\text{x}10^{-6})\ (1\text{x}10^{-6})^2$

$= 4\pi \text{ x } 10^{-15}$ kg

$= 1.26 \text{ x } 10^{-14}$ kg.

Knowing M, we can calculate the moment of inertia:

$I = (2/5)\ Mb^2$

$= (2/5)\ 1.26\text{x}10^{-14} \cdot (1\text{x}10^{-6})^2$

$= 5.03 \text{ x } 10^{-27}$ kg•m^2.

Then a torque of 10^{-18} N•m^2 would generate an angular acceleration of

$\alpha = \tau / I = 10^{-18} / 5\text{x}10^{-27} = 2\text{x}10^8$ rad/s^2.

This is immense! Why doesn't the bacterium reach the speed of light in an hour? Drag.